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# Topological Polygons and Affine Buildings of Rank Three (***).

**Sunto.** – *Si introduce la nozione di n-gono generalizzato topologico e si mostra che il building all'infinito di ogni building affine, localmente finito, di rango 3 è un n-gono compatto, totalmente disconnesso, con* $n \in \{3, 4, 6\}$. *Di consequenza, tutti i piani proiettivi finiti di ordine fissato sono immagini epimorfe continue di un qualche piano proiettivo compatto, totalmente disconnesso (ed analogamente per i quadrangoli e gli esagoni).*

Affine buildings of rank at least 4 are «classical», see Tits [38]. Many examples of non-classical affine buildings of rank 3 have been constructed, see Ronan [28], Kantor [24], [40]. Here we investigate affine buildings of rank 3 and establish some connections with topological polygons. The concept of topological polygons is introduced as a generalization of topological projective planes. Special attention is given to compact totally disconnected polygons and inverse limit representations. By (3.4.3) the geometry «at infinity» of every locally finite affine building of rank 3 is a compact totally disconnected $n$-gon with $n \in \{3, 4, 6\}$. Together with a strong existence theorem of Ronan [28] this yields the following result (4.2): for every natural number $n$ there exists a compact totally disconnected projective plane which admits continuous epimorphisms onto all projective planes of order $n$; there are analogous results for generalized quadrangles and generalized hexagons.

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## 1. – Definition of topological polygons.

An incidence structure (of rank 2) is a triple $(P, \mathfrak{L}, I)$ of non-empty sets with $I \subseteq P \times \mathfrak{L}$ and $P \cap \mathfrak{L} = \emptyset$. Every incidence structure $(P, \mathfrak{L}, I)$ gives rise to a graph $(V, *)$ with vertex set $V = P \cup \mathfrak{L}$ and adjacency relation $*$ defined by

$$x * y \Leftrightarrow (x = y \text{ or } xIy \text{ or } yIx);$$

hence $*$ is reflexive and symmetric (unlike $I$). Let $d\colon V^2 \to \mathbb{N}_0 \cup \{\infty\}$ be the path metric of $(V, *)$, i.e. $d(x, y)$ is the smallest integer $k \in \mathbb{N}_0$ such that there exists a chain of elements $x_1, \ldots, x_{k-1} \in V$ with $x * x_1 * x_2 * \ldots * x_{k-2} * x_{k-1} * y$, and $d(x, y) = \infty$ if $x$ and $y$ are not connected by such a path.

Let $n \geqslant 2$ be an integer. A *generalized n-gon* is an incidence structure $S = (P, \mathfrak{L}, I)$ satisfying the following axioms:

(i) $n = \max\{d(x, y)\colon x, y \in P \cup \mathfrak{L}\}$; in particular, $(V, *)$ is connected.

(ii) If $d(x, y) = k < n$, then there is a unique chain $x_1, x_2, \ldots, x_{k-1}$ such that $x * x_1 * x_2 \ldots x_{k-2} * x_{k-1} * y$.

Furthermore we always assume that $S$ is thick, i.e.

(iii) Every element of $P \cup \mathfrak{L}$ is incident with at least three elements.

Cp. Tits [36] § 11, Kantor [24].

The axioms for a generalized $n$-gon $S = (P, \mathfrak{L}, I)$ can be expressed by saying that $(V, *)$ is a connected bipartite graph of diameter $n$ and girth $2n$ such that every vertex has valency at least three, cp. Kantor [24]. Up to duality, the generalized $n$-gon $S$ can be reconstructed from the graph $(V, *)$.

Every line of a generalized $n$-gon $S$ carries the same (possibly infinite) number $r$ of points; dually, every point is incident with the same number $s$ of lines. The pair $(r, s)$ is called the *order* of $S$.

Remarks. (1) A generalized 2-gon (digon) is a trivial incidence structure, every point being incident with every line; the corresponding graph $(V, *)$ is a complete bipartite graph.

(2) The generalized 3-gons are exactly the projective planes. Later on we shall deal also with generalized 4-gons (quadrangles) and generalized 6-gons (hexagons).

(3) By the Feit-Higman theorem, finite (thick) generalized $n$-gons exist only for $n \in \{2, 3, 4, 6, 8\}$, see Feit-Higman [12], D. G. Higman [22] 3.2, Kilmoyer-Solomon [25].

Let $(V, *)$ be the graph determined by a generalized $n$-gon $S$. If $x, y \in V$ with $d(x, y) = k$ and $2 \leqslant k \leqslant n-1$, then there exist uniquely determined elements $x_1, \ldots, x_{k-1} \in V$ such that $x * x_1 * x_2 * \ldots * x_{k-1} * y$. Hence we have a map

$$F_k \colon \{(x, y) \in V^2 \colon d(x, y) = k\} \to V^{k-1}$$

with $F_k(x, y) = (x_1, x_2, \ldots, x_{k-1})$. The maps $F_k$ may be regarded as the « geometric operations » of $S$; e.g. for $n = 3$, the map $F_2$ combines the two operations of joining distinct points and intersecting distinct lines in the projective plane $S$.

(1.1) DEFINITION. A *topological $n$-gon* is a generalized $n$-gon $S = (P, \mathfrak{L}, I)$ with non-discrete Hausdorff topologies on $P$ and $\mathfrak{L}$ such that the topological sum $V = P \cup \mathfrak{L}$ satisfies

(TOP) all maps $F_k \colon \{(x, y) \in V^2 \colon d(x, y) = k\} \to V^{k-1}$ as defined above with $2 \leqslant k \leqslant n-1$ are continuous.

(1.2) COMMENTS. (1) Let $S = (P, \mathfrak{L}, I)$ be a generalized digon. Then all non-discrete Hausdorff topologies on $P$ and $\mathfrak{L}$ render $S$ a topological digon.

(2) Let $n = 3$, i.e. let $S$ be a projective plane. Then condition (TOP) means that joining distinct points and intersecting distinct lines are continuous operations ; this is the usual requirement in the definition of a topological projective plane, cp. Salzmann [29], Salzmann [30]. We remark that non-trivial topologies on the point set $P$ and on the line set $\mathfrak{L}$ of a projective plane are automatically Hausdorff, if (TOP) is satisfied, cp. Salzmann [29], p. 492, Salzmann [30] (1.7).

(3) Let $n = 4$, i.e. let $S$ be a generalized quadrangle. The definition of a topological quadrangle used in Forst [13], [17] requires only continuity of the map $F_3$; however, this implies continuity of $F_2$, cp. [17] (2.3), and is therefore equivalent to (TOP).

Every locally compact (connected) Laguerre-plane of topological dimension 2 gives rise to a compact (connected) quadrangle of topological dimension 3, see Forst [13] 5.10; in fact, a beautiful result of Schroth [32] says that every compact quadrangle of dimension 3 arises in this fashion. Thus many examples of non-classical compact (connected) quadrangles are obtained from the non-miquelian Laguerre-planes constructed in Groh [15], Artzy-Groh [3], Löwen-Pfüller [27], Steinke [34], [35].

(4) There exist strong connections between the maps $F_k$, and it is conceivable that continuity of $F_{n-1}$ implies continuity of the other maps $F_k$, cp. (3).

(5) Burns-Spatzier [6] define topological buildings by only requiring the set of (ordered) flags of fixed length $k$ to be closed in $V^k$ (hence for topological polygons they only require that the adjacency relation is closed in $V^2$). This concept seems to be too weak in general, as the following examples show. Let $F$ be a field with a ring topology; then in every projective space $PG(m, F)$ of finite dimension $m$ over $F$, every such set of flags is closed with respect to the natural topology of $PG(m, F)$ (obtained from the quotient topology of $F^{m+1}\setminus\{0\} \to$ $\to PG(m, F)$). But there exist ring topologies on the field $\boldsymbol{Q}$ of rational numbers which are not field topologies, see e.g. Weber [42] (1.9), (3.4); with these topologies, $PG(m, \boldsymbol{Q})$ is not a topological projective space in the usual sense.

Burns-Spatzier [6] use their definition only for compact topologies on $V$; cp. (2.1) and Burns-Spatzier [6] 1.12 for this special case. One of the main results of Burns-Spatzier [6] says that the topological automorphism group of every irreducible compact metric building of rank at least 2 is locally compact in the compact-open topology.

## 2. – Compact polygons, inverse limits, examples.

This section contains some simple criteria for compact topological polygons, leading to large classes of classical and non-classical compact polygons.

(2.1) LEMMA. Let $(P, \mathfrak{L}, I)$ be an infinite generalized $n$-gon with compact topologies on $P$ and on $\mathfrak{L}$. Assume that the incidence relation $I$ is closed in $P \times \mathfrak{L}$. Then the following hold:

(*a*) $(P, \mathfrak{L}, I)$ is a compact topological $n$-gon.

(*b*) The distance function $d\colon (P \cup \mathfrak{L})^2 \to \{0, 1, \ldots, n\}$ is lower semi-continuous, i.e. the sets $d^{-1}(\{i, i+1, \ldots, n\})$ are open in $(P \cup \mathfrak{L})^2$ for every $i$.

(*c*) The topology of $P$ (and the topology of $\mathfrak{L}$) is determined uniquely by the topologies induced on a line and on a pencil of lines.

PROOF. $P, \mathfrak{L}$ and $V = P \cup \mathfrak{L}$ are non discrete Hausdorff spaces (being compact), and $* = I \cup \mathrm{Id}_V \cup I^{\mathrm{converse}}$ is closed in $V^2$.

(*a*) Concerning continuity of $F_k\colon d^{-1}(\{k\}) \to V^{k-1}$, it suffices to show that the graph of $F_k$ is closed in $X := d^{-1}(\{k\}) \times V^{k-1}$, cp. Dugundji [10] XI.2.7. This graph can be written as

$$\{(x, y, x_1, x_2, \ldots, x_{k-1}) \in X\colon x * x_1 * x_2 * \ldots x_{k-1} * y\}$$

and is therefore closed in $X$.

(*b*) The set $d^{-1}(\{0\}) = \mathrm{id}_V$ is closed in $V^2$, as $V$ is a Hausdorff space. For $i > 0$, we have

$$d^{-1}(\{0, 1, \ldots, i\}) = \{(x, y) \in V^2\colon d(x, y) \leqslant i\} =$$
$$= (p_0 \times p_i)[\{(x_0, x_1, \ldots, x_i) \in V^{i+1}\colon x_j * x_{j+1} \text{ for } 0 \leqslant j < i\}],$$

where $p_k\colon V^{i+1} \to V$ denotes the (continuous) projection onto the coordinate indexed by $k$. The square bracket [...] is closed in $V^{i+1}$, hence compact. Therefore $d^{-1}(\{0, 1, \ldots, i\})$ is compact, hence closed in $V^2$.

(*c*) Pick an ordinary (non-degenerate) $n$-gon in $(P, \mathfrak{L}, I)$, and denote by $x_1, \ldots, x_n$ its points (for $n$ even) resp. its lines (for $n$ odd). Then

$$P = \bigcup_{i=1}^{n} \{x \in P\colon d(x, x_i) = n\},$$

and each set $\{x \in P\colon d(x, x_i) = n\}$ is open in $P$ by (*b*), hence $P$ has the weak topology determined by these sets, cp. Dugundji [10] III.9.3, VI.8. Each set $\{x \in P\colon d(x, x_i) = n\}$ is geometrically and topologically a product of «punctured» lines and «punctured» pencils, i.e. lines minus a point resp. pencils minus a line (for generalized quadrangles see Forst [13], [18], [17] 2.7). Finally, any two lines (or pencils) are homeomorphic via projectivities (cp. Tits [36] p. 59, [26]). Q.E.D.

(2.2) *Examples.* Let $K$ be a non-discrete locally compact skew field. The point set of the projective space $PG(n, K)$ of finite dimension $n$ over $K$ is compact (with respect to the quotient topology obtained from $K^{n+1}\setminus\{0\} \to PG(n, K)$); furthermore every Graßmann manifold $G_d$ consisting of all subspaces of $K^{n+1}$ of fixed dimension $d$ is compact as well.

Hence every generalized polygon embedded into $PG(n, K)$ as a closed subset (of $G_1 \times G_d$, with natural incidence) is a compact topological polygon by (2.1). This observation yields large classes of compact polygons, connected ones as well as totally disconnected ones. We indicate a few concrete instances.

The desarguesian projective plane $PG(2, K)$ in its usual representation $(G_1, G_2, \subseteq)$ by subspaces of $K^3$ is the most familiar example of this type. The projective Moufang plane over the real Cayley division algebra (octonion algebra) may be represented by closed subsets of $G_1$ and $G_{26}$ in $PG(26, \mathbf{R})$, cp. Freudenthal [14] § 7, Springer [33] p. 82, Faulkner-Ferrar [11] p. 258.

Every continuous polarity of $PG(n, K)$ of Witt index 2 yields a compact topological quadrangle (except if $n = 4$ and the polarity is orthogonal); by Dienst [9] every generalized quadrangle embedded into $PG(n, K)$ is in fact embedded into a polarity.

Let $K$ be commutative. The generalized hexagon belonging to the split Cayley algebra over $K$ is a compact topological hexagon, since it can be realized by closed subsets of $G_1$ and $G_2$ in $PG(7, K)$, see Schellekens [31] p. 202.

In the sections to follow we shall use limits of inverse systems. An inverse system is a sequence $X_i$, $i \in N_0$, of sets with «bonding maps» $f_i \colon X_i \to X_{i-1}$ for $i \in N$. Its inverse limit $X_\infty$ is defined by

$$X_\infty = \varprojlim_i X_i = \varprojlim_i (X_i, f_i) = \Big\{x \in \prod_i X_i \colon f_n(x_n) = x_{n-1} \text{ for all } n \in N\Big\}.$$

The inverse limit of an inverse system $X_i$, $i \in N_0$, of topological spaces with continuous bonding maps $f_i$ is the set $X_\infty$ endowed with the product topology (cp. Hocking-Young [23] p. 91, Christenson-Voxman [7] 6.B, Dugundji [10] p. 427). Let $S_i = (P_i, \mathfrak{L}_i, I_i)$, $i \in N_0$, be an inverse system of incidence structures with bonding maps $f_i \colon S_i \to S_{i-1}$ which are incidence preserving, i.e. $f_i$ is a pair of mappings $(g_i, h_i) \colon P_i \times \mathfrak{L}_i \to P_{i-1} \times \mathfrak{L}_{i-1}$ such that $(g_i, h_i)(I_i) \subseteq I_{i-1}$. The inverse limit is then defined by

$$S_\infty = \underleftarrow{\lim}_i S_i := (P_\infty, \mathfrak{L}_\infty, I_\infty)$$

with $P_\infty = \underleftarrow{\lim}_i (P_i, g_i)$, $\mathfrak{L}_\infty = \underleftarrow{\lim}_i (\mathfrak{L}_i, h_i)$ and

$$I_\infty = \underleftarrow{\lim}_i (I_i, f_i) = (P_\infty \times \mathfrak{L}_\infty) \cap \prod_i I_i .$$

If $F$ is a locally compact local field, i.e. a finite extension of the $p$-adic field $\boldsymbol{Q}_p$ or of the Laurent series field $\boldsymbol{F}_p((x))$, then the natural valuation ring $\mathcal{O}$ of $F$ with maximal ideal $\mathfrak{I}$ can be written as an inverse limit of finite rings:

$$\mathcal{O} = \underleftarrow{\lim}_i \mathcal{O}/\mathfrak{I}^i .$$

This implies that the projective plane over $F$, which coincides with the ring plane over $\mathcal{O}$, can be written as the inverse limit of the finite Hjelmslev-planes coordinatized by the rings $\mathcal{O}/\mathfrak{I}^i$ (cp. Cronheim [8] p. 216).

For $n = 3$, the following lemma is contained in [16] (1.1).

(2.3) LEMMA. Let $(P, \mathfrak{L}, I)$ be an infinite generalized $n$-gon which (as an incidence structure) can be written as an inverse limit of a sequence of finite incidence structures $(P_i, \mathfrak{L}_i, I_i)$, $i \in \boldsymbol{N}$:

$$(P, \mathfrak{L}, I) = \underleftarrow{\lim}_i (P_i, \mathfrak{L}_i, I_i) .$$

Endow $P$ and $\mathfrak{L}$ with the inverse limit topology derived from discrete topologies on $P_i$ and $\mathfrak{L}_i$. Then $I$ is closed in $P \times \mathfrak{L}$, and $(P, \mathfrak{L}, I)$ is a compact totally disconnected topological $n$-gon; in fact, $P$ and $\mathfrak{L}$ are homeomorphic to the Cantor set. The projections of the cartesian product $\prod_i (P_i, \mathfrak{L}_i, I_i)$ yield continuous homomorphisms of $(P, \mathfrak{L}, I)$ into $(P_i, \mathfrak{L}_i, I_i)$; these homomorphisms are surjective on points (lines, flags) if all bonding maps of the inverse system $(P_i, \mathfrak{L}_i, I_i)$, $i \in \boldsymbol{N}$, are surjective on points (lines, flags).

PROOF. The cartesian products $\prod_i P_i$ and $\prod_i \mathfrak{L}_i$ as well as $P$ and $\mathfrak{L}$ are compact and totally disconnected. Furthermore $I = \underleftarrow{\lim}_i I_i$ is compact, hence closed in $P \times \mathfrak{L}$. Therefore $(P, \mathfrak{L}, I)$ is a topological $n$-gon by (2.1). In fact, $P$ and $\mathfrak{L}$ have no isolated points, since the projectivities of a line onto itself act transitively on this line (cp. Knarr [26] 1.2). Hence $P$ and $\mathfrak{L}$ are homeomorphic to the Cantor set by a result of Brouwer, see Hausdorff [21] § 29, XVI, p. 99 and § 35, V, p. 197, Hocking-Young [23] 2-97 and 2-98, or Christenson-Voxman [7] 6.C.11. The projections of the cartesian product are continuous by definition of the product topology; for surjectivity see Hocking-Young [23] 2-84. Q.E.D.

Examples for the application of (2.3) can be found in (3.4.3) and section 4.

## 3. – A class of compact totally disconnected topological $n$-gons ($n = 3$, 4, 6) arising from rank 3 affine buildings.

In this section we show that the geometry at infinity (see (3.4)) of a locally finite rank 3 affine building can be given the structure of a compact totally disconnected $n$-gon in a natural way.

(3.1) *Notation.*

If $X$ is a rank 2 incidence structure [5] of points and lines, then we denote by $P(X)$ its set of points, by $L(X)$ its set of lines, and $I$ will always be the incidence relation. Points and lines are also called *varieties* (as for a general incidence structure), and the *type map* $t$ assignes to any variety its *type*, i.e. either « point » or « line ». Suppose $G = (W_1, W_2, W_3, I)$ is a rank 3 incidence structure and let $x \in W_1$. Denote by $R_x = \big(W_2(R_x), W_3(R_x), I\big)$ the rank 2 incidence structure with

$$W_2(R_x) = \{v \in W_2\colon vIx\}$$

$$W_3(R_x) = \{v \in W_3\colon vIx\}$$

and incidence is the same as in $G$. $R_x$ is called the *residue* of $x$ in $G$. Note that we can view $R_x$ as a point-line geometry in two different ways, i.e.

(1) call $W_2(R_x)$ points and $W_3(R_x)$ lines

(2) call $W_2(R_x)$ lines and $W_3(R_x)$ points

(3.2) *Rank* 3 *affine buildings.*

*Apartments.* Suppose $A$ is the real Euclidean plane and consider three triangles $T_1, T_2, T_3$ with respective angles $(60°, 60°, 60°)$, $(45°, 45°, 90°)$, $(30°, 60°, 90°)$ (we consider each triangle as the convex closure of its vertices). Suppose $L_j^{(i)}$, $j = 1, 2, 3$ support the sides of $T_i$, $i = 1, 2, 3$. Denote by $W_i$ $(i = 1, 2, 3)$ the group of isometries of $A$ generated by the reflections about $L_j^{(i)}$, $j = 1, 2, 3$.

Denote by $L_i$ the set of lines of $A$ for which $W_i$ contains the reflection about it. $L_i$ defines a tessellation of $A$ into triangles isometric to $T_i$. These triangles are called *chambers*. By definition, $(A, L_i)$ is a *standard apartment* and $W_i$ is its *Weylgroup*. Fix now $i \in \{1, 2, 3\}$. The vertices of the chambers are also called the *vertices* of the apartment. Two vertices are called *adjacent* if they lie on a common chamber. Suppose $x$ is a vertex and suppose $L \in L_i$ with $x$ on $L$ in $A$. Then a (topologically) closed half-line $p$ contained in $L$ and bounded by $x$ is called a *panel of a quarter*, or briefly a *panel* (*with source* $x$). Also, $x$ is called *special* if for any $L \in L_i$, there exists $L^* \in L_i$ such that $L^*$ is parallel to $L$ and $x$ is on $L^*$. If $i = 1$, then all vertices are special. If $i = 2$ (resp. 3), then only vertices at the angles of 45° (resp. 30°) of the chambers are special (see [4]). Suppose $x$ is special and $P_x$ is the set of panels with source $x$. The (topological) closure of a connected component of $A - \bigcup P_x$ is called a *quarter with source* $x$. A subset of $A$ is called *chamber convex* if it is convex (in the usual sense) and if it is either a (not necessarily finite) union of chambers, or if it is a panel (of a quarter), or a wall, or an interval contained in a wall and bounded by two vertices. Note that the standard apartment we just described is nothing else than the (geometric realization of a) Coxeter complex of irreducible affine type of rank 3. All of the above definitions are standard concepts (see Bourbaki [4]).

*System of apartments.*

Fix again $i \in \{1, 2, 3\}$ and consider the standard apartment $(A, L_i)$. Suppose $J = \{1, 2, 3\}$. A *rank* 3 *affine building*, also called a *discrete, complete system of apartments* $(\Delta, F)$ *over* $J$ is a set $\Delta$ together with a family $F$ of injections from $A$ into $\Delta$, satisfying axioms (AB1), (AB2), (AB3), (AB4) and (AB5) below. The images of $A$ under the elements of $F$ are called the *apartments* of $\Delta$. Also, the image under any element of $F$ of a chamber, a panel, (adjacent) vertices, a (special) vertex, a quarter is also called resp. a *chamber*, a *panel*, etc...

Elements of $\Delta$ are called points. Keeping $i \in \{1, 2, 3\}$ fixed, we now give the five axioms mentioned above.

(AB1) $F \circ W_i = F$.

(AB2) If $f, f' \in F$, then the set $B = (f^{-1} \circ f')(A)$ is chamber convex in $A$ and there exists $w \in W_i$ such that $f|_B = f' \circ w|_B$.

(AB3) Any two points of $\Delta$ lie in a common apartment.

(AB4) If $f \in F$ and $x \in f(A)$, then there exists a retraction map (i.e. an idempotent surjection) $\varrho: \Delta \to f(A)$ such that $f^{-1} \circ \varrho \circ f'$ diminishes distances in $A$ for every $f' \in F$, and such that $\varrho^{-1}(x) = \{x\}$.

The preceding axioms imply that $\Delta$ can be viewed as a combinatorial abstract affine building (see Tits [38], cp. also an unpublished correction).

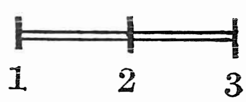
Now it is known that there exists a unique maximal set of apartments for $\Delta$ (see Tits [38], theorème 1). Hence the last axiom:

(AB5) The building $\Delta$ is thick and is endowed with a maximal set $\{f(A): f \in F\}$ of apartments.

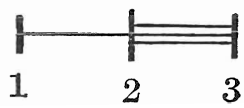
The thickness of $\Delta$ is equivalent to saying that any panel of a quarter belongs to at least three distinct quarters with same source.

There is a type map $t$ from the set of vertices vert $(\Delta)$ of $\Delta$ to $J$ which turns $\Delta$ into a rank 3 geometry. Actually, a similar map $t_A$ can be defined over the set of vertices of the standard apartment $A$, and $t$ can be viewed as the image of $t_A$ under the mappings $f \in F$. Such a mapping $t$ is well defined by (AB2). We denote that geometry also by $\Delta$. The varieties are the vertices and incidence is adjacency. The Buekenhout-diagram [5] is:


$i = 1$, type $\tilde{A}_2$




$i = 2$, type $\tilde{C}_2$




$i = 3$, type $\tilde{G}_2$

So suppose $i = 1$. All residues are projective planes and all vertices are special. Next, suppose $i = 2$. The residues of varieties of type 1 or 3 are generalized quadrangles and these of type 2 are generalized digons. The vertices of type 1 or 3 are special, and we call the vertices of type 2 *anti-special*. Finally, suppose $i = 3$. The residues of varieties of type 1 are generalized hexagons, those of type 2 are generalized digons and those of type 3 are projective planes. The vertices of type 1 are special, and we call the vertices of type 2, resp. 3, *anti-special*, resp. *non-special*.

(3.3) *The n-th floor of a rank 3 affine building.*

Throughout we keep $i \in \{1, 2, 3\}$ fixed.

(3.3.1) The next definition can be generalized to all affine buildings. Suppose $\Delta$ is a rank 3 affine building, $v \in \text{vert}(\Delta)$ and $v$ is special. For any panel $p$ with source $v$, we denote by $v_p$ the vertex in $\Delta$ adjacent to $v$ and lying on $p$. We can consider $R_v$ as a point-line geometry in two distinct ways. In the $\tilde{A}_2$-case the two choices for $P(R_v)$ and $L(R_v)$ are equivalent but we take in the $\tilde{C}_2$-case $P(R_v)$ to be the set of special vertices in $R_v$, and consequently $L(R_v)$ is the set of anti-special vertices in $R_v$. In the $\tilde{G}_2$-case, $P(R_v)$ is the set of anti-special vertices in $R_v$, and hence $L(R_v)$ is the set of non-special vertices in $R_v$. Define

$$P(V_n) = \{v_n \in \text{vert}(\Delta)\text{: there exists a panel } p \text{ with source } v \text{ containing } v_n \text{ such that } v_p \in P(R_v) \text{ and } d(v, v_n) = n\}$$

$$L(V_n) = \{v_n \in \text{vert}(\Delta)\text{: there exists a panel } p \text{ with source } v \text{ containing } v_n \text{ such that } v_p \in L(R_v) \text{ and } d(v, v_n) = n\}$$

where $d$ is the path metric in $\Delta$, i.e. $d(x, y)$ is the smallest integer $k \in \mathbf{N}_0$ such that there exists a chain $(x, v_1, v_2, \dots, v_{k-1}, y)$ of consecutively adjacent vertices of $\Delta$.

*The n*-th *floor of $\Delta$ with basement $v$* is the rank 2 incidence structure $V_n = (P(V_n), L(V_n), I)$ with $v_n I w_n$ $(v_n \in P(V_n), w_n \in L(V_n))$ iff $v_n$ and $w_n$ lie on a common quarter with source $v$. The *type* of the $n$-th floor is the type of the corresponding rank 3 affine building. Note that $R_v \equiv V_1$.

(3.3.2) There is a map $\pi^n_{n-1}\colon V_n \to V_{n-1}$ from the $n$-th floor with basement $v$ onto the $n-1$st floor with basement $v$, mapping the vertex $v_n$ on the panel $p$ with source $v$ onto the vertex $v_{n-1}$ on $p$ with $d(v, v_{n-1}) = n - 1$. This map clearly preserves incidence and is surjective on the set of incident point-line pairs of $V_{n-1}$.

(3.3.3) For the $\tilde{A}_2$-case, the $n$-th floors are investigated in detail in [41], [19]. They are in fact projective Hjelmslev planes of level $n$ (for definitions cp. Artmann [1]). Artmann [2] shows that the inverse limit of the sequence $(V_j, \pi^j_{j-1})_{j\in N}$ is a projective plane. In the next section this result is generalized to sequences of $n$-th floors in buildings of type $\tilde{C}_2$ and $\tilde{G}_2$, see (3.4.2).

(3.4) *The geometry at infinity of a rank* 3 *affine building.*

Suppose $\Delta$ is a rank 3 affine building and keep $i \in \{1, 2, 3\}$ fixed, i.e. we keep the type of $\Delta$ fixed. A *germ of quarters* in $\Delta$ is an equivalence class in the set of quarters of $\Delta$ with respect to the equivalence relation: $Q_1$ is equivalent to $Q_2$ if $Q_1 \cap Q_2$ contains some quarter (see [38]). We say that two panels $p$ and $q$ are *parallel* if they are on bounded distance from one another, i.e. if the set of distances from a variable point of $p$ to $q$ and from a variable point of $q$ to $p$ is bounded in $\boldsymbol{R}$. This defines an equivalence relation, and we denote the class of $p$ by $c(p)$. By Tits [38], proposition 1, there exists a type map

$$t_\infty\colon \{c(p)\colon p \text{ is a panel}\} \to \{1, 2\}$$

such that the geometry $\Delta_\infty = \big(t_\infty^{-1}(1), t_\infty^{-1}(2), I\big)$ defined by

> $c(p) I c(q) \Leftrightarrow$ there exists a germ $G$ of quarters such that for all $Q \in G$, there exist $p' \in c(p)$ and $q' \in c(q)$ with $p' \cup q' \subseteq Q$, for all $c(p)$, $c(q)$

is a generalized $n$-gon, $n = 3, 4, 6$ for resp. $i = 1, 2, 3$.

(3.4.1) PROPOSITION. Let $v$ be a special vertex of $\Delta$. Define a point-line geometry $V_\infty = \big(P(V_\infty), L(V_\infty), I\big)$ as follows:

$$P(V_\infty) = \{p \mid p \text{ is a panel of } \Delta \text{ with source } v \text{ and } v_p \in P(R_v)\}$$

$$L(V_\infty) = \{l \mid l \text{ is a panel of } \Delta \text{ with source } v \text{ and } v_l \in L(R_v)\}\,.$$

$$pIl \Leftrightarrow p \text{ and } l \text{ lie on a common quarter with source } v, \text{ for all } (p, l) \in P(V_\infty) \times L(V_\infty)\,.$$

Then $V_\infty$ is isomorphic to one of the two point-line geometries associated to $\Delta_\infty$.

PROOF. The map $\varphi_v\colon \Delta_\infty \to V_\infty$ which maps $c(p)$ to the unique panel of $c(p)$ with source $v$ is well-defined by Tits [38], proposition 5, and it clearly preserves incidence. The map $\psi_v\colon V_\infty \to \Delta_\infty$ which maps a panel $p$ with source $v$ to $c(p)$ is set-theoretically clearly the inverse of $\varphi_v$. We now show that $\psi_v$ also preserves incidence. So suppose $pIl$ in $V_\infty$ and let $Q$ be a quarter with source $v$ containing $p$ and $l$. Let $G$ be the germ of quarters containing $Q$ and let $Q' \in G$ be arbitrary. Since $Q, Q' \in G$, there exists a quarter $Q'' \subseteq Q \cap Q'$. Note that $Q'' \in G$. Let $v''$ be the source of $Q''$. Let $p''$ and $l''$ be the panels with source $v''$ lying on the boundary of $Q''$. Looking inside $Q$, one can easily see that, up to permutation, $p'' \in c(p)$ and $l'' \in c(l)$. But $p'' \cup l'' \subseteq Q'' \subseteq Q'$, hence $c(p)\, Ic(l)$ in $\Delta_\infty$ and $\varphi_v$ is an isomorphism, Q.E.D.

(3.4.2) Proposition.

$$V_\infty \cong \varprojlim_j \,(V_j, \pi^j_{j-1})$$

Proof. For $i = 1, 2$, this is shown in [41], theorem (4.4.1), resp. [20], theorem (2.3.3.1). For $i = 3$, the proof is similar to that for $i = 1$ or 2, Q.E.D.

(3.4.3) Corollary. Suppose $\Delta$ is locally finite (i.e. one, and hence each, residue is finite), then $\Delta_\infty$ is a compact totally disconnected topological $n$-gon, $n = 3, 4, 6$ for resp. $i = 1, 2, 3$.

Proof. This follows immediately from (2.3), (3.4.1) and (3.4.2), Q.E.D.

We denote the topology on $\Delta_\infty$ induced by the inverse limit of the $n$-th floors with basement $v$ as in (3.4.3) by $\tau_v$. The remainder of this section is devoted to the proof of the fact that $\tau_v$ on $\Delta_\infty$ is independent of the special vertex $v$. This requires the introduction of « trees ».

(3.5) *Trees.*

(3.5.1) Definition and notation. Let $T$ be an infinite tree without (finite) endpoints, and assume for the sake of simplicity that the valency of any vertex is greater than 2, i.e. each vertex is adjacent to at least 3 other vertices. If a vertex $x$ is adjacent to a vertex $y$, then we write $x \sim y$. Following the notation of Tits [38], we call a *half apartment* in $T$ any infinite sequence $(x_0, x_1, \ldots)$ of consecutively adjacent vertices, all different from one another. Two half apartments are called equivalent if they share infinitely many common vertices.

This induces an equivalence relation in the set of half apartments of $T$, and an equivalence class is called an *end*. The set of ends of $T$ is denoted by End $(T)$. We now make End $(T)$ into a topological space by establishing a base $\mathfrak{B}$ for the topology. We define $\mathfrak{B}$ as follows. Let $x$ and $y$ be two adjacent vertices, then the set $B(x, y)$ of ends having a representative (= a half apartment) of the form $(x, y, \ldots)$ is an element of $\mathfrak{B}$, for all vertices $x, y$ of $T$.

(3.5.2) LEMMA. Let $T$ and $\mathfrak{B}$ be as above. Fix an arbitrary vertex $x$ of $T$ and denote, for any vertex $y \neq x$ in $T$, by $E_x(y)$ the set of ends of $T$ having a representative of the form $(x, \ldots, y, \ldots)$. Then $\mathfrak{B}_x = \{E_x(y) : y$ is a vertex of $T\}$ is a base for the topology on End $(T)$ induced by $\mathfrak{B}$.

PROOF. (1) We show first that $\mathfrak{B}_x \subseteq \mathfrak{B}$.

Let $y$ be a vertex of $T$ and $e \in E_x(y)$. Suppose $e$ has a representative $(x, \ldots, y, y_0, y, \ldots)$. Hence $e$ has also a representative of the form $(y_0, y, \ldots)$. Since $T$ is a tree, $y_0$ is independent of $e \in E_x(y)$. Hence $E_x(y) \subseteq B(y_0, y)$. Conversely, any half apartment $(y_0, y, \ldots)$ is equivalent to the half apartment $(x, \ldots, y_0, y, \ldots)$ (note that all vertices are distinct from one another indeed!), hence $B(y_0, y) = E_x(y) \in \mathfrak{B}$.

(2) We now show that every element of $\mathfrak{B}$ is a union of elements of $\mathfrak{B}_x$. So suppose $B(v_1, v_2) \in \mathfrak{B}$. If the unique path from $x$ to $v_2$ passes $v_1$, then as in (1), $B(v_1, v_2) = E_x(v_2) \in \mathfrak{B}_x$. If this is not the case, then the unique path from $x$ to $v_1$ passes $v_2$. Suppose that path is $(x, v_n, v_{n-1}, \ldots, v_3, v_2, v_1)$. One can check that (putting $x = v_{n+1}$)

$$B(v_1, v_2) = \bigcup_{j=2}^{n} \{\cup \{E_x(v) : v \sim v_j, v \neq v_{j-1}, v \neq v_{j+1}\}\} \cup$$
$$\cup \bigcup \{E_x(v) : v \sim x, v \neq v_n\} \qquad \text{Q.E.D.}$$

(3.5.3) THEOREM. $\tau_v$ is independent of the special vertex $v \in \text{vert}(\Delta)$.

PROOF. Let $p$ be any panel and suppose without loss of generality that $p$ has source $v$ and $p \in P(V_\infty)$. We define a graph $G(p)$ by restricting $\Delta$ to the set of all vertices lying on some panel $lIp$ in $V_\infty$. Clearly, there is a unique path form $v$ to any vertex $x$ of $G(p)$, namely the interval joining $v$ to $x$. Hence $G(p)$ is an infinite tree without finite endpoints. Apparently, every end represents a line of $V_\infty$ incident with $p$ in $V_\infty$ and conversely, every line of $V_\infty$ incident with $p$ represents an end of $G(p)$ (the first assertion follows from the completeness of $\Delta$, the second is trivial). In fact $G(p)$ is the discrete version of Tits' tree $\left(I(D^\infty), \{f_D^\infty : f(D) \in D^\infty\}\right)$, $D = c(p)$ (see [38], § 8). Now note that an arbitrary element of the base of the topology $\tau_v$ on $L(V_\infty)$ is the set of panels with source $v$ containing some $L_j \in L(V_j)$, $j \geqslant 1$. If we intersect each element of that base with the pencil on $p \in P(V_\infty)$, we obtain in $G(p)$ sets of ends represented by half apartments of the form $(\ldots, \ldots., L_n, \ldots)$, hence we obtain as a base for the topology $\tau_v$ restricted to the pencil of $p$ the set

$$\mathcal{B} = \{E_v(L_n) : L_n \in L(V_n), n \in \mathbf{N}^*\} = \{E_v(x) : x \text{ is a vertex of } G(p)\} .$$

So the topology on the pencil of $p$ induced by $\tau_v$ is the natural topology induced by the representation of that pencil as a tree. This representation as a tree is independent of $v$ (by the remark above that it is in fact the discrete version of Tits' tree in [38], § 8) and hence the topology on pencils is independent of $v$. The assertion now follows from lemma (2.1) (*c*) Q.E.D.

## 4. – A theorem on epimorphisms.

In our formulation, the next theorem is in fact only a special case of the original theorem by Ronan. Nevertheless it is still quite deep and surprising.

(4.1) THEOREM (M. Ronan [28]). *Suppose $P_r$ is the set of all projective planes of order $r$, $Q_{r,s}$ the set of all generalized quadrangles of order $(r, s)$ and $H_{r,s}$ the set of all generalized hexagons of order $(r, s)$. Suppose $q_1, q_2, q_3, q_4, q_5$ are such that $P_{q_1} \neq \emptyset$, $Q_{q_2,q_3} \neq \emptyset$, $Q_{q_3,q_4} \neq \emptyset$ and $H_{q_1,q_5} \neq \emptyset$, $q_i$ possibly infinite for some $i$'s. Then there exist buildings $\Delta_1, \Delta_2, \Delta_3$ of resp. type $\tilde{A}_2, \tilde{C}_2, \tilde{G}_2$ such that $\Delta_1$ (resp. $\Delta_2, \Delta_3$) has as set of rank 2 residues any subset of $P_{q_1}$ (resp. $Q_{q_2,q_3} \cup Q_{q_3,q_4}$, $P_{q_1} \cup H_{q_1,q_5}$), disregarding the generalized digons.*

From the inverse limit representation of the geometry at infinity of a rank 3 affine building $\Delta$ follows that, for any special vertex $v$, there exists a natural continuous (by lemma (2.3)) epimorphism

$$\pi_1^\infty : \Delta_\infty \cong V_\infty \to V_v \equiv R_v .$$

Combining this observation with Ronan's theorem, we get:

(4.2) THEOREM. (1) There exists a compact totally disconnected projective plane admitting continuous epimorphisms onto any number of finite projective planes of fixed order.

(2) There exists a compact totally disconnected generalized quadrangle admitting continuous epimorphisms onto any number of finite generalized quadrangles of order $(q_1, q_2)$ or $(q_2, q_3)$, where $q_1, q_2, q_3$ are such that $Q_{q_1,q_2} \neq \emptyset \neq Q_{q_2,q_3}$.

(3) There exists a compact totally disconnected generalized hexagon admitting continuous epimorphisms onto any number of finite generalized hexagons of fixed order $(q_1, q_2)$ such that $P_{q_1} \neq \emptyset$.

In particular, we have

(4.3) COROLLARY. Any finite projective plane (resp. generalized quadrangle, generalized hexagon of order $(q_1, q_2)$) is the continuous epimorphic image of an infinite compact projective plane (resp. generalized quadrangle, generalized hexagon provided $P_{q_1} \neq \emptyset$ or $P_{q_2} \neq \emptyset$).

The case of $\tilde{G}_2$ needs our special attention. Suppose $v$ is a non-special vertex of $\Delta$ and $l$ is a panel containing special, non-special and anti-special vertices (then one can check that $c(l)$ is a line in $\Delta_\infty$ after identifying with any $V_\infty$). Since, with our definition, panels can only have special vertices as their source, we must interpret [38], proposition 5 as follows: there exists an apartment $\Sigma$ of $\Delta$ containing $v$ and some element $l' \in c(l)$. Viewing $\Sigma$ as an affine plane, we see that the half line $L$ in $\Sigma$, bounded by $v$ and parallel to $l'$ (« going in the same direction » as $l$) contains a unique vertex $v'$ adjacent to $v$ and $v'$ is special or anti-special. The map $\mu_v\colon c(l) \to v'$ is surjective on the set of varieties of $R_v$ and it maps (by a similar argument as above) concurrent lines of $\Delta^\infty$ onto incident varieties of $R_v$. If we define a *hemimorphism* $\mu\colon X \to Y$ from a generalized hexagon $X$ onto a projective plane $Y$ as a surjective map $\mu\colon P(X) \to P(Y) \cup L(Y)$ or $L(X) \to \to P(Y) \cup L(Y)$ such that $\mu$ maps collinear points or concurrent lines onto incident varieties (where a variety is assumed to be incident with itself), then $\mu_v$ is a hemimorphism. If $X$ and $Y$ are topological $n$-gons (resp. $n = 6, 3$), then a hemimorphism $\mu$ is continuous if $\mu$ is continuous as a map between the two topological spaces $L(X)$, or $P(X)$, and $P(Y) \cup L(Y)$ (topological sum of $P(Y)$ and $L(Y)$). We now show that $\mu_v$ is continuous. So we must prove that $\mu_v^{-1}(v')$ is open in the set of lines of $\Delta_\infty$. Let $\mathfrak{V}$ be the set of all special vertices $s$ with the properties:

(V1) there exists a panel $p$ with source $s$ containing $v$ and $v'$ and $d(x, v) + 1 = d(x, v')$

(V2) If $s' \neq s$ is any special vertex on any panel $p$ with source $s$ containing $v$ and $v'$, then $d(s, s') > d(s, v)$.

In fact, (V2) says that the interval joining $s$ and $v$ contains no other special vertex than $s$ itself. Now let $s \in \mathfrak{V}$ and let $\mathfrak{P}_s$ be the set of all panels of $\Delta$ with source $s$ containing $v$ and $v'$. Then by definition, $\varphi_s(\mathfrak{P}_s)$ is an open set in $\Delta_\infty$. Hence the set

$$\bigcup \{\varphi_s(\mathfrak{P}_s)\colon s \in \mathfrak{V}\}$$

is open in $\Delta_\infty$. But apparently this set equals $\mu_v^{-1}(v')$. Hence the next theorem:

(4.4) Theorem. There exists a compact totally disconnected generalized hexagon admitting continuous epimorphisms onto any number of finite generalized hexagons of some fixed order $(q_1, q_2)$ with $P_{q_1} \neq \emptyset$ and also admitting continuous hemimorphisms onto any number of finite projective planes of order $q_1$.

(4.5) Corollary. Any finite projective plane of order $q_1$ such that there exists $q_2 \in N + 2$ with $H_{q_1, q_2} \neq \emptyset$ is a continuous hemimorphic image of an infinite compact totally disconnected generalized hexagon.

In particular, corollary (4.5) holds for every known finite projective plane.

## 5. – Remarks.

(5.1) Suppose $\Delta$ is a *symmetric* discrete system of apartments, i.e. $\Delta$ satisfies (AB1), (AB2), (AB3), (AB4) and (AB5′) where

(AB5′) $\Delta$ is thick and any two germs of quarters have respective representatives lying in a common apartment.

Any complete discrete systems of apartments is symmetric (see Tits [38]). The geometry at infinity of $\Delta$ (as defined in (3.4)) is again a projective plane, a generalized quadrangle, a generalized hexagon according to $i = 1, 2, 3$. Now, to any symmetric discrete system of apartments $\Delta$ corresponds a unique complete discrete system of apartments $\bar{\Delta}$ such that the set of points of $\Delta$ coincides with the set of points of $\bar{\Delta}$, but $\bar{\Delta}$ has possibly more apartments. So in fact $\Delta = (\Delta, F)$ and $\bar{\Delta} = (\Delta, \bar{F})$ with $F \subseteq \bar{F}$. Hence $\Delta_\infty$ can be embedded in $\bar{\Delta}_\infty$ and so $\Delta_\infty$ can be given the structure of a (not necessarily compact) totally disconnected non-discrete topological $n$-gon.

EXAMPLE. $PG(2, K(t))$ it is a subplane of $PG\big(2, K((t))\big)$ for every finite field $K$.

(5.2) EXAMPLES of non-classical locally finite rank 3 affine buildings of type $\tilde{A}_2$ are provided by [40], where the coordinatizing planar ternary field of the projective plane at infinity is explicitly given. A similar result can be given for type $\tilde{C}_2$, using *quadratic quaternary rings* (see [18], [39]). An example of a non-classical rank 3 affine building of type $\tilde{G}_2$ is given by the universal cover $\Gamma$ of the geometry associated with Lyon's group. In [37], § 6.5, Tits alludes to the topology of the geometry at infinity of $\Gamma$ by saying that it provides an interesting exotic compact generalized hexagon.

(5.3) Suppose $\Delta$ is a (not necessarily locally finite) rank 3 affine building, then it should not be difficult to show the following

(5.3.1) CONJECTURE. $\Delta_\infty$ can be given the structure of a nontrivial totally disconnected topological $n$-gon in a natural way.

The authors actually have a proof of this conjecture for types $\tilde{A}_2$ and $\tilde{C}_2$, using different techniques. Combining this with the existence theorems above and Ronan's theorem one can « construct » a totally disconnected topological projective plane (resp. generalized quadrangle) admitting continuous epimorphisms onto every finite projective plane (resp. generalized quadrangle) and onto any countable number of countable or uncountable connected or disconnected topological projective planes (resp. generalized quadrangles) (without proof).

(5.4) Let $G$ be the group of continuous automorphisms of the topological space of ends of a tree $T$. Not every element of $G$ « extends » to an automorphism of $T$ itself (note that $G$ is $n$-transitive for all $n \in \boldsymbol{N}$ because End $(T)$ is homeomorphic to Cantor's set). In contrast, the group of automorphisms of an affine building of type $\tilde{A}_2$ is isomorphic to the group of continuous automorphisms of its geometry at infinity w.r.t. the natural topology described in the present paper. This will be proved in a forthcoming paper.

(5.5) If $\Delta$ is a classical rank 3 affine building, i.e. $\Delta$ arises from an algebraic group over a local field, then the topology on $\Delta_\infty$ induced by $\Delta$ (as above) coincides with the topology on $\Delta_\infty$ induced by a classical embedding of $\Delta_\infty$ into a projective space (as in example (2.2)), since the topologies on a line (resp. a pencil) coincide (and then use lemma (2.1)).

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