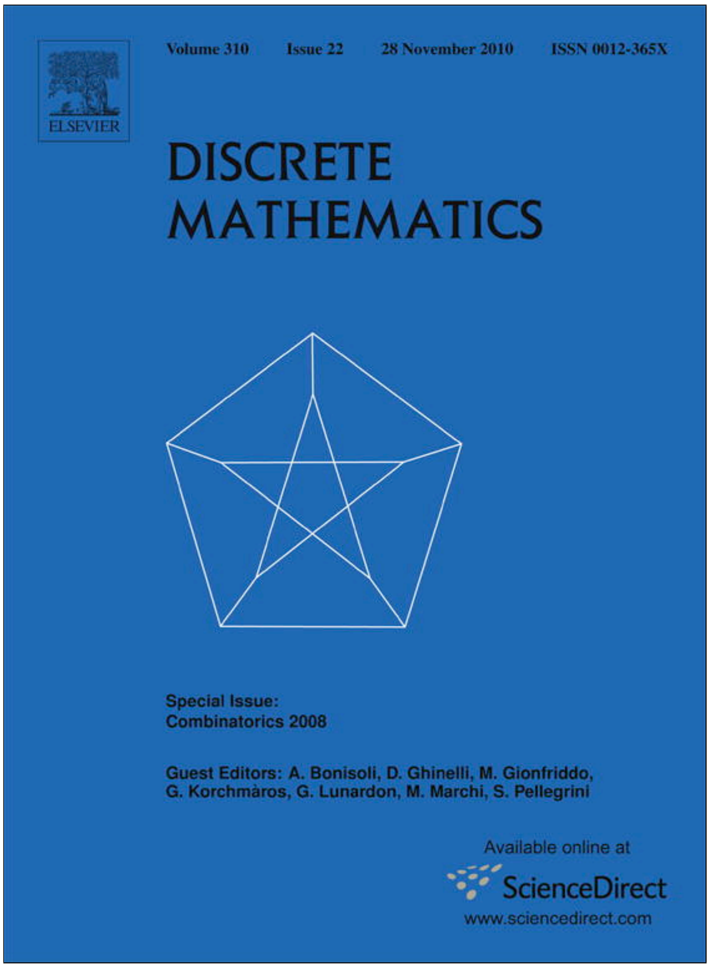
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Transitive bislim geometries of gonality 3, Part II: The group theoretic cases

Hendrik Van Maldeghem[a], Valerie Ver Gucht[b]

[a] Department of Pure Mathematics and Computer Algebra, Ghent University, Galglaan 2, 9000 Gent, Belgium

[b] Department of Applied Mathematics, Biometrics and Process Control, Ghent University, Coupure Links 653, 9000 Gent, Belgium

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ABSTRACT

We consider point–line geometries having three points on every line, having three lines through every point (*bislim geometries*), and containing triangles. We classify such geometries under the hypothesis of the existence of a collineation group acting transitively on the point set.

1. Introduction

In Part I of this paper we classified some *geometrically point homogeneous* bislim geometries of gonality 3. In the present paper we explicitly assume a point transitive group action and classify the corresponding geometries. It will turn out that we can rely on Part I for large classes.

To make this Part II self-contained, we recall some notation. However, for a general introduction and motivation we refer the reader to Part I. Here, we just mention that our result is equivalent to the classification of all cubic bipartite graphs containing a cycle of length 6 and admitting a group acting transitively on one of the bipartition classes (and we do not assume that the graph is finite). However, it seems easier and more convenient to describe this classification in the language of geometries, which is exactly what we will do.

We note that our "classification" is only explicit in "most" cases. In three cases (depending on the local structure), we only describe the geometries in either group theoretic or graph theoretic terms. However, this should provide enough information if one wants to use this classification.

The paper is organized as follows. In Section 2, we recall notation. In Section 3 we describe all geometries that are involved in our Main Result 2, and then we state this result, the proof of which will be given in Section 5.

2. Preliminaries

A *point–line geometry* $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ consists of two disjoint sets $\mathcal{P}$ (the point set) and $\mathcal{L}$ (the line set), together with a symmetric *incidence relation* $\mathtt{I}$ between $\mathcal{P}$ and $\mathcal{L}$. The graph with vertex set $\mathcal{P} \cup \mathcal{L}$, where two vertices are adjacent if they represent an incident point–line pair, is called the *incidence graph* of Γ, and is also denoted by Γ (since this graph unambiguously determines the geometry and vice versa), and we use graph theoretic notation. For instance, if n is any natural number, then $\Gamma_n(x)$ denotes the set of vertices at distance n from the vertex x. The incidence graph is a bipartite

E-mail addresses: hvm@cage.UGent.be (H. Van Maldeghem), Valerie.VerGucht@UGent.be (V. Ver Gucht).

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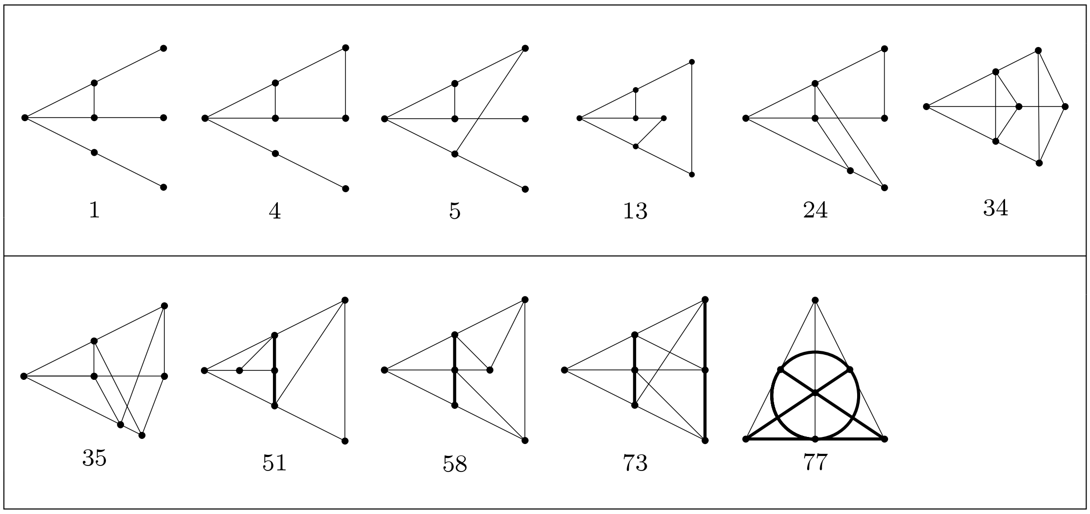

Fig. 1. Local structure in a geometrically point homogeneous bislim geometry of gonality 3.

graph. Every automorphism of that graph fixing the two bipartition classes is a *collineation* of the geometry. Also, if the graph is connected, then we say that the geometry is connected. A geometry where every line carries exactly three points is called *slim*. If also every point is incident with three lines, then the geometry is called *bislim*. The *dual* of a geometry is obtained by interchanging the point and line set; the incidence graph remains unchanged. A *duality* is an automorphism of the incidence graph interchanging the two bipartition classes.

The *gonality* of a geometry is half of the girth of its incidence graph. In this paper, we are only concerned with geometries having gonality distinct from 2 (the so-called *partial linear spaces*, because two points determine at most one line); in fact we will assume gonality 3 all the time (this means that the geometry has triangles). If a geometry admits a collineation group acting transitively on both the point set and the line set, then we say for short that the geometry is *transitive*. A *flag* is an incident point–line pair, or, equivalently, an edge of the incidence graph.

We will also use some obvious notation from incidence geometry like ab is the line incident with the points a and b, if it exists and is unique. We extend this notation to abc to say that the points a, b, c are incident with a common line and to denote that unique line (we sometimes express this by saying that *the line abc exists*).

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a connected point transitive bislim geometry of gonality 3. Let G be a point transitive collineation group. If G acts flag transitively, then all possibilities are given in [2]; see Theorem 4.1. Hence, in this paper, we may assume that G is not flag transitive. Let x be any point of Γ and L any line incident with x. Let x_1, x_2 be the two other points incident with L, and let L_1, L_2 be the two other lines incident with x. The points on L_i, $i = 1, 2$, different from x will be denoted by y_i and z_i. The *local structure at the point* x is the subgeometry Γ_x of Γ with point set $x \cup \Gamma_2(x)$ and line set the elements of $\Gamma_1(x) \cup \Gamma_3(x)$ incident with 2 or 3 of these points. Remark that this subgeometry is not necessarily bislim (in fact, it is only bislim if it coincides with Γ itself!). Denote the set of lines of Γ_x not through x by Γ_x^l. If Γ_x is isomorphic to some geometry Γ', for all points x, then we say that Γ is geometrically point homogeneous and point-locally Γ'. Similarly for geometrical line homogeneity and line-local geometries. If a geometry is point-locally Γ' and line-locally also Γ', then we say that Γ is locally Γ', or Γ has local structure Γ', and Γ is geometrically homogeneous.

A 1-*cover* of a connected bislim geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a connected bislim geometry $\widetilde{\Gamma} = (\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}}, \widetilde{\mathtt{I}})$ together with a (necessarily surjective) incidence preserving mapping $\theta : \widetilde{\mathcal{P}} \to \mathcal{P}; \widetilde{\mathcal{L}} \to \mathcal{L}$ such that the three points on any line $\widetilde{L}$ of $\widetilde{\Gamma}$ are mapped onto the three points of $\widetilde{L}^\theta$, and dually for the three lines through any point $\widetilde{x}$ of $\widetilde{\Gamma}$. Clearly, the local structure of $\widetilde{\Gamma}$ at a point $\widetilde{x}$ can abstractly be viewed as a subgeometry of the local structure of Γ at the point $\widetilde{x}^\theta$. Now, if for all points and lines $\widetilde{A}$ of $\widetilde{\Gamma}$, the local structure at $\widetilde{A}$ is mapped under θ bijectively onto the local structure of Γ at $\widetilde{A}^\theta$, then we say that $\widetilde{\Gamma}$ is a $1\frac{1}{2}$-*cover with covering epimorphism* θ. We now have the usual definition of universal $1\frac{1}{2}$-cover $\overline{\Gamma}$ of Γ with universal covering epimorphism $\overline{\theta}$: for every $1\frac{1}{2}$-cover $\theta : \widetilde{\Gamma} \to \Gamma$, there exists a cover $\widetilde{\theta} : \overline{\Gamma} \to \widetilde{\Gamma}$ such that $\overline{\theta} = \widetilde{\theta}\theta$, and then $\widetilde{\theta}$ is a $1\frac{1}{2}$-cover. Finally we say that Γ is $1\frac{1}{2}$-*connected* if for every $1\frac{1}{2}$-cover the covering epimorphism is an isomorphism. Clearly for every $1\frac{1}{2}$-connected bislim geometry, the identity defines a universal $1\frac{1}{2}$-cover, and every universal $1\frac{1}{2}$-cover is $1\frac{1}{2}$-connected.

3. Examples of point transitive bislim geometries with triangles

For a list of local structures, we refer the reader to the Appendix of [3]. The local structure with number n of that list will be referred to as $\mathsf{LS}(n)$. In Part I of the present work, we proved that this list is complete, and that the local structure at a point x of a geometrically point homogeneous bislim geometry of gonality 3 is one of the 11 possibilities in Fig. 1.

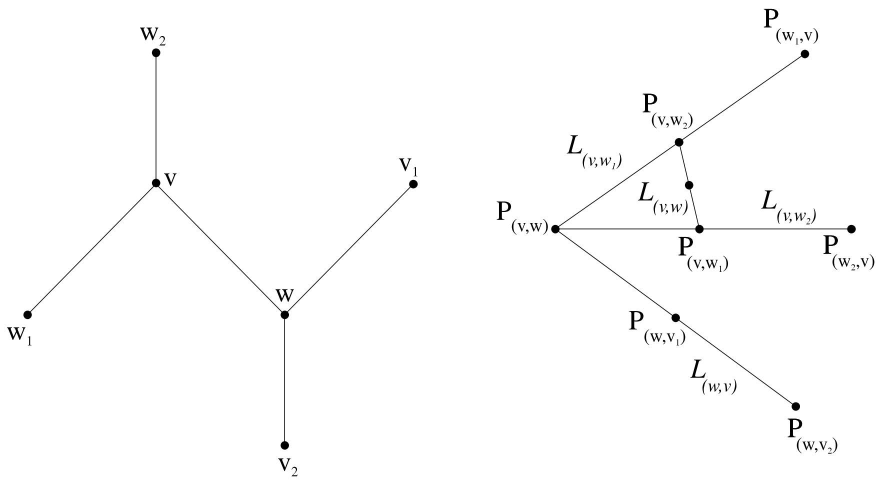

Fig. 2. $\mathcal{G}(V, E)$ does not contain triangles.

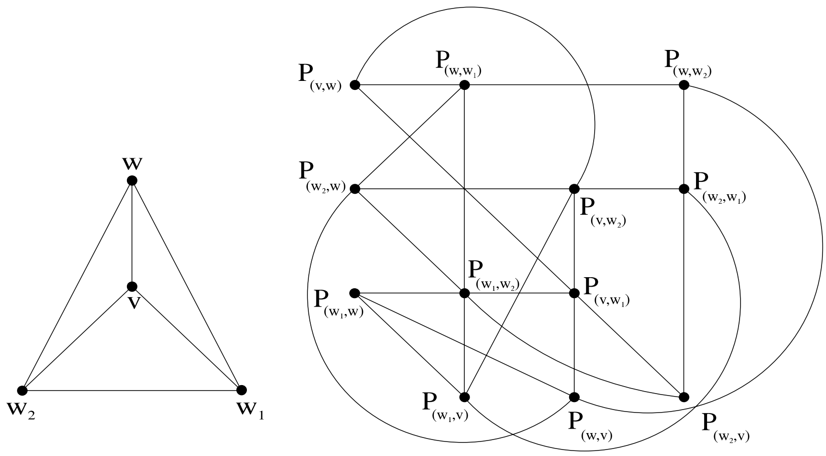

Fig. 3. $\mathcal{G}(V, E)$ does contain triangles.

3.1. *A family associated with symmetric trivalent graphs*

Let there be given a 3-regular (or *trivalent*, or *cubic*) graph $\mathcal{G}(V, E)$ admitting an automorphism group acting transitively on the set of ordered edges $(v, w) \in V \times V$ with $\{v, w\} \in E$ (i.e. a *symmetric* graph). We define a geometry $\Gamma := \Gamma_{\mathcal{G}}$ in the following way. To every ordered edge (v, w) of $\mathcal{G}$ we attach a point $P_{(v,w)}$ and a line $L_{(v,w)}$. If $v \in V$ is adjacent to w, w_1 and w_2, then the point $P_{(v,w)}$ and the line $L_{(v,w)}$ are incident with the lines $L_{(v,w_1)}$, $L_{(v,w_2)}$, $L_{(w,v)}$, and with the points $P_{(v,w_1)}$, $P_{(v,w_2)}$, $P_{(w,v)}$, respectively. It is easily seen that the geometry Γ is bislim, without digons but containing triangles. Indeed, with the above notation, $\{P_{(v,w)}, P_{(v,w_1)}, P_{(v,w_2)}\}$ is a triangle of Γ with sides $L_{(v,w)}$, $L_{(v,w_1)}$ and $L_{(v,w_2)}$. Moreover, it is easy to check that Γ is a transitive bislim geometry. Also, if $\mathcal{G}(V, E)$ does not contain triangles (see Fig. 2), then the local structure in any point or line x of $\Gamma_{\mathcal{G}}$ is isomorphic to $\mathsf{LS}(1)$, which is not symmetric in the three elements incident with x; hence $\Gamma_{\mathcal{G}}$ does not admit a flag transitive collineation group. On the other hand, if $\mathcal{G}(V, E)$ contains triangles (see Fig. 3), then the transitivity on edges implies readily that $\mathcal{G}(V, E)$ is the complete graph on four vertices (this is straightforward to check and left to the reader). In the latter case, $\Gamma_{\mathcal{G}}$ is isomorphic to the Coxeter geometry, introduced by Coxeter [1] and named after him in [2]. This geometry is flag transitive and has local structure $\mathsf{LS}(13)$. But both the alternating group $\mathsf{Alt}(4)$ and the symmetric group $\mathsf{Sym}(4)$ define a transitive but not flag transitive collineation group of the Coxeter geometry.

3.2. *A family attached to certain symmetric (3, 3)-valent digraphs*

Let there be given a symmetric (3, 3)-regular digraph $\mathcal{G}$, i.e., every vertex of $\mathcal{G}$ lies on three incoming edges and three outgoing ones, and the automorphism group G of $\mathcal{G}$ acts transitively on the set of (directed) edges. Moreover we suppose that the valency of the underlying undirected graph is 6 and that the action of the stabilizer G_v of a vertex v on its neighbors is isomorphic to $\mathsf{Sym}(3)$ with natural actions on the three incoming and three outgoing vertices, respectively, in v. We call such a graph a $\mathsf{Sym}(3)$-symmetric (3, 3)-valent digraph. We denote the permutation group induced by G_v on the six edges in v by $\overline{G}_v$.

Each element of order 2 in $\overline{G}_v$ fixes one edge in each orbit. Those two edges will be called opposite (the opposition relation is clearly symmetric). Clearly, this opposition relation is well defined, i.e., opposite edges are mapped onto opposite edges under an automorphism in G.

Now we define the following geometry $\Gamma_{\mathcal{G}}$. To every directed edge (v, w) of $\mathcal{G}$ we attach a point $P_{(v,w)}$ and a line $L_{(v,w)}$. We now define incidence. The point $P_{(v,w)}$ is always incident with the line $L_{(v,w)}$. Furthermore, the point $P_{(v,w)}$ is incident with the line $L_{(u,v)}$ if and only if (v, w) is not opposite (u, v). It is easily seen that the geometry $\Gamma_{\mathcal{G}}$ is bislim, without digons but containing triangles. Indeed, $\{P_{(v,w)}, P_{(v,w')}, P_{(v,w'')}\}$ is a triangle of Γ if (v, w), (v, w') and (v, w'') are the three directed edges leaving from v.

One now checks that $\Gamma_{\mathcal{G}}$ has local structure $\mathsf{LS}(13)$ or $\mathsf{LS}(1)$ depending on whether or not $\mathcal{G}$ contains (directed) triangles whose edges are not mutually opposite.

We remark that directed triangles with opposite edges are allowed and lead to $\mathsf{LS}(1)$ and $\mathsf{LS}(13)$ respectively if $\mathcal{G}$ does not contain and does contain (directed) triangles whose edges are not mutually opposite.

3.3. *Some families attached to classes of groups*

(1) Consider the group $G_{a,b} := \langle a, b : a^3 = \mathrm{id}\rangle$, and let N be a normal subgroup of $G_{a,b}$ not containing b^2, b^3, $baba^2$, $ab^{-1}ab$, $[a, b]$, $ba^2b^2a^2$, $(ba^2)^2$, $(ba^2)^3$, ba^2ba. Then we define the following geometry, $\Gamma_{G_{a,b},N}$. The points are the elements of the quotient group $G_{a,b}/N$ and the lines are the right translates of $\{N, Na, Nb\}$. One can check (we will do this explicitly later on) that the conditions are chosen such that $\Gamma_{G_{a,b},N}$ is a transitive bislim geometry with local structure $\mathsf{LS}(1)$. For every such N, the geometry $\Gamma_{G_{a,b},\{\mathrm{id}\}}$ is the universal $1\frac{1}{2}$-cover of $\Gamma_{G_{a,b},N}$.

(2) Consider the group $G_{s,t} := \langle s, t : s^2 = t^3 = \mathrm{id}\rangle$, and let N be a normal subgroup of $G_{s,t}$ not containing $(st)^3$, $[s, t]st$, $[s, t]^2$, $[s, t](st)^2$, $(st^2)^2(st)^2$, $[s, t]^2st$, $[s, t]^3$. We define the geometry $\Gamma_{G_{s,t},N}$ with points the right cosets in the quotient group $G_{s,t}/N$ of the subgroup $\{N, Ns\}$, and lines the right translates of the 3-set $\{\{N, Ns\}, \{Nt, Nst\}, \{Nts, Nsts\}\}$. One can check that the conditions are chosen such that $\Gamma_{G_{s,t},N}$ is a transitive bislim geometry with local structure $\mathsf{LS}(5)$. For every such N, the geometry $\Gamma_{G_{s,t},\{\mathrm{id}\}}$ is the universal $1\frac{1}{2}$-cover of $\Gamma_{G_{s,t},N}$.

(3) Consider the subgroup $G_{sts,t}$ of $G_{s,t} = \langle s, t : s^2 = t^3 = \mathrm{id}\rangle$ generated by sts and t, and let N be a normal subgroup of $G_{sts,t}$ not containing $[s, t](st)^2$, $[sts, t]$, $[s, t]^2$ and $[s, t]^3$. Then we define the following geometry, $\Gamma_{G_{sts,t},N}$. The points are the elements of the quotient group $G_{sts,t}/N$ and the lines are the right translates of $\{N, Nsts, Nt\}$. One can check (and this is completely similarly to Example (2)) that the conditions are chosen such that $\Gamma_{G_{sts,t},N}$ is a transitive bislim geometry with local structure $\mathsf{LS}(5)$. For every such N, the geometry $\Gamma_{G_{sts,t},\{\mathrm{id}\}}$ is the universal $1\frac{1}{2}$-cover of $\Gamma_{G_{sts,t},N}$ and is isomorphic to the geometry $\Gamma_{G_{s,t},\{\mathrm{id}\}}$ of the previous paragraph.

3.4. *Geometries with two non-isomorphic local structures for lines*

Let $\mathcal{D}$ be the dual of the geometry defined by the vertices and edges of a complete graph K_4 on four vertices. Let $\mathcal{G}(\mathcal{F} \cup \mathcal{B}, E)$ be a connected bipartite (6, 3)-valent graph. We define a geometry $\Gamma := \Gamma_{\mathcal{G}}$ in the following way. To every vertex $f \in \mathcal{F}$ we attach a geometry $f^{\mathcal{D}}$ isomorphic to $\mathcal{D}$. Then every edge of $\mathcal{G}$ containing f corresponds to a unique point of the geometry $f^{\mathcal{D}}$. The points of the geometry $\Gamma_{\mathcal{G}}$ are the edges of the graph $\mathcal{G}$. Let b be a vertex of $\mathcal{B}$ adjacent to the vertices f, f_1 and f_2. Then $\{\{f, b\}, \{f_1, b\}, \{f_2, b\}\}$ is a line of the geometry Γ. The line set of Γ is the union of the line sets of all geometries corresponding to the vertices of $\mathcal{F}$, together with the above mentioned triples. It is clear that the local structure in each point is isomorphic to $\mathsf{LS}(4)$. However, the local structure of a line in some member of $\mathcal{F}$ is isomorphic to $\mathsf{LS}(10)$, while the local structure of all other lines is isomorphic to $\mathsf{LS}(0)$.

Now suppose that the graph $\mathcal{G}$ admits an automorphism group acting transitively on the edges and preserving the incidences in all geometries corresponding to the vertices in $\mathcal{F}$. Then it is clear that the geometry $\Gamma_{\mathcal{G}}$ is point transitive. Remark that, since local structure $\mathsf{LS}(4)$ is not symmetric in the three elements incident with a point, the geometry does not admit a flag transitive collineation group.

3.5. *Quotients of the honeycomb geometry*

In Part I of the present work, we have introduced many geometrically point homogeneous quotients of the honeycomb geometry. We will briefly repeat some of these constructions, adding some comments on collineation groups.

Let $\mathbb{E}$ be the real Euclidean plane, and consider the tiling $\mathcal{T}$ of $\mathbb{E}$ in regular hexagons (a *honeycomb*). The skeleton of this honeycomb is in fact a bipartite graph which divides the vertices into two classes that we will designate as black and white. We define the *honeycomb geometry* $\mathcal{S}_\infty$ as the geometry with points the black vertices and lines the white vertices, and where incidence is adjacency.

Let $W(\widetilde{\mathsf{A}}_2)$ be the full collineation group of $\mathcal{S}_\infty$, or equivalently, the group of isometries of $\mathbb{E}$ preserving the honeycomb tiling $\mathcal{T}$ and stabilizing the set of black vertices (which is the Weyl group of type $\widetilde{\mathsf{A}}_2$, whence the notation).

Let G be a subgroup of $W(\widetilde{\mathsf{A}}_2)$ such that for every vertex v of $\mathcal{T}$, the graph theoretic distance between two distinct vertices of the orbit v^G is at least 8. Then the quotient geometry $\mathcal{S}_\infty/G$ defined in the obvious way by identifying the elements in the same orbit is a geometry with local structure $\mathsf{LS}(13)$.

Very explicitly, we can define the following geometries, where the description is a result of a coordinatization of $\mathbb{E}$, which is reflected in the conditions on the various parameters and the identifications (we always coordinatize with respect to basis vectors forming an angle of sixty degrees; if one were to choose e.g. 120 degrees, then the formulas would look different, but similar).

(HC1) Let r, s be two integers with $0 \le s \le r$ and $r^2 + rs + s^2 \ge 12$. We define a geometry $\mathcal{S}_{(r,s)}$ as follows. The points are the equivalence classes of ordered pairs (i, j), with i, j integers and with respect to the equivalence relation $\sim$ defined as $(i, j) \sim (i', j')$ if $(i-i', j-j') = (kr, ks)$, for some integer k. We denote by $(i, j)/\sim$ the equivalence class containing (i, j). The lines of the geometry are the 3-sets $\{(i, j)/\sim, (i + 1, j)/\sim, (i, j + 1)/\sim\}$, for all integers i, j.

It is clear that the group consisting of the collineations $(x, y) \mapsto (x + a, y + b)$, for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, acts as a transitive collineation group. Also, it is easy to see that the reflection $(x, y) \leftrightarrow (y, x)$ induces a collineation of $\mathcal{S}_{(r,s)}$ if and only if $r = s$. In this case, $\mathcal{S}_{(r,s)}$ admits a point transitive collineation group with point stabilizer of order 2. If $s = 0$, then the reflection $(x, y) \leftrightarrow (-x-y, y)$ is a collineation, and hence also in this case we have a point transitive collineation group with point stabilizer of size 2. In each of these two cases, there is also an extra sharply transitive collineation group containing a so-called glide reflection; we refer to this group as a glide group.

(HC2) Let a, c and d be integers with $a, d > 0$, $0 \le c < a$ and such that for every non-trivial integer linear combination of $(a, 0)$ and (c, d), say (r, s), $r^2 + rs + s^2 \ge 12$. The points of the geometry $\mathcal{M}_{(a,0),(c,d)}$ are the equivalence classes of ordered pairs (i, j), with i, j integers, with respect to the equivalence relation $\approx$, defined as $(i, j) \approx (i', j')$ if $(i - i', j - j') = (ka + \ell c, \ell d)$, for some integers k and ℓ. With similar notation to the previous example, the lines of the geometry are the 3-sets $\{(i, j)/\approx, (i + 1, j)/\approx, (i, j + 1)/\approx\}$, for all integers i, j.

Here, again every translation $(x, y) \mapsto (x+t, y+u)$ induces a collineation of $\mathcal{M}_{(a,0),(c,d)}$, for all $(t, u) \in \mathbb{Z}\times\mathbb{Z}$. The group generated by all these translations will be called the *full translation group*. The rational numbers c/d, a/d and $\frac{d^2-c^2}{ad}$ are integers if and only if $\mathcal{M}_{(a,0),(c,d)}$ admits the reflection $(x, y) \leftrightarrow (y, x)$, and hence it admits a second sharply point transitive collineation group, which we will describe below. Moreover, in this case, the two sharply transitive collineation groups generate a transitive collineation group with point stabilizer of size 2.

Similarly, $\mathcal{M}_{(a,0),(c,d)}$ admits the reflection $(x, y) \leftrightarrow (x, -x - y)$ if and only if the rational numbers c/d, a/d and $\frac{c^2+2cd}{ad}$ are integers, in which case again a second sharply point transitive collineation group exists, and also again a point transitive one with point stabilizer of size 2.

Likewise for the reflection $(x, y) \leftrightarrow (-x - y, y)$ and the condition that the rational number $\frac{d+2c}{a}$ is integer.

Remark that, if at least two of the above reflections act on $\mathcal{M}_{(a,0),(c,d)}$, then it is isomorphic to either $\mathcal{M}_{(a,0),(0,a)}$ or $\mathcal{M}_{(3a,0),(a,a)}$. In this case, the three reflection groups above live in the geometry and there are two more sharply point transitive collineation groups, both containing rotations: one acting line sharply transitively, unlike the other.

In a similar fashion, one can check that a rotation of 120 degrees around the origin preserves the geometry if and only if d divides both a and c, and ad divides $c^2 + cd + d^2$. In this case the full automorphism group of $\mathcal{M}_{(a,0),(c,d)}$ acts flag transitively. The flag transitive bislim geometries with triangles have been classified in [2], and this particular class has been defined there in a slightly different way (this is because the proof in [2] does not use the geometric result of [3]). A direct proof of the equivalence is cumbersome and left to the interested reader. Here, we just recall the construction of the equivalent class of geometries $\mathcal{G}_{(r,s)}$. These geometries depend on two integer parameters r and s with $r \ge s$ and $r + s \ge 4$.

The points of $\mathcal{G}_{(r,s)}$ are the ordered pairs (i, j), with i, j integers and with identification of all pairs $(i, j) + k(r, s) + l(-s, r + s) = (i + kr - ls, j + ks + lr + ls)$ with k, l integers. The lines of the geometry are the 3-sets $\{(i, j), (i + 1, j), (i + 1, j - 1)\}$ consisting of the three points incident with the line, and where for each point the above identification rule holds.

One easily checks that all these geometries are bislim with gonality 3 and they are all locally $\mathsf{LS}(13)$. The geometries in (HC1) are infinite, while the others are finite.

But the above examples also exist for smaller parameters. More exactly, $\mathsf{LS}(n)$ occurs, for $n \in \{24, 35, 51, 58, 73, 77\}$.

Below, we do not care about the restrictions on the parameters a, c, d, which were only required to avoid isomorphic geometries.

All of $\mathcal{M}_{(3,0),(-d,2d+1)}$, $\mathcal{M}_{(3,0),(1-d,2d+1)}$, $\mathcal{M}_{(3,0),(-d,2d)}$ and $\mathcal{M}_{(3,0),(1-d,2d)}$, with $d \ge 2$, are locally $\mathsf{LS}(24)$ and admit a point transitive collineation group (translations). Also $\mathcal{S}_{(3,0)}$ is locally $\mathsf{LS}(24)$ and admits all translations. Moreover, the geometries $\mathcal{M}_{(3,0),(1-d,2d+1)}$, $\mathcal{M}_{(3,0),(-d,2d)}$ and $\mathcal{S}_{(3,0)}$ admit the reflection $(x, y) \leftrightarrow (-x - y, y)$, and hence these geometries also admit a second sharply point transitive collineation group (called a glide group below), and the group generated by the translations and the above reflection acts transitively and has point stabilizer of size 2.

If, in the previous paragraph, $2d+1 = 3$, then $\mathcal{M}_{(3,0),(0,3)}$ is the Pappus geometry and admits two different point transitive groups acting non-flag transitively: a translation group of order 9 and the group generated by the translations and the reflection $(x, y) \leftrightarrow (y, x)$. Also, $\mathcal{M}_{(3,0),(-1,3)} \cong \mathcal{M}_{(3,0),(1,3)}$ is a geometry which is locally $\mathsf{LS}(58)$, admitting a transitive translation group. This geometry is isomorphic to $\mathcal{M}_{(9,0),(2,1)}$.

The geometries $\mathcal{S}_{(2,1)}$ and $\mathcal{M}_{(n,0),(2,1)}$, with $n \geq 10$, are all locally $\mathsf{LS}(51)$. For all these, $\mathcal{S}_{(2,1)}$ is a $1\frac{1}{2}$-cover. The collineation group is in each case a cyclic group acting sharply transitively on the point set.

The Möbius–Kantor geometry, $\mathcal{M}_{(8,0),(2,1)}$, is locally $\mathsf{LS}(73)$ and is only $1\frac{1}{2}$-covered by itself. The cyclic group of order 8 acts sharply point transitively.

Finally, $\mathcal{M}_{(7,0),(2,1)}$, the Fano geometry, is locally $\mathsf{LS}(77)$ and admits a sharply point transitive cyclic collineation group of order 7.

The geometries with local structure $\mathsf{LS}(51)$, $\mathsf{LS}(58)$, $\mathsf{LS}(73)$ and $\mathsf{LS}(77)$ can very easily be described using their cyclic collineation group as follows. Consider the cyclic group $\mathbb{Z}_n$ with $n \in \mathbb{N} \cup \{\infty\}$, $n \geq 7$. Define the point set of the geometry as the elements of $\mathbb{Z}_n$, and the line set are the 3-subsets $\{x, x+1, x+3\}$ with $x \in \mathbb{Z}_n$.

4. Statement of Main Result 2

In the present paper we will prove the following theorem.

Main Result 2. If Γ is a connected bislim geometry of gonality 3 with a point transitive collineation group G which is not flag transitive, then (Γ, G) is one of the examples in the previous section. In particular, either

(i) Γ has local structure $\mathsf{LS}(1)$ and either
  (ia) Γ arises from a symmetric trivalent graph without triangles, while G is a graph automorphism group acting transitively on ordered edges, or
  (ib) Γ arises from a $\mathsf{Sym}(3)$-symmetric $(3,3)$-valent digraph containing no directed triangles with non-opposite directed edges and G is an automorphism group of the graph acting transitively on the directed edges, and with G_v acting as $\mathsf{Sym}(3)$ on the three incoming (respectively outgoing) edges of each vertex v, or
  (ic) $\Gamma \cong \Gamma_{G_{a,b},N}$, with $G_{a,b} = \langle a, b : a^3 = \mathrm{id}\rangle$ and N is a normal subgroup of $G_{a,b}$ not containing b^2, b^3, $baba^2$, $[a,b]$, $ab^{-1}ab$, $ba^2b^2a^2$, $(ba^2)^2$, $(ba^2)^3$, ba^2ba, and $G = G_{a,b}/N$;

or

(ii) Γ has local structure $\mathsf{LS}(4)$ and Γ arises from a symmetric $(6,3)$-valent graph where each 6-valent vertex is associated with a geometry isomorphic to the dual of K_4, while G is a graph automorphism group acting transitively on the edges and preserving the incidences in the geometries corresponding to the 6-valent vertices; or

(iii) Γ has local structure $\mathsf{LS}(5)$ and either
  (iiia) $\Gamma \cong \Gamma_{G_{s,t},N}$ with $G_{s,t} = \langle s, t : s^2 = t^3 = \mathrm{id}\rangle$ and N is a normal subgroup of $G_{s,t}$ not in the subgroup $G_{sts,t}$ generated by sts and t, and not containing $(st)^3$, $[s,t]^2$, $[s,t]^3$, $[s,t](st)^2$, $(st^2)^2(st)^2$, $[s,t]^2st$, $[s,t]st$, and $G = G_{s,t}/N$ with point stabilizer of order 2, or
  (iiib) $\Gamma \cong \Gamma_{G_{sts,t},N}$ with N a normal subgroup of $G_{sts,t}$ and not containing sts, t, $stst^2$, $(stst^2)^2$, $(st)^3st^2$, $(ts)^3t^2s$, $[sts,t]$, $(stst^2)^3$, and $G = G_{sts,t}/N$ with point stabilizer of order 1. If $N^s = N$, then $N \trianglelefteq G_{s,t}$ and $\Gamma \cong \Gamma_{G_{s,t},N}$, but the group $G_{sts,t}/N$ is a sharply point transitive subgroup of $G_{s,t}/N$; the latter induces a point stabilizer of order 2;

or

(iv) Γ has local structure $\mathsf{LS}(13)$ and Γ is isomorphic either to $\mathcal{S}_{(r,s)}$, with $0 \leq s \leq r$ and $r^2+rs+s^2 \geq 12$, or to $\mathcal{M}_{(a,0),(c,d)}$, with a, c and d integers with $a, d > 0$, $0 \leq c < a$ and for every integer linear combination of $(a,0)$ and (c,d), say (r,s), $r^2+rs+s^2 \geq 12$. Or Γ is isomorphic to the honeycomb geometry itself. In the first case, if G acts sharply point transitively, then either it is generated by translations or it is a glide group (only if $r = s$ or $s = 0$). If the point stabilizer has size 2, then G is generated by all translations and a reflection, and $r = s$ or $s = 0$. In the second case, G is the full translation group, except in the following cases:

  (a) $\frac{a}{d}, \frac{c}{d}, \frac{d^2-c^2}{ad} \in \mathbb{Z}$,
  (b) $\frac{a}{d}, \frac{c}{d}, \frac{c^2+2cd}{ad} \in \mathbb{Z}$,
  (c) $\frac{2c+d}{a} \in \mathbb{Z}$,
  (d) $\frac{a}{d}, \frac{c}{d}, \frac{c^2+cd+d^2}{ad} \in \mathbb{Z}$.

If exactly one of (a), (b), (c) holds, then ((d) does not hold and) we have a second sharply point transitive group containing glide reflections, and the group generated by the two sharply point transitive groups has point stabilizer of size 2. If two of (a), (b) and (c) hold, then they all hold and (d) holds and then we have the so called square geometry $\mathcal{M}_{(a,0),(0,a)}$ or the so called triple square geometry $\mathcal{M}_{(3a,0),(a,a)}$, which admits exactly fifteen possible point transitive non-flag transitive collineation groups (all such groups inherited from the honeycomb geometry). If none of (a), (b), (c) holds, but (d) holds, then we have two more sharply point transitive collineation groups (one acting transitively on the lines, the other does not) containing rotations, every other point transitive collineation group is flag transitive, and we have the geometry $\mathcal{G}_{(r,s)}$ with $0 < s < r$ and $r+s \geq 4$. In the third case (honeycomb geometry) there are in total fifteen possible point transitive non-flag transitive collineation groups (seven up to conjugacy), classified in Lemma 5.3; or

(v) Γ has local structure LS(24) and either $\Gamma \cong \mathcal{M}_{(3,0),(-d,2d+1)}$, or $\Gamma \cong \mathcal{M}_{(3,0),(1-d,2d+1)}$, or $\Gamma \cong \mathcal{M}_{(3,0),(-d,2d)}$, or $\Gamma \cong \mathcal{M}_{(3,0),(1-d,2d)}$, $d \geq 2$, or $\Gamma \cong \mathcal{S}_{(3,0)}$. In all cases, G acts sharply point transitively and is generated by translations, except in the second, third and last case, where a glide group can act sharply point transitively, or the group generated by all translations and a reflection $(x, y) \leftrightarrow (-x - y, y)$ acts transitively and has point stabilizer of size 2; or
(vi) Γ has local structure LS(34) and is the Desargues configuration. The group G is the Frobenius group of order 20; or
(vii) Γ has local structure LS(35) and $\Gamma \cong \mathcal{M}_{(3,0),(0,3)}$ is the Pappus configuration. The group G has order 9 (two possibilities), or order 18 (four possibilities) or order 36 (one possibility); or
(viii) Γ has local structure LS(51) and $\Gamma \cong \mathcal{M}_{(n,0),(2,1)}$, $n \geq 10$, or $\Gamma \cong \mathcal{S}_{(2,1)}$, and G is cyclic of order n (first case) or has infinite order (second case); or
(ix) Γ has local structure LS(58) and $\Gamma \cong \mathcal{M}_{(3,0),(1,3)} \cong \mathcal{M}_{(9,0),(2,1)}$, and G is cyclic of order 9; or
(x) Γ has local structure LS(73) and $\Gamma \cong \mathcal{M}_{(8,0),(2,1)}$ is the Möbius–Kantor configuration, and G has order 8 (two possibilities) or order 16 (one possibility); or
(xi) Γ has local structure LS(77) and $\Gamma \cong \mathcal{M}_{(7,0),(2,1)}$ is the Fano plane, and G is cyclic of order 7.

Note that the classes of geometries in (ia), (ib) and (ic) are not at all disjoint, but together with the given group action, there is only some overlap between (ia) and (ic), which can be avoided by requiring that the normal subgroup N in (ic) does not contain $(ab)^2$.

It is interesting to note that all point transitive bislim geometries are point-locally the same as line-locally, except in the case of LS(4).

For the sake of completeness, we also recall the main result of [2], where all flag transitive bislim geometries containing triangles are classified, using completely different methods. We do not want to go into detail about the groups, but just mention their orders.

Theorem 4.1 (*Van Maldeghem & Ver Gucht [2]*). *Let Γ be a (not necessarily finite) connected bislim flag transitive point–line geometry of gonality 3 with a flag stabilizer H. Then one of the following possibilities occurs.*

(i) *Γ is isomorphic to the honeycomb geometry, and $|H| \in \{1, 2\}$;*
(ii) *Γ is isomorphic to the Möbius–Kantor geometry, and again $|H| \in \{1, 2\}$;*
(iii) *Γ is isomorphic to the Desargues geometry, and H is elementary abelian and of order 2 or 4;*
(iv) *Γ is isomorphic to $\mathcal{G}_{(r,s)}$, with $r \geq s \geq 0$ and $r + s \geq 3$. In this case $|H| \in \{1, 2\}$ for $s = 0$ and for $r = s$, except if $(r, s) = (3, 0)$. In the latter case Γ is the Pappus geometry and H is elementary abelian of order 1, 2 or 4. If $(r, s) = (2, 1)$, then Γ is the Fano geometry and either $|H| = 1$ or $|H| = 8$ and H is dihedral. In all the other cases $|H| = 1$.*

We note that the geometry $\mathcal{G}_{(r,0)}$ is isomorphic to $\mathcal{M}_{(r,0),(0,r)}$ and that the geometry $\mathcal{G}_{(r,r)}$ is isomorphic to $\mathcal{M}_{(3r,0),(r,r)}$. This can be shown directly relatively easily.

The proof of Main Result 2 will to a large extent depend on Main Result 1 (contained in Part I of the present work), which we therefore recall in some detail (leaving out unimportant details).

We need a few additional definitions of geometries that are also quotients of the honeycomb geometry.

(HC3a) Let r be an integer with $r \geq 2$. The points of the geometry $\mathcal{S}^*_{(r)}$ are the equivalence classes of ordered integer pairs (i, j) with respect to the equivalence relation $\overset{*}{\sim}$ defined as $(i, j) \overset{*}{\sim} (i', j')$ if either $(i - i', j - j') = (-2kr, 4kr)$, for some integer k, or $(i + i' + r + j, j' - j - 2r) = (-2kr, 4kr)$, for some integer k. One checks that this is indeed an equivalence relation (in particular, it is symmetric!). The lines are again, with similar notation to before, the 3-sets $\{(i, j)/\overset{*}{\sim}, (i + 1, j)/\overset{*}{\sim}, (i, j + 1)/\overset{*}{\sim}\}$, for all integers i, j.
(HC3b) Let r be an integer with $r \geq 1$. The points of the geometry $\mathcal{S}^{**}_{(r)}$ are the equivalence classes of ordered integer pairs (i, j) with respect to the equivalence relation $\overset{**}{\sim}$ defined as $(i, j) \overset{**}{\sim} (i', j')$ if either $(i-i', j-j') = (-k(2r+1), 2k(2r+1))$, for some integer k, or $(i + i' + r + j, j' - j - 2r - 1) = (-k(2r + 1), 2k(2r + 1))$, for some integer k. One checks that this is indeed an equivalence relation (in particular, it is symmetric!). The lines are again, with similar notation to before, the 3-sets $\{(i, j)/\overset{**}{\sim}, (i + 1, j)/\overset{**}{\sim}, (i, j + 1)/\overset{**}{\sim}\}$, for all integers i, j.
(HC4a) Let r, s be two integers with $r \geq 2$ and $s \geq 3$. The points of the geometry $\mathcal{M}^*_{(r),(s,0)}$ are the equivalence classes of ordered integer pairs (i, j) with respect to the equivalence relation $\overset{*}{\approx}$ defined as $(i, j) \overset{*}{\approx} (i', j')$ if either $(i-i', j-j') = (-2kr + \ell s, 4kr)$, for some integers k, ℓ, or $(i + i' + r + j, j' - j - 2r) = (-2kr + \ell s, 4kr)$, for some integers k, ℓ. One again checks that this is indeed an equivalence relation. The lines are, again, with similar notation to before, the 3-sets $\{(i, j)/\overset{*}{\approx}, (i + 1, j)/\overset{*}{\approx}, (i, j + 1)/\overset{*}{\approx}\}$, for all integers i, j.
(HC4b) Let r, s be two integers with $r \geq 1$ and $s \geq 3$. The points of the geometry $\mathcal{M}^{**}_{(r),(s,0)}$ are the equivalence classes of ordered integer pairs (i, j) with respect to the equivalence relation $\overset{**}{\approx}$ defined as $(i, j) \overset{**}{\approx} (i', j')$ if either $(i-i', j-j') = (-k(2r+1)+\ell s, 2k(2r+1))$, for some integers k, ℓ, or $(i+i'+r+j, j'-j-2r-1) = (-k(2r+1)+\ell s, 2k(2r+1))$, for some integers k, ℓ. One checks that this is indeed an equivalence relation. The lines are again, with similar notation to before, the 3-sets $\{(i, j)/\overset{**}{\approx}, (i + 1, j)/\overset{**}{\approx}, (i, j + 1)/\overset{**}{\approx}\}$, for all integers i, j.

We can now repeat Main Result 1 to some lesser extent than in [3].

Main Result 1. If $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a geometrically point homogeneous connected bislim geometry of gonality 3 which is point-locally $\mathsf{LS}(n)$, $1 \le n \le 77$, then $n \in \{1, 4, 5, 13, 24, 34, 35, 51, 58, 73, 77\}$. In particular, we have the following characterizations.

(i) If $n = 1$, then there are always lines with local structure $\mathsf{LS}(1)$, and there are always points incident with three such lines. Also, if Γ is not geometrically homogeneous, then there are lines with local structure $\mathsf{LS}(0)$, and there are points incident with a unique such line.

(ii) If $n = 4$, then Γ arises from a symmetric (6, 3)-valent graph where each 6-valent vertex is associated with a geometry isomorphic to the dual of K_4.

(iii) If $n = 13$, then Γ is isomorphic to a quotient of the honeycomb geometry, which is the universal $1\frac{1}{2}$-cover of Γ (and which is also the only $1\frac{1}{2}$-connected bislim geometry with this local structure). In particular, Γ is isomorphic either to

- (iiia) $\mathcal{S}_{(r,s)}$, with $0 \le s \le r$ and $r^2 + rs + s^2 \ge 12$, or to
- (iiib) $\mathcal{M}_{(a,0),(c,d)}$, with $a, d > 0$, $0 \le c < a$, such that for every integer linear combination of $(a, 0)$ and (c, d), say (r, s), one has $r^2 + rs + s^2 \ge 12$, or to
- (iiic) $\mathcal{S}^*_{(r)}$ or $\mathcal{S}^{**}_{(r)}$, with $r \ge 2$, or to
- (iiid) $\mathcal{M}^*_{(r),(s,0)}$ or $\mathcal{M}^{**}_{(r),(s,0)}$, with $r \ge 2$ and $s \ge 4$.

(iv) If $n = 24$, then Γ is isomorphic to a quotient of the honeycomb geometry. In particular, Γ is isomorphic either to

- $\mathcal{S}_{(3,0)}$, or to
- $\mathcal{M}_{(3,0),(-d,2d+1)}$ or $\mathcal{M}_{(3,0),(1-d,2d+1)}$ or $\mathcal{M}_{(3,0),(-d,2d)}$ or $\mathcal{M}_{(3,0),(1-d,2d)}$, with $d \ge 2$, or to
- $\mathcal{M}^*_{(r),(3,0)}$ or $\mathcal{M}^{**}_{(r),(3,0)}$, with $r \ge 2$.

The geometry $\mathcal{S}_{(3,0)}$ is the unique $1\frac{1}{2}$-connected member of this family and is hence the universal $1\frac{1}{2}$-cover of Γ.

(v) If $n = 34$, then Γ is isomorphic to the Desargues configuration.

(vi) If $n = 35$, then $\Gamma \cong \mathcal{M}_{(3,0),(0,3)}$ is the Pappus configuration.

(vii) If $n = 51$, then either $\Gamma \cong \mathcal{S}_{(2,1)}$, or $\Gamma \cong \mathcal{M}_{(n,0),(2,1)}$, with $n \ge 10$. In any case, $\mathcal{S}_{(2,1)}$ is the universal $1\frac{1}{2}$-cover of Γ, and it is also the only $1\frac{1}{2}$-connected member of this family.

(viii) If $n = 58$, then $\Gamma \cong \mathcal{M}_{(3,0),(1,3)} \cong \mathcal{M}_{(9,0),(2,1)}$.

(ix) If $n = 73$, then $\Gamma \cong \mathcal{M}_{(4,0),(1,2)} \cong \mathcal{M}_{(8,0),(2,1)}$ is the Möbius–Kantor configuration.

(x) If $n = 77$, then $\Gamma \cong \mathcal{M}_{(7,0),(2,1)} \cong \mathsf{LS}(77)$ is the Fano plane.

5. Proof of Main Result 2

From now on, we will assume that all geometries are connected and we will not add this condition in our statements. We state the following part of Main Result 1 as a separate lemma.

Lemma 5.1. *If Γ is a bislim geometry of gonality* 3 *which is geometrically point homogeneous, then, for any point x of Γ, the local structure Γ_x is isomorphic to one of the* 11 *configurations listed in Fig. 1, and referred to as* $\mathsf{LS}(n)$, *with* $n \in \{1, 4, 5, 13, 24, 34, 35, 51, 58, 73, 77\}$.

Since a point transitive bislim geometry of gonality 3 is automatically geometrically point homogeneous, we only have to consider the local structures listed in the previous lemma, and that is exactly what we now do.

5.1. *Local structures* $\mathsf{LS}(34)$, $\mathsf{LS}(35)$, $\mathsf{LS}(58)$, $\mathsf{LS}(73)$ *and* $\mathsf{LS}(77)$

We first take a look at those local structures that give rise to a unique geometry.

Lemma 5.2. *If Γ is a bislim geometry of gonality* 3 *admitting a point transitive group G, and for some point x of Γ, the local structure Γ_x is isomorphic to one of* $\mathsf{LS}(34)$, $\mathsf{LS}(35)$, $\mathsf{LS}(58)$, $\mathsf{LS}(73)$ *or* $\mathsf{LS}(77)$, *then Γ is uniquely determined. If moreover G does not act flag transitively, then G is unique, up to conjugation in the full collineation group of Γ except if $\Gamma_x \cong \mathsf{LS}(35)$, in which case there are precisely seven possibilities for G: two of order* 9, *four of order* 18 *and one of order* 36 *and if $\Gamma_x \cong \mathsf{LS}(73)$, in which case there are precisely three possibilities for G: two of order* 8 *and one of order* 16.

Proof. If $\Gamma_x \cong \mathsf{LS}(34)$, then it is shown in Part I that Γ is the **Desargues geometry**. Since the points of the Desargues geometry can be identified with the pairs of the set $\{1, 2, 3, 4, 5\}$, and the lines with the triples of that set (incidence is natural), we see that a point transitive group is a subgroup of $\mathsf{Sym}(5)$ acting transitively on the pairs of $\{1, 2, 3, 4, 5\}$. Clearly, only $\mathsf{Sym}(5)$ and $\mathsf{Alt}(5)$ qualify – but these act flag transitively – together with the Frobenius group of order 20, which acts point transitively but not flag transitively.

Consider now $\mathsf{LS}(35)$. Then Γ is the **Pappus geometry**, according to Part I. This geometry can be seen as the points of an affine plane of order 3, where the lines are all affine lines except for the lines of exactly one parallel class C. If G contains all nine translations then it contains no other element of order 3; otherwise it acts flag transitively. If G contains an element of order 2 which then normalizes the group of all translations, we obtain three groups of order 18 and one group of order 36. If G does not contain all translations, then it is easily seen that G contains the translations fixing C elementwise. Now we see that G contains a unique Sylow 3-subgroup P (of order 9) and P is completely determined by its action on the points at infinity and the elements of C. But all these actions are conjugate in the full collineation group of Γ. Hence G is contained in the normalizer of P. One now checks that since G cannot contain a Sylow 3-subgroup of order 27, it must have order 18, and the lemma follows in this case.

If $\Gamma_x \cong \mathsf{LS}(58)$, then $\Gamma \cong \mathcal{M}_{(3,0),(1,3)} \cong \mathcal{M}_{(9,0),(2,1)}$ by Part I again. Moreover, one easily sees that there is no non-trivial collineation of $\mathsf{LS}(58)$; hence the point stabilizer in the full automorphism group of Γ is trivial. This implies that the full collineation group of Γ is cyclic of order 9, and hence it is the unique transitive collineation group.

For $\Gamma_x \cong \mathsf{LS}(73)$, we get the **Möbius–Kantor geometry**. But the full collineation group of this geometry has order 48 and acts flag transitively. Hence only groups of order 8 or order 16 can act point transitively but not flag transitively. One checks that there are two groups of order 8 and one group of order 16.

The geometry $\mathsf{LS}(77)$ is itself the **Fano plane**, and it is well known that every subgroup of order 7 of the full collineation group acts transitively on Γ, and all such groups are, being Sylow 7-subgroups, conjugate. No other groups act point transitively but not flag transitively.

The lemma is proved. □

This leaves us with point transitive bislim geometries with local structure isomorphic to one of $\mathsf{LS}(1)$, $\mathsf{LS}(4)$, $\mathsf{LS}(5)$, $\mathsf{LS}(13)$, $\mathsf{LS}(24)$ and $\mathsf{LS}(51)$.

5.2. *Local structure* $\mathsf{LS}(1)$

It is easy to show that a point transitive geometry Γ with local structure $\mathsf{LS}(1)$ in a point x is locally $\mathsf{LS}(1)$. Let Γ be locally $\mathsf{LS}(1)$. We associate a directed graph $\mathcal{G}_\Delta$ to Γ as follows. It is easy to see that every point of Γ belongs to a unique triangle, and likewise for the lines. The vertices of the graph $\mathcal{G}_\Delta$ are the triangles of Γ. A vertex $v = \{p_1, p_2, p_3\}$ is adjacent to a vertex $w = \{q_1, q_2, q_3\}$, with the directed edge (v, w) if $v \neq w$ and if one of the points p_1, p_2, p_3 is incident with one of the lines q_1q_2, q_2q_3, q_1q_3. Note that at most one point of a triangle can be incident with a line of another triangle. Hence we obtain a graph with indegree 3, equal to the outdegree. Hence the bidegree is (3, 3). Point transitivity of the geometry Γ implies that, if in $\mathcal{G}_\Delta$ there are two vertices v, w for which both (v, w) and (w, v) are directed edges, then for every directed edge (v', w'), (w', v') is also a directed edge. In this case we can just forget about the directions and consider $\mathcal{G}_\Delta$ as a cubic undirected graph.

Note that the number of points of Γ is a multiple of 3, and looking at $\mathsf{LS}(1)$, we see that there are at least four triangles; hence there are at least 12 points in Γ.

First we consider the case where $\mathcal{G}_\Delta$ is undirected and cubic. It is clear that in this case this graph is symmetric, as the stabilizer of a triangle in Γ (vertex of $\mathcal{G}_\Delta$) acts transitively on the points of that triangle (edges of $\mathcal{G}_\Delta$ through the vertex). Standard arguments now show that the construction of $\mathcal{G}_\Delta$ out of Γ and the construction of a geometry out of a cubic symmetric graph $\mathcal{G}(V, E)$ as given in Section 3.1 are inverse to each other. This is (ia) of Main Result 2.

Now consider the case where $\mathcal{G}_\Delta$ is directed and has valency (3, 3). The point transitivity of G on the geometry Γ implies that G acts edge transitively on $\mathcal{G}_\Delta$. Clearly the stabilizer G_v of any vertex v induces exactly two orbits on the edges containing v.

Now there are two possibilities. First we consider the case where the action of G_v on the incoming (or, equivalently, on the outgoing) edges is equivalent to the natural action of $\mathsf{Sym}(3)$.

Then we clearly have a $\mathsf{Sym}(3)$-symmetric graph, and one can check that the construction of the graph out of the geometry and the construction of the geometry out of a graph as given in Section 3.2 are mutually inverse. From our discussion in Section 3.2 we conclude that $\mathcal{G}_\Delta$ had no directed triangles consisting of non-opposite edges.

Hence we have Case (ib) of Main Result 2.

Secondly, we consider the case where the action of G_v on the incoming edges is equivalent to the natural action of $\mathbb{Z}\mathsf{mod}(3)$.

In this case G acts sharply transitively on the set of directed edges, or, equivalently, G acts sharply transitively on the set of points and on the set of lines of Γ.

Hence we can identify the set of points of Γ with G, and the action of G on this set is given by right multiplication. Let e, a and b be different elements of G on one line L_e (where e is the identity in G). Then a^{-1}, e and ba^{-1} are also on one line L_a. The third line L_b through e is then given by the points b^{-1}, ab^{-1} and e. The 12 conditions arising from requiring that those lines contain seven different points (namely, no point on $L_e \setminus \{e\}$ coincides with a point on $(L_a \cup L_b) \setminus \{e\}$ and similarly for the points on $L_a \setminus \{e\}$ and $L_b \setminus \{e\}$) reduce to the six conditions $a^2 \neq e, a^2 \neq b, ab \neq e, b^2 \neq e, b^2 \neq a, ba^{-1}b \neq a$.

Without loss of generality we may assume that e and a belong to the same triangle. Then $\{a, a^2\}$ and $\{a^2, a^3\}$ form the other sides of the same triangle (by right multiplication with a). Hence $a^3 = e$.

Now we have to require that $\Gamma_e \cong \mathsf{LS}(1)$. This is done by requiring that no point collinear with b, but not on the line L_e, is collinear with e, and similarly for b^{-1}, ba^{-1}, ab^{-1}. For the point a, we require that only the points ba, $b^{-1}a$, $ab^{-1}a$ are not among $\Gamma_2(e)$. Similarly for the point a^{-1}. We leave the details to the reader, and only note the final conditions: $a^2 \neq e, b^2 \neq e, b^3 \neq e, b^2 \neq a, a \neq ba^2b, b^2 \neq a^2, a \neq bab, ab \neq ba, ba^2 \neq ab, a \neq b^3, a \neq ba^2b^2, a^2 \neq ba^2b$ and

$a^2ba^2 \neq b^{-1}ab^{-1}$. Note that $a^2 \neq b$ follows from $b^3 \neq e$, that $a^2 \neq e$, $b^2 \neq a$, $b^2 \neq a^2$ and $a \neq b^3$ follow from $ab \neq ba$, and that $a^2 \neq ba^2b$ follows from $a \neq bab$. Because of point transitivity of the group it follows that we have the right local structure in each point.

Since every point collinear with e is the image of e under an element of $\langle a, b\rangle$, we deduce that $G = \langle a, b\rangle$.

It is now easy to see that G is as described in Section 3.3(1). We have found Case (ic) of Main Result 2, and this finishes Case (i) completely.

Note that, still in Case (ic), and with the above description, the triangle through Nb has vertices Nb, Nab, Na^2b, and hence the resulting geometry is already in (ia) if and only if $\{Na^2, Nab, Na^2b\}$ forms a line. In view of the local structure, this line must coincide with $\{Na^2, Nab^{-1}a^2, Nb^{-1}a^2\}$. Since $Nab \neq Nab^{-1}a^2$ (indeed, otherwise $Na = Nb^{-2}$ and hence $[a, b]N = N$, a contradiction), this happens if and only if $Nab = Nb^{-1}a^2$, which reduces to $abab \in N$. This proves our remark after the statement of Main Result 2.

5.3. *Local structure* LS(4)

Let x be a point of a point transitive bislim geometry Γ which is point-locally LS(4). According to previous notation, we let x_1y_1 and x_2z_1 be the lines of Γ_x in $\Gamma_3(x)$, and y_2, z_2 are the other points in $\Gamma_2(x)$, with xy_2z_2 a line of Γ. Considering Γ_{x_1}, we see that x_2 is collinear with the "third point" u_1 on the line x_1y_1. But if x_2u_1 is distinct from x_2z_1, then we cannot have local structure LS(4) in x_2; hence $x_2z_1u_1$ is a line. The subgeometry defined by $x, x_1, x_2, y_1, z_1, u_1$ and the lines xx_1x_2, xy_1z_1, $u_1x_1y_1$ and $u_1x_2z_1$ is isomorphic to the dual of K_4. Moreover, the points y_2 and z_2 are not contained in a common triangle of Γ, as this would imply, looking in Γ_{z_2}, that x and z_2 are collinear with a point distinct from y_2. Hence we see that Γ_{L_2} is isomorphic to LS(0), and that every point is contained in a unique such line. Also, Γ_L is easily seen to be isomorphic to LS(10). Removing all lines with local structure LS(0) from Γ, we obtain a disjoint union of a family $\mathcal{F}$ of geometries isomorphic to the dual of K_4.

We now associate a graph $\mathcal{G}$ to Γ as follows. The vertex set of the graph $\mathcal{G}$ is the set $\mathcal{F} \cup \mathcal{B}$ where $\mathcal{F}$ is as mentioned above, and where $\mathcal{B}$ consists of vertices representing the lines with local structure LS(0). A vertex $f \in \mathcal{F}$ is adjacent to a vertex $b \in \mathcal{B}$ if the line corresponding to b contains a point of the geometry represented by vertex f. It is easy to see that $\mathcal{G}$ is a bipartite (6, 3)-valent graph where every vertex of $\mathcal{F}$ has degree 6 and every vertex of $\mathcal{B}$ degree 3. Every point of Γ clearly corresponds to a unique edge $\{f, b\}$ of $\mathcal{G}$. A point transitive collineation group of Γ induces a graph automorphism group acting transitively on the edges and preserving the incidences in the geometries corresponding to the vertices of $\mathcal{F}$. One can check that the construction of the graph out of the geometry and the construction of the geometry out of a graph as given in Section 3.4 are mutually inverse. Now (ii) of Main Result 2 is clear.

5.4. *Local structure* LS(5)

Let x be a point of Γ and suppose that LS(5) is the local structure in x. We use the same notation as before. We first claim that G_x has at most order 2. Whenever a collineation in G fixes a point u and two lines incident with u, or one point collinear with u, then it fixes all points collinear with u and hence is the identity. Suppose $\phi, \psi \in G_x$ and $\phi \neq \mathtt{Id}$, $\psi \neq \mathtt{Id}$. Clearly ϕ and ψ fix L and $\phi\psi$ also fixes L_1 and L_2. Hence $\phi\psi$ is the identity and our claim follows.

So we distinguish two cases. The first case is $|G_x| = 1$, the second $|G_x| = 2$.

The line L is, with respect to x, unique with the property that it is concurrent with two elements of Γ_x^l. We call L *principal* with respect to x. Every element of Γ_x^l meets the line principal with respect to x.

The local structure of the line L is LS(13) if L is principal with respect to x_1 or LS(5) if x_1y_1 is principal with respect to x_1. Two concurrent lines cannot both have local structure LS(13). Now suppose that L has local structure LS(13). Since Γ is point transitive every principal line has local structure LS(13). Either xy_1z_1 or either x_1y_1 is principal with respect to y_1 and has local structure LS(13). Because of concurrency with L this leads to a contradiction. Hence L has local structure LS(5) and x_1y_1 is principal with respect to x_1. Analogously x_2y_2 is principal with respect to x_2. If a line is principal with respect to some point it is principal with respect to a unique point. It is easily seen that L_1 (resp. L_2) is principal with respect to y_1 (resp. y_2). We conclude that every line of the geometry is principal with respect to a unique point. It follows that point transitivity induces line transitivity.

Suppose first $|G_x| = 1$. Clearly $|G_M| = 1$, for every line M of Γ. Denote by θ the unique collineation of G taking x to x_1. The line L is then mapped onto the line x_1y_1a (defining the point a). Indeed, from the previous paragraph we infer that L is not principal with respect to x_1 and that the line x_1y_1 is principal with respect to x_1, so $L^\theta = x_1y_1$. This implies that x_1 is taken onto either y_1 or a. We claim that $x_1^\theta = y_1$. Indeed, suppose on the contrary that $x_1^\theta = a$. The inverse image of x is a point contained together with x as vertices in a triangle, but clearly cannot be incident with L. Hence $y_2^\theta = x$. Now consider the collineation θ' mapping x to x_2 (and which maps, similarly to before, L onto x_2y_2). If it mapped x_2 onto y_2, then y_2 would be mapped onto x, and so $\theta^{-1}\theta' \in G_x$ contradicting $|G_x| = 1$. Hence x_2 is mapped under θ' onto b, the "third point" on the line x_2y_2, and x_1 is mapped onto y_2. Hence $\theta'\theta\theta'$ fixes x and hence is the identity. Similarly $\theta\theta'\theta$ is the identity. This implies that θ has order 3, contradicting $y_2 \neq a$.

Hence θ maps x onto x_1 and x_1 onto y_1 and y_1 onto x. Likewise, $\theta' : x \mapsto x_2 \mapsto y_2 \mapsto x$.

We can now identify the point set of Γ with G, acting on the right on itself. We re-denote the element θ by a and θ' by b. Then $a^3 = b^3 = e$, and the lines through e are $\{e, a, b\}$, $\{e, a^{-1}, ba^{-1}\}$ and $\{e, b^{-1}, ab^{-1}\}$. This again already implies $G = \langle a, b\rangle$.

In order to obtain the examples of Section 3.3(3), we have to derive the necessary and sufficient conditions on a and b that guarantee Γ to have gonality 3 and local structure $\mathsf{LS}(5)$. This is straightforward and can be done completely like in the previous case. We obtain $a^2 \neq b$, $a \neq ba^2b$, $a \neq bab$, $ab \neq ba$, $ba^2 \neq ab$ and $ab^2a \neq ba^2b$. After slightly rewriting these conditions (putting $b = t$ and s the (outer) automorphism of order 2 interchanging a and b), we see that we obtain the examples of Section 3.3(3) (the point transitivity of the group implies that we have the right local structure in each point).

Now suppose $|G_x| = 2$. We put $G_x := \{e, a\}$, with a an involution. Then we can identify the right cosets of $\{e, a\}$ in G with the points of the geometry Γ, and the action of G on the geometry is given by right multiplication. Let $\{e, a\}$, $\{b, ab\}$ and $\{c, ac\}$ be different right cosets of G_x incident with a common line. This implies that e, a, b and c are mutually different and $c \neq ab$. It follows that $\{b^{-1}, ab^{-1}\}$, $\{e, a\}$ and $\{cb^{-1}, acb^{-1}\}$ are also on one line. The third line through $\{e, a\}$ is then given by the points $\{c^{-1}, ac^{-1}\}$, $\{bc^{-1}, abc^{-1}\}$ and $\{e, a\}$. One can check that the conditions arising from requiring that those lines contain seven different points are equivalent to $b^2 \neq e$, $b^2 \neq a$, $b^2 \neq c$, $b^2 \neq ac$, $bc \neq e$, $bc \neq a$, $b \neq ac$, $bc \neq ab$, $cb \neq a$, $cb \neq ac$, $c^2 \neq e$, $c^2 \neq a$, $c^2 \neq b$, $c^2 \neq ab$, $b \neq ca$, $bab \neq c$, $b \neq cac$, $cb^{-1} \neq bc^{-1}$ and $cb^{-1} \neq abc^{-1}$.

Without loss of generality we assume that $\{e, a\}$ and $\{b, ab\}$ belong to a triangle and also $\{e, a\}$ and $\{c, ac\}$ belong to a triangle. Since the involution a interchanges the two points $\{b, ab\}$ and $\{c, ac\}$, it follows that $\{ba, aba\} = \{c, ac\}$. Since ba cannot be equal to c we have that $c = aba$. The two collineations mapping the point $\{e, a\}$ onto the point $\{b, ab\}$ are b and ab. The inverse images of $\{e, a\}$ under these two collineations give the third points of the two triangles with vertex $\{e, a\}$: $\{b^{-1}, ab^{-1}\}$ and $\{b^{-1}a, ab^{-1}a\}$. We now must distinguish two possibilities: either $\{e, a\}$, $\{b, ab\}$ and $\{b^{-1}, ab^{-1}\}$ are the vertices of a triangle, or $\{e, a\}$, $\{b, ab\}$ and $\{b^{-1}a, ab^{-1}a\}$ are the vertices of a triangle.

1. $\{e, a\}$, $\{b, ab\}$ and $\{b^{-1}, ab^{-1}\}$ belong to a triangle.

   The second triangle in $\{e, a\}$ is then given by the points $\{e, a\}$, $\{aba, ba\}$ and $\{b^{-1}a, ab^{-1}a\}$. The collineation b takes $\{b^{-1}, ab^{-1}\}$ onto $\{e, a\}$ and $\{e, a\}$ onto $\{b, ab\}$. Since there is at most one triangle containing two given points, we see that b must map $\{b, ab\}$ onto $\{b^{-1}, ab^{-1}\}$, and from our previous considerations, this implies that b has order 3. Hence $b^3 = e$.

   Now one has to require that no points of $\Gamma_2(\{e, a\})$ are collinear other than those on a common line through $\{e, a\}$, and the two collinear pairs in the two triangles through $\{e, a\}$.

   This is done similarly to above, with the additional difficulty of dealing with cosets instead of single elements. However, we leave the details to the reader. The eventual conditions read: a has order 2, b has order 3, ab has order bigger than 3 and $aba \neq b$, $aba \neq bab$, $aba \neq bab^2ab$, $aba \neq babab$, $aba \neq b^2abab$, $aba \neq bab^2abab$ and $aba \neq bab^2abab^2ab$.
2. $\{e, a\}$, $\{b, ab\}$ and $\{b^{-1}a, ab^{-1}a\}$ belong to a triangle.

   This case is completely similar to the previous one. One shows that ab has order 3, and that the lines through the point $\{e, a\}$ are $\{\{e, a\}, \{b, ab\}, \{aba, ba\}\}$, $\{\{e, a\}, \{ababa, baba\}, \{ab^2aba, b^2aba\}\}$ and $\{\{e, a\}, \{bab, abab\}, \{b^2ab, ab^2ab\}\}$.

   Eventually, one finds the following conditions on the group elements e, a and b for obtaining the right local structure at the point $\{e, a\}$: a has order 2, b has order bigger than 3, ab has order 3 and $a \neq b^4$, $a \neq b^5$, $a \neq b^3ab^3$, $a \neq b^2ab^4$, $a \neq b^3ab^4$ and $a \neq b^3ab^3ab^3$. Now redefining b as ab, we obtain again the previous case.

Let $G_{s,t} := \langle s, t : s^2 = t^3 = \mathrm{id}\rangle$. Consider the geometry $\Gamma_{G_{s,t},\{\mathrm{id}\}}$ with points the right cosets in the group $G_{s,t}$ of the subgroup $\{\mathrm{id}, s\}$ and with lines the right translates of $\{\{\mathrm{id}, s\}, \{t, st\}, \{ts, sts\}\}$. Clearly $\Gamma_{G_{s,t},\{\mathrm{id}\}}$ is a bislim geometry which is point-locally $\mathsf{LS}(5)$ with $G_{s,t}$ as point transitive collineation group with point stabilizer of order 2. Let N be a normal subgroup of the group $G_{s,t}$ not containing the words $(st)^3$, $[s, t]st$, $[s, t]^2$, $[s, t](st)^2$, $(st^2)^2(st)^2$, $[s, t]^2st$, $[s, t]^3$. We define the quotient geometry $\Gamma_{G_{s,t},N}$ of $\Gamma_{G_{s,t},\{\mathrm{id}\}}$: the points are the right cosets of $\{N, Ns\}$ in $G_{s,t}/N$, the lines are the right translates of $\{\{N, Ns\}, \{Nt, Nst\}, \{Nts, Nsts\}\}$. Then $\Gamma_{G_{s,t},N}$ is a bislim geometry which is point-locally $\mathsf{LS}(5)$, with $G_{s,t}/N$ as point transitive collineation group with point stabilizer of order 2. We obtain the examples of Section 3.3(2).

Left to prove are the additional claims made in (iiib) of Main Result 2. We shall do this now.

Let $G_{a,b} := \langle a, b : a^3 = b^3 = \mathrm{id}\rangle$. Consider the geometry $\widetilde{\Gamma} = \Gamma_{G_{a,b},\{\mathrm{id}\}}$ with points the elements of the group $G_{a,b}$ and lines the right translates of $\{\mathrm{id}, a, b\}$. From above we know that $\widetilde{\Gamma}$ is a sharply point transitive bislim geometry which is point-locally $\mathsf{LS}(5)$.

Let σ interchange the letters a and b in every word of $G_{a,b}$, with $G_{a,b} = \langle a, b : a^3 = b^3 = \mathrm{id}\rangle$. It is easily seen that σ is well defined and is a group automorphism. Let N be a normal subgroup of $G_{a,b}$ not containing $(ab^2)^2$, $baba^2$, $[a, b]$, $abab^2$, $(ab^2)^3$. We define the quotient geometry $\Gamma = \Gamma_{G_{a,b},N}$ of $\widetilde{\Gamma}$: the points are the elements of the quotient group $G_{a,b}/N$, the lines are the right translates of $\{N, Na, Nb\}$. From the foregoing we know that Γ is a bislim geometry which is point-locally $\mathsf{LS}(5)$, with $G_{a,b}/N$ as sharply point transitive collineation group. If σ stabilizes N then it is clear that σ induces a collineation of Γ and in this case the geometry Γ admits a sharply point transitive collineation group as well as a point transitive one with point stabilizer of order 2. Furthermore, suppose that θ is an involution of Γ fixing the point N. It is easy to prove that $(Ng)^\theta = Ng^\sigma$ for all $g \in G_{a,b}$. Thus, if σ does not stabilize N, then Γ only admits a sharply point transitive group.

Let $\Gamma = \Gamma_{G_{a,b},N}$ be a geometry as in the previous paragraph, where σ stabilizes N. Consider the semidirect product of $G_{a,b}$ with $\{\mathrm{id}, \sigma\}$. This group is generated by (a, id), (b, id) and (id, σ). Since $(\mathrm{id}, \sigma)(b, \mathrm{id})(\mathrm{id}, \sigma) = (a, \mathrm{id})$, this group is equal to $G_{\sigma,b} = \langle (b, \mathrm{id}), (\mathrm{id}, \sigma) : (\mathrm{id}, \sigma)^2 = (b, \mathrm{id})^3 = (\mathrm{id}, \mathrm{id})\rangle$, or to $G_{s,t} = \langle s, t : s^2 = t^3 = \mathrm{id}\rangle$ when using s, t and id as shorter notation for (id, σ), (b, id) and $(\mathrm{id}, \mathrm{id})$ respectively. The group $N \cong \{(n, \mathrm{id}) : n \in N\}$ is a normal subgroup of the group $G_{\sigma,b}$.

Now consider the geometry $\Gamma_{G_{\sigma,b},N}$ with points the right cosets of the subgroup $\{N(\mathrm{id},\mathrm{id}), N(\mathrm{id},\sigma)\}$ in $G_{\sigma,b}/N$. The lines are the right translates of

$$\{\{N(\mathrm{id},\mathrm{id}), N(\mathrm{id},\sigma)\}, \{N(b,\mathrm{id}), N(\mathrm{id},\sigma)(b,\mathrm{id})\}, \{N(b,\mathrm{id})(\mathrm{id},\sigma), N(\mathrm{id},\sigma)(b,\mathrm{id})(\mathrm{id},\sigma)\}\}.$$

It is easy to check that the geometries $\Gamma_{G_{a,b},N}$ and $\Gamma_{G_{\sigma,b},N}$ are isomorphic. The group $G_{\sigma,b}/N$ acts on the right on the points of the geometry Γ and is thus the full collineation group of Γ, with point stabilizer of order 2. It is straightforward to check that the conditions of Section 3.3(2) are satisfied. We have thus proved that every geometry of the type described in Section 3.3(3) satisfying $N^\sigma = N$ is isomorphic to a geometry of type mentioned in Section 3.3(2).

Conclusion. Let $G_{s,t} := \langle s, t : s^2 = t^3 = \mathrm{id}\rangle$. The geometry $\Gamma_{G_{s,t},\{\mathrm{id}\}}$ with points the right cosets in $G_{s,t}$ of $\{\mathrm{id}, s\}$ and lines the right translates of $\{\{\mathrm{id}, s\}, \{t, st\}, \{ts, sts\}\}$ is a $1\frac{1}{2}$-connected $1\frac{1}{2}$-cover of every point transitive geometry which is point-locally $\mathsf{LS}(5)$. The group $G_{s,t}$ is a point transitive group with point stabilizer of order 2, while the group $G_{sts,t} := \langle sts, t : (sts)^3 = t^3 = \mathrm{id}\rangle$ is a sharply point transitive collineation group.

Let N be a normal subgroup of $G_{s,t}$ not in $G_{sts,t}$, and not containing $(st)^3$, $[s,t]st$, $[s,t]^2$, $[s,t](st)^2$, $(st^2)^2(st)^2$, $[s,t]^2st$, $[s,t]^3$. The geometry $\Gamma_{G_{s,t},N}$ with points the right cosets of $\{N, Ns\}$ in $G_{s,t}/N$ and with lines the right translates of $\{\{N, Ns\}, \{Nt, Nst\}, \{Nts, Nsts\}\}$ only admits a point transitive collineation group $G_{s,t}/N$ with point stabilizer of order 2. This is (iiia) of Main Result 2.

Let N be a normal subgroup of $G_{sts,t}$ not containing $(stst^2)^2$, $[sts,t]$, $(st)^3st^2$, $(stst^2)^3$. The points of the geometry $\Gamma_{G_{sts,t},N}$ are the elements of $G_{sts,t}/N$, the lines are the right translates of $\{N, Nsts, Nt\}$. If $sNs = N^s = N$, then $\Gamma_{G_{sts,t},N}$ admits a sharply point transitive collineation group $G_{sts,t}/N$ as well as a point transitive one $G_{s,t}/N$ with point stabilizer of order 2. If $N^s \neq N$ then $\Gamma_{G_{sts,t},N}$ has only a sharply point transitive group $G_{sts,t}/N$. This is (iiib) of Main Result 2 and concludes the case of $\mathsf{LS}(5)$ and (iii) of Main Result 2.

5.5. *Local structure* $\mathsf{LS}(13)$

It is shown in Part I that every bislim geometry with $\Gamma_x \cong \mathsf{LS}(13)$, for all points x of Γ, is covered by the honeycomb geometry $\mathcal{S}_\infty$ and an explicit list is available. Hence Γ is a quotient geometry of $\mathcal{S}_\infty$. A standard topological argument shows that the point transitive automorphism group of Γ lifts to a point transitive collineation group of $\mathcal{S}_\infty$ (including all *deck transformations*).

Hence we first classify the point transitive collineation groups of $\mathcal{S}_\infty$, which do not act flag transitively. We introduce some notation.

We may identify the points and lines of $\mathcal{S}_\infty$ with the vertices of the "honeycomb" tiling of the real Euclidean plane $\mathbb{E}$ in regular hexagons. Let e be a vertex corresponding to a line of $\mathcal{S}_\infty$, and let a, d, f be the points incident with e (and hence the vertices adjacent to e). Let b be the unique point of $\mathcal{S}_\infty$ contained in a triangle together with a and f (hence a, f, b are vertices of the same hexagon in the tiling) and let c be the vertex corresponding with the line ab of $\mathcal{S}_\infty$. Denote by h the center of the hexagon containing a, b, f. Let $W(\widetilde{\mathsf{A}}_2)$ be the full collineation group of $\mathcal{S}_\infty$, or equivalently, the group of isometries of $\mathbb{E}$ preserving the honeycomb tiling and stabilizing each bipartition class (which is the Weyl group of type $\widetilde{\mathsf{A}}_2$, whence the notation). To avoid confusion, we will call the lines of the Euclidean plane $\mathbb{E}$ *Euclidean lines* and abbreviate this to *E-lines*.

There are two rotations $r^+_{ab}, r^-_{ab} \in W(\widetilde{\mathsf{A}}_2)$ mapping a to b, and they have centers c and h, respectively. The rotation r^+_{ab} fixes the line c of $\mathcal{S}_\infty$, while r^-_{ab} acts freely on the lines of $\mathcal{S}_\infty$. We call every conjugate of r^+_{ab} a *hyperbolic rotation*, and every conjugate of r^-_{ab} an *elliptic* one. Let r^+_{df} and r^-_{df} be the hyperbolic and elliptic rotations, respectively, mapping d to f. Then we call the group $\langle r^+_{ab}, r^+_{df}\rangle \trianglelefteq W(\widetilde{\mathsf{A}}_2)$ the *transitive hyperbolic collineation group* of $\mathcal{S}_\infty$, and likewise $\langle r^-_{ab}, r^-_{df}\rangle \trianglelefteq W(\widetilde{\mathsf{A}}_2)$ is the *transitive elliptic collineation group* of $\mathcal{S}_\infty$. For two points x, y denote by t_{xy} the unique translation mapping x to y. Then $\langle t_{ab}, t_{ad}\rangle \trianglelefteq W(\widetilde{\mathsf{A}}_2)$ is called the *transitive parabolic collineation group* of $W(\widetilde{\mathsf{A}}_2)$.

Let, in the Euclidean plane $\mathbb{E}$, p be the intersection of the E-lines through a, e and d, f, respectively. Also, let p' be the mid-point of the segment $[e, d]$. Let L be the E-line through p' parallel to $\overrightarrow{ae}$. Then the composition $t_{ap}r_L = r_L t_{ap}$ of the translation t_{ap} with the reflection r_L about L belongs to $W(\widetilde{\mathsf{A}}_2)$ and maps a to d. It is a so-called *glide reflection* (the axis of the reflection is parallel to the translation vector). We call the group $\langle t_{ap}r_L, t_{ab}\rangle$ the *transitive glide collineation group* of $\mathcal{S}_\infty$.

Finally, let ρ be the reflection in $\mathbb{E}$ about the E-line through the vertices a and e.

Lemma 5.3. *If $H \le W(\widetilde{\mathsf{A}}_2)$ acts transitively on the point set of $\mathcal{S}_\infty$, but not flag transitively, then either H acts sharply transitively and coincides with the transitive hyperbolic, elliptic or parabolic collineation group or one of the three conjugates of the glide collineation group, or H_a has order 2, for any point a of $\mathcal{S}_\infty$ and H is one of the three conjugates of the subgroup generated by ρ and the transitive hyperbolic, elliptic, parabolic, respectively, collineation group.*

Proof. If H acts sharply transitively, then there are unique collineations in H taking a to b and a to d. These are either translations, rotations or glide reflections (a reflection is impossible since it fixes points of $\mathcal{S}_\infty$). A tedious but rather straightforward case-by-case study implies the assertion.

If H does not act sharply transitively, then the point stabilizer H_a must have order exactly 2 (as otherwise we have a flag transitive action). Hence H contains *indirect isometries* (these are isometries whose matrix has determinant equal to -1). The normal subgroup N of H consisting of *direct isometries* (the matrix has determinant equal to $+1$) has index 2 in H

and acts sharply transitively on the point set of $\mathcal{S}_\infty$, by a standard argument. The assertion now follows from the sharply transitive case. □

From now on we may assume that Γ is not isomorphic to the honeycomb geometry.

Now, in Part I of the present work, we classified all quotients of the honeycomb geometry. Referring to that list, it is clear that the examples involving a glide reflection g (with minimal translation vector) do not admit a point transitive group. Indeed,by the above lemma, every point transitive collineation group of $\mathcal{S}_\infty$ different from the three glide collineation groups contains a translation t in any of the three directions given by the edges of the tiling of $\mathbb{E}$ in regular hexagons mapping a point onto another point at graph theoretic distance 4. Choosing this direction not parallel to the axis of the glide reflection, and choosing a vertex x as close as possible to the axis, we see that $(x^t)^g$ and $(x^g)^t$ are at graph theoretic distance 6 from each other, and the line joining them is perpendicular to the axis of the glide reflection. These vertices represent the same line in the local structure of the point y at graph theoretic distance 3 from both $(x^t)^g$ and $(x^g)^t$; yet this line is distinct from the one defined by the vertex in any hexagon containing y. So the local structure in y cannot be LS(13), a contradiction. Likewise, every glide collineation group contains a translation t' in the direction perpendicular to the direction of its glide reflections mapping a point onto another point at graph theoretic distance 2. This direction is then also perpendicular to the axis of the glide reflection g, and choosing a point x as close as possible to the axis, we see that, similarly to above, $(x^{t'})^g$ and $(x^g)^{t'}$ can never be identified without violating the local structure, a contradiction.

Hence Γ is isomorphic either to $\mathcal{S}_{(r,s)}$, with $0 \leq s \leq r$ and $r^2 + rs + s^2 \geq 12$, or to $\mathcal{M}_{(a,0),(c,d)}$, with a, c and d integers with $a, d > 0$, $0 \leq c < a$ and for every integer linear combination of $(a, 0)$ and (c, d), say (r, s), $r^2 + rs + s^2 \geq 12$. We determine the point transitive groups which do not act flag transitively.

Clearly, the transitive parabolic group always induces a sharply point transitive collineation group of Γ. It is also clear that, in the case of $\Gamma \cong \mathcal{S}_{(r,s)}$, every collineation θ of the honeycomb geometry that induces a collineation in Γ has to fix $\pm(r, s)$ "as a set of two vectors", i.e., $(r, s)^\theta - (0, 0)^\theta = \pm(r, s)$. We denote (r, s) as a vector by $\overrightarrow{(r,s)}$. Hence G cannot be induced by the elliptic or hyperbolic transitive collineation group, nor any extension of these. If θ is a (glide) reflection which induces an element of G, then clearly the axis of θ must be either parallel or perpendicular to $\overrightarrow{(r,s)}$. Parallel axes occur only for $r = s$, while perpendicular axes only occur if $s = 0$ (this follows from our choice of coordinates in [2,3]). In all these cases, the transitive glide collineation group (with appropriate axes of the glide reflections) induces a sharply point transitive collineation group of Γ, and the group generated by the transitive parabolic collineation group and an appropriate reflection induces a point transitive collineation group in Γ with size of the point stabilizer equal to 2.

Now suppose $\Gamma \cong \mathcal{M}_{(a,0),(c,d)}$. If there is some rotation in $W(\widetilde{\mathsf{A}}_2)$ preserving the identification giving rise to Γ, then we have a geometry $\mathcal{G}_{(r,s)}$ (see [2]) and then the assertions follow from the observations that all collineations of the honeycomb geometry induce collineations of the square and triple square geometry, and that no (glide) reflection survives in a geometry $\mathcal{G}_{(r,s)}$ which is not a square or triple square geometry. Hence we may suppose that only (glide) reflections of the honeycomb geometry give rise to extra collineations in Γ, and all the axes are parallel. This gives us three possibilities, depending on the direction of the axes. Also, since we have the full translation group, we only have to assume that some reflection acts on Γ. Without loss of generality, we may assume that the axis contains the point $(0, 0)$.

Now we observe that a reflection ρ with axis containing $(0, 0)$ preserves the identification induced by $(a, 0)$ and (c, d) if and only if $(a, 0)^\rho$ and $(c, d)^\rho$ are identified with $(0, 0)$. In other words, they are linear combinations with integer coefficients of $(a, 0)$ and (c, d). It is now an easy exercise to express this for the different reflections $(x, y) \leftrightarrow (y, x)$, $(x, y) \leftrightarrow (x, -x-y)$ and $(x, y) \leftrightarrow (-x - y, y)$. We obtain the conditions stated in Main Result 2(iv).

5.6. *Local structure* LS(24)

As in this case, the geometry is again a quotient of the honeycomb geometry with only a limited number of possibilities (see Part I), the proof is completely analogous to that for the previous case of LS(13).

It is easy to prove that all the geometries are mutually non-isomorphic. Indeed, of those having the same number of points, there is exactly one admitting a point transitive collineation group with point stabilizer of order 2 and the other one has as full collineation group a sharply point transitive one.

5.7. *Local structure* LS(51)

There is a unique example for each countable cardinality of the point set and it is shown in Part I that this example admits a sharply point transitive cyclic group. Furthermore, since LS(51) has no non-trivial symmetries, this cyclic group must be the full collineation group and we are done.

This completes the proof of Main Result 2.

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References

[1] H.S.M. Coxeter, Self dual configurations and regular graphs, Bull. Amer. Math. Soc. 56 (1950) 413–455.
[2] H. Van Maldeghem, V. Ver Gucht, Bi-slim flag-transitive geometries of gonality 3: Construction and classification, Ars Combin. 75 (2005) 75–96.
[3] H. Van Maldeghem, V. Ver Gucht, Transitive bislim geometries of gonality 3. I. The geometrically homogeneous cases, European J. Combin. 28 (2007) 1530–1551.