Ann. Comb. 15 (2011) 69–80
DOI 10.1007/s00026-011-0083-8
Published online March 17, 2011

Annals of Combinatorics

Local Sharply Transitive Actions on Finite Generalized Quadrangles

Theo Grundhöfer[1] and Hendrik Van Maldeghem[2*]

[1]Institut für Mathematik, Universität Würzburg, Am Hubland, D-97074 Würzburg, Germany
grundh@mathematik.uni-wuerzburg.de

[2]Department for Pure Mathematics and Computer Algebra, Ghent University, Krijgslaan 281, S22, B-9000 Gent, Belgium
hvm@cage.UGent.be

Received March 6, 2008

Mathematics Subject Classification: 51E12

Abstract. We classify the finite generalized quadrangles containing a line L such that some group of collineations acts sharply transitively on the ordered pentagons which start with two points of L. This can be seen as a generalization of a result of Thas and the second author [22] classifying all finite generalized quadrangles admitting a collineation group that acts transitively on all ordered pentagons, although the restriction to sharp transitivity is essential in our arguments. However, the conclusion is exactly the same family of classical generalized quadrangles (the orthogonal quadrangles and their duals). Our main result thus provides a local group theoretic characterization of these classical quadrangles.

Keywords: Moufang panel, root elations, classical generalized quadrangles

1. Introduction

It is still an open problem to determine the finite generalized quadrangles admitting a collineation group acting transitively on the ordered ordinary quadrangles without using the classification of finite simple groups. When the group acts transitively on the ordered pentagons, then Thas & Van Maldeghem [22] showed that only the classical quadrangles with orders q, $\left(q, q^2\right)$ and $\left(q^2, q\right)$ arise. While we are yet unable to generalize this by weakening the hypothesis to ordinary quadrangles, we can generalize it by making the hypothesis more local, but requiring sharp transitivity instead, and that is what we do in the present paper.

This problem fits into a sequence of results that classify generalized polygons admitting a group of automorphisms acting sharply transitively on a class of substructures. Let us review some of these results, and then it will become clear that the present paper is a logical sequel.

* The second author's research is partially supported by the Fund for Scientific Research – Flanders (Belgium).

In [4], the first author classifies all projective planes admitting a group of collineations acting sharply transitively on the set of all ordinary quadrangles. Then, the second author generalized this result to all generalized $(2n-1)$-gons (with a group acting sharply transitively on ordinary $2n$-gons), and to self dual generalized quadrangles and hexagons. Subsequently, the authors considered the family of ordered ordinary $(2n-1)$-gons instead of $2n$-gons and classified in [5] the projective planes admitting a collineation group acting sharply transitively on ordered ordinary triangles, and mutatis mutandis for the generalized $(2n-1)$-gons. A logical next step was to consider triangles in *affine* planes. This yields a local version of the result of the first author [4]. In [6], the authors classify all affine planes admitting a collineation group acting sharply transitively on ordered triangles. In other words, they classify projective planes admitting a collineation group acting sharply transitively on the set of all ordered quadrangles which contain a fixed (first) line. They then go on proving that there are no generalized $(2n-1)$-gons admitting a group acting sharply transitively on the ordered $2n$-gons containing a fixed (first) line.

All these results, except for the ones mentioned above about the self dual quadrangles and hexagons, are about generalized polygons with odd diameter (*generalized odd-gons*). This is not so surprising, since the main techniques use properties of involutions, and these are better manageable when the diameter is odd (in terms of fixed points). In generalized even-gons, involutions can have no or many fixed points, as any other collineation, and this makes the study of sharply transitive actions in these structures very hard. However, if we restrict to the class of finite generalized quadrangles, then we can generalize the local results of [6]. This is exactly what we do in the present paper. As will become apparent, the techniques are completely different from those used before, and involve typically "finiteness" arguments, both on the geometric and group-theoretic level. In particular, the classification of finite split BN-pairs is used in the proof of Proposition 3.6. Moreover, also the proofs of the technical results Lemma 4.3 and 4.4 require some group theory.

2. Notation and Main Result

If $\mathcal{P}$ and $\mathcal{L}$ are two disjoint sets and $\mathtt{I}$ is a symmetric relation whose graph is connected, then we say that the triple $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a *point-line geometry* and we call $\mathcal{P}$ the *set of points* and $\mathcal{L}$ the *set of lines*. We use common terminology such as *collinear* points to denote points that are incident with one line; *concurrent lines* for lines that are incident with a common point. A geometry is called a *generalized quadrangle with order* (s,t), where s,t are cardinal numbers, if

(i) every line of Γ is incident with exactly $s+1$ points and every point with exactly $t+1$ lines,

(ii) no two different points are incident with two different lines, and

(iii) every given point x is collinear with a unique point $\mathrm{proj}_L x$ incident with a given line L, with x not incident with L.

For an introduction to (finite) generalized quadrangles, we refer to [15]. Let us mention that deletion of Axiom (i) would have as only consequence the inclusion of some trivial geometries where most points and/or lines are incident with at most two elements. Hence we will mainly be interested in the case where both s and t are strictly bigger than 1. Also, we sometimes view a generalized quadrangle Γ as a graph (the *incidence graph*) with vertex set $\mathcal{P} \cup \mathcal{L}$ and adjacency relation $\mathtt{I}$. For this graph, we use some graph theoretic notions such as cycles and distance. In particular, we denote, for $x \in \mathcal{P} \cup \mathcal{L}$, by $\Gamma_i(x)$ the set of elements of Γ at distance i from x (with distances measured in the incidence graph, and the distance between x and y is denoted by $d(x,y)$). For two distinct elements x, y at distance at most 3 from each other, there is a unique element incident with x and at distance $d(x,y)-1$ from y. We denote that element by $\mathrm{proj}_x y$, and call it the *projection of y onto x*. Elements at distance 4 from one another will be called *opposite*. If we interchange the roles of $\mathcal{P}$ and $\mathcal{L}$ then we obtain again a generalized quadrangle, of order (t,s), called the *dual of* Γ.

A *collineation* θ of a generalized quadrangle Γ is a pair of permutations, both denoted by the symbol θ, of the point set and the line set, respectively, such that $x\mathtt{I}L$ if and only if $x^\theta \mathtt{I} L^\theta$ (we use exponential notation for permutations).

The main examples of generalized quadrangles arise from pseudo-quadratic forms of Witt index 2 in arbitrary vector spaces. In the present paper we are interested in the finite examples, and especially in the case where the quadrangle arises from a quadratic form of Witt index 2. In this case, there is a simple geometric description. Indeed, any nonsingular quadric $\mathsf{Q}(4,q)$ and $\mathsf{Q}(5,q)$ with projective index 1 (i.e., containing lines but not planes) in the projective spaces $\mathsf{PG}(4,q)$ and $\mathsf{PG}(5,q)$, respectively, is a generalized quadrangle when considered as point-line geometry in the natural way. The order is (q,q), (q,q^2), respectively.

In this paper we prove the following main result.

Main Result. *Let Γ be a finite generalized quadrangle with a line L_∞ and a collineation group G satisfying the following condition.*

(LST) *The group G fixes L_∞ and acts sharply transitively on the ordered ordinary pentagons (a,b,c,d,e) in Γ such that a, b are incident with L_∞.*

Then Γ is isomorphic to $\mathsf{Q}(4,q)$ or its dual, or to $\mathsf{Q}(5,q)$ or its dual, for some prime power q. In each case, the group G contains all root elations of Γ that fix the line L_∞.

Each of the generalized quadrangles mentioned in the theorem above actually admits a group satisfying (LST) for any line L_∞; this follows from Theorems 4.6.2 and 4.6.3 of [26] (which imply that, in the cases of the dual of $\mathsf{Q}(4,q)$ in $\mathsf{PG}(3,q)$ and the dual of $\mathsf{Q}(5,q)$ in $\mathsf{PG}(3,q^2)$, the group of collineations induced by the linear group of the ambient projective space acts regularly on the set of pentagons). However, we point out that the group G is in general not uniquely determined by Γ. Indeed, let $\Gamma = \mathsf{Q}(4,q)$, with q an odd square, and let L_∞ be an arbitrary line of Γ. Then the group N generated by all root elations of Γ that fix the line L_∞ has order $q^4(q-1)$. Moreover, if x and y are distinct points on L_∞ and L and M are lines, distinct from L_∞, incident with x and y, respectively, then the subgroup of N fixing L and M pointwise has order $q-1$ and consequently acts sharply transitively on the set of lines through x different from L_∞ and from L (this follows from the fact, proven in [12] and [23], that the elementwise stabilizer in N of the set of lines meeting both L and M is isomorphic to $\mathsf{SL}_2(q)$). Moreover, N is normalized by all collineations that fix L_∞. For $z\,\mathtt{I}\,L$, $z \neq x$, let Z be the line through z meeting M. Then there is a cyclic group H of order $q-1$ fixing L, M, Z and L_∞, fixing all lines through x, and acting sharply transitively on $\Gamma_1(L) \setminus \{x, z\}$. The group $G := HN$ satisfies the conditions of the theorem. But we can replace the cyclic group H of order $q-1$ by a nonabelian group H^* of that order (by involving the involution of the Galois group, as in the construction of the Dickson nearfields); the resulting group $G^* := H^*N$ is different from G and also satisfies the conditions of the theorem.

3. Proof of the Main Result if L_∞ Is not Regular

We assume that $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a finite generalized quadrangle with order (s,t) with a distinguished line L_∞ and a collineation group G satisfying Condition (LST).

Lemma 3.1. *For each triple $(i, j, k) \in \{(1,2,3), (1,4,5), (3,2,3), (3,2,5), (3,4,3), (3,4,5), (5,2,5), (5,4,3)\}$, the group G acts sharply transitively on the set of triples $(\mathcal{A}, x, L)$, where $\mathcal{A}$ is an apartment containing L_∞, x is a point not belonging to $\mathcal{A}$ but incident with a line of $\mathcal{A}$, L is a line not belonging to $\mathcal{A}$ but incident with a point of $\mathcal{A}$, and $d'(L_\infty, x) = i$, $d'(L_\infty, L) = j$, $d'(x, L) = k$, where d' denotes the distance in the configuration $\mathcal{A} \cup \{x, L\}$.*

Proof. This follows directly from condition (LST). ∎

A *panel* is a set $\{x, y, z\}$ with $x\mathtt{I}y\mathtt{I}z$ and $x \neq z$. A panel $\{x, y, z\}$ is called *Moufang* if the pointwise stabilizer in G of $\Gamma_1(x) \cup \Gamma_1(y) \cup \Gamma_1(z)$ is a group of order s (if $x \in \mathcal{P}$) or t (if $x \in \mathcal{L}$). This group will be referred to as the *root group belonging to the panel*. If this root group is elementary abelian, then we say that the corresponding panel is *elementary abelian Moufang*. If, for some point x, the stabilizer in G of all points collinear with x has order t, then we say that x is a *center of symmetry*.

We will frequently use the following almost trivial observation.

Lemma 3.2. *Let G be a finite group acting on a finite set X. Suppose $H \leq G$ is such that all nontrivial elements of H are conjugate in G and suppose also that $|X| < |H|$. Then H acts trivially on X.*

Proof. Put $|H| = h$. By possibly adding abstract new elements to X on which each element of G acts trivially, we may assume that $|X| = h-1$. Since all nontrivial elements of H are conjugate under G, all these elements of H have the same number, say n, of fixed points in X. Suppose H has m orbits in X; then Burnside's orbit counting theorem states $mh = (h-1) + n(h-1) = (n+1)(h-1)$. This implies that $h-1$ divides m, and so $m = h-1 = n$. Consequently H fixes X pointwise. ∎

Lemma 3.3. (i) *If $s \leq t$, then every panel $\{L_\infty, x, L\}$ with $L\mathtt{I}x\mathtt{I}L_\infty$ and $L \neq L_\infty$ is elementary abelian Moufang.*

(ii) *If $t \leq s$, then every panel $\{x, L_\infty, y\}$ with $x\mathtt{I}L_\infty\mathtt{I}y$ and $x \neq y$ is elementary abelian Moufang. Also, every panel $\{y, L, z\}$ with $L_\infty\mathtt{I}y\mathtt{I}L\mathtt{I}z$, $L \neq L_\infty$ and $y \neq z$ is elementary abelian Moufang.*

In any case, both s and t are prime powers.

Proof. All these assertions are proved similarly, so we prove one of them, e.g., (i). So let x and L as in (i) above, and choose two additional arbitrary points x', x'' on L_∞, $x \neq x' \neq x'' \neq x$. Also, let y be some point on L, $y \neq x$. By Lemma 3.1, the stabilizer H in G of the set $\{x, x', x'', y\}$ acts sharply 2-transitively on the set of apartments containing $\{x, x', y\}$. Since there are exactly t of them, t is a prime power. Let F be the Frobenius kernel of H. Then F is an elementary abelian group of order t and all nontrivial elements of F are conjugate in H. Let X be the set of lines through x different from L and from L_∞, or the set of points on either L_∞ or L, different from x, x' and from y. Then H acts on X and $|X| < |F|$. Lemma 3.2 implies that F fixes all elements of X. This proves (i).

As mentioned above, the proof of (ii) is completely similar. We end the proof of 3.3 by showing that s is a prime power (note that in the above argument that showed that t is a prime power we did not use the inequality $s \leq t$). Let x, x' be two distinct points on L_∞, and let M, M' be two lines incident with x, x', respectively, with $M \neq L_\infty \neq M'$. Let K be a third line through x, $L_\infty \neq K \neq M$. Then the stabilizer in G of the set $\{M, M', K\}$ acts sharply 2-transitively on the set of apartments containing $\{L_\infty, M, M'\}$. Since there are exactly s of these, it follows that s is a prime power. ∎

For a subset B of points of Γ, we denote by $B^\perp$ the set of points of Γ collinear with all of B. The set $\{x, y\}^\perp$, for two noncollinear points x, y, is called a *trace* (*in both* $x^\perp$ *and* $y^\perp$).

A point x of Γ is called *regular* if traces in $x^\perp$ that do not coincide meet in at most one point. A point x is called *antiregular* if $s = t$, if $\{x, y\}^\perp \neq \{x, z\}^\perp$ for $y \neq z$ (and y, z are two points not collinear with x) and if two traces in $x^\perp$ meet in at most 2 points.

Dual definitions hold for regular and antiregular lines.

Lemma 3.4. (i) *If $s = t$, then L_∞ is either a regular line, or an antiregular line. Also, every point on L_∞ is either regular or antiregular.*
(ii) *If $s < t$, then L_∞ is regular.*
(iii) *If $t < s$, then every point on L_∞ is regular.*

Proof. Suppose $s \leq t$ and that L_∞ is not regular. We show that necessarily $s = t$ and L_∞ is antiregular.

Our assumption implies the existence of two distinct lines M_1 and M_2 not concurrent with L_∞ such that $2 \leq |\{L_\infty, M_1, M_2\}^\perp| \leq s$. Hence there are distinct lines $L, L' \in \{L_\infty, M_1, M_2\}^\perp$, and there is a point $x \mathtt{I} L_\infty$ such that $N_1 := \mathrm{proj}_x M_1 \neq \mathrm{proj}_x M_2 =: N_2$.

Let z be the intersection of L and M_1. The stabilizer T in G of $L, L_\infty, L', M_1, M_2$ acts sharply transitively on $\Gamma_1(z) \setminus \{L, M_1\}$, cf. Lemma 3.1. The stabilizer in T of x is trivial for otherwise there are at least two lines through z meeting N_2. Hence the orbit of x under T contains exactly $t - 1$ elements. This implies firstly $t - 1 \leq s - 1$, hence $s = t$. Secondly, we now see that $\{L_\infty, M_1, M_2\} = \{L, L'\}$. The transitivity of G on the set of lines not collinear with L_∞, and the double transitivity of the stabilizer in G of some line L opposite L_∞ on the set of points incident with L_∞, imply that L_∞ is antiregular.

We have shown (ii) and the first assertion of (i). In order to show the other assertions, we can appeal to the dual arguments, except that the stabilizer in G of any point on L_∞ fixes L_∞. So, dualizing the above arguments, we can show that, given some point $x\mathtt{I}L_\infty$ and two points y, y' opposite x with $\mathrm{proj}_{L_\infty} y = \mathrm{proj}_{L_\infty} y'$ and $2 \le |\{x, y, y'\}^\perp| \le t$, this forces $s = t$ and $|\{x, y, y'\}| = 2$. We refer to this as the dual of the first part of the proof.

We must show that (1) the existence of y and y' as just stated is implied by the assumption that x is not regular, and (2) the property of y, y' just stated implies that x is antiregular.

We start with (2). If x is not antiregular, then, by the dual of the first part of the proof, there exist z, z' opposite x with $|\{x, z, z'\}^\perp| \ge 3$ and either $\mathrm{proj}_{L_\infty} z \ne \mathrm{proj}_{L_\infty} z'$ or $|\{x, z, z'\}^\perp| = t+1$. Suppose first $|\{x, z, z'\}^\perp| = t+1$. Put $\mathrm{proj}_{L_\infty} z = u$ and let u' be another arbitrary element of $\{x, z, z'\}^\perp$. Then the first part of the proof implies that $|\{x, z, z', z''\}^\perp| = t+1$ for all $z'' \in \{u, u'\}^\perp \setminus \{x\}$. Transitivity of G_x on the set of points opposite x implies that x is regular. This contradicts our assumption on x. So we may assume that $3 \le |\{x, z, z'\}^\perp| \le t$ and $u := \mathrm{proj}_{L_\infty} z \ne \mathrm{proj}_{L_\infty} z' =: u'$. Let $\{v, v', v''\} \subseteq \{x, z, z'\}^\perp$, with $|\{v, v', v''\}| = 3$. The pointwise stabilizer in G of $\{x, u, u', z, v\}$ has order $t-1$ and acts transitively on $\Gamma_1(v) \setminus \{vx, vz\}$. Hence there are $t-1$ traces in $x^\perp$, containing u' and v, and sharing at least two elements with $\{x, z\}^\perp \setminus \{u, v\}$. It now follows that either two of these traces coincide (contradicting the first part of this paragraph), or two of these traces share at least three elements, amongst which is u'. This contradicts the dual of the first part of the proof.

Now we prove (1). If y and y', both opposite x, and with the properties $\mathrm{proj}_{L_\infty} y = \mathrm{proj}_{L_\infty} y'$ and $2 \le |\{x, y, y'\}^\perp| \le t$ do not exist, then we certainly can find $y, y' \in \Gamma_4(x)$ with $|\{x, y, y'\}^\perp| = t+1$ (indeed, just consider arbitrary $u\mathtt{I}L_\infty$, $u \ne x$, and arbitrary $v \in \Gamma_2(x) \setminus \Gamma_1(L_\infty)$, and take $y, y' \in \{u, v\}^\perp \setminus \{x\}$). But the argument in the previous paragraph now leads to a contradiction. ∎

Lemma 3.5. *If L_∞ is not regular, then each point on L_∞ is a center of symmetry.*

Proof. Note that by Lemma 3.4(ii) the assumption of L_∞ being not regular implies $t \le s$.

Let $x\mathtt{I}L_\infty$. We must show (1) that x is regular and (2) that, for any line $L\mathtt{I}x$, $L \ne L_\infty$, the panel $\{L_\infty, x, L\}$ is Moufang.

If $s = t$, then (2) follows from Lemma 3.3. If $s > t$, then (1) follows from Lemma 3.4.

Choose $M_2\mathtt{I}x_2\mathtt{I}L_\infty\mathtt{I}x_1\mathtt{I}M_1\mathtt{I}y$, with M_1 opposite M_2 and $y \ne x_1$. Let U_1 and U_3 be the root groups belonging to the panels $\{y, M_1, x_1\}$ and $\{x_1, L_\infty, x_2\}$, respectively.

Since L_∞ is not regular, it is not an axis of symmetry (which is the dual of a center of symmetry). Hence there is some $u_3 \in U_3$ and some point $z\mathtt{I}L_\infty$ such that u_3 does not fix every element of $\Gamma_1(z)$. Let $u_1 \in U_1$ be such that it maps x_2 on z. Then $[u_3, u_1]$ does not act trivially on $\Gamma_1(x_2)$, by construction. It is now easy to see that $[u_3, u_1]$ fixes all elements incident with M, with x_1 and with L_∞. Conjugating $[u_3, u_1]$ with the stabilizer in G of y and M_2, we see that $\{M_1, x_1, L_\infty\}$ is an elementary abelian Moufang panel. Similarly $\{L_\infty, x_2, M_2\}$ is elementary abelian Moufang. Moreover, we have shown that, if U_2 is the root group belonging to $\{M_1, x_1, L_\infty\}$, then $[U_1, U_3] = U_2$.

We have thus proved the lemma for $s \neq t$. If $s = t$, then we only need to show that x_1 and x_2 are regular. If not, then similarly as above we have $[U_2, U_4] = U_3$, where U_4 is the root group belonging to $\{L_\infty, x_2, M_2\}$.

It is easy to see that the group U^+ generated by U_1, U_2, U_3, U_4 has order s^4, and hence is a p-group for some prime p. Consequently U^+ is nilpotent. But clearly $[U^+, U_2U_3]$ contains U_2U_3, contradicting nilpotency. ∎

Proposition 3.6. *If L_∞ is not regular, then Γ is isomorphic to either the dual of $\mathsf{Q}(4,s)$ with s odd, or to the dual of $\mathsf{Q}(5,t)$, and G contains all root elations of Γ that fix the line L_∞.*

Proof. By the previous lemma, each point on L_∞ is a center of symmetry, hence L_∞ is a translation line with translation group $U_2U_3U_4$ (with notation as in the previous proof). Note that it is indeed easy to show that $\langle U_2, U_3, U_4\rangle = U_2U_3U_4$ and that $|U_2| = |U_4| = t$, $|U_3| = s$ and hence $|U_2U_3U_4| = st^2$.

Let M be an arbitrary line opposite L_∞, and let $M_1, M_2 \in \{L_\infty, M\}^\perp$ be distinct. Let x_1 be the intersection of L_∞ and M_i, $i = 1,2$, and y_i the intersection of M and M_i, $i = 1,2$. Noticing that the root group U_i belonging to $\{x_i, M_i, y_i\}$, $i = 1,2$, is normal in the stabilizer in G of y_i, we see that $J := \langle U_1, U_2\rangle$ generates a split BN-pair of rank one on $\{L_\infty, M\}^\perp$. Lemma 3.1 readily implies that this BN-pair has a 3-transitive automorphism group on $\{L_\infty, M\}^\perp$, hence it easily follows, by the classification of split BN-pairs of rank one (see, [13] and [18]), that the action of J on $\{L_\infty, M\}^\perp$ is the natural action of $\mathsf{PSL}_2(s)$ on the projective line $\mathsf{PG}(1,s)$. We deduce that the action of G_M on $\Gamma_1(L_\infty)$ can be identified with the natural action of a subgroup of $\mathsf{P\Gamma L}_2(s)$ on $\mathsf{PG}(1,s)$. Let K be the kernel of that action on $\Gamma_1(L_\infty)$. Then, if $s = p^h$, with p prime,

$$s(s^2-1)h|K| \geq |G_M| = s(s^2-1)(t-1),$$

implying $|K| \geq \frac{t-1}{h} \geq \sqrt{t}+1$.

Now the kernel of the translation generalized quadrangle is a subfield of $\mathsf{GF}(t)$. Hence, since K is a multiplicative subgroup of that kernel, this implies that the kernel has order t and, by [21, Theorem 3.5.7], that Γ is isomorphic to a generalized quadrangle $T_i(O)$ of Tits, with either $s = t$, $i = 2$ and O an oval of the projective plane $\mathsf{PG}(2,s)$, or $s = t^2$, $i = 3$ and O an ovoid of the projective space $\mathsf{PG}(3,t)$. If $i = 2$, then, since L_∞ is not regular, s must be odd, and hence O is a conic (by a famous result of Segre [17]), implying that Γ is isomorphic to $\mathsf{Q}(4,s)$. If $i = 3$, then the 3-transitivity of G on the set of points of L_∞ and the above observation concerning the split BN-pair imply that O is an orbit in $\mathsf{PG}(3,t)$ under a subgroup of $\mathsf{PGL}_4(t)$ isomorphic to $\mathsf{PSL}_2(t^2)$, acting sharply 3-transitively on O. It follows that each plane section is an oval admitting a sharply 3-transitive group of automorphisms, and hence each plane section is a conic (see e.g., [19, Proposition 15]). But then O is an elliptic quadric by Barlotti's result [1] and so Γ is isomorphic to the dual of $\mathsf{Q}(5,t)$. ∎

4. The Case Where L_∞ Is Regular

As before, $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a finite generalized quadrangle with order (s,t) with a distinguished line L_∞ and a collineation group G satisfying Condition (LST). We now study the situation where L_∞ is regular. By [15, 1.3.6(i)] we have $s \le t$.

Proposition 4.1. *If L_∞ is regular and $s = t$, then $\Gamma \cong \mathsf{Q}(4,s)$, and G contains all root elations of Γ that fix the line L_∞.*

Proof. Let M be opposite L_∞, and take two lines M_1, M_2 in $\{M, L_\infty\}^\perp$. Let again x_i be the intersection of L_∞ and M_i, $i = 1,2$, and let y_i be the intersection of M and M_i, $i = 1,2$. Consider the root group U_i belonging to $\{x_i, M_i, y_i\}$, $i = 1,2$. All its nontrivial elements are conjugate, as before (since U_i arises as Frobenius kernel of a sharply 2-transitive group), and they all fix the lines of $\{M_1, M_2\}^\perp$. Hence, by Lemma 3.2 we deduce that U_i fixes all lines concurrent with M_i, and so M_i is an axis of symmetry, $i = 1,2$. Thus Γ is span-symmetric and hence isomorphic to $\mathsf{Q}(4,s)$ by a result of Kantor [12] and independently Thas [23].

The assertion on G follows directly from Lemma 3.3. ∎

It remains to consider the case where $s < t$ $\big($note that $t \le s^2\big)$. As L_∞ is regular, Γ is a skew translation generalized quadrangle (the fact that L_∞ is an axis of symmetry follows as before from considering the commutator $[U_2, U_4]$ and keeping in mind that no point on L_∞ is a center of symmetry, hence some member of U_2 does not fix all points collinear to x_1). This implies in particular that st is a power of some prime p.

Proposition 4.2. *Let L_∞ be regular and $s < t$. Consider a panel $\{x, L, y\}$ with $x \in L_\infty$ and $L \ne L_\infty$, and a point $y' \in L \setminus \{x, y\}$. If the stabilizer $T := G_{L,y,y'}$ does not act faithfully on $\Gamma(y)$, then $\Gamma \cong \mathsf{Q}(5,s)$, and G contains all root elations of Γ that fix the line L_∞.*

Proof. The group T has order $st(t-1)$ and contains the root group V belonging to the panel $\{L_\infty, x, L\}$. Clearly $V \trianglelefteq T$ and, since V is elementary abelian by 3.3, it is a vector space over the field with p elements. Note that we can identify V with $\Gamma(y) \setminus \{L\}$.

By assumption, there exists an element $g \in T \setminus \{\mathrm{id}\}$ acting trivially on $\Gamma(y)$. Thus g centralizes V. It is easy to see that g acts freely on the set of points of any line $M \ne L$ through y, where we remove y.

Now let M and M' be two such lines, $M \ne M'$, and consider the group $G_{L,M,M'}$, which has order $s(s-1)$ and acts sharply 2-transitively on $\Gamma(M) \setminus \{y\}$. The corresponding Frobenius kernel is a group U acting sharply transitively on $\Gamma(M) \setminus \{y\}$, and using Lemma 3.2 we conclude that U fixes $\Gamma(L)$ pointwise. In particular it fixes y' and so U is a subgroup of T. Since $g \in G_{L,M,M'}$, and since g acts freely on the set of points of M different from y, it belongs to the Frobenius kernel of $G_{L,M,M'}$, and so $g \in U$. Since all nontrivial elements of U are conjugate in $G_{L,M,M'}$, we conclude that U fixes all lines through y.

Now let K be any line meeting L, but not incident with x. Choose a point $x' \ne x$ on L_∞. Let K' be the unique line through x' meeting K and redefine $M := \mathrm{proj}_y K'$. Then an arbitrary element u of U maps K' onto some line K'' meeting both L_∞ and M. By the regularity of L_∞, it also meets K. Since u fixes the intersection of K and L, it now also fixes K. So we have shown that U fixes pointwise the set $\Gamma_2(L) \setminus \Gamma(x)$.

Now let J be any line incident with x, $L_\infty \ne J \ne L$. Let $J' \ne L$ be any line meeting both J and M (with M, as above, a line distinct from L through y). Since u fixes all lines meeting both L and J', we see that the s points of J' different from $z := \mathrm{proj}_{J'}x$ are paired up with the s points of J'^u different from $z^u = \mathrm{proj}_{J'^u}x$ by the relation "being collinear". Hence also z is collinear with z^u (as the only remaining possibility) and we see that $J' \neq J'^u$ leads to a triangle.

Thus U fixes $\Gamma_2(L)$ pointwise, and U consists of symmetries about L; hence L is a regular line. We have shown that every line concurrent with L_∞ is an axis of symmetry and that every point on L_∞ is a translation point. Now Theorem 10.6.4 of Thas, Thas & Van Maldeghem [21] implies that every pentagon of Γ containing L_∞ is contained in a unique subquadrangle of order s. Since G obviously is transitive on the set of such subquadrangles, a theorem of Thas [24] implies that Γ is isomorphic to $\mathsf{Q}(5,s)$.

The conjugates of V, the conjugates of U and the group of symmetries about L_∞ are the root groups of Γ that fix L_∞, and by the above arguments they all belong to G. ∎

In order to show that T as in 4.2 cannot act faithfully on $\Gamma(y)$, see 4.5, we use the following two results. For a finite group H, we denote by $O(H)$ the largest normal subgroup of H of odd order.

Lemma 4.3. *Let $p = 2$ and choose a line M not concurrent with L_∞, a line $M' \in \{L_\infty, M\}^{\perp\perp}$ and a point $x \in L_\infty$. Then the quotient $G_{M,M',x}/O(G_{M,M',x})$ is solvable and has an elementary abelian subgroup of order s.*

Proof. The stabilizer G_M has order $(s+1)s(s-1)(t-1)$ and acts triply transitively on the set $\{L_\infty, M\}^\perp$ of size $s+1$. By a theorem of Holt [8], the triply transitive group induced on $\{L_\infty, M\}^\perp$ contains $\mathsf{PSL}_2(s) = \mathsf{PGL}_2(s)$ and is contained in $\mathsf{P\Gamma L}_2(s)$ in its natural permutation representation on the projective line (note that $\mathsf{P\Gamma L}_2(s) = \mathsf{Sym}_{s+1}$ for $s = 2,\ 4$; for $s \geq 8$, alternating and symmetric groups are excluded by their large orders, as $t \leq s^2$). Therefore, $G_{M,x}$ induces on $\{L_\infty, M\}^\perp$ a subgroup of $\mathsf{A\Gamma L}_1(s)$ that contains $\mathsf{AGL}_1(s)$. This subgroup is solvable and has an elementary abelian subgroup of order s (the group of translations). The order of the kernel of the action on $\{L_\infty, M\}^\perp$ divides $t-1$, which is odd. Since $G_{M,M',x} \leq G_{M,x}$, the assertion follows. ∎

The following technical result on linear groups is true also without the solvability condition for $p = 2$, but then the proof requires deeper group theory.

Lemma 4.4. *Let p be a prime, $m, n \in \mathbb{N}$, and let $S_0 < T_0 \leq \mathsf{GL}_n(p)$ be linear groups such that S_0 is sharply transitive on the non-zero vectors and $|T_0 \colon S_0| = p^m \geq p^{n/2}$. For $p = 2$ assume also that $T_0/O(T_0)$ is solvable. Then either $n = 2$, $p \in \{2, 3, 5, 7, 11\}$ and $T_0 = \mathsf{SL}_2(p)$, or $n = 4$, $p = 2$ and $T_0 \cong \Gamma\mathsf{L}_1(16) \leq \mathsf{GL}_4(2)$ (and $m = n/2$ in each case).*

Proof. Let P be a Sylow p-subgroup of T_0. Then $T_0 = PS_0 = S_0P$.

First we deal with the case $n = 2$. Here P is a Sylow p-subgroup of $\mathsf{SL}_2(p)$, hence P fixes a unique one-dimensional subspace of $\mathbb{F}_p^2$. The transitivity of S_0 implies that T_0 contains all Sylow p-subgroups of $\mathsf{SL}_2(p)$. Thus $\mathsf{SL}_2(p) \leq T_0$, and from $|T_0| = |S_0|p = (p^2-1)p$ we infer that $T_0 = \mathsf{SL}_2(p)$. The restrictions on the prime p follow from Dickson's list of all subgroups of $\mathsf{PSL}_2(p)$; see Huppert [10, II 8.27 and 8.28], or Suzuki [20, Chapter 3 §6].

For the rest of the proof, we may assume that $n \geq 3$. The group S_0 is the multiplicative group of a nearfield of order p^n. We use the classification of all finite nearfields, which is due to Zassenhaus; compare Passman [16, 20.3], Huppert & Blackburn [11, XII.9.2 and XII.9.4], or Hering [7, Theorem 2]. Since $n \geq 3$, this classification implies that $S_0 \leq \Gamma\mathsf{L}_1(p^n)$. Hence S_0 is metacyclic, and therefore supersolvable.

We claim that $T_0 = PS_0$ is solvable. If p is odd, then this follows from a result of Berkovic [2] (see also Finkel & Ward [3]) which says that each product of a nilpotent group of odd order with a supersolvable group is solvable (in this result, supersolvability cannot be replaced by solvability, since the alternating group A_5 is the product of a cyclic group of order 5 with A_4). For $p = 2$, we have $O(T_0) \leq S_0$, hence $O(T_0)$ is metacyclic, and our assumption for $p = 2$ implies that T_0 is solvable.

The solvable subgroups $T_0 \leq \mathsf{GL}_n(p)$ that are transitive on the non-zero vectors have been classified by Huppert [9]; see also Passman [16, 19.10], Lüneburg [14, 37.3], or Huppert & Blackburn [11, XII.7.3]. As $n \geq 3$, we obtain from this classification that either $T_0 \leq \Gamma\mathsf{L}_1(p^n)$, or $n = 4$, $p = 3$ and $T_0 = \left(3^4 - 1\right) 2^e$ with $e \in \{1, 2, 3\}$, which is a contradiction to our assumption that $|T_0\colon S_0| = p^m$. Hence it remains to consider the case $T_0 \leq \Gamma\mathsf{L}_1(p^n)$. We infer that p^m divides n. Since $n \leq 2m$, this occurs only if $p = 2$ and $n = m/2 \in \{2, 4\}$. Thus $T_0 \leq \mathsf{GL}_2(2) = \mathsf{SL}_2(2)$ or $T_0 \leq \Gamma\mathsf{L}_1\left(2^4\right)$, and equality holds for order reasons. ∎

Proposition 4.5. *Let L_∞ be regular and $s < t$. Consider a panel $\{x, L, y\}$ with $x \in L_\infty$ and $L \neq L_\infty$, and a point $y' \in L \setminus \{x, y\}$. Then the stabilizer $T := G_{L,y,y'}$ does not act faithfully on $\Gamma(y)$.*

Proof. We assume that $T := G_{L,y,y'}$ acts faithfully on $\Gamma(y)$ and aim for a contradiction. As in 4.2, we identify $\Gamma(y) \setminus \{L\}$ with the elementary abelian root group V that belongs to the panel $\{L_\infty, x, L\}$. Then $T \leq \mathsf{AGL}(V)$. First we determine the possibilities for Γ.

Choose a point $x' \neq x$ on L_∞. The stabilizer $S := T_{x'}$ has order $t(t-1)$ and acts sharply 2-transitively on $\Gamma(y) \setminus \{L\}$. Choose a line M opposite L_∞ incident with y, and put $T_0 = T_M$ and $S_0 = S_M$. By Lemma 4.3, the quotient $T_0/O(T_0)$ is solvable if $p = 2$. Since $t \leq s^2$, we can apply Lemma 4.4, which yields $t = s^2$. Since $T_0/O(T_0)$ contains an elementary abelian subgroup of order s by 4.3, the case with $n = 4$ in 4.4 cannot occur. The cases $s = 2, 3$ lead to the quadrangles $\mathsf{Q}(5, 2)$ and $\mathsf{Q}(5, 3)$ of order $(2, 4)$ and $(3, 9)$, respectively, since these quadrangles are uniquely determined by their orders; see Payne & Thas [15, 5.3.2]. By 4.4, in the remaining cases we have

$$t = s^2 \text{ and } T_0 \cong \mathsf{SL}_2(s), \quad \text{with } s \in \{5, 7, 11\}.$$

By condition (LST), the group T_0 is transitive on $\Gamma(L_\infty) \setminus \{x\}$. The normal structure of $\mathsf{SL}_2(s)$ implies that the permutation group $\overline{T}_0$ induced by T_0 on $\Gamma(L_\infty) \setminus \{x\}$ is the group $\mathsf{PSL}_2(s)$ in an unnatural transitive action of degree s (these actions are uniquely determined, up to automorphisms of $\mathsf{SL}_2(11)$ for $s = 11$). In fact, $\overline{T}_0 = \mathsf{Alt}_5 \cong \mathsf{PSL}_2(5)$ for $s = 5$, and $\overline{T}_0 = \mathsf{GL}_3(2) \cong \mathsf{PSL}_2(7)$ acting on non-zero vectors for $s = 7$.

The group T_0 acts trivially on $\Gamma(L)$, because T_0 fixes x, y, y' and $\mathsf{SL}_2(s)$ has no proper subgroup of index $s-2$ or smaller (compare [10, Satz 8.28, p. 214]). Hence T_0 is the kernel of the action of $G_{L,M}$ on $\Gamma(L)$.

Let $M' \in \Gamma(x) \setminus \{L, L_\infty\}$ and $x' \in \Gamma(L_\infty) \setminus \{x\}$. The stabilizer $G_{L,M,M'}$ acts faithfully and sharply 2-transitively on the set $\Gamma(L_\infty) \setminus \{x\}$ of size s. Since s is a prime, the group $G_{L,M,M',x'}$ is cyclic. Pick a generator g of $G_{L,M,M',x'}$. Then the permutation $\overline{g}$ induced by g on $\Gamma(L_\infty)$ acts on $\Gamma(L_\infty)$ as a cycle of length $s-1$, fixing two points (x and x'). Thus $\overline{g}$ is a cycle of length $s-1$, hence an odd permutation, and $\overline{T}_0$ is normalized by $\overline{g}$, in view of $T_0 \trianglelefteq G_{L,M}$. But for $s=7$ and $s=11$, such a cycle does not exist: The groups $\overline{T}_0 = \mathsf{GL}_3(2) \leq \mathsf{Sym}_7$ and $\overline{T}_0 = \mathsf{PSL}_2(11) \leq \mathsf{Sym}_{11}$ consist of even permutations and coincide with their normalizers in Sym_7 and Sym_{11}, respectively.

Now we consider the case $s=5$. The set $Y := \{L_\infty, M\}^{\perp\perp}$ has size $s+1=6$, since L_∞ is regular. Denote by $G_{[Y]}$ the elementwise stabilizer of Y in G. Then $G_{[Y],L} = G_{M,[\Gamma(L)]} = T_0$. Varying L in $\{L_\infty, M\}^\perp$, we see that $G_{[Y]}$ is transitive on $\Gamma(L_\infty)$ and induces Alt_6 on $\Gamma(L_\infty)$. Moreover, $|G_{[Y]}| = 6 \cdot |T_0| = 2 \cdot |\mathsf{Alt}_6|$, hence the kernel $G_{[Y],[\Gamma(L_\infty)]}$ has order 2 and coincides with the center of T_0. Thus $G_{[Y]}$ is a perfect central extension of $\mathsf{Alt}_6 \cong \mathsf{PSL}_2(9)$. We infer that $G_{[Y]} \cong \mathsf{SL}_2(9)$, because the Schur multiplier of $\mathsf{PSL}_2(9)$ is a cyclic group of order 6, compare [10, Satz 25.7, p. 646]. The element g acts by conjugation on T_0 and on $G_{[Y]}$, inducing automorphisms of order 4 (since $\overline{g}^2 \neq 1$). The automorphism group of $\mathsf{SL}_2(q)$ is the group $\mathsf{P\Gamma L}_2(q)$. One easily shows by calculation that g centralizes a cyclic subgroup J of order 4 in $T_0 \cong \mathsf{SL}_2(5)$. Likewise, g centralizes a cyclic subgroup J^* of order 8 in $G_{[Y]} \cong \mathsf{SL}_2(9)$. Clearly, the elements of $J^* \setminus J$ interchange L and L'. If we denote the projection of $\mathrm{proj}_M L'$ onto M' by z, then this implies that g fixes z^{J^*}. Since g cannot fix a quadrangle in Γ, the fixed point structure of g is a dual (6×6)-grid. Varying L, x, M and M', this implies that every point of L_∞ is 3-regular in the sense of [15, Section 1.3]. But then [15, 5.3.3(i)] says that Γ is isomorphic to $\mathsf{Q}(5,5)$.

Since $s = 2,3,5$ is a prime, the group T of order $st(t-1)$ coincides with the group of all automorphisms of $\Gamma \cong \mathsf{Q}(5,s)$ that fix L_∞ and x, y, y'. This group does not act faithfully on $\Gamma(y)$, as the root group belonging to $\{x, L, y\}$ shows, and we have reached a contradiction. ∎

The Main Result is a consequence of Propositions 3.6, 4.1, 4.2, and 4.5.

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