Codistances of 3-spherical buildings

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Abstract

We show that a 3-spherical building in which each rank 2 residue is connected far away from a chamber, and each rank 3 residue is simply 2-connected far away from a chamber, admits a twinning (i.e., is one half of a twin building) as soon as it admits a codistance, i.e., a twinning with a single chamber.

1 Introduction

Twin buildings have been introduced by M. A Ronan and J. Tits in the late 1980's. Their definition is motivated by the theory of Kac-Moody groups over fields. Kac-Moody groups are infinite-dimensional generalizations of Chevalley groups and the buildings associated with the latter are spherical. Spherical buildings have been classified by J. Tits in [Ti74]. This classification relies heavily on the fact that there is an opposition relation on the set of chambers of a spherical building. The idea in the definition of a twin building is to extend the notion of an opposition to non-spherical buildings: instead of taking one building, one starts with two buildings $\mathcal{B}_+, \mathcal{B}_-$ of the same type and defines an opposition relation between the chambers of the two buildings in question. Technically, this is done by requiring a *twinning function* between the chamber sets of the two buildings that takes its values in the Weyl group W. Two chambers x, y of $\mathcal{B}_+$ and $\mathcal{B}_-$ are then defined to be opposite, if their twinning is the identity in W.

There are variations of the idea of a twinning. For instance, one can introduce 'by restriction' a twinning between one chamber of $\mathcal{B}_+$ and the building $\mathcal{B}_-$, seen as an application from the set of chambers of $\mathcal{B}_-$ to the Weyl group. A function from the set of chambers of a building $\mathcal{B}$ to its Weyl group and satisfying similar properties to those of this 'twinning to a chamber' will be called *a codistance on* $\mathcal{B}$. This idea occurs at various places in the literature (see for instance [Mu98] and [Ro08]). In particular, [Ro08] deals with the question to which extent the existence of a codistance of a building $\mathcal{B}$ restricts its structure. The main result of the present paper ensures that any 3-spherical building admitting a codistance, and satisfying some local condition, is in fact one 'half' of a twin building. It is known that the local condition in question is satisfied if the diagram is simply laced and if each panel contains at least 4 chambers (see the final section of this paper).

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Here is the precise statement of our main result. For the definitions and notation we refer to Sections 2 and 3.

Main result: *Let $\mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be a thick building of 3-spherical type (W, S). Assume that the following two conditions hold.*

(lco) *If R is a rank 2 residue of $\mathcal{B}_-$ containing a chamber c, then the chamber system defined by the set of chambers opposite c inside R is connected.*

(lsco) *If R is a rank 3 residue of $\mathcal{B}_-$ containing a chamber c, then the chamber system defined by the set of chambers opposite c inside R is simply 2-connected.*

If there exists a codistance function $f : \mathcal{C}_- \to W$, then there exists a building $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+)$ and a mapping $\delta_* : (\mathcal{C}_- \times \mathcal{C}_+) \cup (\mathcal{C}_+ \times \mathcal{C}_-) \to W$ such that the following two statements hold.

a) *$(\mathcal{B}_-, \mathcal{B}_+, \delta_*)$ is a twin building.*

b) *There exists a chamber $c \in \mathcal{C}_+$ such that $\delta_*(c, x) = f(x)$ for all $x \in \mathcal{C}_-$.*

We would like to mention that the Conditions (lco) and (lsco) are 'almost always' automatic in 3-spherical buildings. We will explain this in the final section of this paper. In view of the discussion there the following corollary is a consequence of our main result.

Corollary 1: *Let $\mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be a thick, irreducible building of 3-spherical type (W, S) whose rank is at least 3. Then the conclusions of the main result hold as soon as one of the following conditions is satisfied:*

(1) *(W, S) is simply laced and all panels contain at least 4 chambers.*

(2) *Any residue of type A_2 corresponds to a Desarguesian projective plane and any panel contains at least 17 chambers.*

Some general remarks on 3-spherical buildings

The most impressive results in the theory of abstract buildings are the classifications of the irreducible spherical buildings of rank at least 3 and the irreducible affine buildings of rank at least 4 by Tits. In the 1980's it was an open question whether Tits' classification could be extended to irreducible affine buildings of rank 3. By independent work of Ronan and the third author constructions for such buildings were given, which showed that such a classification cannot be expected. Especially, Ronan's construction could be extended in order to show that there is a sort of free construction for buildings of type (W, S) for a lot of Coxeter systems (this is the Ronan-Tits construction, see [RT87]). However, if a Coxeter system contains spherical subsystems of rank 3, the degree of freeness of the Ronan-Tits-construction is considerably reduced. In fact, if all rank 3 subsystems are spherical, the only known choice of parameters in that construction yields buildings coming from Kac-Moody groups—hence buildings of algebraic origin. Note that an irreducible affine building of rank at least 4 is 3-spherical and that there are affine buildings which do not come from Kac-Moody groups. At present, the only known irreducible 3-spherical buildings of rank at least 4 which are not of affine type are those coming from groups of Kac-Moody-type. Whether these are all, appears to be an interesting open question in the theory of abstract buildings. Our main result provides a positive answer to that question under the additional assumption that the building admits a codistance. We make this more precise for the *simply laced* case (i.e. if all entries are 2 or 3).

Simply laced 3-spherical buildings

Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be an irreducible twin building of rank at least 3 whose diagram is simply laced. Then it is known that $\mathcal{B}$ is Moufang (see for instance [AB08]) and therefore each of its spherical residues is Moufang. If we assume in addition that $\mathcal{B}$ is 3-spherical, then all its A_2-residues are (up to duality) isomorphic to the building associated to a projective plane over a division ring K. If there is a D_4-subdiagram, then K is commutative and those buildings have been classified in [Mu99a]; in particular, they are of 'algebraic origin' in the sense that they can be constructed as 'k-forms' of certain Kac-Moody groups. If there is no D_4-subdiagram, then $\mathcal{B}$ is of type A_n or $\tilde{A}_n$ for some $n \geq 3$. Those buildings are also known by [Ti74] and [Ti84] and of algebraic origin. Putting together all this information, we get the following corollary of our main result.

Corollary 2: *Let $\mathcal{B}_-$ be an irreducible, 3-spherical and simply laced building of rank at least 3 in which each panel contains at least 4 chambers. If $\mathcal{B}_-$ admits a codistance, then it is known and in particular of algebraic origin.*

Content

The paper is organized as follows. In Section 2, we collect the definitions, known results and preliminaries that we need. In Section 3, we prove some basic properties of a codistance; most properties are known to be valid for a twinning, but we need to reprove them here for a codistance. In Section 4, we show that, under the assumptions of our main result, the complex of chambers with codistance the identity is simply 2-connected (for *any* codistance!). In Section 5 we study parallel panels and in Section 6, we construct bijections between panels that are contained in a chamber of codistance the identity. These bijections will then be used in Section 7 to define codistances adjacent to a given codistance. Finally, in Section 8, we prove that all the codistances thus obtained constitute the second half of a twinning, the first half of which is the original building.

Acknowledgement: We thank the referee for his detailed comments and valuable suggestions which improved the presentation of our work considerably.

2 Preliminaries

In this section, we recall basic definitions and results.

Chamber systems

Let I be a set. A *chamber system* over I is a pair $\mathcal{C} = (C, (\sim_i)_{i\in I})$ where C is a set whose elements are called *chambers* and where $\sim_i$ is an equivalence relation on the set of chambers for each $i \in I$, such that if $c \sim_i d$ and $c \sim_j d$ then either $i = j$ or $c = d$.

We refer to [AB08, DM07] for the definitions of *i-adjacent chambers*, *galleries*, *J-galleries* *J-residues*, *i-panels*. The J-residue containing the chamber c is denoted by $R_J(c)$.

Two galleries $G = (c_0, \dots, c_k)$ and $H = (c'_0, \dots, c'_{k'})$ with $c_0 = c'_0$ and $c_k = c'_{k'}$ are said to be *elementary* 2*-homotopic* if there exist two galleries X, Y and two J-galleries G_0, H_0 for some $J \subset I$ of cardinality at most 2 such that $G = XG_0Y$, $H = XH_0Y$. Two galleries G, H are said to be 2*-homotopic* if there exists a finite sequence $G_0, G_1, \dots, G_l$ of galleries such that $G_0 = G, G_l = H$ and such that $G_{\mu-1}$ is elementary 2-homotopic to G_μ for all $1 \le \mu \le l$. The chamber system $\mathcal{C}$ is called *simply* 2*-connected* if it is connected and if each closed gallery is 2-homotopic to a trivial gallery.

Coxeter systems

A *Coxeter system* is a pair (W, S) consisting of a group W and a set $S \subset W$ such that $\langle S \rangle = W$, $s^2 = 1_W \neq s$ for all $s \in S$ and such that the set S and the relations $((st)^{o(st)})_{s,t\in S}$ constitute a presentation of W, where $o(g)$ denotes the order of g.

Let (W, S) be a Coxeter system. The matrix $M(S) := (o(st))_{s,t\in S}$ is called the *type* or the *diagram* of (W, S). For an element $w \in W$ we put $l(w) := \min\{k \in \mathbf{N} \mid w = s_1 s_2 \dots s_k$ where $s_i \in S$ for $1 \le i \le k\}$. The number $l(w)$ is called the *length* of w. For a subset J of S we put $W_J := \langle J \rangle$ and we call it *spherical* if W_J is finite.

The following proposition collects several basic facts on Coxeter groups. These facts will be used without reference throughout the paper.

Proposition 2.1.: *Let (W, S) be a Coxeter system.*

a) For $w \in W, s \in S$ we have $\{l(ws), l(sw)\} \subset \{l(w) - 1, l(w) + 1\}$.

b) For $w \in W, s, t \in S$ with $l(sw) = l(w) + 1 = l(wt)$ we have $l(swt) = l(w) + 2$ or $swt = w$.

c) For $J \subset S$ the pair (W_J, J) is a Coxeter system and if $l_J : W_J \to \mathbf{N}$ is its length function, then $l_J = l\mid_{W_J}$.

d) Let $w \in W$ and $J \subset S$. Then there exists a unique element $w_J \in wW_J$ such that $l(w_J t) = l(w_J) + 1$ for all $t \in J$. Moreover, we have $l(x) = l(w_J) + l_J(w_J^{-1}x)$ for all $x \in wW_J$.

e) If $J \subset S$ is spherical, then there is a unique element $r_J \in W_J$ such that $l(r_J w) + l(w) = l(r_J)$ for all $w \in W_J$; the element r_J is a non-trivial involution if $J \neq \emptyset$. Moreover, we have $r_J J r_J = J$.

f) Let $w \in W$ and let $J \subset S$ be spherical. Then there exists a unique element $w^J \in wW_J$ such that $l(w^J t) = l(w^J) - 1$ for all $t \in J$ and we have $w^J = w_J r_J$. Moreover we have $l(x) = l(w^J) - l_J((w^J)^{-1}x)$ for all $x \in wW_J$; in particular, $l(w_J) + l(r_J) = l(w^J)$.

Proof: Parts a) and b) follow from the Deletion and the Folding Condition (see Sections 2.1 and 2.3 in [AB08]) and Part c) is an immediate consequence of the Deletion Condition. Part d) is a reformulation of [Hu90, Proposition 1.10] and Part e) follows from [We03, Proposition 5.7]. Finally, Part f) is a consequence of Parts d) and e). □

Let (W, S) be a spherical Coxeter system and let $r := r_S$ be the longest element in W. Then $rSr = S$ and hence conjugation by r induces an involutory permutation op of S; we say that $J \subset S$ is opposite $K \subset S$ if $\mathrm{op}(J) = K$. More generally, if (W, S) is an arbitrary Coxeter system and if $J \subset S$ is a spherical subset, then we say that two sets $K, L \subset J$ are opposite with respect to J if $r_J K r_J = L$.

Buildings

Let (W, S) be a Coxeter system. A *building* of type (W, S) is a pair $\mathcal{B} = (C, \delta)$ where C is a set and where $\delta : C \times C \to W$ is a *distance function* satisfying the following axioms, where $x, y \in C$ and $w = \delta(x, y)$:

(Bu 1) $w = 1$ if and only if $x = y$;

(Bu 2) if $z \in C$ is such that $\delta(y, z) = s \in S$, then $\delta(x, z) = w$ or ws, and if, furthermore, $l(ws) = l(w) + 1$, then $\delta(x, z) = ws$;

(Bu 3) if $s \in S$, there exists $z \in C$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

For a building $\mathcal{B} = (C, \delta)$ we define the chamber system $\mathbf{C}(\mathcal{B}) = (C, (\sim_s)_{s \in S})$ where two chambers $c, d \in C$ are defined to be *s-adjacent* if $\delta(c, d) \in \langle s \rangle$. The *rank* of a building $\mathcal{B}$ of type (W, S) is $|S|$.

In this paper all buildings are assumed to be of finite rank and *thick* (which means that for any $s \in S$ and any chamber $c \in C$ there are at least three chambers being s-adjacent to c).

For any two chambers x and y we set $l(x, y) = l(\delta(x, y))$. We say that a gallery $x_0, x_1, \ldots, x_n$ is *minimal* if $n = l(x_0, x_n)$.

In the following proposition we collect several basic facts about buildings.

Proposition 2.2.: *Let (W, S) be a Coxeter system and let $\mathcal{B} = (C, \delta)$ be a building of type (W, S).*

a) The chamber system $\mathbf{C}(\mathcal{B}) = (C, (\sim_s)_{s \in s})$ uniquely determines $\mathcal{B}$; in other words, the s-adjacency relations on C determine the distance function δ.

b) For $c \in C$ and $J \subset S$ we have $R_J(c) = \{x \in C \mid \delta(c, x) \in W_J\}$.

c) *If* $d : C \times C \to \mathbf{N}$ *is the numerical distance between two chambers in* $(C, (\sim_s)_{s\in s})$*, then* $d = l$*.*

d) *Let* $c \in C$ *and let* $R \subset C$ *be a* J*-residue for some* $J \subset S$*. Then there exists a unique chamber* $x \in R$ *such that* $\delta(c, x) = (\delta(c, x))_J$*. Moreover, for all* $y \in R$ *one has* $\delta(c, y) = \delta(c, x)\delta(x, y)$ *and in particular,* $l(c, y) = l(c, x) + l(x, y)$*.*

Proof: Parts a), b) and c) follow from the fact that the distance function can be characterized in terms of types of galleries (see [We03, Definition 7.1] or [Ti81]). For Part d) we refer to [We03, Proposition 8.24]. □

Given $c \in C$ and a J-residue R of $\mathcal{B}$ as in Assertion d) of the previous proposition, then the chamber x in that statement is called the *projection of* c *onto* R and it is denoted by $\mathrm{proj}_R\, c$.

Given two residues R and R', we define $\mathrm{proj}_R\, R'$ by the set $\{\mathrm{proj}_R\, c | c \in R'\}$.

Two residues R_1 and R_2 of a building are called *parallel* if $\mathrm{proj}_{R_1} : R_2 \to R_1$ and $\mathrm{proj}_{R_2} : R_1 \to R_2$ are adjacency-preserving bijections inverse to each other.

Proposition 2.3.: *Let* R, Q *be two residues of a building. Then the following holds:*

a) $\mathrm{proj}_R\, Q$ *is a residue contained in* R*.*

b) *The residues* $R' := \mathrm{proj}_R\, Q$ *and* $Q' := \mathrm{proj}_Q\, R$ *are parallel.*

Proof: Part a) follows from the first statement of [DS87, Proposition 3] while Part b) follows from Part a of the main Theorem in [DS87]. □

Let R be a spherical J-residue of a building of type (W, S). Two chambers x, y of R are *opposite* in R whenever $\delta(x, y) = r_J$. Two residues R_1 of type K_1 and R_2 of type K_2 in R are *opposite* in R if R_1 contains a chamber opposite to a chamber of R_2 and if $K_1 = r_J K_2 r_J$ (which means that K_1 and K_2 are opposite with respect to J as defined earlier).

Proposition 2.4.: *Let* R *be a spherical* J*-residue of a building of type* (W, S) *and let* R_1, R_2 *be two residues which are opposite in* R*. Then* R_1 *and* R_2 *are parallel.*

Proof: This is a consequence of Theorem 3.28 of [Ti74]. □

Proposition 2.5.: *Let* R_I, R_J, R_K *be residues of respective type* I, J, K *of a building of type* (W, S)*. Assume that* $R_I \subseteq R_J$*. Then we have* $\mathrm{proj}_{R_I} R_K = \mathrm{proj}_{R_I} \mathrm{proj}_{R_J} R_K$*.*

Proof: This is a consequence of Proposition 2 in [DS87] □

Twin buildings

Let $\mathcal{B}_+ = (C_+, \delta_+), \mathcal{B}_- = (C_-, \delta_-)$ be two buildings of the same type (W, S), where (W, S) is a Coxeter system. A *twinning* between $\mathcal{B}_+$ and $\mathcal{B}_-$ is a mapping $\delta_* : (C_+ \times C_-) \cup (C_- \times C_+) \to W$ satisfying the following axioms, where $\epsilon \in \{+, -\}, x \in C_\epsilon, y \in C_{-\epsilon}$ and $w = \delta_*(x, y)$:

(Tw 1) $\delta_*(y, x) = w^{-1}$;

(Tw 2) if $z \in C_{-\epsilon}$ is such that $\delta_{-\epsilon}(y, z) = s \in S$ and $l(ws) = l(w) - 1$, then $\delta_*(x, z) = ws$;

(Tw 3) if $s \in S$, there exists $z \in C_{-\epsilon}$ such that $\delta_{-\epsilon}(y, z) = s$ and $\delta_*(x, z) = ws$.

A *twin building of type* (W, S) is a triple $(\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ where $\mathcal{B}_+, \mathcal{B}_-$ are buildings of type (W, S) and where δ_* is a twinning between $\mathcal{B}_+$ and $\mathcal{B}_-$.

Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be a twin building. Then $x \in \mathcal{C}_+$ and $y \in \mathcal{C}_-$ are called *opposite* if $\delta_*(x, y) = 1_W$. For each chamber c in one of the two buildings, c^{op} denotes the set of chambers in the other building that are opposite c.

Here is a lemma the proof of which is left to the reader (it follows directly from the definition above and an easy induction on the length of $\delta_+(x, y)$).

Lemma 2.6.: *Let $((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta_*)$ be a twin building of type (W, S). Let $x, y \in \mathcal{C}_+$ and $z \in \mathcal{C}_-$ be such that $\delta_+(x, y) = \delta_*(x, z)$. Then y and z are opposite. In particular, if $x^{\mathrm{op}} = y^{\mathrm{op}}$, then $x = y$.*

3 Codistances

In this section, we take (W, S) a Coxeter system and $\mathcal{B} = (\mathcal{C}, \delta)$ a building of type (W, S).

Definition 3.1.: A *codistance* on $\mathcal{B}$ is a function $f : \mathcal{C} \to W$ such that, for all $s \in S$ and P an s-panel of $\mathcal{C}$, there exists $w \in W$ with $f(x) \in \{w, ws\}$ for all $x \in P$ and P contains a unique chamber with f-value the longer word of the two. If the latter is satisfied for a fixed panel P, then we say that P *satisfies the codistance condition for* f.

As an example, if $\mathcal{B}$ is half of a twin building and x is a chamber in the other half, the twinning to x is a codistance on $\mathcal{B}$.

For the rest of this section $f : \mathcal{C} \to W$ is a codistance of $\mathcal{B}$.

Lemma 3.2.: *Let R be a J-residue of $\mathcal{B}$ and x be a chamber of R. Then the image of f restricted to R is $f(x)W_J$.*

Proof: By the definition of f, the image is contained in $f(x)W_J$. Let w be a word of W_J written as a reduced word as $s_1 s_2 \ldots s_k$. Using the fact that for all $s \in J$ and all chambers $y \in R$, there exists at least one chamber s-adjacent to y with f-value $f(y)s$, it follows by induction on k that there exists a chamber in R with f-value $f(x)w$. □

Proposition 3.3.: *Let R be a spherical J-residue of $\mathcal{B}$. Then there exists a unique chamber c in R such that $l(f(c)) > l(f(y))$ for all $y \in R \setminus \{c\}$. This unique chamber will be denoted by $\operatorname{proj}_R f$. Moreover for all $y \in R$, we have $f(y) = f(c)\delta(c, y)$.*

Proof: Let y be a chamber in R and $w := f(y)$. By Lemma 3.2, f takes on R its values in wW_J. Since R is spherical, wW_J contains a unique longest word w^J by Part f) of Proposition 2.1. Moreover $l(x) = l(w^J) - l((w^J)^{-1}x)$ for all $x \in wW_J$. By Lemma 3.2, there exists a chamber $c \in R$ with $f(c) = w^J$.

Let y be a chamber in R. The distance $\delta(c, y)$ is in W_J and so can be written as a reduced word as $t_1 t_2 \dots t_k$. Therefore there is a gallery $c = y_0 \sim_{t_1} y_1 \sim_{t_2} \dots \sim_{t_k} y_k = y$. Using the fact that $l(w^J t_1 \dots t_i) = l(w^J) - i$, it follows by induction that $f(y) = w^J \delta(c, y)$. Therefore $l(f(y)) = l(w^J) - l(\delta(c, y)) = l(f(c)) - l(c, y) \leq l(f(c))$ with equality only if $y = c$. □

Proposition 3.4.: *Let R be a J-residue of $\mathcal{B}$. Put $l_f(R) := \min\{l(f(x)) \mid x \in R\}$ and $A_f(R) := \{x \in R \mid l(f(x)) = l_f(R)\}$.*

a) Let $x \in R$. Then $x \in A_f(R)$ if and only if $f(x)$ is the unique shortest word of $f(x)W_J$. Moreover, if $x, y \in A_f(R)$, then $f(x) = f(y)$.

b) Let $y \in R$. Then there exists $x \in A_f(R)$, such that $f(y) = f(x)\delta(x, y)$.

c) If J is spherical, then $A_f(R)$ is the set of all chambers opposite $\operatorname{proj}_R f$ in R.

Proof: Let $y \in R$ and put $w := f(y)$.

By Lemma 3.2, $\{f(x) \mid x \in R\} = wW_J$. By Part d) of Proposition 2.1 there exists a unique shortest element $w_J \in wW_J$. It follows that $A_f(R) = \{x \in R \mid f(x) = w_J\}$. This proves Part a) of the proposition.

Now let $t_1 t_2 \dots t_k$ be a reduced representation of $w_J^{-1} w$ and let $x = y_0 \sim_{t_1} y_1 \sim_{t_2} \dots \sim_{t_k} y_k = y$ be a reduced gallery ending in y. Using the fact that $l(w t_k t_{k-1} \dots t_{i+1}) = l(w_J t_1 t_2 \dots t_i) = l(w_J) + i$, it follows by induction on k that $x \in A_f(R)$. By construction $\delta(x, y) = t_1 t_2 \dots t_k = w_J^{-1} w = f(x)^{-1} f(y)$. This finishes Part b).

Let J be spherical. Let $c = \operatorname{proj}_R f$ so that $f(c) = w^J$ as in Proposition 3.3. We have seen that $f(x) = w^J \delta(c, x)$ for all $x \in R$. Since $w^J = w_J r_J$, where r_J is the unique longest word of W_J, we can conclude that $x \in A_f(R)$ if and only if $\delta(c, x) = r_J$, that is, if and only if x is opposite c in R.

□

Definition 3.5.: We denote by f^{op} the set of chambers of $\mathcal{C}$ with f-value 1_W.

Let $R \subset \mathcal{C}$ be a J-residue. By Part a) of the previous proposition we have $f(x) = f(y)$ for all $x, y \in A_f(R)$. We denote this common value by R^f. Note that $A_f(R) = \{x \in R \mid f(x)_J = f(x)\}$ by Part a) of the previous proposition.

Lemma 3.6.: *Let c be a chamber of $\mathcal{C}$. Then a shortest gallery from c to a chamber in f^{op} has length $l(f(c))$.*

Proof: It is obvious from the definition of the codistance f that no chamber at distance strictly less than $l(f(c))$ from c can be in f^{op}. Now by Part b) of Proposition 3.4 with $J = S$, there exists $x \in A_f(\mathcal{C}) = f^{\text{op}}$ such that $f(c) = \delta(x, c)$. Hence a minimal gallery from c to x will have length $l(f(c))$. $\square$

For $c \in \mathcal{C}$, we define $f_c^{\text{op}} = \{x \in f^{\text{op}} | \delta(x, c) = f(c)\}$ (that is the set of chambers of f^{op} closest to c), which is non-empty, by Lemma 3.6.

Lemma 3.7.: *The following statements are equivalent:*

a) the chamber x is in f_c^{op},

b) for any minimal gallery $x = x_0, x_1 \ldots, x_n = c$ we have $l(f(x_i)) = i$ for all $0 \leq i \leq n$,

c) there exists a minimal gallery $x = x_0, x_1 \ldots, x_n = c$ with $l(f(x_i)) = i$ for all $0 \leq i \leq n$.

Proof: Assume $x \in f_c^{\text{op}}$. Let $x = x_0, x_1 \ldots, x_n = c$ be any minimal gallery from x to c. Since $\delta(x, c) = f(c)$, $n = l(f(c))$. By the axioms of codistance, the f-values of two adjacent chambers are either equal or have length difference one, hence we must have $l(f(x_i)) = l(f(x_{i-1})) + 1$ for all $1 \leq i \leq n$, which implies b).

Obviously b) implies c).

Assume that there exists a minimal gallery $x = x_0, x_1 \ldots, x_n = c$ with $l(f(x_i)) = i$ for all $0 \leq i \leq n$. Then $l(f(x)) = 0$, so $f(x) = 1_W$. Assume that $f(x_i) = \delta(x, x_i)$, then $f(x_{i+1}) = \delta(x, x_{i+1})$. Indeed $x_i \sim_{s_i} x_{i+1}$ for some $s_i \in S$ and so $f(x_{i+1}) = f(x_i)$ or $f(x_i)s_i$. Since $l(f(x_{i+1})) \neq l(f(x_i))$, we are in the second case and $f(x_{i+1}) = \delta(x, x_i)s_i = \delta(x, x_{i+1})$. This proves by induction that $f(x_i) = \delta(x, x_i)$ for all $0 \leq i \leq n$, and so $f(c) = \delta(x, c)$, which yields a). $\square$

Lemma 3.8.: *Let $x \in \mathcal{C}$ and $w \in W$ such that $l(f(x)w) = l(f(x)) + l(w)$. Then there exists a unique chamber c of $\mathcal{C}$ with $f(x)^{-1}f(c) = w = \delta(x, c)$.*

Proof: Let $s_1 s_2 \ldots s_k$ be a reduced word for w. Since $l(f(x)w) = l(f(x)) + l(w)$, we have $l(f(x)s_1 s_2 \ldots s_i) = l(f(x)) + i$. Consider the s_1-panel on x, it follows from the axioms of codistance that this panel contains a unique chamber with f-value $f(x)s_1$, namely the projection of f on it. Continuing by induction on k, we can build a unique gallery $x = x_0 \sim_{s_1} x_1 \sim s_2 \ldots \sim_{s_k} x_k = c$ such that $f(x_i) = f(x)s_1 s_2 \ldots s_i$ for all i. Hence $w = \delta(x, c)$ and $f(c) = f(x)w$, and so c exists.

Assume there exists another chamber c' with $f(x)^{-1}f(c') = w = \delta(x, c')$. Hence, on the one hand, there exists a minimal gallery $x = x'_0, x'_1 \ldots, x'_k = c'$ with $l(f(x_i)) = f(x) + i$ for all $0 \leq i \leq k$, of type $t_1, t_2, \ldots, t_k$ where $t_1 t_2 \ldots t_k = w$. On the other hand, since $\delta(x, c) = w$, there is a minimal gallery of type $t_1, t_2, \ldots, t_k$ from x to c. Because the length of the f-value has to increase at each step, we see by induction that this gallery coincides with $x = x'_0, x'_1 \ldots, x'_n = c'$, and so $c = c'$. $\square$

Lemma 3.9.: *Let R be a J-residue of $\mathcal{B}$ and c a chamber of R. If $x \in f_c^{\mathrm{op}}$ then $\mathrm{proj}_R\, x \in A_f(R)$, $l(x, \mathrm{proj}_R\, x) = l_f(R)$ and $\delta(x, \mathrm{proj}_R\, x) = R^f$.*

Proof: Let $w = f(c) = \delta(x, c)$. We have $l(w) = l(w_J) + l(w_J^{-1}w)$. Hence, if $s_1 s_2 \dots s_k$ is a reduced word for w_J and $s_{k+1}s_{k+2}\dots s_n$ is a reduced word for $w_J^{-1}w \in W_J$, then $s_1 s_2 \dots s_n$ is a reduced word for w. Consider the gallery $x = x_0 \sim_{s_1} x_1 \sim_{s_2} \dots \sim_{s_n} x_n$ and such that $f(x_i) = s_1 s_2 \dots s_i$. In particular $f(x_n) = w$ and $f(x_k) = w_J$. By construction, since $s_1 s_2 \dots s_i$ is a reduced word, we also have $\delta(x, x_i) = s_1 s_2 \dots s_i$ and in particular $\delta(x, x_n) = w$. A chamber satisfying $f(x_n) = w = \delta(x, x_n)$ is unique by Lemma 3.8 and therefore $x_n = c$. As $w_J^{-1}w \in W_J$, $s_i \in W_J$ for $i \geq k+1$, and so $x_i \in R$ for $i \geq k$. Since $l(x, x_k) = l(\delta(x, x_k)) = l(w_J)$, which is the shortest possible length for under the restriction $x_k \in R$, we have $x_k = \mathrm{proj}_R\, x$ and $x_k \in A_f(R)$ by Proposition 3.4 a). This shows in particular that $\delta(x, \mathrm{proj}_R\, x) = R^f$ because $R^f = w_J$, and so $l(x, \mathrm{proj}_R\, x) = l_f(R)$. $\square$

Lemma 3.10.: *The set f^{op} uniquely determines f.*

Proof: Assume there exists a codistance $f' \neq f$ on $\mathcal{B}$ with $f'^{\mathrm{op}} = f^{\mathrm{op}}$. Then consider c at minimal distance from f^{op} under the condition that $f'(c) \neq f(c)$. Of course, c is not in f^{op}. Let $c = c_0, c_1, \dots, c_m$ be a shortest gallery from c to f^{op}. This minimal gallery has length $l(f(c))$ by Lemma 3.6. It is also a shortest gallery to f'^{op}, and so has length $l(f'(c))$. Therefore $l(f(c)) = l(f'(c))$. Now c_1 is closer to f^{op} than c, and so $f(c_1) = f'(c_1)$. By the definition of codistance, $f(c) = f(c_1)$ or $f(c_1)t$ (where t is such that $c_0 \sim_t c_1$). This holds also with f' in place of f. Since $l(f(c)) = l(f'(c))$, it implies that $f(c) = f'(c)$. This contradiction proves that $f = f'$. $\square$

4 Simple connectivity of f^{op}

In this section we will apply a result proved in [DM07] using filtrations.

Let I be a set and let $\mathcal{C} = (C, (\sim_i)_{i \in I})$ be a chamber system over I. In the following we denote the set of non-negative integers by $\mathbf{N}$ and the set of positive integers by $\mathbf{N}_0$.

A *filtration* of $\mathcal{C}$ is a family $\mathcal{F} = (C_n)_{n \in \mathbf{N}}$ of subsets of C such that the following holds.

(F1) $C_n \subset C_{n+1}$ for all $n \in \mathbf{N}$,

(F2) $\bigcup_{n \in \mathbf{N}} C_n = C$,

(F3) for each $n > 0$ if $C_{n-1} \neq \emptyset$ then there exists an index $i \in I$ such that for each chamber $c \in C_n$ there exists a chamber $c' \in C_{n-1}$ which is i-adjacent to c.

A filtration $\mathcal{F} = (C_n)_{n \in \mathbf{N}}$ is called *residual* if for each $\emptyset \neq J \subset I$ and each J- residue R the family $(C_n \cap R)_{n \in \mathbf{N}}$ is a filtration of the chamber system $\mathcal{R} := (R, (\sim_j)_{j \in J})$.

For each $x \in C$ we put $|x| := \min\{\lambda \in \mathbf{N} \mid x \in C_\lambda\}$. For a subset X of C we put $|X| := \min\{|x| \mid x \in X\}$ and $\mathrm{aff}(X) := \{x \in X \mid |x| = |X|\}$. Note that $C_0 = \mathrm{aff}(C)$ if we assume that $C_0 \neq \emptyset$.

We say that a filtration *satisfies Condition* (lco) if for every rank 2 residue R, $\mathrm{aff}(R)$ is a connected subset of the chamber system $\mathcal{R}$.

We say that a filtration *satisfies Condition* (lsco) if for every rank 3 residue R, $\mathrm{aff}(R)$ is a simply 2-connected subset of the chamber system $\mathcal{R}$.

Theorem 4.1 (see [DM07]).**:** *Suppose that the residual filtration* $\mathcal{F} = (C_n)_{n\in\mathbf{N}}$ *of the chamber system* $\mathcal{C}$ *satisfies* (lco), (lsco) *and that* $C_0 \neq \emptyset$. *Then the following are equivalent:*

a) $\mathcal{C}$ *is simply 2-connected;*

b) $(C_n, (\sim_i)_{i\in I})$ *is simply 2-connected for all* $n \in \mathbf{N}$.

The filtration $\mathcal{F}_f$

We choose an injection $w \mapsto |w|$ from W into $\mathbf{N}$ such that $l(x) < l(y)$ implies $|x| < |y|$ for all $x, y \in W$ and such that $|1_W| = 0$. Such an injection exists because $\mathcal{B}$ is of finite rank. Let f be a codistance. We define C_n by setting $C_n := \{x \in C \mid |f(x)| \leq n\}$.

The goal of this subsection is to show the following proposition.

Proposition 4.2.: *With the definitions above, the family* $\mathcal{F}_f := (C_n)_{n\in\mathbf{N}}$ *is a residual filtration of the chamber system* $\mathcal{C}$.

Proof: It is obvious that $\mathcal{F}_f$ satisfies the axioms (F1) and (F2) and from this it follows that these axioms also hold 'residually'.

Let R be a J-residue of $\mathcal{C}$ with $J \neq \emptyset$ and let $|R| := \min\{k \mid C_k \cap R \neq \emptyset\}$. It follows from the definition of $\mathcal{F}_f$ and by Proposition 3.4 that $\mathrm{aff}(R) = C_{|R|} \cap R = A_f(R) = \{x \in R \mid f(x) = f(x)_J\}$.

Let $0 < n \in \mathbf{N}$ be such that $C_{n-1} \cap R \neq \emptyset$. We have to show that there is $t \in J$ with the property that each chamber x in $R \cap C_n$ is t-adjacent to a chamber $x' \in R \cap C_{n-1}$. If $C_n \cap R = C_{n-1} \cap R$ we can choose $t \in J$ arbitrarily and set $x' := x$ for each $x \in R \cap C_n$. Suppose now that $C_{n-1} \cap R$ is properly contained in $C_n \cap R$, choose $y \in R \cap C_n \setminus C_{n-1}$ and put $w := f(y)$. Since $|\cdot|$ injects W into $\mathbf{N}$, it follows from the definition of $\mathcal{F}_f$ that, on the one hand, $f(y') = w$ for all $y' \in C_n \setminus C_{n-1}$. On the other hand, there exists $x \in A_f(R)$ such that $w = f(y) = f(x)\delta(x, y)$ by Assertion b) of Proposition 3.4. As $C_{n-1} \cap R \neq \emptyset$ it follows that $y \notin A_f(R)$ and hence $\delta(x, y) \in W_J \setminus \{1_W\}$. Let $t \in J$ be such that $l(\delta(x, y)t) = l(\delta(x, y)) - 1$. As $f(x) = f(x)_J = w_J$ and $\delta(x, y) \in W_J$, it follows that $l(wt) = l(w_J\delta(x, y)t) = l(w_J) + l(\delta(x, y)t) = l(w_J) + l(\delta(x, y)) - 1 = l(w) - 1$, by Part d) of Proposition 2.1. For any chamber $z \in R \cap C_n$ we choose a chamber $z' \in R$ as follows. If $z \in C_{n-1}$ then we put $z' := z$. If $z \in C_n \setminus C_{n-1}$ then we know that $f(z) = w$ and we choose $z' \in R$ such that $z \sim_t z' \neq z$. In the first case, it is obvious that z' is in $R \cap C_{n-1}$; in the second case we have $f(z') = wt$ by the definition of f, as wt is shorter than w. It follows that $|wt| < |w| = n$ and therefore $z' \in C_{n-1}$. As $t \in J$ we have also $z' \in R$.

The case $J = S$ is a special case of the consideration above. This shows that $\mathcal{F}_f$ satisfies Axiom (F3). Hence $\mathcal{F}_f$ is a residual filtration. $\square$

Theorem 4.3.: *Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type (W, S) and f a codistance on $\mathcal{B}$. Suppose that the following conditions are satisfied:*

(3-sph.) *If $J \subseteq S$ is of cardinality at most 3, then J is spherical.*

(lco) *If J is of cardinality 2, if $R \subset \mathcal{C}$ is a J-residue and if $x \in R$, then the chamber system $(\{y \in R \mid \delta(x,y) = r_J\}, (\sim_t)_{t \in J})$ is connected.*

(lsco) *If J is of cardinality 3, if $R \subset \mathcal{C}$ is a J-residue and if $x \in R$, then the chamber system $(\{y \in R \mid \delta(x,y) = r_J\}, (\sim_t)_{t \in J})$ is simply 2-connected.*

Then the chamber system f^{op} is simply 2-connected.

Proof: Let $\mathcal{F}_f = (C_n)_{n \in \mathbf{N}}$ be the residual filtration of Proposition 4.2. Note first that $C_0 = f^{\mathrm{op}}$.

Given a spherical J-residue R of $\mathcal{B}$, then $\mathrm{aff}(R) = A_f(R)$ as we have proved above. By Assertion c) of Proposition 3.4, we have therefore $\mathrm{aff}(R) = \{x \in R \mid \delta(\mathrm{proj}_R f, x) = r_J\}$, where r_J is the longest word of W_J.

Now $\mathcal{F}_f$ satisfies (lco) and (lsco). As it is well-known that $\mathcal{C}$ is simply 2-connected (see for instance Theorem (4.3) in [Ro89]), the claim follows now from Theorem 4.1. $\square$

5 Parallel panels in buildings

In this section (W, S) is a Coxeter system and $\mathcal{B} = (\mathcal{C}, \delta)$ is a building of type (W, S).

Definition 5.1.: For $w \in W$ we put $S^-(w) := \{s \in S \mid l(ws) = l(w) - 1\}$ and $S^+(w) := \{s \in S \mid l(ws) = l(w) + 1\}$.

Lemma 5.2.: *Let $w \in W$ and $J \subset S$. Then $w = w_J$ if and only if $J \subset S^+(w)$.*

Proof: This is a consequence of Part d) in Proposition 2.1. $\square$

Lemma 5.3.: *Let $w \in W$ and $J \subset S$. Then the following are equivalent:*

a) $J \subset S^-(w)$,

b) *J is spherical and* $l(w) = l(wr_J) + l(r_J)$,

c) *J is spherical and* $w = w_J r_J$,

d) *J is spherical and* $w = w^J$.

Proof: This is Lemma 2.8 in [Mu92]. $\square$

Definition 5.4.: For $w_1, w_2 \in W$, we denote $w_1 \prec w_2$ if $l(w_1^{-1} w_2) = l(w_2) - l(w_1)$.

Lemma 5.5.: *Let $w \in W$ and $J \subset S$. Then $w_J \prec w$ and if J is spherical, then $w \prec w^J$.*

Proof: This follows from Parts d) and f) of Proposition 2.1. □

Let $R \subset \mathcal{C}$ be a J-residue for some subset J of S. We recall that for each chamber $c \in \mathcal{C}$ there is a unique chamber $\operatorname{proj}_R c$ in R satisfying $\delta(c, \operatorname{proj}_R c) = \delta(c, \operatorname{proj}_R c)_J$ and that we have $\delta(c,d) = \delta(c, \operatorname{proj}_R c)\delta(\operatorname{proj}_R c, d)$ for any $d \in R$. Hence we have a mapping $\operatorname{proj}_R : \mathcal{C} \to R$.

Lemma 5.6.: *Let R be a residue of $\mathcal{B}$ and let $c, d \in \mathcal{C}$. Then $l(\operatorname{proj}_R c, \operatorname{proj}_R d) \leq l(c,d)$.*

Proof: This follows from Lemma 1 in [DS87]. □

Let $R, T \subset \mathcal{C}$ be two residues. We recall that $\operatorname{proj}_R T$ is a residue contained in R and that the residues R and T are called parallel if $\operatorname{proj}_R T = R$ and $\operatorname{proj}_T R = T$. By Part b) of Proposition 2.3 we know that $\operatorname{proj}_R T$ and $\operatorname{proj}_T R$ are parallel residues. We denote the restriction of $\operatorname{proj}_R : \mathcal{C} \to R$ to the residue T by proj_R^T.

Lemma 5.7.: *Let P be a panel and let R be a residue. Then $\operatorname{proj}_R P$ is either a singleton or a panel contained in R. In particular, if $|\operatorname{proj}_R P| \geq 2$, then $\operatorname{proj}_R P$ is a panel contained in R.*

Proof: We know that $\operatorname{proj}_R P$ is a residue in which any two chambers have distance at most 1 by Lemma 5.6. The claim follows. □

Lemma 5.8.: *Two panels P_1 and P_2 are parallel if and only if $|\operatorname{proj}_{P_2} P_1| \geq 2$.*

Proof: If P_1 and P_2 are parallel, then $|\operatorname{proj}_{P_2} P_1| = |P_2| \geq 2$ because $\operatorname{proj}_{P_2}^{P_1}$ is a bijection from P_1 onto P_2.

Suppose now that $|\operatorname{proj}_{P_2} P_1| \geq 2$. Then $\operatorname{proj}_{P_2} P_1$ is a panel contained in P_2 by the previous lemma and therefore $P_2 = \operatorname{proj}_{P_2} P_1$. As $\operatorname{proj}_{P_1} P_2$ is parallel to $\operatorname{proj}_{P_2} P_1 = P_2$ by Assertion b) of Proposition 2.3, it follows that $|\operatorname{proj}_{P_1} P_2| = |P_2| \geq 2$. Using the same argument as before we obtain $P_1 = \operatorname{proj}_{P_1} P_2$. Hence P_1 and P_2 are parallel. □

Lemma 5.9.: *Let P_1 and P_2 be two parallel panels of type s_1 and s_2, respectively. Then $s_2 = w^{-1} s_1 w$, where $w := \delta(x, \operatorname{proj}_{P_2} x)$ does not depend on the choice of x in P_1.*

Conversely, if x and y are chambers with $\delta(x,y) = w$, where w satisfies $s_2 = w^{-1} s_1 w$ and $l(s_1 w) = l(w) + 1$, then the s_1-panel on x is parallel to the s_2-panel on y.

Proof: We know by Part b) of Proposition 2.1 that for $w \in W$ and $s_1, s_2 \in S$ such that $l(s_1 w) = l(w) + 1 = l(w s_2)$, we have $l(s_1 w s_2) = l(w) + 2$ or $s_1 w s_2 = w$. The result follows. □

Definition 5.10.: For two parallel panels P_1 and P_2 we put $\delta(P_1, P_2) := \delta(x, \operatorname{proj}_{P_2} x)$, where x is a chamber in P_1; by the previous lemma $\delta(P_1, P_2)$ does not depend on the choice of $x \in P_1$.

Definition 5.11.: For $s \in S$, we define $X_s := \{w \in W | w^{-1} s w \in S \text{ and } l(sw) = l(w) + 1\}$.

Lemma 5.12.: *For $w \in X_s$ and a given s-panel P, there exists an $w^{-1}sw$-panel P' parallel to P and with $\delta(P, P') = w$. Let J be a spherical subset of S containing s and let r_J be the longest word of W_J, then $x_J := sr_J$ is in X_s. Moreover, if $w \in W_J$ is in X_s, then $w \prec x_J$.*

Proof: The first statement is a corollary of Lemma 5.9. We have $x_J^{-1}sx_J = r_J s r_J$, which has length $l(r_J) - l(sr_J) = 1$ and so is an element of S, and $l(sx_J) = l(r_J) = l(x_J) + 1$, hence the second statement. Finally $l(w^{-1}x_J) = l(w^{-1}sr_J) = l(r_J sw) = l(r_J) - l(sw) = l(r_J) - l(w) - 1 = l(x_J) - l(w)$, hence the third statement. □

Lemma 5.13.: *Let $s \in S$, let $w \in X_s$ and put $t := w^{-1}sw \in S$. Then $t \in S^+(w)$ and $S^-(wt) = S^-(w) \cup \{t\}$. In particular $S^-(w) \cup \{t\}$ is a spherical subset of J.*

Proof: Note first that $l(w)+1 = l(sw) = l(wt)$ and therefore $t \in S^+(w)$ and $t \in S^-(wt)$. Now let $u \in S$ be unequal to t. Suppose first that $u \in S^-(w)$. Then

$$l(wtu) = l(swu) \leq l(s) + l(wu) = 1 + l(w) - 1 = l(w) = l(wt) - 1$$

and so we have $l(wtu) = l(wt) - 1$ and $u \in S^-(wt)$. Hence $S^-(wt) \supset S^-(w) \cup \{t\}$.

Suppose now that $u \in S^-(wt)$ but $u \notin S^-(w) \cup \{t\}$. Then $l(wu) = l(w) + 1$ and $u \neq t$. By Part b) of Proposition 2.1, $swu = w$ or $l(swu) = l(w) + 2$. In the first case, $swu = wtu = w$ implies $t = u$ a contradiction, so, on the one hand, $l(swu) = l(w) + 2$. On the other hand $l(wtu) = l(wt) - 1 = l(w)$, so we also get a contradiction. Therefore $S^-(wt) \subset S^-(w) \cup \{t\}$. This proves the first assertion. The second assertion is now a consequence of Lemma 5.3. □

Definition 5.14.: Let Γ be the graph whose vertices are the panels of $\mathcal{B}$ with panels adjacent if there exists a rank 2 residue in which the two panels are opposite. For two adjacent panels P, Q, there exists a unique rank 2 residue containing P and Q, that will be denoted by $R(P, Q)$. A path $\Pi = (P_0, P_1, \ldots P_k)$ in Γ is called *compatible* if $\operatorname{proj}_{R(P_{i-1}, P_i)} P_0 = P_{i-1}$ for all $1 \leq i \leq k$. The number k is the *length* of that path Π. The sequence $(J_1, \ldots, J_k)$ where J_i is the type of $R(P_{i-1}, P_i)$ will be called the *type* of Π.

Lemma 5.15.: *Let P, Q be two parallel panels of $\mathcal{B}$ and let R be a residue containing Q. Then $\operatorname{proj}_R P$ is a panel parallel to both P and Q. Moreover, if $P = P_0, P_1, \ldots P_k = \operatorname{proj}_R P$ and $\operatorname{proj}_R P = T_0, T_1, \ldots T_l = Q$ are compatible paths in Γ, the second one contained in R, then $P = P_0, P_1, \ldots P_k = T_0, T_1, \ldots T_l = Q$ is a compatible path in Γ.*

Proof: The projection of a residue on a residue is a residue, so $P' := \operatorname{proj}_R P$ is either a chamber or a panel. Since $\operatorname{proj}_Q = \operatorname{proj}_Q \operatorname{proj}_R$ by Proposition 2.5, we have $Q = \operatorname{proj}_Q P = \operatorname{proj}_Q P'$, and so P' cannot be reduced to a chamber and is parallel to Q by Lemma 5.8. Since $\operatorname{proj}_P R \supseteq \operatorname{proj}_P Q = P$, we have $\operatorname{proj}_P R = P$ and hence P' is parallel to P by Proposition 2.3.

We already have $\operatorname{proj}_{R(P_{i-1}, P_i)} P_0 = P_{i-1}$ for all $1 \leq i \leq k$ by hypothesis. For all $1 \leq i \leq l$, we have $\operatorname{proj}_{R(T_{i-1}, T_i)} P = \operatorname{proj}_{R(T_{i-1}, T_i)} \operatorname{proj}_R P = T_{i-1}$ by Proposition 2.5 and because $T_0, T_1, \ldots, T_l$ is a compatible path. This concludes the proof. □

Lemma 5.16.: *Let R be a rank 2 residue and let P, Q be two parallel panels contained in R. Then either $P = Q$ or R is spherical and P and Q are opposite in R.*

Proof: Let J be the type of R and let P be an s-panel and let Q be a t-panel Then $w := \delta(P, Q) \in X_s \cap W_J$. If $l(w) = 0$, then $w = 1_W$ and $P = Q$. Suppose that $w \neq 1_W$ and let $u \in S$ be such that $l(wu) = l(w) - 1$. As $w \in W_J$ it follows that $u \in J$; moreover $u \neq t$ because $l(wt) = l(w) + 1$ which implies that $J = \{t, u\}$. By Lemma 5.13 it follows that $J \subset S^-(wt)$. By Lemma 5.3 it follows that J is a spherical subset of S and, as $wt \in W_J$, that $wt = r_J$. Now $r_J s r_J = r_J^{-1} s r_J = (wt)^{-1} swt = tw^{-1}swt = t^3 = t$ and therefore the panels P and Q are of opposite type with respect to J.

Let $c \in P$, put $d := \mathrm{proj}_Q c$ and choose $d' \in Q$ distinct from d. Then $\delta(d, d') = t$ and $\delta(c, d') = \delta(c, d)\delta(d, d') = wt = r_J$. Hence c and d' are chambers which are opposite inside the residue R and, as $c \in P$ and $d' \in Q$, it follows that the panels P and Q are opposite inside R. □

Lemma 5.17.: *Two panels are parallel if and only if there exists a compatible path in Γ from one to the other.*

Proof: The right to left implication will be proved by an induction on the length of the path. If the path has length one, the result is obvious since opposite panels in a residue are parallel. Assume we have proved the result for all paths of length strictly less than k, and assume $P = P_0, P_1, \ldots P_k = Q$ is a compatible path in Γ. By induction P is parallel to P_{k-1}. We have $\mathrm{proj}_Q = \mathrm{proj}_Q \mathrm{proj}_{R(P_{k-1}, P_k)}$ by Proposition 2.5, and so $\mathrm{proj}_Q P = \mathrm{proj}_Q P_{k-1}$ which is equal to Q since P_{k-1} and Q are parallel. By Lemma 5.8, that means P and Q are parallel.

The left to right implication will be proved by an induction on the numerical distance between the two panels. Let P, Q be two parallel panels. If $l(\delta(P, Q)) = 0$ then $P = Q$ and the trivial path $P = P_0 = Q$ is compatible. Suppose $l(\delta(P, Q)) = l > 0$ and the result is proved for all parallel panels at distance strictly less than l. Choose $c \in P$ and let $d = \mathrm{proj}_Q c$. There exists a chamber e adjacent to d such that $l(c, d) = l(c, e) + 1$. Let R be the unique rank 2 residue containing Q and e. By Lemma 5.15, $\mathrm{proj}_R P = Q'$ is a panel parallel to P and to Q. Since there is a chamber in R closer to P than d, Q cannot be equal to Q' and so they are opposite in R by the previous lemma. Moreover $l(\delta(P, Q')) < l(\delta(P, Q))$. By induction, there exists a compatible path $P = P_0, P_1, \ldots P_k = Q'$. Since $R = R(Q', Q)$, the path $P = P_0, P_1, \ldots P_k, Q$ is compatible. □

Lemma 5.18.: *Let P, Q be parallel panels of type s and t, respectively, and let $u \in S \setminus \{t\}$. Then the following are equivalent:*

a) $u \in S^-(\delta(P, Q))$;

b) There exists a compatible path $P = P_0, \ldots, P_k = Q$ from P to Q such that $R(P_{k-1}, Q)$ has type $\{t, u\}$.

Moreover, if this is the case, then $\delta(P, P_{k-1}) = wtr_{\{u,t\}}$ and in particular $l(\delta(P, P_{k-1})) < l(\delta(P, Q))$.

Proof: Let $w = \delta(P, Q)$ and $c \in P$, $J := \{u, t\}$ and R the J-residue containing Q. We put $d := \operatorname{proj}_Q c$, $e := \operatorname{proj}_R c$ and $T := \operatorname{proj}_R P$. Note that T is parallel to P and to Q by Lemma 5.15. As $e \in T$ we have $e = \operatorname{proj}_T e = \operatorname{proj}_T \operatorname{proj}_R c = \operatorname{proj}_T c$ by Proposition 2.3. Note that Proposition 2.3 also implies $d = \operatorname{proj}_Q \operatorname{proj}_R c = \operatorname{proj}_Q e$.

We first show that Assertion a) implies Assertion b). Suppose $l(wu) = l(w) - 1$, let U be the u-panel containing d and $x := \operatorname{proj}_U c$. Then $x \neq d$ because $\delta(c, d) = w$ and $l(wu) = l(w) - 1$. It follows that $\operatorname{proj}_U e = \operatorname{proj}_U \operatorname{proj}_R c = \operatorname{proj}_U c = x \neq d$ and in particular $e \neq d$. As $d = \operatorname{proj}_Q e$, it follows that $\delta(T, Q) \neq 1_W$. Hence T and Q are opposite in R by Lemma 5.16. As T is parallel to P, Lemma 5.17 implies that there exists a compatible path $P = P_0, \ldots, P_l = T$; setting $l = k + 1$ and $P_k := Q$ yields Assertion b).

We now prove that Assertion b) implies Assertion a) and the remaining assertions. Suppose that there exists a compatible path $P = P_0, \ldots, P_k = Q$ such that $R = R(P_{k-1}, P_k)$. Then $T = P_{k-1}$. As Q is opposite T in R, we have $\delta(T, Q) = r_J t$. Note that $l(r_J tu) = l(r_J t) - 1$ and that $l(r_J t) \geq 1$. We also have that $l(c, d) = l(c, e) + l(e, d)$ and $w = \delta(P, Q) = \delta(c, d) = \delta(c, e)\delta(e, d) = w_J \delta(T, Q) = w_J r_J t$. We recall that $l(w_J w') = l(w_J) + l(w')$ for all $w' \in W_J$. Hence we have $l(wu) = l(w_J r_J tu) = l(w_J) + l(r_J tu) = l(w_J) + l(r_J t) - 1 = l(w_J r_J t) - 1 = l(w) - 1$ which yields Assertion a). We also have $\delta(P, T) = \delta(c, e) = w_J = w_J r_J t t r_J = w t r_J$ and $l(\delta(P, T)) = l(w_J) < l(w_J) + l(r_J t) = l(w_J r_J t) = l(\delta(P, Q))$. This finishes the proof. □

Lemma 5.19.: *Let P and Q be parallel panels such that $P \cap Q \neq \emptyset$. Then $P = P_0 = Q$ is the only compatible path from P to Q.*

Proof: Let s be the type of P and let $c \in P \cap Q$. Then $\operatorname{proj}_Q c = c$ and therefore $\delta(P, Q) = 1_W$. It follows from Lemma 5.9 that Q is also of type s and therefore $P = Q$. Let $P = P_0, \ldots, P_k = Q$ be a compatible path and suppose that $k \geq 1$. Put $R := R(P_{k-1}, P_k)$. By the compatibility of the path we know that $\operatorname{proj}_R P = P_{k-1}$ and that P_{k-1} is opposite $P_k = Q$ in R. As $P = Q$ is contained in R it follows that $P = \operatorname{proj}_R P$, and so $P = Q = P_{k-1}$. This is a contradiction since a panel cannot be opposite to itself in a rank 2 residue. We conclude that $k = 0$. □

Lemma 5.20.: *Let P and Q be parallel panels of type s and t respectively. Let $c \in P$ and put $d := \operatorname{proj}_Q c$ and let $E_1(d)$ be the union of all panels containing d. If $l(c, e) \leq l(c, d)$, for all $e \in E_1(d) \setminus Q$, then $\mathcal{B}$ is spherical and P and Q are opposite panels.*

Proof: Let $w := \delta(P, Q) = \delta(c, d)$. Let $u \in S$ be distinct from t and let U be the u-panel containing d. We claim that $x := \operatorname{proj}_U c \neq d$. Indeed, if $x = d$ and $y \neq d$ is a chamber in U, then $l(c, y) = l(c, d) + 1$ and $y \in E_1(d) \setminus Q$ which contradicts our assumption. Hence $x \neq d$ and therefore $u \in S^-(w)$. It follows now from Lemma 5.13 that $S \subset S^-(wt)$. Hence (W, S) is a spherical Coxeter system and wt is the longest element in W, by Lemma 5.3. Thus $\mathcal{B}$ is a spherical building.

Let $d' \neq d$ be a chamber in Q. Then $\delta(c, d') = wt$ and therefore d and d' are opposite chambers in $\mathcal{B}$. Moreover, since wt is an involution, we have $(wt)t(wt) = wtttw^{-1} = s$ and therefore P and Q are of opposite types. Hence they are opposite. □

Lemma 5.21.: *Let R be a spherical rank 3 residue in $\mathcal{B}$ and let P, Q be two parallel panels in R. If there is more than one compatible path contained in R from P to Q, then P and Q are opposite in R and there are exactly two such paths. Moreover these two paths have the same length.*

Proof: Let $P = P_0, P_1, \dots P_k = Q$ and $P = P'_0, P'_1, \dots P'_l = Q$ be two distinct compatible paths in R. We know by Lemma 5.19, that $P \neq Q$ and that $k, l \geq 1$. Let $Q' = P_{k-i} = P'_{l-i}$ such that $P_{k-j} = P'_{l-j}$ for all $0 \leq j \leq i$ and $P_{k-i-1} \neq P'_{l-i-1}$. Therefore $R(P_{k-i-1}, P_{k-i}) \neq R(P'_{l-i-1}, P'_{l-i})$.

Choose $c \in P$ and let $d = \operatorname{proj}_{Q'} c$. Suppose that P and Q' are not opposite in R. It follows from the previous lemma that there exists a chamber e not in Q' adjacent to d such that $l(c, e) = l(c, d) + 1$. Since there are only two rank 2 residues in R containing a given panel, the rank 2 residue R' containing Q' and e must be either $R(P_{k-i-1}, P_{k-i})$ or $R(P'_{l-i-1}, P'_{l-i})$. Without loss of generality we can assume $R' = R(P_{k-i-1}, P_{k-i})$. Then $\operatorname{proj}_{R'} P = P_{k-i-1}$, which is opposite Q' in R'. Let $c' = \operatorname{proj}_{R'} c$. It follows from Part d) of Proposition 2.2 that $l(c', e) = l(c', d) + 1$, in contradiction with the fact that P_{k-i-1} is opposite to Q' in R'. Therefore P and Q' are opposite. Since Q' cannot be the projection of P on any rank 2 residue containing it, Q' must be equal to Q. Using this and Lemma 5.17 in the building R, we conclude that for two non-opposite parallel panels of R, there is exactly one compatible path in R from one to the other.

Let P and Q be opposite in R and let R' be a rank 2 residue in R containing Q. Then there is exactly one compatible path $P = P_0, P_1, \dots P_k = Q$ such that $R' = R(P_{k-1}, P_k)$. Indeed $P_{k-1} = \operatorname{proj}'_R P$ is determined and there is only one compatible path between P and P_{k-1} since they are not opposite. Since there are two rank 2 residues containing Q in R, there are exactly two compatible paths in R from P to Q.

A rank 3 spherical residue of a thick building is of type A_3, C_3, $A_1 \oplus A_1 \oplus A_1$ or $A_1 \oplus I_n$. Knowing the distance between two opposite panels in R and in all rank 2 residues of R, it is easy to determine the length of compatible paths between opposite panels and see that the two compatible paths have the same length. That length is given in the table below.

type J	$o(s_1 s_2)$	$o(s_2 s_3)$	$o(s_1 s_3)$	$l(r_J)$	length of compatible paths between opposite panels
A_3	3	3	2	6	3
C_3	3	4	2	9	4
$A_1 \oplus A_1 \oplus A_1$	2	2	2	3	2
$A_1 \oplus I_n$	2	n	2	$n+1$	n for s_1-panels 2 for s_2-panels and s_3-panels

$\square$

Lemma 5.22.: *Let P, Q be parallel panels contained in a common residue R and let $P = P_0, P_1, \dots, P_k = Q$ be a compatible path. Then $P_i \subset R$ for all $0 \leq i \leq k$.*

Proof: We use induction on k and remark that the assertion trivially holds for $k \le 1$. Suppose $k \ge 2$ and put $T := P_{k-1}$ and $R' := R(T, Q)$. Then $R' \cap R$ is a residue or rank at most 2 containing Q. If it is strictly smaller than 2 we have $Q = R' \cap R$ and $\mathrm{proj}_{R'} R = Q$. Since $P \subset R$ it follows that $\mathrm{proj}_{R'} P = Q \ne T$, which contradicts the compatibility of the path. We conclude that $R' \subset R$, which implies $T \subset R$. Applying induction to the compatible path $P = P_0, \dots, P_{k-1} = T$ yields the claim. □

Lemma 5.23.: *Let P, Q be two parallel panels of $\mathcal{B}$. Then all compatible paths from P to Q have the same length.*

Proof: Let P be an s-panel and let Q be a t-panel and let $w := \delta(P, Q)$. We will prove the lemma by induction on $l(w)$.

If $l(w) = 0$, then $P = Q$ and the trivial path $P = P_0 = Q$ is the only compatible path from P to Q, by Lemma 5.19. Assume $l(w) = L > 0$ and we have proved the result for all parallel panels at distance strictly less than L. Take two compatible paths from P to Q: $P = P_0, P_1, \dots, P_k = Q$ and $P = P'_0, P'_1, \dots, P'_l = Q$. If $P_{k-1} = P'_{l-1} = Q'$, then $l(\delta(P, Q')) < L$ and so $k - 1 = l - 1$ and we are done.

Assume now $P_{k-1} \ne P'_{l-1}$, so that $R_1 := R(P_{k-1}, P_k) \ne R(P'_{l-1}, P'_l) =: R_2$, and let R be the rank 3 residue containing these two rank 2 residues. As $Q \subset R_i$ the residue R_i has type $\{t, u_i\}$ for $i = 1, 2$; thus we have $u_1 \ne u_2 \in S$ and $\{t, u_1, u_2\}$ is the type of R. By Lemma 5.18 it follows that $\{u_1, u_2\} \subset S^-(w)$ and therefore, by Lemma 5.13, the residue R is spherical.

Let Q' be the projection of P on R. By Lemma 5.15, Q' is parallel to P and Q. As there exists a compatible path from P to P_{k-1}, the panel P_{k-1} is parallel to P by Lemma 5.17 and as $P_{k-1} \subset R$, it is also parallel to Q' by Lemma 5.15. Hence there exists a compatible path $Q' = T_0, T_1, \dots, T_m = P_{k-1}$ from Q' to P_{k-1} and all the T_i are contained in R by the previous lemma. Similarly we have a compatible path $Q' = T'_0, T'_1, \dots, T'_n = P'_{l-1}$ in R. We have $P_{k-1} = \mathrm{proj}_{R(P_{k-1}, P_k)} P = \mathrm{proj}_{R(P_{k-1}, P_k)} \mathrm{proj}_R P = \mathrm{proj}_{R(P_{k-1}, P_k)} Q'$, and so $Q' = T_0, T_1, \dots, T_m, P_k = Q$ is a compatible path. By similar arguments, $Q' = T'_0, T'_1, \dots, T'_n, P'_l = Q$ is also a compatible path. As $P_{k-1} \ne P'_{l-1}$, these paths are distinct. It follows from Lemma 5.21 that Q' and Q are opposite panels in R and that these two paths in R have the same length, hence $m = n$.

By Lemma 5.17, there is a compatible path from P to Q', denoted by $P = S_0, S_1, \dots, S_j = Q'$. By Lemma 5.15, we have, on the one hand, that the paths $P = S_0, S_1, \dots, S_j = Q' = T_0, T_1, \dots, T_m = P_{k-1}$ and $P = S_0, S_1, \dots, S_j = Q' = T'_0, T'_1, \dots, T'_m = P'_{l-1}$ are both compatible of length $j + m$. On the other hand $P = P_0, P_1, \dots, P_{k-1}$ and $P = P'_0, P'_1, \dots, P'_{l-1}$ are also compatible paths. Since $l(\delta(P, P_{k-1})) < L$ and $l(\delta(P, P'_{l-1})) < L$, we can use the hypothesis of induction, and so $k - 1 = j + m$ and $l - 1 = j + m$. We conclude that $k = l$. □

Definition 5.24.: By Lemma 5.23, we can define the *compatible distance between two parallel panels P and Q* as the length of a compatible path joining them. It will be denoted by $L(P, Q)$.

Lemma 5.25.: *Let $w \in X_s$ and let P, P' be s-panels and Q, Q' be $w^{-1}sw$-panels such that $\delta(P,Q) = w = \delta(P',Q')$. Then $L(P,Q) = L(P',Q')$. Moreover, if $(J_1, \ldots, J_k)$ is the type of a compatible path from P to Q, then there exists a compatible path from P' to Q' of the same type.*

Proof: This follows by induction on $l(w)$ using Lemma 5.18. $\square$

Definition 5.26.: Let $w \in X_s$. Then we define its *s-compatible length*, denoted by $L_s(w)$, as the compatible distance between an s-panel P and an $w^{-1}sw$-panel Q such that $\delta(P,Q) = w$.

Proposition 5.27.: *Let $s \in S$ and $w_1, w \in X_s$ such that $w_1 \prec w$. Put $w_2 := w_1^{-1}w, u := w_1^{-1}sw_1, t := w^{-1}sw$. Let c, e, d be chambers such that $\delta(c,e) = w_1$ and $\delta(e,d) = w_2$ and let P be the s-panel containing c, U the u-panel containing e and Q be the t-panel containing d. Then the following holds.*

a) $w_2 \in X_u$;

b) the three panels are pairwise parallel and we have $\delta(P,U) = w_1, \delta(U,Q) = w_2$ *and* $\delta(P,Q) = w$;

c) $\mathrm{proj}_Q^P = \mathrm{proj}_Q^U \circ \mathrm{proj}_U^P$.

Proof: First note that u and t are in S, since $w_1, w \in X_s$. Since $w_1 \prec w$, we also have that $l(w) = l(w_1) + l(w_2)$. We easily see that $uw_2 = w_2t$, so $w_2^{-1}uw_2 \in S$. Since $w \in X_s$ we also have $l(wt) = l(sw) = l(w) + 1$. Therefore $l(w_1) + l(w_2) + 1 = l(w_1w_2t) \leq l(w_1) + l(w_2t) \leq l(w_1) + l(w_2) + 1$, and so $l(w_2t) = l(w_2) + 1$. Hence Part a) holds. Part b) follows from Part a) and the second assertion of Lemma 5.9. Let $x \in P$. Then $\delta(x, \mathrm{proj}_U x) = w_1$ and $\delta(\mathrm{proj}_U x, \mathrm{proj}_Q \mathrm{proj}_U x) = w_2$, by Part b). As $l(w_1w_2) = l(w_1) + l(w_2)$ it follows that $\delta(x, \mathrm{proj}_Q \mathrm{proj}_U x) = w_1w_2 = w$. Now $\mathrm{proj}_Q x$ is the unique chamber y in Q such that $\delta(x,y) = \delta(P,Q) = w$ and therefore $\mathrm{proj}_Q x = \mathrm{proj}_Q \mathrm{proj}_U x$. Hence Part c) holds. $\square$

6 Projectivities between panels

Throughout this section, $\mathcal{B} = (\mathcal{C}, \delta)$ is a building of type (W,S) and f is a codistance on $\mathcal{B}$. Moreover, it is always assumed that $\mathcal{B}$ satisfies the Conditions (3-sph), (lco) and (lsco) of Theorem 4.3. By the latter result we have in particular that f^{op} is simply connected.

We first recall some facts about codistances and fix further notation.

Let $c \in \mathcal{C}$ and $w \in W$ be such that $l(f(c)w) = l(f(c)) + l(w)$. By Lemma 3.8 there is a unique chamber $d \in \mathcal{C}$ such that $\delta(c,d) = w$ and $f(d) = f(c)w$. We denote this chamber by $\pi(c,w)$. Note that $\pi(c,w)$ is defined for all w if $c \in f^{\mathrm{op}}$.

The following observation is immediate.

Lemma 6.1.: *Let $c \in f^{\mathrm{op}}$, $w_1 \prec w \in W$ and put $w_2 := w_1^{-1}w$. Then $\pi(\pi(c,w_1),w_2) = \pi(c,w)$.*

Definition 6.2.: We will say that a residue R is *in* f^{op}, or in f_c^{op}, respectively, if it contains a chamber in f^{op}, or in f_c^{op}, respectively. For $s \in S$, let $\mathcal{P}_s^{\text{op}}(f)$ and $\mathcal{P}_{s,c}^{\text{op}}(f)$, respectively, be the set of all s-panels in f^{op} and f_c^{op}, respectively.

Notice that all chambers of a panel P in f^{op} are in f^{op} except for one, namely $\operatorname{proj}_P f$.

Proposition 6.3.: *Let $P \in \mathcal{P}_s^{\text{op}}(f)$, $w \in X_s$ and $t = w^{-1}sw$. Let P' be a t-panel with $\delta(P, P') = w$. Then the following conditions are equivalent:*

a) P' contains a chamber with f-value w;

b) $f(x) \in \{w, wt\}$ for $x \in P'$ and exactly one chamber of P' has f-value wt;

c) $P \in \mathcal{P}_{s,x}^{\text{op}}(f)$ for all chambers x of P';

d) $P \in \mathcal{P}_{s,x}^{\text{op}}(f)$ for some chamber x of P';

There exists exactly one panel P' satisfying these conditions.

Proof: Conditions a) and b) are equivalent by the definition of a codistance. Assume P' satisfies b). Let x be a chamber with f-value w in P'. Then $\delta(\operatorname{proj}_P x, x) = w = f(x)$. Since $\operatorname{proj}_P x$ cannot be equal to $\operatorname{proj}_P f$ (otherwise the chamber in P' with f-value wt would be at distance $l(w)$ from a chamber in f^{op} yielding a contradiction in view of Lemma 3.6), $\operatorname{proj}_P x \in f_x^{\text{op}}$ and $P \in \mathcal{P}_{s,x}^{\text{op}}(f)$. Now let $z = \operatorname{proj}_{P'} f$ be the unique chamber in P' with f-value wt. If y is any chamber of P in f_{op}, $\delta(y, z) = wt$, so $y \in f_z^{\text{op}}$ and $P \in \mathcal{P}_{s,z}^{\text{op}}(f)$. Obviously c) implies d). Now assume P' satisfies d). Then P contains $y \in f^{\text{op}}$ and $\delta(y, x) = f(x)$. If $y = \operatorname{proj}_P x$, then $\delta(y, x) = w$; if $y \neq \operatorname{proj}_P x$, then $\delta(y, x) = sw = wt$. In both cases, P' contains a chamber with f-value w.

We prove the existence of such a panel. Let p be the unique chamber of P not in f^{op} (so $f(p) = s$). As $l(f(p)w) = l(sw) = l(w) + 1$, Lemma 3.8 implies the existence of a unique chamber c with $sf(c) = w = \delta(p, c)$. Let P' be the t-panel on c. By Lemma 5.9, P' is parallel to P and $\delta(P, P') = w$. It obviously satisfies a).

Now we want to show that P' is unique. Let Q be a t-panel with $\delta(P, Q) = w$ satisfying b). Let x be the chamber of Q with f-value $wt = sw = f(p)w$. We have $\operatorname{proj}_P x = p$, so $\delta(p, x) = w$. By Lemma 3.8, a chamber with that property is unique. Therefore $x = c$ and $Q = P'$. $\square$

Definition 6.4.: For $P \in \mathcal{P}_s^{\text{op}}(f)$, $w \in X_s$ and $t = w^{-1}sw$, we denote the unique t-panel P' with $\delta(P, P') = w$ satisfying the equivalent conditions a) up to d) of Proposition 6.3 by $\pi(P, w)$.

Lemma 6.5.: *Let $P \in \mathcal{P}_s^{\text{op}}(f)$, $w \in X_s$ and $t = w^{-1}sw$ and $P' := \pi(P, w)$. Then $\operatorname{proj}_P \operatorname{proj}_{P'} f = \operatorname{proj}_P f$.*

Proof: As $l(\operatorname{proj}_{P'} f) = l(w) + 1$ and $l(\operatorname{proj}_P \operatorname{proj}_{P'} f, \operatorname{proj}_{P'} f) = l(w)$, it follows that $l(f(\operatorname{proj}_P \operatorname{proj}_{P'} f)) \geq 1$ and hence $f(\operatorname{proj}_P \operatorname{proj}_{P'} f) = s$. The claim follows because $\operatorname{proj}_P f$ is the unique chamber in P having f-value s. $\square$

Lemma 6.6.: *Let Q be a t-panel of $\mathcal{B}$ and let w be the shortest word of $\{f(x)|x \in Q\}$. Suppose $wtw^{-1} := s \in S$. Then there exists an s-panel $P \in \mathcal{P}_s^{\mathrm{op}}(f)$ such that $Q = \pi(P, w)$.*

Proof: Since $w^{-1}sw = t$ and $l(sw) = l(wt) = l(w) + 1$, we have $w \in X_s$. Let x be a chamber of Q with $f(x) = w$. Let $y \in f_x^{\mathrm{op}}$ so that $\delta(y, x) = w = f(x)$. Let P be the s-panel on y. By construction P is a panel in $\mathcal{P}_{s,x}^{\mathrm{op}}(f)$ which is parallel to Q by Lemma 5.9. Moreover $\delta(P, Q) = w$, hence by Proposition 6.3, $Q = \pi(P, w)$. □

Lemma 6.7.: *Let $c \in f^{\mathrm{op}}$, $s \in S$, $w \in X_s$ and put $t := w^{-1}sw$. Let P be the s-panel containing c and let Q be the t-panel containing $d := \pi(c, w)$. Then $Q = \pi(P, w)$ and $d = \mathrm{proj}_Q c$. Moreover, $\mathrm{proj}_P \mathrm{proj}_Q f = \mathrm{proj}_P f$ and $\mathrm{proj}_Q \mathrm{proj}_P f = \mathrm{proj}_Q f$.*

Proof: By its description, Q is a t-panel containing a chamber d such that $\delta(c, d) = w = f(d)$. Since $w \in X_s$, it follows that Q is parallel to P and $\delta(P, Q) = w$; hence $Q = \pi(P, w)$ by Condition a) of Proposition 6.3. As $\delta(c, d) = w = \delta(P, Q)$ we have $d = \mathrm{proj}_Q c$.

Let $x \in Q$. Then, by Condition b) of Proposition 6.3, we have $f(x) \in \{w, wt\}$. As $l(wt) = l(w) + 1$, we have $x = \mathrm{proj}_Q f$ if and only if $f(x) = wt$. Let $q := \mathrm{proj}_Q f$ and $p := \mathrm{proj}_P q$. Then $\delta(p, q) = w$ and therefore $l(\delta(p, q)) < l(f(q))$. It follows that p is not in f^{op} by Lemma 3.6. As $f(y) \in \{1_W, s\}$ for all $y \in P$, we conclude that $f(p) = s$ and therefore $p = \mathrm{proj}_P f$. The second equality follows from the fact that proj_Q^P and proj_P^Q are inverse to each other. □

Definition 6.8.: For $P, Q \in \mathcal{P}_s^{\mathrm{op}}(f)$ and $w \in X_s$, we write $P \equiv_w Q$ if $\pi(P, w) = \pi(Q, w)$. This is an equivalence relation on $\mathcal{P}_s^{\mathrm{op}}(f)$. For $P \equiv_w Q$, we put $\beta(P, Q, w)$ the bijection from P to Q defined by $\mathrm{proj}_Q \mathrm{proj}_{\pi(P,w)}$.

Notice that $\beta(Q, P, w)\beta(P, Q, w) = 1_P$ and that, by Lemma 6.5, $\beta(P, Q, w)$ maps $\mathrm{proj}_P f$ onto $\mathrm{proj}_Q f$ via $\mathrm{proj}_{\pi(P,w)} f$.

Proposition 6.9.: *Let $s \in S$, $w_w, w \in X_s$, and suppose $w_1 \prec w$. Let $P, P' \in \mathcal{P}_s^{\mathrm{op}}(f)$ such that $P \equiv_{w_1} P'$. Then $P \equiv_w P'$ and $\beta(P, P', w) = \beta(P, P', w_1)$.*

Proof: Let $U = \pi(P, w_1) = \pi(P', w_1)$, let $Q = \pi(P, w)$ and let $c \in P \cap f^{\mathrm{op}}$. Put $t := w^{-1}sw, u := w_1^{-1}sw_1, w_2 := w_1^{-1}, e := \pi(c, w_1)$ and $d := \pi(c, w)$. By Lemma 6.7 U is the u-panel containing e and Q is the t-panel containing d. By Lemma 6.1, we have $\pi(e, w_2) = d$, and in particular $\delta(e, d) = w_2$. It now follows from Proposition 5.27 that U is parallel to Q and that $\mathrm{proj}_Q^P = \mathrm{proj}_Q^U \circ \mathrm{proj}_U^P$.

Put $c' := \mathrm{proj}_{P'} e$. As $\delta(c', e) = \delta(P', U) = w_1 = f(e)$, we have $c' \in f^{\mathrm{op}}$ and $e = \pi(c', w_1)$. Now $\pi(c', w) = \pi(\pi(c', w_1), w_2) = \pi(e, w_2) = d$ which implies that $Q = \pi(P', w)$ and shows that $P \equiv_w P'$. We now apply Proposition 5.27 again to see that $\mathrm{proj}_Q^{P'} = \mathrm{proj}_Q^U \circ \mathrm{proj}_U^{P'}$. As proj_Q^U and proj_U^Q are mutually inverse bijections, it follows that $\beta(P, P', w) = \beta(P, P', w_1)$. □

Lemma 6.10.: *Let $c \in f^{\mathrm{op}}$, $w \in W$ and let R be a spherical J-residue containing $\pi(c, w)$. Then $\mathrm{proj}_R f = \pi(c, w^J)$ and $\mathrm{proj}_R c = \pi(c, w_J)$.*

Proof: Let $d := \pi(c, w)$. As $d \in R$ we have $f(R) = f(d)W_J = wW_J$ by Lemma 3.2. It follows that $\delta(c, \mathrm{proj}_R c) = w_J$ and $\delta(\mathrm{proj}_R c, d) = w_J^{-1}w$ and that $\mathrm{proj}_R c$ is on a minimal gallery joining c and d. As $d = \pi(c, w)$, it follows that $\mathrm{proj}_R c = \pi(c, w_J)$. Now $\pi(c, w^J) = \pi(c, w_J r_J)$ and, as $w_J \prec w^J$, we have $\delta(\pi(c, w_J), \pi(c, w^J)) = r_J$. Consequently, $\pi(c, w^J) \in R$ and $\pi(c, w^J)$ is the unique element in R having f-value w^J, which yields $\pi(c, w^J) = \mathrm{proj}_R f$. □

Lemma 6.11.: *Let $P \in \mathcal{P}_s^{\mathrm{op}}(f)$, $w \in X_s$, and let R be a spherical J-residue containing $\pi(P, w)$. Then $w_J, sw^J \in X_s$, $\pi(P, w_J) = \mathrm{proj}_R P$, and $w_J \prec w \prec sw^J$. Also, $\pi(P, sw^J)$ is the t-panel containing $\mathrm{proj}_R f$ where $t = (w^J)^{-1}sw^J$.*

Proof: We put $Q := \pi(P, w)$. Let $c \in P \cap f^{\mathrm{op}}$ and put $T_1 := \mathrm{proj}_R P$ and $c_1 := \mathrm{proj}_R c \in T_1$. First note that T_1 is parallel to both P and Q by Lemma 5.15 and that $w_J = \delta(c, c_1)$ by Part d) of Proposition 2.2. As $c_1 \in T_1 \subset R$, we have $c_1 = \mathrm{proj}_R c = \mathrm{proj}_{T_1} c$ and therefore $w_J = \delta(c, c_1) = \delta(P, T_1) \in X_s$. By the previous lemma we know that $c_1 = \pi(c, w_J)$ and as $c_1 \in T_1$ we obtain $T_1 = \pi(P, w_J)$.

Let $u := w_J^{-1}sw_J$. As $w_J \in X_s$, it follows that $u \in S$ and hence that T_1 is a u-panel. As $T_1 \subset R$, we obtain $u \in J$. We put $t := r_J u r_J \in J$ and recall that $w^J = w_J r_J$. This yields $t = (w^J)^{-1}sw^J$ and in particular $sw^J = w^J t$. As $t \in J$, we have $l(r_J t) = l(r_J) - 1$ and therefore $l(sw^J) = l(w^J t) = l(w_J) + l(w_J^{-1}w^J t) = l(w_J) + l(r_J t) = l(w_J) + l(r_J) - 1 = l(w_J r_J) - 1 = l(w^J) - 1$, hence $sw^J \in X_s$. Let $T_2 := \pi(P, sw^J)$ and put $c_2 := \mathrm{proj}_{T_2} c$. Then T_2 is a t-panel and $c_2 = \pi(c, sw^J)$ by Lemma 6.7. As $\delta(c_1, c_2) = \delta(\pi(c, w_J), \pi(c, w^J t) = r_J t \in W_J$, it follows that $c_2 \in R$ and therefore T_2 is contained in R. Now T_2 is a t-panel contained in R which contains a chamber having f-value $w^J t$, hence it contains also the unique chamber in R having f-value w^J which is in fact the projection of f onto R.

It remains to show that $w \prec sw^J$. As $v := w_J^{-1}w \in W_J$, we have $t' := w^{-1}sw = v^{-1}uv \in S \cap W_J = J$ and therefore $sw = wt' \prec w^J$. As $l(s(sw)) = l(sw) - 1$ and $l(sw^J) = l(w^J) - 1$ the assertion follows. □

Corollary 6.12.: *Let $P, Q \in \mathcal{P}_s^{\mathrm{op}}(f)$, $w \in X_s$ and let $J \subset S$ be spherical. If $\pi(P, w)$ and $\pi(Q, w)$ are contained in the same J-residue R, then $P \equiv_{sw^J} Q$.*

Proof: Put $t := (w^J)^{-1}sw^J$ and let T be the t-panel containing $\mathrm{proj}_R f$. Then we have $\pi(P, sw^J) = T = \pi(Q, sw^J)$ by the previous lemma. □

Definition 6.13.: For $P, Q \in \mathcal{P}_s^{\mathrm{op}}(f)$, we say that P and Q are *t-adjacent*, denoted by $P \sim_t Q$, if there exist $p \in P \cap f^{\mathrm{op}}$ and $q \in Q \cap f^{\mathrm{op}}$ with $p \sim_t q$. Let $P \sim_t Q$, both in $\mathcal{P}_s^{\mathrm{op}}(f)$.

Let $J = \{s, t\}$ and let R be the J-residue containing P and Q. As $\mathcal{B}$ is assumed to be 3-spherical, the residue R is spherical and we put $x_J := sr_J \in X_s$. By the previous corollary we have $P \equiv_{x_J} Q$ and we put $\alpha(P, Q) := \beta(P, Q, x_J)$. If $s = t$, $x_J = 1_W$, $P = Q = \pi(P, x_J)$ and $\alpha(P, Q) = \mathrm{id}_P$.

Notice that if $P, Q \in \mathcal{P}_s^{\mathrm{op}}(f)$ are t-adjacent, then $\alpha(P, Q)(\mathrm{proj}_P f) = \mathrm{proj}_Q f$ in view of Lemma 6.5.

Lemma 6.14.: *Let R be a spherical J-residue in f^{op} and let $c, d \in R \cap f^{\mathrm{op}}$. Let $s \in J$ and let P and Q be the s-panels containing c and d, respectively. Let $c = c_0, \ldots, c_m = d$ be a gallery in $R \cap f^{\mathrm{op}}$ and for each $0 \leq i \leq m$ let P_i be the s-panel containing c_i. Then $\alpha(P_{m-1}, P_m)\alpha(P_{m-2}, P_{m-1}) \ldots \alpha(P_1, P_2)\alpha(P_0, P_1) = \beta(P, Q, sr_J)$.*

Proof: We put $x_J := sr_J$ and observe that $T := \pi(P, x_J) = \pi(P_1, x_J) = \ldots = \pi(P_{m-1}, x_J) = \pi(Q, x_J)$ by Corollary 6.12.

Let $1 \leq i \leq m$. If c_{i-1} is s-adjacent to c_i, we put $J_i := \{s\}$; if they are not s-adjacent, then they are t_i-adjacent for a unique $t_i \in J$, $t_i \neq s$, and we put $J_i := \{s, t_i\}$ in this case. Furthermore, we put $x_i := sr_{J_i} \in X_s$ and observe that $x_i \prec x_J$ for all $1 \leq i \leq m$ by Lemma 6.11.

We can now apply Proposition 6.9 to see that $\alpha(P_{i-1}, P_i) = \beta(P_{i-1}, P_i, x_i) = \beta(P_{i-1}, P_i, x_J) = \mathrm{proj}^T_{P_{i-1}} \mathrm{proj}^{P_i}_T$, for all $1 \leq i \leq m$. As $\mathrm{proj}^T_{P_i}$ and $\mathrm{proj}^{P_i}_T$ are inverse bijections for $1 \leq i \leq m-1$, we obtain $\alpha(P_{m-1}, P_m)\alpha(P_{m-2}, P_{m-1}) \ldots \alpha(P_1, P_2)\alpha(P_0, P_1) = \mathrm{proj}^T_Q \mathrm{proj}^P_T = \beta(P, Q, sr_J)$. $\square$

Theorem 6.15.: *There exists a unique system of bijections $\beta(P, Q) : P \to Q$ where P, Q in $\mathcal{P}^{\mathrm{op}}_s(f)$, such that the following conditions are satisfied for all $P, Q, R \in \mathcal{P}^{\mathrm{op}}_s(f)$:*

- *a)* $\beta(P, P) = 1_P$;
- *b)* $\beta(Q, P)\beta(P, Q) = 1_P$;
- *c)* $\beta(Q, R)\beta(P, Q) = \beta(P, R)$;
- *d)* $\beta(P, Q)(\mathrm{proj}_P f) = \mathrm{proj}_Q f$;
- *e) if P and Q are t-adjacent for some $t \in S$, then $\beta(P, Q) = \alpha(P, Q)$.*

Proof:

Let $P, Q \in \mathcal{P}^{\mathrm{op}}_s(f)$ and choose $p \in P \cap f^{\mathrm{op}}$ and $q \in Q \cap f^{\mathrm{op}}$. By Theorem 4.3, f^{op} is connected, and so there exists a gallery γ from p to q contained in f^{op}. Set $\gamma = (x_0 = p, x_1, x_2, \ldots, x_n = q)$ and let X_i be the s-panel containing x_i. By definition these panels are in $\mathcal{P}^{\mathrm{op}}_s(f)$ and $X_i \sim_{t_i} X_{i+1}$ for some $t_i \in S$. We define $\beta(\gamma, P, Q) := \alpha(X_{n-1}, Q) \ldots \alpha(X_1, X_2)\alpha(P, X_1)$. By the above comment $\beta(\gamma, P, Q)$ maps $\mathrm{proj}_P f$ onto $\mathrm{proj}_Q f$. Note also that, if a system of bijections satisfying the conditions of the theorem exists, then $\beta(P, Q)$ has to coincide with $\beta(\gamma, P, Q)$ in view of Condition e). This yields already the uniqueness and it remains to show that the bijection $\beta(P, Q)$ defined by $\beta(\gamma, P, Q)$ does not depend on the choice of γ.

We will now show that if γ_1 and γ_2 are two galleries in f^{op} from a chamber of P to a chamber of Q, then $\beta(\gamma_1, P, Q) = \beta(\gamma_2, P, Q)$. As $\beta(P, P) = \mathrm{id}_P$, we can assume that γ_1 and γ_2 start and finish with the same chamber. Hence this is equivalent to showing that for a closed gallery γ in f^{op}, $\beta(\gamma, P, P) = \mathrm{id}_P$.

We recall that the assumptions on the buildings considered in this section allow us to apply Theorem 4.3; hence f^{op} is simply 2-connected. Therefore there exists a finite sequence of elementary homotopies from the closed gallery γ to a trivial gallery based in $p \in P$ such that all intermediate galleries are contained in f^{op}. Since two galleries differing by an elementary homotopy are equal except in a rank 2 residue, it is enough to show that $\beta(\gamma, P, P) = \mathrm{id}_P$ for a closed galley γ in a rank 2 residue in f^{op}.

Let $\gamma = (x_0, x_1, x_2, \ldots, x_n = x_0)$ be a closed gallery in f^{op} contained in a rank 2 residue R of type $\{t, u\}$ (where t or u could be equal to s). Let the X_i's be defined as above and put $J = \{s, t, u\}$. In view of Lemma 6.14 we now have $\beta(\gamma, X_0, X_0) = \alpha(X_{n-1}, X_0) \ldots \alpha(X_1, X_2)\alpha(X_0, X_1) = \beta(X_0, X_0, sr_J) = \mathrm{id}_{X_0}$.

It is obvious that a), b) and c) are satisfied. Since d) is satisfied for adjacent panels, it will be satisfied, by induction, for any two panels. Finally, Condition e) is satisfied by the construction of the system of bijections $\beta(P, Q)$. ☐

Theorem 6.16.: *Let $(\beta(P, Q))_{P,Q \in \mathcal{P}_s^{\mathrm{op}}(f)}$ be the unique system of bijections satisfying the conditions of the previous theorem. Let $P, P' \in \mathcal{P}_s^{\mathrm{op}}(f)$ with $P \equiv_w P'$ for $w \in X_s$. Then $\beta(P, P') = \beta(P, P', w)$.*

Proof: We first consider the special case where $w \in W_J$ for some spherical subset J of S containing s. Let R be the spherical J-residue containing P and P'. Let $p \in P \cap f^{\mathrm{op}}$ and $p' \in P' \cap f^{\mathrm{op}}$. As $R \cap f^{\mathrm{op}}$ is connected, there is a gallery $p = x_0, \ldots, x_n = p'$ in $R \cap f^{\mathrm{op}}$ and we let X_i denote the s-panel containing x_i. Now, by Property e) of the system of bijections, we have $\beta(P, P') = \alpha(X_{n-1}, X_n) \ldots \alpha(X_0, X_1) = \beta(P, P', sr_J)$, where the last equality follows from Lemma 6.14. As $w \in W_J$, it follows that $w \prec sr_J$ and therefore the claim follows for this special case in view of Proposition 6.9.

We prove the claim for an arbitrary $w \in X_s$ by induction on $L_s(w)$ and observe that the case $L_s(w) \leq 1$ is covered by the special case already considered before. We put $t := w^{-1}sw$ and $Q := \pi(P, w) = \pi(P', w)$ and remark that Q is a t-panel.

Assume $L_s(w) = k > 1$ and assume that the result is proved for all $w' \in X_s$ with $L_s(w') < k$. Let $P = P_0, P_1, \ldots, P_k = Q$ be a compatible path from P to Q, which exists by Lemma 5.17, and let $P' = P'_0, P'_1, \ldots, P'_k = Q$ be a compatible path from P' to Q with residues $R(P_i, P_{i+1})$ and $R(P'_i, P'_{i+1})$ of the same type for all $0 \leq i \leq k-1$. The existence of such a path follows from Lemma 5.25.

Let R be the rank 3 residue containing $R(P_{k-1}, P_k) = R(P'_{k-1}, P'_k)$ and $R(P_{k-2}, P_{k-1})$. Let J be the type of R, which contains t because $Q \subset R$. Let $T = \mathrm{proj}_R P$ and $T' = \mathrm{proj}_{R'} P'$, which are panels by Lemma 5.15. Let $c \in Q$. By Proposition 6.3, $P \in \mathcal{P}_{s,c}^{\mathrm{op}}(f)$, so there exists $x \in P \cap f_c^{\mathrm{op}}$. By Lemma 3.9, $\mathrm{proj}_R x \in A_f(R)$. This means that T contains chambers in $A_f(R)$, so whose f-value is w_J. We also have $\delta(x, \mathrm{proj}_R x) = w_J$ by Part d) of Proposition 2.2 and as $\mathrm{proj}_R x \in \mathrm{proj}_R P = T$ we have $\mathrm{proj}_T x = \mathrm{proj}_R x$. It follows that $\delta(x, \mathrm{proj}_T x) = w_J$, so that $T = \pi(P, w_J)$. By the same argument, $T' = \pi(P', w_J)$ and so T' contains chambers in $A_f(R)$. Therefore T and T' are s'-panels where $s' = w_J^{-1} s w_J \in J$.

Since P_{k-2} is in a compatible path from P (to Q), it is parallel to P. By Lemma 5.15, T is a panel parallel to P and P_{k-2} and there exists a compatible path from P to P_{k-2} containing T. Since all compatible paths between two given panels have the same length, the length of a compatible path from P to T is less or equal to $k-2$. Hence $L_s(w_J) \leq L_s(w) - 2$.

Choose $q \in T \cap A_f(R)$ and $q' \in T' \cap A_f(R)$. Because of the hypothesis on $\mathcal{B}$, there exists a gallery γ from q to q' contained in $A_f(R)$. If $\gamma = (q = x_0, x_1, x_2, \dots, x_n = q')$, let X_i be the s'-panel containing x_i and $X_i \sim_{t_i} X_{i+1}$ for some $t_i \in J$ for $0 \le i \le n-1$. By Lemma 6.6, there exists an s-panel $Q_i \in \mathcal{P}_s^{\mathrm{op}}(f)$ such that $X_i = \pi(Q_i, w_J)$ for all $0 \le i \le n$. Of course we take $Q_0 = P$ and $Q_n = P'$. Since $\delta(Q_i, X_i) = w_J$, we have $\mathrm{proj}_R Q_i = X_i$ for all $0 \le i \le n$. By Property $c)$ of Theorem 6.15, $\beta(P, P') = \beta(Q_{n-1}, Q_n) \dots \beta(Q_1, Q_2)\beta(Q_0, Q_1)$.

Let $J_i = \{s', t_i\} \subset J$, let R_i be the J_i-residue containing X_i and X_{i+1} and let $w_i := w_J s' r_{J_i} = s w^{J_i}$. Now, $X_i = \pi(Q_i, w_J)$ and $X_{i+1} = \pi(Q_{i+1}, w_J)$ are contained in the same spherical J_i-residue and therefore $Q_i \equiv_{w_i} Q_{i+1}$ by Corollary 6.12.

Since $\mathrm{proj}_{R_i} Q_i = \mathrm{proj}_{R_i} \mathrm{proj}_R Q_i = \mathrm{proj}_{R_i} X_i = X_i$, a compatible path from Q_i to X_i (of length $L_s(w_J) \le k-2$) completed by the panel $\pi(Q_i, w_i)$ is a compatible path of length $L_s(w_i) \le k-1$. By induction, this means that $\beta(Q_i, Q_{i+1}) = \beta(Q_i, Q_{i+1}, w_i)$.

Let $\tilde{w} := w_J s' r_J = s w^J$. By Lemma 6.11 we have $\tilde{w} \in X_s$ and $w_i \prec \tilde{w}$. It follows from Proposition 6.9 that $Q_i \equiv_{\tilde{w}} Q_{i+1}$ and that $\beta(Q_i, Q_{i+1}, w_i) = \beta(Q_i, Q_{i+1}, \tilde{w})$. Hence

$$\begin{aligned} \beta(P, P') &= \beta(Q_{n-1}, Q_n) \dots \beta(Q_1, Q_2)\beta(Q_0, Q_1) \\ &= \beta(Q_{n-1}, Q_n, \tilde{w}) \dots \beta(Q_1, Q_2, \tilde{w})\beta(Q_0, Q_1, \tilde{w}) \\ &= \beta(P, P', \tilde{w}). \end{aligned}$$

We have $l(w^{-1}\tilde{w}) = l(w^{-1} w_J s' r_J) = l(r_J s' w_J^{-1} w) = l(r_J) - l(s' w_J^{-1} w)$ because $s' w_J^{-1} w \in W_J$. Moreover $l(s' w_J^{-1} w) = l(w_J^{-1} s w) = l(w_J^{-1} w t) = l(wt) - l(w_J) = l(w) + 1 - l(w_J)$ because $wt \in wW_J$. Hence, on the one hand, $l(w^{-1}\tilde{w}) = l(r_J) + l(w_J) - 1 - l(w)$. On the other hand, $l(\tilde{w}) - l(w) = l(w_J s' r_J) - l(w) = l(w_J) + l(s' r_J) - l(w) = l(w_J) + l(r_J) - 1 - l(w)$ since $w_J s' r_J \in wW_J$. Therefore $w \prec \tilde{w}$, and so $\beta(P, P', w) = \beta(P, P', \tilde{w})$. This concludes the proof. $\square$

Corollary 6.17.: *Let R be a rank 2 residue of $\mathcal{B}$, let w_R be the shortest element in $f(R)$ and suppose that $w_R \in X_s$. Put $t := w_R^{-1} s w_R \in J$. Let $c \in R$ and let $P, P' \in \mathcal{P}_{s,c}^{\mathrm{op}}(f)$. Then $\beta(P, P')(\mathrm{proj}_P c) = \mathrm{proj}_{P'} c$.*

Proof: Let $w = f(c)$, J the type of R, so $w_R = w_J$. Let $d = \mathrm{proj}_R f$, so that $f(d) = w^J$, where w^J is the unique longest word of wW_J. As $w_J \in X_s$ and $w_J^{-1} s w_J \in J$ it follows that $s w^J \in X_s$ and $u := (s w^J)^{-1} s (s w^J) \in J$. Note that the u-panel Q through d is parallel to both P and P'. Since $\delta(P, Q) = s w^J = w^J u = \delta(P', Q)$ and Q contains a chamber with f-value $w^J u$, we have $Q = \pi(P, s w^J) = \pi(P', s w^J)$. Therefore $P \equiv_{s w^J} P'$ and, by Theorem 6.16, $\beta(P, P') = \beta(P, P', s w^J) = \mathrm{proj}_{P'} \mathrm{proj}_Q$.

Let $x \in P \cap f_c^{\mathrm{op}}$ and let $x' \in P' \cap f_c^{\mathrm{op}}$. By Lemma 3.7, there exist minimal galleries $x = x_0, x_1, \dots, x_n = c$ and $x' = x'_0, x'_1, \dots, x'_n = c$, with $l(f(x_i)) = i = l(f(x'_i))$ for all $0 \le i \le n$, containing $\mathrm{proj}_P c$ and $\mathrm{proj}_{P'} c$, respectively. Obviously $\mathrm{proj}_P c = x_1$ if $x_1 \in P$ and x_0 otherwise. By Proposition 3.3, $f(c) = f(d)\delta(d, c)$; moreover $l(f(c)) = l(w) = l(w^J) - l(\delta(d, c)) = l(f(d)) - l(d, c)$. Hence there is a minimal gallery $c = y_0, y_1, \dots, y_m = d$ with $l(f(y_i)) = l(f(c)) + i$ for all $0 \le i \le m$ and containing $\mathrm{proj}_Q c$, where $m = l(c, d)$. Obviously $\mathrm{proj}_Q c = y_{m-1}$ if $y_{m-1} \in Q$ and $y_m = d$ otherwise. We have that $x =$

$x_0, x_1, \ldots, x_n = y_0, y_1, \ldots, y_m = d$ is a minimal gallery, and so there is a minimal gallery (which is a subgallery of the previous one) from $\mathrm{proj}_P\, c$ to $\mathrm{proj}_Q\, c$ containing c. Therefore $\mathrm{proj}_P\, c = \mathrm{proj}_P \mathrm{proj}_Q\, c$. By a similar argument, $\mathrm{proj}_{P'}\, c = \mathrm{proj}_{P'} \mathrm{proj}_Q\, c$.

Putting everything together, $\beta(P, P')(\mathrm{proj}_P\, c) = \mathrm{proj}_{P'} \mathrm{proj}_Q \mathrm{proj}_P \mathrm{proj}_Q\, c = \mathrm{proj}_{P'} \mathrm{proj}_Q\, c = \mathrm{proj}_{P'}\, c$, because P and Q are parallel. $\square$

Theorem 6.18.: *Let c be a chamber of $\mathcal{B}$ and let $P, P' \in \mathcal{P}^{\mathrm{op}}_{s,c}(f)$. Then $\beta(P, P')(\mathrm{proj}_P\, c) = \mathrm{proj}_{P'}\, c$.*

Proof: Throughout the proof we denote, for any residue R of $\mathcal{B}$, by w_R the unique shortest element in the coset $f(R)$; this means that, if R is a J-residue, and if $w \in f(R)$, then $w_R = w_J$. Furthermore, we put $l_f(R) := l(w_R) = \min\{l(f(x)) \mid x \in R\}$.

We will prove the assertion by induction on $l(f(c))$.

Assume $l(f(c)) = 0$. If $x \in f^{\mathrm{op}}_c$, then $\delta(x, c) = f(c) = 1_W$, so $f^{\mathrm{op}}_c = \{c\}$. Hence $P = P'$ contains c, and the statement is obvious since $\beta(P, P') = 1_P$.

Assume $l(f(c)) = 1$. If $f(c) = s$, then f^{op}_c consists of all chambers s-adjacent to c (except for c itself). Hence $P = P'$ contains c, and the statement is again obvious. We now consider the case $f(c) = t \neq s$. Let R be the $\{s, t\}$-residue containing c. Then $w_R = 1_W \in X_s$ and we are done by Corollary 6.17.

Assume now $l(f(c)) = l \geq 2$ and assume that the theorem is proved for all chambers c' with $l(f(c')) < l$. Let u, t be the last two letters in a reduced word for $f(c)$, so that $l(f(c)ut) = l(f(c)) - 2$. Let R be the $\{u, t\}$-residue containing c. Then $l_f(R) \leq l(f(c)) - 2$. If $\mathrm{proj}_R\, P$ is a panel, then $w_R \in X_s$ and we are done by Corollary 6.17. Hence we are left with the case where $\mathrm{proj}_R\, P$ is a chamber p and $\mathrm{proj}_R\, P'$ is a chamber p'. Since P contains a chamber x in f^{op}_c and $\mathrm{proj}_R\, P = \mathrm{proj}_R\, x$, we have by Lemma 3.9 that $p \in A_f(R)$, and similarly $p' \in A_f(R)$. Moreover, there exists a minimal gallery from x to c containing p such that the length of the f-value strictly increases at each step, and so $x \in f^{\mathrm{op}}_p$ by Lemma 3.7. Similarly $x' \in f^{\mathrm{op}}_{p'}$. Recall that our general assumptions on $\mathcal{B}$ imply that the sets $A_f(R)$ are connected. Hence there exists a gallery $p = p_0, p_1, \ldots, p_n = p'$ (without repetitions) entirely contained in $A_f(R)$. For all $1 \leq j \leq n$, let Q_j be the unique panel containing p_{j-1} and p_j, and let $z_j = \mathrm{proj}_{Q_j}\, f$. Since $l(f(z_j)) = l_f(R) + 1$, we have $l(f(z_j)) < l$ for all $1 \leq j \leq n$. For each $1 \leq j \leq n-1$, we can choose $x_j \in f^{\mathrm{op}}_{p_j}$. We put $x_0 := x, x_n := x'$ and denote the s-panel containing x_i by P_i for all $0 \leq i \leq n$.

By Lemma 3.7, there exists a gallery from x_j to p_j such that the length of the f-value strictly increases at each step, for all $1 \leq j \leq n-1$. Since $l(f(z_j)) = l(f(p_j)) + 1 = l(f(z_{j+1}))$, $z_j, p_j \in Q_j$ and $z_{j+1}, p_j \in Q_{j+1}$, by adding a chamber at the end of the previous gallery, we get two minimal galleries from x_j to z_j and from x_j to z_{j+1}, both such that the length of the f-value strictly increases at each step. Hence $x_j \in f^{\mathrm{op}}_{z_j}$ and $x_j \in f^{\mathrm{op}}_{z_{j+1}}$ for all $1 \leq j \leq n-1$. Therefore $P_j \in \mathcal{P}^{\mathrm{op}}_{s,z_j}(f)$ and $P_j \in \mathcal{P}^{\mathrm{op}}_{s,z_{j+1}}(f)$ for all $1 \leq j \leq n-1$. For a similar reason $P = P_0 \in \mathcal{P}^{\mathrm{op}}_{s,z_1}(f)$ and $P' = P_n \in \mathcal{P}^{\mathrm{op}}_{s,z_n}(f)$. We conclude that $P_{i-1}, P_i \in \mathcal{P}^{\mathrm{op}}_{s,z_i}(f)$ for all $1 \leq i \leq n$. Hence, by induction, $\beta(P_{i-1}, P_i)(\mathrm{proj}_{P_{i-1}}\, z_i) = \mathrm{proj}_{P_i}\, z_i$ for all $1 \leq i \leq n$.

As w_R is not in X_s, it follows that $\operatorname{proj}_R P_i$ is a chamber for all $1 \leq i \leq n$. Because of this and since projections of residues on one another are parallel, $\operatorname{proj}_{P_i} R$ is also a chamber, hence $\operatorname{proj}_{P_i} z_i = \operatorname{proj}_{P_i} R = \operatorname{proj}_{P_i} c$ and $\operatorname{proj}_{P_{i-1}} z_i = \operatorname{proj}_{P_{i-1}} R = \operatorname{proj}_{P_{i-1}} c$. Therefore we have $\beta(P_{i-1}, P_i)(\operatorname{proj}_{P_{i-1}} c) = \operatorname{proj}_{P_i} c$.

By Theorem 6.15, $\beta(P, P') = \beta(P', P_{n-1}) \dots \beta(P_1, P_2)\beta(P, P_1)$ and therefore $\beta(P, P') \operatorname{proj}_P c = \operatorname{proj}_{P'} c$. □

7 Adjacent codistances

In this section we will need the following facts.

Lemma 7.1.: *Let $\mathcal{B}$ be a spherical building of type (W, S). For each residue R let R^{op} denote the set of all residues in $\mathcal{B}$ opposite R. If R and T are two residues with $R^{\mathrm{op}} = T^{\mathrm{op}}$, then $R = T$.*

Proof: Let J be the type R and consider a residue R^* opposite R. Let $r \in W$ be the longest element. of (W, S). Then R^* has type $K := rJr$. As R^* is opposite T, we have that T is of type J.

Let $c \in R$ and put $d := \operatorname{proj}_T c$. Put $w := \delta(c, d)$ and choose a chamber e in $\mathcal{B}$ such that $\delta(d, e) = w_1 := w^{-1}r$ and consider the K-residue R' containing e. Note first that $\delta(c, e) = r$ and that R' is therefore opposite R and hence also opposite T by our assumption. We put $d' := \operatorname{proj}_T e$. Now there exist a minimal gallery from c to d' passing through d, a minimal gallery from d to e passing through d', and a minimal gallery from c to e passing through d. We conclude that there is a minimal gallery from c to e passing through d and d'. As T is opposite R' and $e \in R'$ it follows that $\delta(e, d') = rr_J$ and therefore $\delta(d', c) = r_J \in W_J$ because $\delta(e, c) = r$. This means that the J-residue containing d' contains also c and hence $R = T$. □

The previous lemma has a very simple proof when considering buildings as simplical complexes, as originally defined by Tits [Ti74]. Indeed, the residues R and T (which are just simplices in this setting) are, by definition of a 'simplicial building', contained in an apartment, in which every simplex has a unique opposite.

Lemma 7.2.: *Let f be a codistance on a building $\mathcal{B}$, let R be a spherical residue and let T be a residue contained in R. Then $\operatorname{proj}_T f = \operatorname{proj}_T \operatorname{proj}_R f$.*

Proof: Let $c := \operatorname{proj}_T f$ and $d := \operatorname{proj}_R f$. Then we can find a minimal gallery from c to d such that the length of the f-value of the chambers in that gallery (strictly) increases at each step. Hence $l(c, d) = l(f(d)) - l(f(c))$. If $\operatorname{proj}_T d \neq c$, then we would have a chamber e in T such that $l(d, e) < l(d, c)$ which implies $l(f(e)) > l(f(c))$, which yields a contradiction. □

Definition 7.3.: Two codistances f and g on $\mathcal{B}$ are called *s-adjacent* if $\mathcal{P}_s^{\mathrm{op}}(f) = \mathcal{P}_s^{\mathrm{op}}(g)$. We denote this by $f \sim_s g$.

Lemma 7.4.: *Let f, g be two codistances on a building $\mathcal{B}$. Let R be a spherical J-residue in f^{op}. Let $s \in J$. If f and g are s-adjacent, then $\operatorname{proj}_R f$ and $\operatorname{proj}_R g$ are $r_J s r_J$-adjacent in $\mathcal{B}$.*

Proof: Suppose f and g are s-adjacent. Then $\mathcal{P}_s^{\mathrm{op}}(f) = \mathcal{P}_s^{\mathrm{op}}(g)$, which means that the s-panels of R in f^{op} and in g^{op} coincide. The $r_J s r_J$-panel P containing $d := \operatorname{proj}_R f$ is opposite in R to all s-panels of R in f^{op}. Similarly the $r_J s r_J$-panel P' containing $d' := \operatorname{proj}_R g$ is opposite in R to all s-panel of R in g^{op}. By Lemma 7.1 it follows that $P = P'$ and therefore d and d' are $r_J s r_J$-adjacent. $\square$

Lemma 7.5.: *Let f be a codistance on a building $\mathcal{B}$, and let g be a codistance s-adjacent to f. Let R be a J-residue in f^{op} with $s \in J$. Then R is in g^{op}.*

Proof: Since R is in f^{op}, R contains a chamber x in f^{op}. The s-panel containing x is in f^{op}, and so by hypothesis, it is in g^{op}. Since this panel is in R, it means R is in g^{op}. $\square$

Lemma 7.6.: *Let f be a codistance on a k-spherical building $\mathcal{B}$ such that f^{op} is connected, and let g be a codistance s-adjacent to f. Let R be a J-residue of rank $\leq k-1$ in f^{op}, with $s \in J$. Then $\operatorname{proj}_R g$ determines g uniquely.*

Proof: Suppose that g_1 and g_2 are two codistances s-adjacent to f with $\operatorname{proj}_R g_1 = \operatorname{proj}_R g_2$. By hypothesis, $\mathcal{P}_s^{\mathrm{op}}(f) = \mathcal{P}_s^{\mathrm{op}}(g_1) = \mathcal{P}_s^{\mathrm{op}}(g_2)$.

We claim that $g_1^{\mathrm{op}} \subseteq g_2^{\mathrm{op}}$. Let $x \in g_1^{\mathrm{op}}$. The J-residue R_x containing x is in g_1^{op}. By Lemma 7.5, R_x is also in f^{op}. Since f^{op} is connected, there is a gallery $x_0, x_1, \ldots, x_n$ in f^{op} with $x_0 \in R$ and $x_n \in R_x$. We will show by induction on n that $\operatorname{proj}_{R_x} g_1 = \operatorname{proj}_{R_x} g_2$. If $n = 0$, then $R_x = R$ and we are done. Assume that we have shown that for every J-residue at "distance" (in the sense described above) at most $n-1$ of R, the projections of g_1 and g_2 coincide. Let R' be the J-residue containing x_{n-1}. By the induction hypothesis, $\operatorname{proj}_{R'} g_1 = \operatorname{proj}_{R'} g_2$. We have $x_{n-1} \sim_t x_n$. If $t \in J$ then $R_x = R'$ and we are done. So assume $t \notin J$, let $K = J \cup \{t\}$ (which is spherical by hypothesis) and let $\tilde{R}$ be the K-residue containing R_x and R'. Then, by Lemma 7.1, the residue of type $\mathrm{op}_K(J) := \{r_K u r_K | u \in J\}$ containing $\operatorname{proj}_{\tilde{R}} f$ is the unique residue of $\tilde{R}$ opposite in $\tilde{R}$ to all J-residues of $\tilde{R}$ in f^{op}. Similarly, the $\mathrm{op}_K(J)$-residue containing $\operatorname{proj}_{\tilde{R}} g_i$ is the only residue of $\tilde{R}$ opposite in $\tilde{R}$ to all J-residues in g_i^{op}, for $i = 1, 2$. By Lemma 7.5, the sets of J-residues in f^{op} and in g_i^{op} ($i = 1, 2$) coincide. Therefore these three $\mathrm{op}_K(J)$-residues coincide; let us name it T. As $f(y) = f(\operatorname{proj}_{\tilde{R}} f)\delta(\operatorname{proj}_{\tilde{R}} f, y)$ for $y \in \tilde{R}$, by Lemma 3.3, we have $l(f(y)) = r_K - l(\operatorname{proj}_{\tilde{R}} f, y)$ for $y \in \tilde{R}$, and so $\operatorname{proj}_{R'} f = \operatorname{proj}_{R'} \operatorname{proj}_{\tilde{R}} f$. Similarly for g_1, g_2 and for R_x. We have $\operatorname{proj}_{R'} g_1 = \operatorname{proj}_{R'} g_2$, and so $\operatorname{proj}_{R'} \operatorname{proj}_{\tilde{R}} g_1 = \operatorname{proj}_{R'} \operatorname{proj}_{\tilde{R}} g_2$, with $\operatorname{proj}_{\tilde{R}} g_1, \operatorname{proj}_{\tilde{R}} g_2 \in T$. As T contains $\operatorname{proj}_{\tilde{R}} f, \operatorname{proj}_{\tilde{R}} g_i (i = 1, 2)$ and since R' and T are parallel we have $\operatorname{proj}_{\tilde{R}} g_1 = \operatorname{proj}_{\tilde{R}} g_2$. Now $\operatorname{proj}_{R_x} g_1 = \operatorname{proj}_{R_x} \operatorname{proj}_{\tilde{R}} g_1 = \operatorname{proj}_{R_x} \operatorname{proj}_{\tilde{R}} g_2 = \operatorname{proj}_{R_x} g_2$ Since $x \in g_1^{\mathrm{op}}$ and, for any $y \in R_x$, $g_1(y) = g_1(\operatorname{proj}_{R_x} g_1)\delta(\operatorname{proj}_{R_x} g_1, y)$ by Lemma 3.3, we have $1_W = r_J \delta(\operatorname{proj}_{R_x} g_1, x)$. Therefore $r_J = \delta(\operatorname{proj}_{R_x} g_1, x) = \delta(\operatorname{proj}_{R_x} g_2, x)$, which implies that $x \in g_2^{\mathrm{op}}$. By symmetry, we get $g_1^{\mathrm{op}} = g_2^{\mathrm{op}}$. We now conclude by Lemma 3.10. $\square$

Proposition 7.7.: *Let $\tilde{\mathcal{C}}$ be the set of all codistances on a 3-spherical building $\mathcal{B}$. Then $(\tilde{\mathcal{C}}, (\sim_s)_{s\in S})$ is a chamber system.*

Proof: It follows from the definition that $\sim_s$ is an equivalence relation on $\tilde{\mathcal{C}}$ for all $s \in S$. Suppose $f \sim_s g$ and $f \sim_t g$ for $s, t \in S$ and $f \neq g$. Let $J = \{s, t\}$, which is spherical. Let R be a J-residue in f^{op}. By Lemma 7.6, $\mathrm{proj}_R f$ and $\mathrm{proj}_R g$ are distinct. By Lemma 7.4, the chambers $\mathrm{proj}_R f$ and $\mathrm{proj}_R g$ are $r_J s r_J$-adjacent and also $r_J t r_J$-adjacent in $\mathcal{B}$. Since the chambers of $\mathcal{B}$ form a chamber system, it means $r_J s r_J = r_J t r_J$, and hence $s = t$. □

From now on, we again assume that $\mathcal{B} = (\mathcal{C}, \delta)$ is a 3-spherical building of type (W, S) satisfying (lco) and (lsco) and that f is a codistance on $\mathcal{B}$. Let $\mathcal{B}^* = (\mathcal{C}^*, (\sim_s)_{s\in S})$ be the chamber system on the connected component of f.

Fix $s \in S$ and $\tilde{P}$ in $\mathcal{P}_s^{\mathrm{op}}(f)$. For each chamber p of $\tilde{P}$ in f^{op}, we will define another codistance on $\mathcal{B}$. Let $\beta(p) := \{\beta(\tilde{P}, Q)(p) | Q \in \mathcal{P}_s^{\mathrm{op}}(f)\}$. By Theorem 6.15, this set contains exactly one chamber in each panel of $\mathcal{P}_s^{\mathrm{op}}(f)$, none of which is the projection of f on it.

Theorem 7.8.: *For $c \in \mathcal{B}$, choose $P \in \mathcal{P}_{s,c}^{\mathrm{op}}(f)$, and put*

$$g(c) = \begin{cases} sf(c) & \textit{if } \mathrm{proj}_P c \in \{\mathrm{proj}_P f, \beta(p) \cap P\} \\ f(c) & \textit{otherwise.} \end{cases}$$

Then g is a codistance on $\mathcal{B}$. Moreover g is s-adjacent to f and, for $P \in \mathcal{P}_s^{\mathrm{op}}(f)$, $\mathrm{proj}_P g = \beta(p) \cap P$.

Proof: The function $g : \mathcal{C} \to W$ is independent of the choice of P by Theorem 6.18 and by statement d) of Theorem 6.15. Let Q be a t-panel, so that $f(x) \in \{w, wt\}$ for all $x \in Q$ and Q contains a unique chamber $q := \mathrm{proj}_Q f$ with f-value the longest word of the two, which we can assume to be wt. We first show that Q satisfies the codistance condition for f. We distinguish two cases.

Case 1. Assume $w^{-1}sw = t$. Then $w \in X_s$ and there exists $P \in \mathcal{P}_s^{\mathrm{op}}(f)$ parallel to Q with $\delta(P, Q) = w$. By Proposition 6.3, $P \in \mathcal{P}_{s,x}^{\mathrm{op}}(f)$ for all chambers x of Q. Since P and Q are parallel, proj_P^Q and proj_Q^P are inverse bijections between P and Q. Hence $g(x) = f(x)$ for all $x \in Q$, except for q whose g-value is $sf(q) = swt = w$ and for $\mathrm{proj}_Q(\beta(p) \cap P)$ whose g-value is $sf(\mathrm{proj}_Q(\beta(p) \cap P)) = sw = wt$. Hence $g(x) \in \{w, wt\}$ and Q contains a unique chamber with g-value wt.

Case 2. Now assume $w^{-1}sw \neq t$. We claim that either $g(x) = f(x)$ for all $x \in Q$ or $g(x) = sf(x)$ for all $x \in Q$. Suppose we proved the claim. In the first case, it is obvious that Q will satisfy the codistance condition for g. Suppose we are in the second case. Then $g(x) \in \{sw, swt\}$ for all $x \in Q$ and Q contains a unique chamber with g-value swt. We just need to show that $l(swt) = l(sw) + 1$ to conclude that Q satisfies the codistance condition for g. If $l(sw) = l(w) + 1$, it follows from Assertion b) of Proposition 2.1 that either $l(swt) = l(w) + 2$ or $swt = w$. Since the second case is excluded, we have $l(swt) = l(w) + 2 = l(sw) + 1$. If $l(sw) = l(w) - 1$ and $l(swt) = l(sw) - 1$, then

$l(swt) = l(w) - 2 = l(wt) - 3$, and we get a contradiction, hence if $l(sw) = l(w) - 1$ we also get $l(swt) = l(sw) + 1$.

We now prove the claim. Let $x \in Q$ with f-value w, $y \in f_x^{\mathrm{op}}$ and P the s-panel containing y, so that $P \in \mathcal{P}_{s,x}^{\mathrm{op}}(f)$. If we add the chamber q to a minimal gallery from y to x, we get a minimal gallery from y to q with the required condition on f, and so, by Lemma 3.7, $y \in f_q^{\mathrm{op}}$ and $P \in \mathcal{P}_{s,q}^{\mathrm{op}}(f)$. Let x' be another chamber of Q with f-value w and $P' \in \mathcal{P}_{s,x'}^{\mathrm{op}}(f)$. By the same argument, $P' \in \mathcal{P}_{s,q}^{\mathrm{op}}(f)$. By Theorem 6.18, this means $\beta(P, P')(\mathrm{proj}_P\, q) = \mathrm{proj}_{P'}\, q$. Since P and Q (P' and Q, respectively) are not parallel, $\mathrm{proj}_P\, Q$ ($\mathrm{proj}_{P'}\, Q$, respectively) is a chambers, and so $\mathrm{proj}_P\, q = \mathrm{proj}_P\, x$ ($\mathrm{proj}_{P'}\, q = \mathrm{proj}_{P'}\, x'$, respectively). Therefore $\beta(P, P')(\mathrm{proj}_P\, x) = \mathrm{proj}_{P'}\, x'$, and so $\mathrm{proj}_P\, x \in \{\mathrm{proj}_P\, f, \beta(p) \cap P\}$ if and only if $\mathrm{proj}_{P'}\, x' \in \{\mathrm{proj}_{P'}\, f, \beta(p) \cap P'\}$ by Theorem 6.15 d). Moreover we also have $\mathrm{proj}_P\, x \in \{\mathrm{proj}_P\, f, \beta(p) \cap P\}$ if and only if $\mathrm{proj}_P\, q \in \{\mathrm{proj}_P\, f, \beta(p) \cap P\}$. Therefore the claim is proved.

Hence we have shown that g is a codistance. We now show that g is s-adjacent to f.

Let $P \in \mathcal{P}_s^{\mathrm{op}}(f)$. Then P contains chambers in f^{op} and one chamber p with $f(p) = s$. Obviously $P \in \mathcal{P}_{s,p}^{\mathrm{op}}(f)$, hence $\mathrm{proj}_P\, p = \mathrm{proj}_P\, f = p$ and so $g(p) = sf(p) = 1_W$. Hence $p \in g^{\mathrm{op}}$ and $P \in \mathcal{P}_s^{\mathrm{op}}(g)$. Let P be a s-panel not in $\mathcal{P}_s^{\mathrm{op}}(f)$. Then $f(x) \in \{w, ws\}$ for $x \in P$ with $s \neq w \neq 1$. Hence $g(x) \in \{w, sw, ws, sws\}$ for $x \in P$. Since $1_W \notin \{w, sw, ws, sws\}$, no chamber of P is in g^{op}, and so $P \notin \mathcal{P}_s^{\mathrm{op}}(g)$. This proves that $f \sim_s g$.

Finally, let $P \in \mathcal{P}_s^{\mathrm{op}}(f)$. Then for any $c \in P$, $P \in \mathcal{P}_{s,c}^{\mathrm{op}}(f)$, therefore $g(c) = f(c)$ unless $c \in \{\mathrm{proj}_P\, f, \beta(p) \cap P\}$. Hence the only chamber of P with g-value s is $\beta(p) \cap P$. $\square$

Proposition 7.9.: *Let $\mathcal{B}$ be a 3-spherical building of type (W, S) satisfying* (lco) *and* (lsco). *Let $J \subseteq S$ be spherical, and let f be a codistance on $\mathcal{B}$. Let R be a J-residue of $\mathcal{B}$ in f^{op} and let $\tilde{R}$ be the J-residue containing f in $\mathcal{B}^*$. Then $\alpha : \tilde{R} \to R : g \to \mathrm{proj}_R\, g$ is a bijection such that:*
(i) $\forall g_1, g_2 \in \tilde{R}, s \in J$, we have $g_1 \sim_s g_2$ if and only if $\alpha(g_1) \sim_{r_J s r_J} \alpha(g_2)$,
(ii) $\forall g \in \tilde{R}, c \in R$, we have $c \in g^{\mathrm{op}}$ if and only if $\delta(\alpha(g), c) = r_J$.

Proof: By Lemma 7.6, α is injective. Let $d = \mathrm{proj}_R\, f$. We will show by induction on the numerical distance $l(x, d)$ that $x \in \alpha(\tilde{R})$. First notice that $\alpha(f) = d$, so $x \in \alpha(\tilde{R})$ if $l(x, d) = 0$. Suppose we have proved that $x \in \alpha(\tilde{R})$ for all x satisfying $l(x, d) < l$ and suppose $(ly, d) = l$. Let $y = y_0, y_1, \ldots, y_l = d$ be a minimal gallery. By hypothesis, there exists $g_1 \in \tilde{R}$ with $\alpha(g_1) = y_1$. Let T be the t-panel containing y_0 and y_1 for some $t \in S$. Let $s = \mathrm{op}_J(t) = r_J t r_J \in J$. By Lemma 7.5, R is in g^{op} for any $g \in \tilde{R}$ and so in particular for g_1. Therefore there exists $c \in R \cap g_1^{\mathrm{op}}$ and the s-panel P containing c is in $\mathcal{P}_s^{\mathrm{op}}(g_1)$. By construction P and T are opposite and hence parallel. Let $p := \mathrm{proj}_P\, y$. Using Theorem 7.8, we can construct a codistance g which is s-adjacent to g_1 with $\mathrm{proj}_P\, g = p$. By Lemma 7.4, $\mathrm{proj}_R\, g$ and $\mathrm{proj}_R\, g_1$ are t-adjacent, and so $\mathrm{proj}_R\, g \in T$. Since $\mathrm{proj}_P\, g = \mathrm{proj}_P\, \mathrm{proj}_R\, g$ by Lemma 7.2, we must have $\mathrm{proj}_R\, g = y$. Therefore $\alpha(g) = y$ and α is surjective.

By Lemma 7.4, if g_1 and g_2 are s-adjacent in $\tilde{R}$, then $\mathrm{proj}_R\, g_1$ and $\mathrm{proj}_R\, g_2$ are $r_J s r_J$-adjacent in R.

Now assume g_1 and g_2 are codistances in $\tilde{R}$ with $\mathrm{proj}_R\, g_1 \sim_{r_J s r_J} \mathrm{proj}_R\, g_2$ for some $s \in J$. Let P be the $r_J s r_J$-panel containing them and put $e := \mathrm{proj}_P\, d$. As α is surjective, there exists $g \in \tilde{R}$ with $\alpha(g) = e$. We have shown above that there exist codistances g_1' and g_2', both s-adjacent to g, with $\mathrm{proj}_P\, g_1' = \mathrm{proj}_P\, g_1$ and $\mathrm{proj}_P\, g_2' = \mathrm{proj}_P\, g_2$. By the injectivity of α, $g_1' = g_1$ and $g_2' = g_2$, and so g_1 and g_2 are both s-adjacent to g. Since $\mathcal{B}^*$ is a chamber system, this means $g_1 \sim_s g_2$. This proves (i).

We now prove (ii). Let $g \in \tilde{R}$. By Lemma 3.3, for all $c \in R$, $g(c) = g(\alpha(g))\delta(\alpha(g), c)$. Since $R \in g^{\mathrm{op}}$ as noticed above, g takes on R its values in W_J, and so $g(\alpha(g)) = r_J$. Hence $c \in g^{\mathrm{op}}$ if and only if $g(c) = 1_W$ if and only if $\delta(\alpha(g), c) = r_J$. □

Corollary 7.10.: *The chamber system $\mathcal{B}^*$ has the same diagram as $\mathcal{B}$.*

Proof. Let M be the diagram of $\mathcal{B}$, which mean that each rank 2 J-residue is a generalized M_J-gon. Let $\tilde{R}$ be a J-residue of rank 2 of $\mathcal{B}^*$. Let g be a codistance in $\tilde{R}$ and let R be a J-residue in g^{op}. Then, by Proposition 7.9, $\tilde{R}$ is a building of the same type as R, hence a generalized M_J-gon. Therefore $\mathcal{B}^*$ has diagram M. □

8 Construction of the twinning

In order to construct a twinning we apply the main result of [Mu98] which we recall below and whose statement requires some preparation.

Let (W, S) be a Coxeter system and let $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+), \mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be two buildings of type (W, S). An *opposition relation* between $\mathcal{B}_+$ and $\mathcal{B}_-$ is a non-empty subset $\mathcal{O}$ of $\mathcal{C}_+ \times \mathcal{C}_-$ such that there exists a twinning δ_* of $\mathcal{B}_+$ and $\mathcal{B}_-$ with the property that $\mathcal{O} = \{(x, y) \in \mathcal{C}_+ \times \mathcal{C}_- \mid \delta_*(x, y) = 1_W\}$.

A *local opposition relation* between $\mathcal{B}_+$ and $\mathcal{B}_-$ is a non-empty subset $\mathcal{O}$ of $\mathcal{C}_+ \times \mathcal{C}_-$ such that for each $(x, y) \in \mathcal{O}$ and each subset $J \subseteq S$ of cardinality at most 2 the set $\mathcal{O} \cap (R_J(x) \times R_J(y))$ is an opposition relation between the J-residues of x and y. Note that the definition of a local opposition relation makes perfect sense for two chamber systems of type (W, S) as well.

Here is the main result of [Mu98].

Theorem 8.1.: *Let (W, S) be a Coxeter system and let $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+), \mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be two thick buildings of type (W, S) and let $\mathcal{O}$ be a non-empty subset of $\mathcal{C}_+ \times \mathcal{C}_-$. Then $\mathcal{O}$ is an opposition relation between $\mathcal{B}_+$ and $\mathcal{B}_-$ if and only if it is a local opposition relation between the two buildings.*

The following corollary of the previous theorem has been proved in [Mu99, p.28]. We paraphrase that proof here.

Corollary 8.2.: *Let (W, S) be a Coxeter system and let $(\mathcal{C}_+, (\sim_s)_{s \in S}), (\mathcal{C}_-, (\sim_s)_{s \in S})$ be two connected, thick chamber systems of type (W, S) whose universal 2-covers are buildings. Suppose that there exists a local opposition relation $\mathcal{O} \subseteq (\mathcal{C}_+ \times \mathcal{C}_-)$ between them. Then the chamber systems are buildings. In particular, there exist unique distances $\delta_+ : \mathcal{C}_+ \times \mathcal{C}_+ \to W$, $\delta_- : \mathcal{C}_- \times \mathcal{C}_- \to W$ and $\delta_* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$ such that $((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta_*)$ is a twin building of type (W, S) and such that $\mathcal{O} = \{(x, y) \in \mathcal{C}_+ \times \mathcal{C}_- \mid \delta_*(x, y) = 1_W\}$.*

Proof: Let $\overline{\mathcal{B}}_\epsilon = (\overline{\mathcal{C}}_\epsilon, (\sim_s)_{s\in S})$ be the universal 2-cover of $(\mathcal{C}_\epsilon, (\sim_s)_{s\in S})$, which is a building by hypothesis, with covering morphism $\phi_\epsilon : \overline{\mathcal{C}}_\epsilon \to \mathcal{C}_\epsilon$, for $\epsilon = +, -$. Let $\overline{\mathcal{O}} = \{(x,y) \in \overline{\mathcal{C}}_+ \times \overline{\mathcal{C}}_- | (\phi_+(x), \phi_-(y) \in \mathcal{O}\}$. Obviously $\overline{\mathcal{O}}$ is a local opposition relation between $\overline{\mathcal{B}}_+$ and $\overline{\mathcal{B}}_-$. By the previous theorem, this means that $\overline{\mathcal{O}}$ is the opposition relation of a twin building $(\overline{\mathcal{B}}_+, \overline{\mathcal{B}}_-, \overline{\delta}_*)$.

Let $\overline{x} \neq \overline{y} \in \overline{\mathcal{C}}_-$. By Lemma 2.6, $\overline{x}^{\mathrm{op}} \neq \overline{y}^{\mathrm{op}}$. Hence there exists $\overline{z} \in \overline{\mathcal{C}}_+$ such that $(\overline{z}, \overline{x}) \in \overline{\mathcal{O}}$ but $(\overline{z}, \overline{y}) \notin \overline{\mathcal{O}}$. If $\phi_-(\overline{x}) = v = \phi_-(\overline{y})$, then we have both $(\phi_+(\overline{z}), v) \in \mathcal{O}$ and $(\phi_+(\overline{z}), v) \notin \mathcal{O}$, a contradiction. This shows that ϕ_- is injective and hence is the identity. The same argument shows that ϕ_+ is the identity. Therefore $\overline{\mathcal{C}}_\epsilon = \mathcal{C}_\epsilon$ for $\epsilon = +, -$ and the result follows. $\square$

In order to apply the corollary above we need the following lemma.

Lemma 8.3.: *Let $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+), \mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be two buildings of spherical type (W,S), let $r \in W$ be the longest element in W and let $\mathcal{O}$ be a non-empty subset of $\mathcal{C}_+ \times \mathcal{C}_-$. Then the following are equivalent.*

a) $\mathcal{O}$ is an opposition relation between $\mathcal{B}_+$ and $\mathcal{B}_-$.

b) There exists a bijection $\alpha : \mathcal{C}_+ \to \mathcal{C}_-$ such that the following two conditions are satisfied:

(i) For all $x, y \in \mathcal{C}_+$ and all $s \in S$ we have $x \sim_s y$ if and only if $\alpha(x) \sim_{rsr} \alpha(y)$;

(ii) $\mathcal{O} = \{(x,y) \in \mathcal{C}_+ \times \mathcal{C}_- \mid \delta_-(\alpha(x), y) = r\}$.

Proof: Suppose $\mathcal{O}$ is an opposition relation between $\mathcal{B}_+$ and $\mathcal{B}_-$. Then there exists a twinning $\delta_* : \mathcal{C}_+ \times \mathcal{C}_- \to W$ inducing the opposition relation $\mathcal{O}$. Let x be a chamber in $\mathcal{C}_+$ and let $f_x : \mathcal{C}_- \to W$ be defined by $f_x(y) := \delta_*(x,y)$. Then f is a codistance on $\mathcal{B}_-$ and as $\mathcal{B}_-$ is spherical, $\mathrm{proj}_{\mathcal{C}_-} f_x$ makes sense. It is the unique chamber in $\mathcal{C}_-$ at codistance r to x, where r denotes the longest element in W. Define $\alpha : \mathcal{C}_+ \to \mathcal{C}_-$ by $\alpha(x) = \mathrm{proj}_{\mathcal{C}_-} f_x$. One checks that α is a bijection and satisfies (i) and (ii).

Now suppose there exists a bijection α satisfying (i) and (ii). We define a mapping δ_* from $(\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+)$ into W by $\delta(x,y) := r\delta_-(\alpha(x), y)$ and $\delta(y,x) := \delta(x,y)^{-1}$, for $x \in \mathcal{C}_+$ and $y \in \mathcal{C}_-$. Using the axioms of buildings, it can easily be checked that δ_* is a twinning. Moreover $\mathcal{O} = \{(x,y) \in \mathcal{C}_+ \times \mathcal{C}_- \mid \delta_*(x,y) = 1_W\}$, so $\mathcal{O}$ is an opposition relation between $\mathcal{B}_+$ and $\mathcal{B}_-$.

$\square$

End of the proof of the main result

Let $\mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be a thick building of type (W,S) satisfying all necessary properties and let $f : \mathcal{C}_- \to W$ be a codistance.

Consider the chamber system of all codistances of $\mathcal{B}_-$ which is a chamber system over S. Let $\mathcal{C}_+$ be the connected component containing f and consider the chamber system

$(\mathcal{C}_+, (\sim_s)_{s\in S})$ which is a connected chamber system of type (W, S) by Corollary 7.10. It readily follows from Proposition 7.9 that all J-residues of rank at most 3 are spherical buildings and in particular that $(\mathcal{C}_+, (\sim_s)_{s\in S})$ is thick . By a result of Tits [Ti81] it follows that the universal 2-cover of this chamber system is a building.

We define $\mathcal{O} \subseteq \mathcal{C}_+ \times \mathcal{C}_-$ by setting $\mathcal{O} := \{(g, c) \in \mathcal{C}_+ \times \mathcal{C}_- \mid g(c) = 1_W\}$. Using Lemma 8.3 and Proposition 7.9 we see that $\mathcal{O}$ is a local opposition between the chamber systems $(\mathcal{C}_+, (\sim_s)_{s\in S})$ and $(\mathcal{C}_-, (\sim_s)_{s\in S})$, which are both thick chamber systems of type (W, S) whose universal covers are buildings. Therefore, Corollary 8.2 yields the twin building.

Now we have $f' := \delta_*(f, .)$ is a codistance on $\mathcal{B}_-$ with

$$f'^{\mathrm{op}} = \{c \in \mathcal{C}_- | \delta_*(f, c) = 1_W\} = \{c \in \mathcal{C}_- | (f, c) \in \mathcal{O}\} = \{c \in \mathcal{C}_- | f(c) = 1_W\} = f^{\mathrm{op}}.$$

By Lemma 3.10, we have $f' = f$ and so $\delta_*(f, x) = f(x)$ for all $x \in \mathcal{C}_-$.

9 Remarks on the conditions in the main result

The purpose of this section is to provide some additional information about the conditions on the buildings in our main result. In the discussion below we always assume the buildings are of irreducible type which is not a serious restriction, because the general case can be reduced to the irreducible case.

3-sphericity: If we drop the 3-sphericity condition (together with conditions (lco) and (lsco)), the conclusion of our main result is not always true. Indeed it is fairly easy to construct examples of buildings admitting a codistance which cannot be realized as a 'half of a twin building'. For instance, it is a trivial fact that each thick building $\mathcal{B}_-$ of type $\tilde{A}_1$ admits a codistance f. Moreover, it can be shown that $\mathcal{B}_-$ can be realized as a 'half of a twin building' if and only if panels of the same type have the same cardinality (see [AB99], [RT99]).

It is an interesting question to wonder which buildings admitting a codistance can or cannot be realized as a 'half of a twin building'. It is most likely that all right-angled buildings admit a codistance, and that they can be realized as a 'half of a twin building' if and only if panels of the same type have the same cardinality. If there are finite entries different from 2 in the diagram, the question becomes more delicate. Nevertheless, we expect a behavior similar to the case of right-angled buildings if there are 'enough' infinities in the diagram. Hence, for the conclusion of our main result to hold, it is natural to assume that the diagram is *2-spherical* (i.e. there are no infinities in the diagram), in which case panels of the same type always have the same cardinality. By the following remarks, the conditions asked in addition to 3-sphericity are 'almost always' satisfied and therefore it remains to consider 2-spherical buildings which are not 3-spherical. We have no idea about what to expect in this case. On the one hand, the methods used in the proof of our main result completely fail in this more general context. On the other hand we could not manage to construct counter-examples in the $\tilde{A}_2$-case — a case which is well understood in a lot of respects.

Condition (lco): By the 3-sphericity assumption, all entries in the diagram are equal to 2, 3 or 4, if the rank is at least 3. It follows from an observation of Cuypers, see [Br93], that Condition (lco) is satisfied if there is no rank 2 residue isomorphic to the building associated with $B_2(2)$. In particular, Condition (lco) is satisfied if the diagram is simply laced.

Condition (lsco): It follows from [Ti86] Corollaire 2 that Condition (lsco) is satisfied if the diagram is simply laced and if each panel contains at least 4 chambers. If there are subdiagrams of type B_2 we have to consider buildings of type B_3. For those the relevant results concerning Condition (lsco) may be found in [Ab96]. They imply that Condition (lsco) is satisfied if each residue of type B_3 comes from an embeddable polar space and if each panel contains at least 17 chambers. The first condition is equivalent to the fact that any A_2-residue corresponds to a Desarguesian projective plane, and it is very likely that it can be dropped. Moreover, it is expected that the bound 17 is not optimal.

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