# Generalized Veronesean embeddings of projective spaces

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**Abstract**

We classify all embeddings $\theta : \mathsf{PG}(n, q) \longrightarrow \mathsf{PG}(d, q)$, with $d \geq \frac{n(n+3)}{2}$, such that $\theta$ maps the set of points of each line to a set of coplanar points and such that the image of $\theta$ generates $\mathsf{PG}(d, q)$. It turns out that $d = \frac{1}{2}n(n+3)$ and all examples are related to the quadric Veronesean of $\mathsf{PG}(n, q)$ in $\mathsf{PG}(d, q)$ and its projections from subspaces of $\mathsf{PG}(d, q)$ generated by sub-Veroneseans (the point sets corresponding to subspaces of $\mathsf{PG}(n, q)$). With an additional condition we generalize this result to the infinite case as well.

## 1 Introduction

Quadric Veroneseans are important varieties in classical algebraic geometry, and play also an important role in combinatorial geometry, see [1]. In particular, they turn up in many characterization and classification problems of certain objects in Galois spaces. It is then important to be able to recognize them. There exist a number of characterizations of quadric Veroneseans, and of their injective images under projection from suitable subspaces. However, there do not seem to exist characterizations of projections of the Veronesean $\mathfrak{V}_n$ of $\mathsf{PG}(n, q)$ from subspaces generated by sub-Veroneseans (remark that for $(m, q) \neq (1, 2)$, every sub- Veronesean $\mathfrak{V}_m$ of $\mathfrak{V}_n$ is the image under the Veronesean mapping of $\mathsf{PG}(n, q)$ onto $\mathfrak{V}_n$ of a subspace $\mathsf{PG}(m, q)$ of $\mathsf{PG}(n, q)$, see [1]). These objects have even not been studied in detail, although they are natural generalizations of normal rational cubic scrolls. The present paper presents a common characterization of quadric Veroneseans and unions of such projections thereof (for a precise statement, see below).

For the ease of notation, the symbol $\mathsf{PG}(2, q)$ will from now on denote *any* projective plane of order $q$, and not just the Desarguesian one. Of course, only the latter admits a Veronesean. Our main result below will imply that it is also the only one admitting a *generalized* Veronesean embedding, see below.

This common characterization we mentioned above will be achieved by introducing the notion of a *generalized Veronesean embedding*. But first we start with recalling the classical Veronesean map $\alpha : \mathsf{PG}(n,q) \longrightarrow \mathsf{PG}(\frac{n(n+3)}{2},q)$, which maps a point with coordinates $(x_i)_{0\leq i\leq n}$ onto the point with coordinates $(x_ix_j)_{0\leq i\leq j\leq n}$. Notice that the image of a line under $\alpha$ is a plane conic. Next, we define the (new) notion of an $(i+1)$-*Veronesean embedding* $\theta : \mathsf{PG}(n,q) \longrightarrow \mathsf{PG}(\frac{n(n+3)}{2},q)$, with $0\leq i+1\leq n$. We use induction on $n$. For $n=0$, a 0-Veronesean map is trivial and equal to the classical Veronesean map. For $n=1$, a 0-Veronesean embedding is any injective map from $\mathsf{PG}(1,q)$ to $\mathsf{PG}(2,q)$ such that the image spans $\mathsf{PG}(2,q)$. Now let $\max\{n,i+2\}>1$. Let $U$ be an $i$-dimensional subspace of $\mathsf{PG}(n,q)$. Put $d'=\frac{i(i+3)}{2}$. Then the image of $U$ under $\alpha$ spans a $d'$-dimensional subspace $V$ of $\mathsf{PG}(d,q)$. Let $W$ be a $(d-d'-1)$-dimensional subspace of $\mathsf{PG}(d,q)$ skew to $V$ and let $\theta' : U\rightarrow V$ be a $j$-Veronesean embedding of $U$ (defined inductively), for some $j$, with $0\leq j\leq i$. Then define $\theta : \mathsf{PG}(n,q) \longrightarrow \mathsf{PG}(d,q)$ as $\theta(x)=\theta'(x)$ for $x\in U$, and $\theta(x)=\langle\alpha(x),V\rangle\cap W$ for $x\in\mathsf{PG}(n,q)\setminus U$. The latter means the projection of $\alpha(x)$ from $V$ onto $W$.

The subspace $U$ will be referred to as the *lid* of the embedding. Hence a 0-Veronesean embedding for $n>1$ is an ordinary quadric Veronesean embedding and has empty lid.

Now, notice that the image of any line of an $i$-Veronesean embedding, $0\leq i\leq n$, of $\mathsf{PG}(n,q)$ is a set of points contained in a plane. More generaly, we now define a *generalized Veronesean embedding* as a map $\theta : \mathsf{PG}(n,q) \longrightarrow \mathsf{PG}(d,q)$, with $d\geq\frac{n(n+3)}{2}$, such that $\theta$ maps the set of points of each line of $\mathsf{PG}(n,q)$ to a set of coplanar points of $\mathsf{PG}(d,q)$ and such that the image of $\theta$ generates $\mathsf{PG}(d,q)$. The image in $\mathsf{PG}(d,q)$ is then called a *generalized Veronesean*. We will prove that $d=\frac{n(n+3)}{2}$ and that each such embedding $\theta$ is an $i$-Veronesean embedding for some $i$, $0\leq i\leq n$.

For a given embedding $\theta : \mathsf{PG}(n,q) \longrightarrow \mathsf{PG}(d,q)$, we will from now on identify each point of $\mathsf{PG}(n,q)$ with its image under $\theta$. For $\theta$ an $i$-Veronesean embedding we call the image under $\theta$ an $i$-*Veronesean*. We can then formulate our main result as follows.

**Main Result.** *Let $\mathcal{S}=(\mathcal{P},\mathcal{L},\in)$ be isomorphic to the geometry of points and lines of $\mathsf{PG}(n,q)$, $n\geq 2$, $q>2$, with $\mathcal{P}\subseteq\mathsf{PG}(d,q)$, $\langle\mathcal{P}\rangle=\mathsf{PG}(d,q)$, $d\geq\frac{n(n+3}{2}$, and such that every member $L$ of $\mathcal{L}$ is a subset of points of a plane in $\mathsf{PG}(d,q)$. Then $\mathcal{P}$ is an $i$-Veronesean, for some $i\in\{0,1,\ldots,n\}$.*

The case $q=2$ is a true exception, since every injective mapping from $\mathsf{PG}(n,2)$ into $\mathsf{PG}(d,q)$, with $d\geq n$, is a generalized Veronesean embedding as soon as the image set generates $\mathsf{PG}(d, q)$. And this can be achieved whenever $d \leq 2^{n+1} - 2$.

The rest of the paper is devoted to the proof of the Main Result. Since we will need to identify a single projection of a quadric Veronesean of $\mathsf{PG}(n, q)$ in $\mathsf{PG}(d, q)$, $d = \frac{1}{2}n(n+3)$, from a subspace of $\mathsf{PG}(d, q)$ spanned by a subspace of $\mathsf{PG}(n, q)$, we first show how one can recognize this situation easily. It provides a nice connection between quadric Veronesean varieties and Segre varieties.

In the last section, we generalize to finite-dimensional projective spaces over infinite (skew) fields.

## 2 Projections of quadric Veroneseans

Consider coordinate systems of the projective spaces $\mathsf{PG}(n, q)$, $\mathsf{PG}(m, q)$ and $\mathsf{PG}(nm+n+m, q)$, where we write the coordinate tuples of the points of $\mathsf{PG}(n, q)$ as column matrices, those of $\mathsf{PG}(m, q)$ as row matrices, and those of $\mathsf{PG}(nm + n + m, q)$ as $(n + 1) \times (m + 1)$-matrices. In practice, $n$ is allowed to be equal to $m$, in which case we still view $\mathsf{PG}(n, q)$ and $\mathsf{PG}(m, q)$ as distinct projective spaces (this does not cause any notational confusion). With a pair of points $(p_1, p_2)$, where $p_1 \in \mathsf{PG}(n, q)$ and $p_2 \in \mathsf{PG}(m, q)$, we can associate a point $p_1 * p_2$ of $\mathsf{PG}(nm + n + m, q)$ by simply multiplying the column matrix associated with $p_1$ with the row matrix associated with $p_2$.

Let $\mathfrak{S}_{n,m}(q)$ be the *Segre variety* of the pair of projective spaces $(\mathsf{PG}(n, q), \mathsf{PG}(m, q))$, i.e., $\mathfrak{S}_{n,m}(q)$ is the subset of points $p_1 * p_2$ of $\mathsf{PG}(nm + n + m, q)$, with $p_1 \in \mathsf{PG}(n, q)$ and $p_2 \in \mathsf{PG}(m, q)$. This variety has some nice properties, which are proved in [1]. We write down some of them.

(1) There are two classes $\Sigma_1$ and $\Sigma_2$ of maximal subspaces contained in $\mathfrak{S}_{n,m}(q)$. All members of $\Sigma_1$ have dimension $n$, those of $\Sigma_2$ dimension $m$. Also, every pair of members of $\Sigma_i$, $i \in \{1, 2\}$, is disjoint and every member of $\Sigma_1$ meets every member of $\Sigma_2$ in precisely one point.

(2) Let $U$ and $V$ be two arbitrary members of $\Sigma_1$. Define the following map $\sigma : U \to V$: a point $u \in U$ is mapped onto the point $v \in V$ if $u$ and $v$ are contained in a common member of $\Sigma_2$. Then $\sigma$ is a projectivity (i.e., a collineation preserving the cross-ratio with respect to the given coordinate system in $\mathsf{PG}(nm+n+m, q)$). The same property holds interchanging the roles of $\Sigma_1$ and $\Sigma_2$.

(3) Let $U \in \Sigma_1$ be arbitrary. Let $P$ be the set of $n+2$ points of a skeleton of $U$. Let $\Pi$ be the set of $n+2$ members of $\Sigma_2$ containing a point of $P$. Then $\Pi$ is projectively equivalent to any set of $n+2$ subspaces of $\mathsf{PG}(nm+n+m,q)$ of dimension $m$ such that every $n+1$ of that set generate $\mathsf{PG}(nm+n+m,q)$. Also, for each point $x$ belonging to the union of the members of $\Pi$, there exists a unique subspace $U'$ in $\mathsf{PG}(nm+n+m,q)$ of dimension $n$ meeting every member of $\Pi$ (necessarily in unique points), and $U'$ belongs to $\Sigma_1$.

The last property allows for the following characterization of $\mathfrak{S}_{n,m}(q)$.

**Proposition 2.1** *Let $\Sigma_1$ and $\Sigma_2$ be two families of subspaces of $\mathsf{PG}(d,q)$, with $d \geq nm+n+m$, where $n$ is the dimension of every member of $\Sigma_1$, and $m$ is the dimension of every member of $\Sigma_2$, and assume that $\Sigma_1$ and $\Sigma_2$ cover the same point set of $\mathsf{PG}(d,q)$. Suppose that every member of $\Sigma_1$ meets every member of $\Sigma_2$. Suppose also that $\Sigma_2$ contains a set of $n+2$ members each $n+1$ of which generate a space of dimension $nm+n+m$. Then $\Sigma_1$ and $\Sigma_2$ form the two sets of maximal subspaces of a Segre variety $\mathfrak{S}_{n,m}(q)$.*

Note also that (2) above implies the following property.

**Proposition 2.2** *Let $\Sigma_1$ and $\Sigma_2$ be the two families of maximal subspaces of the Segre variety $\mathfrak{S}_{n,m}(q)$, where the dimension of each member of $\Sigma_1$, $\Sigma_2$ is equal to $n$, $m$, respectively. If we call a block of $\Sigma_2$ a set of $q+1$ members of $\Sigma_2$ intersecting some (and hence every) member of $\Sigma_1$ in $q+1$ points of a line, then this gives $\Sigma_2$ the structure of (the point-line geometry associated to) $\mathsf{PG}(n,q)$.*

Now let $\mathfrak{V}_n(q) \subseteq \mathsf{PG}(d,q)$ be the quadric Veronesean of $\mathsf{PG}(n,q)$, $d = \frac{1}{2}n(n+3)$. Embed $\mathsf{PG}(d,q)$ in a $\mathsf{PG}(d',q)$, with $d' = d+nm+n+m+1$, with $m \geq 0$ arbitrary. Choose a subspace $\mathsf{PG}(nm+n+m,q)$ in $\mathsf{PG}(d',q)$ skew to $\mathsf{PG}(d,q)$ and consider a Segre variety $\mathfrak{S}_{n,m}(q)$ in $\mathsf{PG}(nm+n+m,q)$ with $\Sigma_2$ the family of $m$-spaces partitioning $\mathfrak{S}_{n,m}(q)$. Choose any isomorphism $\sigma$ between $\Sigma_2$ (viewed as projective space of dimension $n$, see Proposition 2.2) and the point set of $\mathfrak{V}_n(q)$, also viewed as the point set of $\mathsf{PG}(n,q)$. Assume that $\sigma$ preserves the cross-ratio. Let $\mathfrak{P}_{n,m}(q)$ be the set of points of $\mathsf{PG}(d',q)$ obtained from $\sigma$ by considering the union of all sets $\langle X, X^\sigma\rangle \setminus X$, with $X \in \Sigma_2$. Then it is clear that the isomorphism class of this object inside $\mathsf{PG}(d',q)$ is independent of the choice of the embeddings of $\mathfrak{V}_n(q)$ and $\mathfrak{S}_{n,m}(q)$, and also independent of $\sigma$. We call it an *SV-combination (over $\mathsf{GF}(q)$) with parameters $(n,m)$.*

Now, it is easy to verify that the following set of points of $\mathsf{PG}(d',q)$ is an SV-combination over $\mathsf{GF}(q)$ with parameters $(n,m)$:

$$\{((x_ix_j)_{0\le i\le n,0\le j\le n},(x_iy_j)_{0\le i\le n,0\le j\le m}):x_i,y_j\in\mathsf{GF}(q),0\le i\le n,0\le j\le m\}.$$

But this is clearly also the set of points obtained by projecting $\mathfrak{V}_{n+m+1}(q)\setminus\mathfrak{V}_m(q)$ from the subspace generated by the points of a canonical sub-Veronesean $\mathfrak{V}_m(q)$. This follows from an easy coordinate calculation.

Hence we can summarize:

**Proposition 2.3** *Let $\mathfrak{V}_m(q)$ be a sub-Veronesean of the Veronesean $\mathfrak{V}_{n+m+1}(q)$, except that for $(q,m)=(2,1)$ we take for $\mathfrak{V}_1(2)$ the image of a line of $\mathsf{PG}(n+2,2)$. Then the projection of $\mathfrak{V}_{n+m+1}(q)\setminus\mathfrak{V}_m(q)$ from the subspace generated by the points of $\mathfrak{V}_m(q)$ is projectively equivalent with an SV-combination over $\mathsf{GF}(q)$ with parameters $(n,m)$.*

Notice that the case $n=0$ corresponds with an affine space of dimension $m+1$. Also, Proposition 2.3 is fundamental to the (inductive) proof of the Main Result.

# 3 Generalized Veronesean embeddings of projective planes

We aim to prove our Main Result using induction on $n$. Hence we first consider the case $n=2$. In the present section, we will show that any generalized Veronesean embedding of a projective plane of order $q$ is an $i$-Veronesean of the classical projective plane of order $q$, with $i\in\{0,1,2\}$.

Hence, throughout this section, we assume that $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathtt{I})$ is isomorphic to a projective plane of order $q$, which we denote by $\mathsf{PG}(2,q)$. We assume that $\mathcal{P}\subseteq\mathsf{PG}(d,q)$, $\langle\mathcal{P}\rangle=\mathsf{PG}(d,q)$, $d\ge 5$, and that every member $L$ of $\mathcal{L}$ is a subset of points of a plane in $\mathsf{PG}(d,q)$, which we denote by $\pi_L$ if it is unique; if it is not unique, then $\pi_L$ is the intersection of all such planes (and so $\pi_L$ is the line of $\mathsf{PG}(d,q)$ containing all points of $L$).

We denote the line of $\mathsf{PG}(d,q)$ spanned by two points $a,b\in\mathcal{P}$ by $\langle a,b\rangle$, while the line of $\mathcal{S}$ through $a,b$ is denoted by $ab$. More generally, we use the symbol $\langle A\rangle$ to denote the subspace of $\mathsf{PG}(d,q)$ generated by the elements of $A$.

We will assume that $q>2$.

We begin with a general lemma.

**Lemma 3.1** *Let $S_1, S_2, S_3$ be three sets, of at least three lines each, in* $\mathsf{PG}(m,q)$*,* $m \geq 3$*, such that each member of $S_i$ meets every member of $S_j$ in a unique point, for $i \neq j$, for all $i, j \in \{1,2,3\}$. Then there are distinct indices $i, j \in \{1,2,3\}$ such that either all lines of $S_i \cup S_j$ are contained in a plane, or they contain a common point.*

**Proof** Without loss of generality we may suppose that there are two skew lines $L_1, M_1 \in S_1$. As soon as two members $L_2, M_2$ of $S_2$ are skew, every member of $S_3$ must either contain $L_1 \cap L_2$ and $M_1 \cap M_2$, or $L_1 \cap M_2$ and $M_1 \cap L_2$, and so $|S_3| \leq 2$, a contradiction. Hence all members of $S_2$ and, likewise, of $S_3$ meet each other. By assumption all members of $S_2$ meet all members of $S_3$, so all members of $S_2 \cup S_3$ meet each other. The lemma is now clear. $\square$

**Lemma 3.2** *If $L, M$ are two distinct lines of $\mathcal{S}$, meeting in the point $z \in \mathcal{P}$, and $x \in \mathcal{P}$ is a point off $L \cup M$ not contained in $\langle \pi_L, \pi_M \rangle$, then every point $y \in \mathcal{P}$ off $xz$ is contained in the space $W := \langle \pi_L, \pi_M, x \rangle$.*

**Proof** The line $xy$ meets $L \cup M$ in two distinct points $u, v$, with $u \in L$ and $v \in M$. So the space $\pi_{xy}$ has a line in common with $\langle \pi_L, \pi_M \rangle$ and contains $x$. Hence $\pi_{xy} = \langle x, u, v \rangle \subseteq W$. $\square$

**Lemma 3.3** *If $d \geq 5$ and if a proper subspace $H$ of* $\mathsf{PG}(d,q)$ *contains all points of $\mathcal{S}$ off a certain line $L \in \mathcal{L}$, then $d = 5$ and either $\mathcal{S}$ is 2-Veronesean, or $\mathcal{S}$ is a 1-Veronesean.*

**Proof** Suppose some proper subspace $H$ contains all points of $\mathcal{S}$ off a line $L \in \mathcal{L}$. We may assume that $H$ is spanned by the points of $\mathcal{S}$ off $L$. Then there is a least one point $x \in L$ not contained in $H$. For each line $M \in \mathcal{L}$ containing $x$, with $M \neq L$, the space $\pi_M$ is a plane and intersects $H$ in a line. It follows that all points of $M$ except $x$ are contained in a line $L_M$ of $H$.

Suppose first that there is a second point $x' \in L$ not contained in $H$, then we obtain likewise for each line $M' \in \mathcal{L}$ through $x'$, with $M' \neq L$, a line $L'_{M'}$ in $H$ containing all points of $M'$ except for $x'$. Each of the lines $L_M$ meets each of the lines $L'_{M'}$. It follows easily that $H$ is at most 3-dimensional.

If no other point of $L$ lies outside $H$, then, as $q \geq 3$, at least two points of $L$ are contained in $H$, and so $\pi_L$ is a plane meeting $H$ in a line. This implies that $H$ is a hyperplane, contradicting our observation above. Hence, at least three points $x, x', x''$ of $L$ lie outside $H$. In $H$, there arise three sets of $q$ lines, and two lines of different sets always intersect.

It follows that all points of $\mathcal{S}$ off $L$ lie in a plane, which by definition coincides with $H$. Hence, deleting $L$ and its points, there arises an affine plane $\mathsf{AG}(2,q)$ in $H$, and $\mathcal{S}$ is Desarguesian. As $\mathcal{S}$ generates a $d$-dimensional space, with $d \geq 5$, $\pi_L$ is a plane which is disjoint from the plane $H$, and $d = 5$.

Note that with what we already proved, we can derive the following property:

(*) If all points of $\mathcal{S}$ except some contained in a fixed line are contained in a 3-space, then $\mathcal{S}$ is Desarguesian and $\theta$ is a 2-Veronesean.

Also, since two lines generate at most a 4-space, our previous arguments and Lemma 3.2 imply that, if $d > 5$, then we have a 2-Veronesean, and hence $d = 5$ after all, a contradiction. Hence $d = 5$ in any case.

Next suppose that only one point $x$ of $L$ lies outside $H$. Then $H$ is a hyperplane, and hence 4-dimensional. Let $L_1$ and $L_2$ be two lines of $\mathcal{S}$ through $x$, and let $L_1'$ and $L_2'$ be the respective lines of $H$ containing $q$ points of these (we shall adopt this notation throughout this proof). Suppose that $L_1'$ and $L_2'$ are contained in a plane $\pi$. Take a point $y$ of $\mathcal{S}$ outside $L_1 \cup L_2$ and not belonging to $\pi$, and consider any line $M$ through $y$ not through $x$. Then $\pi_M$ meets $\pi$ in a line and hence $\pi_M$ is contained in the 3-space $\langle \pi, y \rangle$. It follows that all points of $\mathcal{S}$ off $xy$ are contained in a 3-space. Now (*) above implies that we have a 2-Veronesean, contradicting the hypothesis of this paragraph. Hence $L_1'$ and $L_2'$ are skew.

Let $z \in \mathcal{P}$ be distinct from $x$. Let $M_1$ and $M_2$ be two lines of $\mathcal{S}$ through $z$ not incident with $x$. We claim that $Z = \langle M_1, M_2 \rangle$ is 4-dimensional. Assume that $Z$ is 3-dimensional (if $\langle M_1, M_2 \rangle$ is a plane, then $\langle L_1', L_2' \rangle$ is a plane, a contradiction). Every line $N$ through $x$, with $N \neq xz$, has two points $N \cap M_i$, $i = 1, 2$, in common with $Z$, and none of these points coincides with $x$. It follows that $\mathcal{P} \setminus xz$ is contained $Z$. Again (*) implies that $\mathcal{P}$ is a 2-Veronesean, a contradiction. The claim follows.

Now let $L_1$ and $L_2$ again be two arbitrary lines through $x$, and let $z$ be a point distinct from $x$ off $L_1 \cup L_2$. We claim that $Z' = \langle L_1', L_2', z \rangle$ is 4-dimensional. Indeed, we already know that $\langle L_1', L_2' \rangle$ is a 3-space, so we assume by way of contradiction that $z \in \langle L_1', L_2' \rangle = Z'$. We already showed that $z$ is not contained in $L_1' \cup L_2'$, otherwise $\langle M_1, M_2 \rangle$ is not 4-dimensional for any two lines $M_1, M_2$ on $z$ not containing $x$; hence there is a unique line of $\mathsf{PG}(5,q)$ containing $z$ and meeting both $L_1'$ and $L_2'$. Since $q \geq 3$, it follows that at least two lines of $\mathcal{S}$ through $z$ not through $x$ have three non-collinear points in common with $Z'$, and so these two lines generate $Z'$, contradicting our previous clam. Our present claim follows.

Now let $L_3$ be a third line through $x$. Denote the unique point of $\mathsf{PG}(5,q)$ on $L'_i$ which does not belong to $\mathcal{P}$ by $x_i$, $i = 1,2,3$. From our previous claim we infer that $x_i \in \langle L'_j, L'_k \rangle$, for $\{i,j,k\} = \{1,2,3\}$. It is easy to see that, if $x_1, x_2, x_3$ were not collinear, then either two members of $\{L'_1, L'_2, L'_3\}$ would be coplanar, or $\langle L'_1, L'_2, L'_3 \rangle$ would be 3-dimensional, both contradictions. Hence $x_1, x_2, x_3$ are collinear. We conclude that there is a line $X$ in $\mathsf{PG}(5,q)$ containing no point of $\mathcal{S}$, but meeting every line of $\mathsf{PG}(5,q)$ containing $q$ collinear points of $\mathcal{S}$ which form, together with $x$, a line of $\mathcal{S}$.

Now we project $\mathcal{P} \setminus \{x\}$ from an arbitrary point $z \in \mathcal{P}$, $z \neq x$, onto a suitable 3-space. Our previous claims imply that the $q$ projections of the planes supporting the lines of $\mathcal{S}$ through $z$ distinct from $xz$, together with the projection of $X$, form a full set of generators of a hyperbolic quadric $\mathcal{H}$ for which the complementary regulus contains the $q$ projections of the lines $\langle L \rangle \cap H$, with $L$ a line of $\mathcal{S}$ through $x$ not containing $z$. It now follows that every line of $\mathcal{S}$ not through $x$ nor $z$ is projected onto a plane conic lying on $\mathcal{H}$. Consequently every line of $\mathcal{S}$ not through $x$ forms a plane conic in $\mathsf{PG}(5,q)$. Moreover, the complementary regulus of the above quadric $\mathcal{H}$ defines a projectivity between the points of the projection of such a conic $C$ (provided $z \notin C$) and the points of the projection of $X$ where the projection of the line $\langle xz \rangle \cap H$ is a fixpoint). It follows that the mapping $\beta : C \longrightarrow X$ defined by $\beta(u)$ is the intersection of $X$ with the plane generated by the line $ux$ of $\mathcal{S}$, is a projectivity. We conclude now that $\mathcal{S} \cap H$ is a normal rational cubic scroll, and consequently that $\mathcal{S}$ is a 1-Veronesean (since any normal rational cubic scroll is the projection from a point of a conic Veronesean onto a 4-space). In particular, $\mathcal{S}$ is Desarguesian.

This completes the proof of the lemma. $\square$

So we may, from now on, assume that no line $L$ of $\mathcal{S}$ has the property that $\mathcal{P} \setminus L$ is contained in a proper subspace. In view of Lemma 3.2, this implies that every pair of lines generates a 4-space.

We now show that no three collinear points of $\mathcal{S}$ are collinear in $\mathsf{PG}(5,q)$. Assume by way of contradiction that $L \in \mathcal{L}$ contains three points $x_1, x_2, x_3$ such that $X := \langle x_1, x_2, x_3 \rangle$ is a line (and note that $X$ does not contain points of $\mathcal{S}$ off $L$). We now project all planes spanned by lines of $\mathcal{S}$ through one of the $x_i$, $i = 1,2,3$, except for $L$, from $X$ onto a suitable 3-space $U$ and thus obtain three sets $S_1, S_2, S_3$ of at least three lines each, such that each member of $S_i$ meets every member of $S_j$, for $i,j \in \{1,2,3\}$, $i \neq j$. Note that no two of such projections coincide, since otherwise the corresponding planes live in a solid, contradicting our observation in the previous paragraph. Hence we can apply Lemma 3.1 and deduce without loss of generality that either $S_2 \cup S_3$ is contained in a plane (but then all points of $\mathcal{S}$ off $L$ are contained in a hyperplane, contradicting our hypothesis), or all members of $S_2 \cup S_3$ contain a common point $z$. But then every point of $\mathcal{S}$ off $L$ is even contained in a plane, again a contradiction.

Consequently, it follows that every line $L$ of $\mathcal{S}$ is an oval in $\pi_L$, and it follows from [3] that $\mathcal{S}$ is Desarguesian and isomorphic to a 0-Veronesean.

# 4 Generalized Veronesean embeddings of projective spaces of dimension at least 3

Here we assume that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is isomorphic to $\mathsf{PG}(n, q)$, $n > 2$, with $\mathcal{P} \subseteq \mathsf{PG}(d, q)$, $\langle \mathcal{P} \rangle = \mathsf{PG}(d, q)$, $d \geq \frac{1}{2}n(n+3)$, and such that every member $L$ of $\mathcal{L}$ is a subset of points of a plane in $\mathsf{PG}(d, q)$, which we again denote by $\pi_L$ if it is unique; if it is not unique, then $\pi_L$ is again the line of $\mathsf{PG}(d, q)$ containing all points of $L$. In the present section, we finish the proof of our Main Result.

We use the same notation as before to distinguish lines of $\mathcal{S}$ from lines of $\mathsf{PG}(d, q)$: we denote the line of $\mathsf{PG}(d, q)$ spanned by two points $a, b \in \mathcal{P}$ by $\langle a, b \rangle$, while the line of $\mathcal{S}$ through $a, b$ is denoted by $ab$. More generally, we use the symbol $\langle A \rangle$ to denote the subspace of $\mathsf{PG}(d, q)$ generated by the elements of $A$, and we use $\langle A \rangle_{\mathcal{S}}$ to denote the subspace of $\mathcal{S}$ spanned by $A$.

We will assume that $q > 2$ as for $q = 2$ every injective map from $\mathcal{S}$ to $\mathsf{PG}(d, 2)$ such that the image of $\mathcal{S}$ spans $\mathsf{PG}(d, 2)$ is a generalized Veronesean embedding, for every $d$.

Our proof proceeds by induction on $n$. The result for $n = 2$ has been proved in Section 3, and we assume that the result is true for any generalized Veronesean embedding of $\mathsf{PG}(n', q)$ in $\mathsf{PG}(d', q)$, with $n' < n$ and $d' \geq \frac{1}{2}n'(n'+3)$.

We first prove that $d$ must necessarily be equal to $\frac{1}{2}n(n+3)$. This will be a consequence of the following lemma, in which we abandon for once our restrictive assumption on $d$.

**Lemma 4.1** *Let $H \cong \mathsf{PG}(n-1, q)$ be a hyperplane of $\mathcal{S}$, and let $H$ generate (in $\mathsf{PG}(d, q)$) a $d'$-dimensional subspace. Then $d - d' \leq n + 1$.*

**Proof** We prove this by induction on $n$. For $n = 2$, this is proved in Section 3, so we may suppose that $n > 2$ (notice that, if both $H$ and $\langle H \rangle$ are 1-dimensional, then by our classification above, $d < 5$ and hence the assertion follows). Choose an arbitrary subspace $U \subseteq H$ of dimension $n - 2$ in $H$, and let $H'$ be a second hyperplane of $\mathcal{S}$ containing $U$.

Using the induction hypothesis applied to $U \subseteq H'$, we see that $\dim\langle H, H'\rangle \leq d' + n$. If $d - d' > n + 1$, then we can select two points $x_1, x_2 \in \mathcal{P}$ such that $\langle x_1, x_2\rangle$ is disjoint from $\langle H, H'\rangle$. For an arbitrary point $z \in \mathcal{P}$ not in $\langle U, x_1\rangle_{\mathcal{S}}$, let $y$ and $y'$ be the intersection points of $x_1 z$ with $H$ and $H'$, respectively. Then $z \in \langle x_1, y, y'\rangle$. We conclude that, in particular, $x_1 x_2$ meets $U$, and, interchanging the roles of $x_1$ and $x_2$, that all points of $\mathcal{S}$ off $\langle U, x_1\rangle_{\mathcal{S}}$ are contained in $\langle H, H'\rangle$.

We can consider any hyperplane $H'' \neq H$ of $\mathcal{S}$ through $U$, but not containing $x_1$ and $x_2$, let it play the role of $H'$ in the previous paragraph, select the same points $x_1$ and $x_2$ outside $\langle H, H''\rangle$, argue as before and obtain that $\langle H'\rangle$ is contained in $\langle H, H''\rangle$; hence $\langle H, H'\rangle = \langle H, H''\rangle$. Now, we can repeat the same argument starting with $H^* = \langle U, x_1\rangle_{\mathcal{S}}$ instead of $H'$, and using $q > 2$ obtain that $H^*$ is contained in at least one of $\langle H, H'\rangle$ and $\langle H, H''\rangle$, a contradiction. The lemma is proved. □

A repetitive use of the previous lemma, using an appropriate maximal chain of nested subspaces of $\mathcal{S}$, easily shows the following proposition:

**Proposition 4.2** *If $d \geq \frac{1}{2}n(n+3)$, then $d = \frac{1}{2}n(n+3)$ and every $i$-dimensional subspace $U$ of $\mathcal{S}$, $i \leq n-1$, generates in $\mathsf{PG}(d,q)$ a subspace of dimension $\frac{1}{2}i(i+3)$. Hence the induction hypothesis implies that $U$ is an $\ell$-Veronesean, for some nonnegative integer $\ell \leq i$. In particular, for every line $L \in \mathcal{L}$ it holds that $\pi_L$ is 2-dimensional.*

Our next task is to identify the (future) lid of the embedding. Therefore, we introduce the following notions. Let $L \in \mathcal{L}$ be arbitrary. Then we say that $L$ is a *semiaffine line* if there is a unique point $x$ on $L$ such that $\langle L \setminus \{x\}\rangle$ is 1-dimensional. The point $x$ is called a *lid point*, or the *lid of L*. The line $L$ is called a *box for x*. Clearly, the lid of a semiaffine line is unique, but a lid point can have several boxes. The *lid* of $\mathcal{S}$ is the set of the lid points of all semiaffine lines.

The lid of an $i$-Veronesean of $\mathsf{PG}(n,q)$ is a proper subspace of $\mathsf{PG}(n,q)$, and being a proper subspace is exactly the first thing we will prove for the lid of $\mathcal{S}$, in three lemmas.

Let $\mathfrak{L}$ be the lid of $\mathcal{S}$.

**Lemma 4.3** *The set $\mathfrak{L}$ is a subspace of $\mathcal{S}$.*

**Proof** The lemma is trivial if $\mathfrak{L}$ either is empty, or contains a unique point. Hence we may assume that at least two elements belong to $\mathfrak{L}$. Let $x_1$ and $x_2$ be two lid points, and let $L_1$ and $L_2$ be boxes of $x_1$ and $x_2$, respectively. If $L_1$ and $L_2$ generate a plane $\pi$ of

$\mathcal{S}$, then $\pi$ induces an $i$-Veronesean in some 5-dimensional subspace $\mathsf{PG}(5,q)$ of $\mathsf{PG}(d,q)$, $i \in \{0,1,2\}$, by Proposition 4.2. But only a 2-Veronesean has two distinct lid points, and clearly, all points on $x_1x_2$ are lid points for $\pi$, and hence for $\mathcal{S}$.

So we may assume that $U := \langle L_1, L_2 \rangle_{\mathcal{S}}$ is a 3-space. Let $\pi_i$ be a plane of $\mathcal{S}$ containing $L_i$, but not containing $x_1x_2$, $i = 1,2$. Then $\pi_i$ is a $j_i$-Veronesean with $j_i \in \{1,2\}$. If $j_i = 1$, then all lines in $\pi_i$ through $x_i$ are boxes for $x_i$; if $j_i = 2$, then at most one line in $\pi_i$ through $x_i$ is not a box for $x_i$, $i = 1,2$. It follows that there are at most two points $z$ on the intersection line $\pi_1 \cap \pi_2$ for which $x_iz$ is a not a box for $x_i$, for some $i \in \{1,2\}$. Hence there is at least one point $y \in \pi_1 \cap \pi_2$ such that $x_iy$ is a box for $x_i$, for all $i \in \{1,2\}$. Since the lines $x_1y$ and $x_2y$ meet, we deduce from the first paragraph of the present proof that all points of $x_1x_2$ are lid points.

This proves the lemma. □

**Lemma 4.4** *Let $x$ be a lid point of $\mathcal{S}$. Then the set of lines $L$ of $\mathcal{S}$ containing $x$ and such that $L$ is not a box for $x$ is the set of lines through $x$ of a subspace of $\mathcal{S}$ through $x$. (Lemma 4.6 will clearly imply that this subspace is contained in $\mathfrak{L}$.)*

**Proof** Suppose two lines $L_1, L_2$ through $x$ are not boxes for $x$. Then inspecting all possibilities in the $i$-Veronesean defined by the plane $\langle L_1, L_2 \rangle_{\mathcal{S}}$, $i = 0,1,2$, we see that all lines through $x$ inside that plane are not boxes for $x$. The assertion now follows easily. □

**Lemma 4.5** *The set $\mathfrak{L}$ does not coincide with $\mathcal{S}$.*

**Proof** We assume, by way of contradiction, that all points of $\mathcal{S}$ are lid points. Then, let $H$ be a hyperplane of $\mathcal{S}$ containing some semiaffine line. Then $H$ is an $i$-Veronesean, for $i > 0$ (use the induction hypothesis). Let $\mathfrak{L}_H$ be the lid of $H$; then $\mathfrak{L}_H$ is $(i-1)$-dimensional, $0 \leq i-1 \leq n-2$ (use induction). Hence we can pick a point $y \in H \setminus \mathfrak{L}_H$. Notice that, since $y$ is not a lid point for $H$, the subspace of $\mathcal{S}$ generated by all lines through $y$ which are not boxes for $y$, contains $H$. But, by assumption, this subspace does not coincide with $\mathcal{S}$ (as $y$ is a lid point for $\mathcal{S}$). Hence every line of $\mathcal{S}$ through $y$ not in $H$ is a box for $y$. Since $y$ is arbitrary, and since lid points are unique for any of their boxes, no point off $H$ is a lid point for a line of $\mathcal{S}$ missing $\mathfrak{L}_H$. Hence, if $z \in \mathcal{P}$ is some point off $H$, then there exists a point $x$ in $\mathfrak{L}_H$ such that $xz$ is a box for $z$. By definition of $\mathfrak{L}_H$ and the induction hypothsis, $xy$ is a box for $x$. So, in the plane $\langle x,y,z \rangle_{\mathcal{S}}$, we have a triangle $\{x,y,z\}$ of lid points for this plane, a contradiction (as in this case every point is a lid point for that plane, clearly impossible).

This contradiction completes the proof of the lemma. $\square$

So now we know that the set of lid points is a proper subspace of $\mathsf{PG}(d,q)$, say of dimenson $\ell$. Next thing we want to verify is that every line intersecting $\mathfrak{L}$ in a unique point $x$ is a box for $x$.

**Lemma 4.6** *Let $L$ be a line intersecting $\mathfrak{L}$ in a unique point $x$. Then $L$ is a box for $x$.*

**Proof**

Assume, by way of contradiction, that $L$ is not a box for $y$. Let $M$ be a box for $y$. By considering the plane of $\mathcal{S}$ defined by $L$ and $M$ (which clearly defines a 2-Veronesean), one sees that $L$ consists of lid points, implying $L \subseteq \mathfrak{L}$, a contradiction. The lemma is proved. $\square$

We now treat three special cases.

**Proposition 4.7** *If $\mathfrak{L}$ is empty, then $\mathcal{S}$ is a 0-Veronesean.*

**Proof** We can include any line of $\mathcal{S}$ in a plane of $\mathcal{S}$, which, by the fact that there are no lid points, is a 0-Veronesean in some 5-subspace of $\mathsf{PG}(d,q)$. It follows that every line of $\mathcal{S}$ is a plane conic of $\mathsf{PG}(d,q)$. The Main Result of [3] completes the proof of the proposition. $\square$

Let $L \in \mathcal{L}$, with $L$ not a line of $\mathfrak{L}$, be a box with lid $x$. Then there is a unique line $L'$ of $\mathsf{PG}(d,q)$ containing at least three points of $L$, and there is a unique point $x'$ of $\mathsf{PG}(d,q)$ incident with $L' \setminus L$. We call $L'$ and $x'$ the *bearer* and the *cap*, respectively, of $L$. We will also sometimes denote $x'$ by $\widehat{L}$. By considering a plane of $\mathcal{S}$ containing $L$, and the corresponding generalized Veronesean, one sees that $x'$ does not belong to $\mathcal{S}$.

**Proposition 4.8** *If $\mathfrak{L}$ is a point, then $\mathcal{S}$ is a 1-Veronesean.*

**Proof** We put $\{x\} = \mathfrak{L}$. Let $\mathfrak{C}$ be the set of all caps. We claim that $\mathfrak{C}$ is an $(n-1)$-subspace of $\mathsf{PG}(d,q)$. Consider two points $c_1, c_2$ of $\mathfrak{C}$, and let $L'_1, L'_2$ be two (distinct) corresponding bearers. Then the corresponding lines $L_1, L_2$ of $\mathcal{S}$ span a plane $\pi$ which induces a 1-Veronesean (because $\pi$ contains $x$). Hence all points of the line $\langle c_1, c_2 \rangle$ are caps. This also shows that different bearers contain different caps. Hence, (all lines of $\mathcal{S}$ through $x$ of) planes of $\mathcal{S}$ through $x$ correspond with (all points in $\mathsf{PG}(5,q)$ of) lines in $\mathfrak{C}$. This easily implies that (all lines of $\mathcal{S}$ through $x$ of) solids of $\mathcal{S}$ through $x$ correspond with (all points in $\mathsf{PG}(d,q)$ of) planes in $\mathfrak{C}$. An easy inductive argument now shows the claim. Moreover, the previous argument shows that coordinates can be chosen such that, for an arbitrary hyperplane $H$ of $\mathcal{S}$ not containing $x$, the mapping $\mu : H \rightarrow \mathfrak{C} : z \mapsto \widehat{xz}$ is a projectivity (in the sense that $\mu$ is a collineation between projective spaces preserving the cross-ratio). Indeed, the assertion about the cross-ratio follows from the fact that this is obviously true for the case $n = 2$ by the previous subsection.

Now, since $H$ does not contain any lid point, it corresponds with a 0-Veronesean. Hence the above collineation $\mu$ can be considered as a bijection from this 0-Veronesean to the $(n-1)$-subspace $\mathfrak{C}$ such that all points of any line of $\mathsf{PG}(d,q)$ joining corresponding points, except for the point in $\mathfrak{C}$, belong to $\mathcal{S}$, and all points of $\mathcal{S} \setminus \{x\}$ can be obtained this way. Also, looking in a plane of $\mathcal{S}$ through $x$, we see that this bijection maps plane conics to lines and preserves the cross-ratio. It follows from the equality $\frac{1}{2}n(n+3) = 1+((n-1)+\frac{1}{2}(n-1)(n+2)+1)$ that $\mathfrak{C}$ and $\langle H \rangle$ are disjoint, and that $x$ is not contained in $\langle \mathfrak{C}, H \rangle$ either. Hence $\mathcal{S} \setminus \{x\}$ is the projection from $x$ of $\mathfrak{V}_n(q) \setminus \{x\}$, with $\mathfrak{V}_n(q)$ a 0-Veronesean of $\mathsf{PG}(d,q)$ containing $x$, and so $\mathcal{S}$ is a 1-Veronesean. □

**Proposition 4.9** *If $\mathfrak{L}$ is a hyperplane, then $\mathcal{S}$ is an $n$-Veronesean.*

**Proof** By Proposition 4.2, the space $\langle \mathfrak{L} \rangle$ has dimension $\frac{1}{2}(n-1)(n+2)$, and by induction, $\mathfrak{L}$ is an $i$-Veronesean, $0 \leq i \leq n-1$. Also, Lemma 4.6 combined with the 2-Veronesean structure of any plane of $\mathcal{S}$ not contained in $\mathfrak{L}$ implies that the inclusion map $\iota : \mathcal{S} \setminus \mathfrak{L} \subseteq \mathsf{PG}(d,q)$ induces an isomorphism between affine spaces. Since $\frac{1}{2}n(n+3) = 1 + \frac{1}{2}(n-1)(n+2)+n$, we see that the subspace of $\mathsf{PG}(d,q)$ generated by the image of $\iota$ and the one generated by $\mathfrak{L}$ are disjoint. Since the projection of $\mathfrak{V}_n(q)\setminus\mathfrak{V}_{n-1}(q)$, with $\mathfrak{V}_{n-1}(q) \subseteq \mathfrak{V}_n(q)$ and $\langle \mathfrak{V}_n(q) \rangle = \mathsf{PG}(n(n+3)/2, q)$, from $\langle \mathfrak{V}_{n-1}(q) \rangle$ onto a $\mathsf{PG}(n,q) \subseteq \mathsf{PG}(n(n+3)/2, q)$ skew to $\langle \mathfrak{V}_{n-1}(q) \rangle$, is an $\mathsf{AG}(n,q)$ (cf. Proposition 2.3), the proposition is proved. □

We can now finish the proof of our Main Result. With our current notation, and in view of the previous propositions, we may assume $1 \leq m \leq n-1$, with $m$ the dimension of the subspace $\mathfrak{L}$ of $\mathcal{S}$. Now let $W \cong \mathsf{PG}(n-m-1, q)$ be a fixed subspace of $\mathcal{S}$ skew to $\mathfrak{L}$. Then $W$ is a quadric Veronesean and $\langle W \rangle$ has dimension $\frac{1}{2}(n-m-1)(n-m+2)$. For each point $x \in \mathfrak{L}$, the set $\langle x, W \rangle_{\mathcal{S}}$ induces a 1-Veronesean of an $(n-m)$-dimensional projective space with as set of caps a subspace $\mathfrak{C}_x$ of dimension $n-m-1$ of $\mathsf{PG}(d,q)$. Also, for a point $y \in W$, the set $\langle y, \mathfrak{L} \rangle_{\mathcal{S}}$ induces an $(m+1)$-Veronesean of an $(m+1)$-dimensional projective space with as set of caps a subspace $\mathfrak{C}_y$ of dimension $m$ of $\mathsf{PG}(d,q)$. Now let $\Sigma_1$ be the set of all subspaces $\mathfrak{C}_x$, with $x \in \mathfrak{L}$, and $\Sigma_2$ is the set of all subspaces $\mathfrak{C}_y$, with $y \in W$. It is clear that each member of $\Sigma_1$ meets each member of $\Sigma_2$ in at least one point, namely, with the above notation, $\mathfrak{C}_x$ and $\mathfrak{C}_y$ have the point in common that corresponds with the cap of the line $xy = \langle x, W\rangle_{\mathcal{S}} \cap \langle y, \mathfrak{L}\rangle_{\mathcal{S}}$. Also, it is easy to see that $\mathfrak{C}$ is the union of all $\mathfrak{C}_x$, and also the union of all $\mathfrak{C}_y$. Moreover, since $\mathsf{PG}(d, q)$ is generated by $W, \mathfrak{C}$ and $\mathfrak{L}$, we see that the dimension of $\langle \mathfrak{C}\rangle$ is at least

$$\frac{1}{2}n(n+3)-\frac{1}{2}(n-m-1)(n-m+2)-1-\frac{1}{2}m(m+3)-1 = (n-m-1)m+(n-m-1)+m,$$

which is precisely the natural dimension of the ambient space of a Segre variety $\mathfrak{S}_{n-m-1,m}(q)$. If $n = 3$ and $m = 1$, then it is easy to see that $\mathfrak{C}$ is a hyperbolic quadric in a 3-space. Considering in this case a plane of $\mathcal{S}$ through the line $W$, we see that the correspondence $y \mapsto \mathfrak{C}_y$, $y \in W$, is a projectivity (preserving the cross-ratio). Since $\langle \mathfrak{C}\rangle$ is disjoint from from $\langle W\rangle$ by the above relation between the dimensions of $\langle W\rangle$, $\langle \mathfrak{C}\rangle$ and $\langle \mathfrak{L}\rangle$, it follows from Proposition 2.3 that $\mathcal{S}$ is a 2-Veronesean.

Since two lines of $\mathcal{S}$ are always contained in some 3-subspace of $\mathcal{S}$, we see, using the previous paragraph, that the caps of two semiaffine lines $L_1, L_2$ coincide if and only if the plane $\langle L_1, L_2\rangle_{\mathcal{S}}$ intersects $\mathfrak{L}$ in a line of $\mathcal{S}$. This now implies that every member of $\Sigma_1$ meets every member of $\Sigma_2$ in a unique point, and two members of either $\Sigma_1$ or $\Sigma_2$ are disjoint. Looking at appropriate 3-dimensional subspaces, we also see that Property (2) of Section 2 holds for $\Sigma_1$ and $\Sigma_2$. This now implies that, considering $U_1 \in \Sigma_1$, each $(n-m)$-subset of the $n - m + 1$ members of $\Sigma_2$ containing a point of a given skeleton of $U_1$ generates $\langle \mathfrak{C}\rangle$. It also follows that the dimension of $\langle \mathfrak{C}\rangle$ is equal to $(n-m-1)m + (n-m-1) + m$, and hence that each of $\langle \mathfrak{C}\rangle$, $\langle W\rangle$ and $\langle \mathfrak{L}\rangle$ is disjoint from the subspace of $\mathsf{PG}(d, q)$ generated by the other two. Furthermore, as before, considering appropriate planes of $\mathcal{S}$, we deduce that the correspondence $y \mapsto \mathfrak{C}_y$, $y \in W$, is a projectivity (preserving the cross-ratio). Proposition 2.3 now finishes the proof of our Main Result.

# 5 What about finite dimensional infinite projective spaces?

One can easily check that the above proof holds verbatim to prove the following result (with a similar definition of $i$-Veronesean as for the finite case, and where $\mathsf{PG}(2, \mathbb{K})$ denotes *any* projective plane), except that one has to use [2] instead of [3] in the last line of Section 3, and in the proof of Theorem 4.7:

**Main Result—General Version** *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be isomorphic to $\mathsf{PG}(n, \mathbb{K})$, $n \geq 2$, $\mathbb{K}$ a skew field with at least 3 elements, with $\mathcal{P} \subseteq \mathsf{PG}(d, \mathbb{K})$, $\langle \mathcal{P}\rangle = \mathsf{PG}(d, \mathbb{K})$, $d \geq \frac{n(n+3}{2}$, and such that every member $L$ of $\mathcal{L}$ is a subset of points of a plane in $\mathsf{PG}(d, \mathbb{K})$. Assume* *also that for each line $L$ of $\mathcal{S}$, and each point $x \in L$, whenever the map $y \mapsto \langle x, y\rangle$, $y \in L \setminus \{x\}$, is injective, then there is a unique line $T$ of* $\mathsf{PG}(d, \mathbb{K})$ *in $\langle L\rangle$ through $x$ such that $T \cap L = \{x\}$. Then $\mathcal{P}$ is an $i$-Veronesean, for some $i \in \{0, 1, \ldots, n\}$.*

Notice that, for $\mathbb{K}$ commutative, $i$-Veroneseans of $\mathsf{PG}(n, \mathbb{K})$ always exist for $0 \leq i \leq n$, but for $\mathbb{K}$ noncommutative, an $i$-Veronesean of $\mathsf{PG}(n, \mathbb{K})$, $n \geq 2$, exists only if $i = n$. The reason is that, as soon as $i < n$ and $\mathbb{K}$ is noncommutative, there exists a sub-1-Veronesean of a plane $\mathsf{PG}(2, \mathbb{K})$, and this cannot exist, as an appropriate projection of that object gives rise to two complementary sets of reguli. On the other hand, building up inductively (on $n$), one can always construct an $n$-Veronesean of $\mathsf{PG}(n, \mathbb{K})$. It consists of the union of affine subspaces $\mathsf{AG}(i, \mathbb{K})$, for all $i \in \{2, 3, \ldots, n\}$ (and the projective closure of each such subspace is disjoint from the subspace generated by the other affine subspaces), and a generalized Veronesean of the line $\mathsf{PG}(1, \mathbb{K})$ satisfying the additional condition of the General Version of the Main Result above (and the plane generated by this generalized Veronesean is disjoint from the subspace generated by all affine subspaces $\mathsf{AG}(i, \mathbb{K})$, $i = 2, 3, \ldots, n$). This 1-Veronesean certainly is not projectively unique, but ignoring this, generalized Veronesean embeddings of $\mathsf{PG}(n, \mathbb{K})$ in $\mathsf{PG}(d, \mathbb{K})$, for noncommutative $\mathbb{K}$, with the above assumptions, are projectively unique.

# References


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