

# Characterizations of Segre Varieties

J. A Thas                  H. Van Maldeghem

## Abstract

In this paper several characterizations of Segre varieties and their projections are given. The first two characterization theorems are in terms of (projections of) the subspaces of the Segre variety; the third characterization is in terms of the 3-spaces intersecting the given set in hyperbolic quadrics. The first two theorems are over a finite field, the last theorem over an arbitrary field.

Keywords: Segre variety, Segre pair, Segre geometry, projective space, embedding

## 1 Introduction

Let  $\Sigma$  and  $\tilde{\Sigma}$  be two sets of  $q + 1$  mutually disjoint lines of  $\text{PG}(3, q)$ , such that any line of  $\Sigma$  intersects any line of  $\tilde{\Sigma}$ . Then it is well known that  $\Sigma$  and  $\tilde{\Sigma}$  are the sets of generators of a hyperbolic quadric of  $\text{PG}(3, q)$ . The problem is whether or not this can be generalized to systems of  $m$ - and  $n$ -dimensional subspaces of  $\text{PG}(d, q)$ . Since Segre varieties of pairs of projective spaces contain two such sets of subspaces, they provide generalizations. But also suitable projections of Segre varieties contain such systems of subspaces.

In this paper we first introduce *Segre pairs with parameters*  $(q; m, n, d)$ , consisting of a set of  $m$ -dimensional subspaces of  $\text{PG}(d, q)$  and a set of  $n$ -dimensional subspaces of  $\text{PG}(d, q)$  which satisfy similar properties as the two sets of generators of a hyperbolic quadric of  $\text{PG}(3, q)$ .

In the First Main Theorem it is shown that for  $d \geq mn + m + n$  each Segre pair generating  $\text{PG}(d, q)$  arises from a Segre variety generating  $\text{PG}(d, q)$  and hence  $d = mn + m + n$ . So for  $d \geq 3$  each Segre pair with parameters  $(q; 1, 1, d)$  arises from a hyperbolic quadric in  $\text{PG}(3, q)$ .

In the second part, conditions are given under which the union of the subspaces of  $\Sigma$  and  $\tilde{\Sigma}$ , with  $(\Sigma, \tilde{\Sigma})$  a Segre pair, is the projection of a Segre variety. Here, there is no restriction on  $d$ , but other conditions are assumed.

The last part of the paper is again a characterization of projections of Segre varieties, but this time in projective spaces over any field  $\mathbb{K}$ . Here one starts from a spanning set

$X$  of  $\text{PG}(N, \mathbb{K})$ ,  $\mathbb{K}$  any field, and a family of 3-dimensional subspaces intersecting  $X$  in hyperbolic quadrics. It is remarkable that a similar characterization exists for projections of quadric and Hermitian Veroneseans, replacing the hyperbolic quadrics with ovals and ovoids, respectively [4, 5].

Finally it should be noticed that the proofs of the three theorems rely on characterizations of Segre varieties by Herzer [1], see also Melone and Olanda [3] and Zanella [6]. Herzer [1] also starts from what we will call below a Segre pair, but adds an extra condition, called *linearity* (see the proof below of our First Main Result). Our first theorem gets rid of this extra assumption in the finite case if the dimension of the ambient space is large enough. This answers Problemi 2.13.2 in [1] negatively in this case (in our notation, Problemi 2.13.2 in [1] asks whether there are Segre pairs which do not satisfy the linearity condition).

## 2 Segre pairs and Segre varieties

Let  $d, m, n$  be three positive integers and  $q > 1$  a prime power. Let  $\Sigma$  and  $\tilde{\Sigma}$  be two families of subspaces of  $\text{PG}(d, q)$ , where  $\Sigma$  contains exactly  $q^n + q^{n-1} + \dots + q + 1$  subspaces of dimension  $m$ , and  $\tilde{\Sigma}$  contains  $q^m + q^{m-1} + \dots + q + 1$  subspaces of dimension  $n$ . Suppose that each member of  $\Sigma$  intersects each member of  $\tilde{\Sigma}$  in precisely one point, that any two distinct members of  $\Sigma$ , respectively  $\tilde{\Sigma}$ , are disjoint, and that both families generate  $\text{PG}(d, q)$ . We call  $(\Sigma, \tilde{\Sigma})$  a *Segre pair* with parameters  $(q; m, n, d)$ .

Consider coordinate systems of the projective spaces  $\text{PG}(n, q)$ ,  $\text{PG}(m, q)$  and  $\text{PG}(nm + n + m, q)$ , where we write the coordinate tuples of the points of  $\text{PG}(n, q)$  as column matrices, those of  $\text{PG}(m, q)$  as row matrices, and those of  $\text{PG}(nm + n + m, q)$  as  $(n + 1) \times (m + 1)$ -matrices. In practice,  $n$  is allowed to be equal to  $m$ , in which case we still view  $\text{PG}(n, q)$  and  $\text{PG}(m, q)$  as distinct projective spaces (this does not cause any notational confusion). With a pair of points  $(p_1, p_2)$ , where  $p_1 \in \text{PG}(n, q)$  and  $p_2 \in \text{PG}(m, q)$ , we can associate a point  $p_1 * p_2$  of  $\text{PG}(nm + n + m, q)$  by simply multiplying the column matrix associated with  $p_1$  with the row matrix associated with  $p_2$ .

Then we denote the *Segre variety* of the pair of projective spaces  $(\text{PG}(n, q), \text{PG}(m, q))$  by  $\mathfrak{S}_{n,m}(q)$ , i.e.,  $\mathfrak{S}_{n,m}(q)$  is the subset of points  $p_1 * p_2$  of  $\text{PG}(nm + n + m, q)$ , with  $p_1 \in \text{PG}(n, q)$  and  $p_2 \in \text{PG}(m, q)$ , see [2].

If  $p_1$  is fixed and  $p_2$  varies, then the set of corresponding points  $p_1 * p_2$  is an  $m$ -dimensional subspace of  $\text{PG}(nm + n + m, q)$ ; if  $p_2$  is fixed and  $p_1$  varies, then there arises an  $n$ -dimensional subspace of  $\text{PG}(nm + n + m, q)$ . This way there arise families of  $m$ -spaces and  $n$ -spaces entirely contained in the Segre variety  $\mathfrak{S}_{n,n}(q)$  and forming a Segre pair (if  $n = m$ , then the Segre pair consists of two families of  $n$ -spaces each consisting of  $q^n + q^{n-1} + \dots + q + 1$  mutually disjoint elements); these families are the so-called *families of generators* of the variety.

Our First Main Result says that, under a mild but obviously necessary condition, also the converse is true, and hence one obtains a combinatorial characterization of (finite) Segre varieties.

**First Main Result.** *Let  $q$  be a prime power and let  $d \geq mn + m + n$ . Then the set of points covered by a Segre pair with parameters  $(q; m, n, d)$  is a Segre variety (and hence  $d = mn + m + n$ ).*

Our Second Main Result also characterizes appropriate projections of (finite) Segre varieties.

**Second Main Result.** *Let  $(\Sigma, \tilde{\Sigma})$  be a Segre pair with parameters  $(q; m, n, d)$ , where  $m, n, d \in \mathbb{N}$  and  $q$  a prime power. Further, assume that*

- (i) *there is a set  $\mathcal{L}$  whose elements are subsets of size  $q + 1$  of  $\Sigma$ , and such that  $\Sigma$  provided with  $\mathcal{L}$  is the design of points and lines of an  $n$ -dimensional projective space  $\Delta$  of order  $q$ ;*
- (ii) *each element of  $\mathcal{L}$  is the family of  $m$ -dimensional generators of a Segre variety  $\mathfrak{S}_{m,1}(q)$  (a family of generators if  $m = 1$ );*
- (iii) *the common points of an element of  $\tilde{\Sigma}$  and the  $q + 1$  spaces of an element of  $\mathcal{L}$  are collinear, that is, they form a 1-dimensional generator of the corresponding Segre variety  $\mathfrak{S}_{m,1}(q)$ .*

*Then the set of points covered by the Segre pair is the projection of some Segre variety of  $\text{PG}(mn + m + n, q) \supseteq \text{PG}(d, q)$  onto  $\text{PG}(d, q)$  from some subspace  $\text{PG}(mn + m + n - d - 1, q)$  of  $\text{PG}(mn + m + n, q)$  skew to  $\text{PG}(d, q)$ . Also,  $\Sigma$  and  $\tilde{\Sigma}$  are the projections of the families of generators of the Segre variety.*

Our Third Main Result characterizes projections of not necessarily finite Segre varieties. We first need some preliminaries.

A *hypo*  $H$  in a 3-dimensional projective space  $\Sigma$  is the set of points of  $\Sigma$  on some hyperbolic quadric. For every point  $x \in H$ , the unique plane  $\pi$  through  $x$  intersecting  $H$  in two intersecting lines contains all lines through  $x$  that meet  $H$  only in  $x$ . The plane  $\pi$  is called the *tangent plane* at  $x$  to  $H$  and denoted  $T_x(H)$ .

Note that, in order for  $H$  to exist, the underlying field of  $\Sigma$  has to be abelian.

Let  $X$  be a spanning point set of  $\text{PG}(N, \mathbb{K})$ , with  $\mathbb{K}$  any field and  $N \in \mathbb{N}$ , and let  $\Xi$  be a nonempty collection of 3-dimensional projective subspaces of  $\text{PG}(N, \mathbb{K})$ , called the *hyperbolic spaces* of  $X$ , such that, for any  $\xi \in \Xi$ , the intersection  $\xi \cap X$  is a hypo  $X(\xi)$  in  $\xi$ ; then, for  $x \in X(\xi)$ , we sometimes denote  $T_x(X(\xi))$  simply by  $T_x(\xi)$ . We say that two points of  $X$  are *collinear in  $X$*  if all points of the joining line belong to  $X$ . We call  $X$  a *Segre geometry* if the following properties hold :

- (S1) Any two points  $x$  and  $y$  in  $X$  lie in an element of  $\Xi$ , denoted by  $[x, y]$  if it is unique.

(S2) If  $\xi_1, \xi_2 \in \Xi$ , with  $\xi_1 \neq \xi_2$ , then  $\xi_1 \cap \xi_2 \subseteq X$ .

From (S2) it follows that the element of  $\Xi$  in (S1) is unique as soon as  $x$  and  $y$  are not collinear in  $X$ .

(S3) If  $x \in X$  and  $L$  is a line entirely contained in  $X$  such that no point on  $L$  is collinear in  $X$  with  $x$ , then the planes  $T_x([x, y])$ ,  $y \in L$ , are contained in a common 3-dimensional subspace of  $\text{PG}(N, \mathbb{K})$ , denoted by  $T(x, L)$ .

**Third Main Result.** *A Segre geometry is the projection of a Segre variety.*

### 3 Proof of the First Main Result

Let  $(\Sigma, \tilde{\Sigma})$  be a Segre pair with parameters  $(q; m, n, d)$ , with  $d \geq mn + m + n$ .

First we note that, by the numbers, the set of points covered by  $\Sigma$  coincides with that covered by  $\tilde{\Sigma}$ , which means that each point that lies on a member of  $\Sigma$  is contained in a unique member of  $\tilde{\Sigma}$ .

We will need the following number theoretic result. Denote, for  $i$  a nonnegative integer,  $g_i := \frac{q^i - 1}{q - 1}$ . Note that  $g_0 = 0$ . Let  $s_2, s_3, \dots, s_t$ ,  $t \geq 2$ , be nonnegative integers and let  $m$  be an integer not smaller than any  $s_i$ ,  $i = 2, 3, \dots, t$ . Let  $C = \sum_{i=2}^t s_i$ . Then we claim that

$$\sum_{i=2}^t g_{s_i} \leq \frac{C}{m} g_m.$$

Indeed, let  $s$  and  $s'$  be integers with  $1 \leq s \leq s' < m$ . From  $0 < (q - 1)(q^{s'} - q^{s-1})$  follows  $q^s + q^{s'} < q^{s-1} + q^{s'+1}$ , and this implies  $g_s + g_{s'} < g_{s-1} + g_{s'+1}$ . If we apply this over and over again to the sum  $\sum_{i=2}^t g_{s_i}$  and replace each time  $\{s, s'\}$  by  $\{s - 1, s' + 1\}$  in the multiset  $\{s_2, s_3, \dots, s_t\}$ , then  $C$  remains constant and hence, if  $C = am + b$ , with  $0 \leq b < m$ , then we can push exactly  $a$  of the  $s_i$  to  $m$ , one other to  $g_b$  and the others will be pushed to 0. After this manipulation, we obtain

$$\sum_{i=2}^t g_{s_i} \leq ag_m + g_b.$$

Now noting  $g_b \leq \frac{b}{m} g_m$  leads to our claim.

Now, since  $d \geq mn + m + n = (m + 1)(n + 1) - 1$ , we can pick members  $\Pi_1, \Pi_2, \dots, \Pi_{n+1} \in \Sigma$  such that for each  $i = 1, 2, \dots, n$ , the space  $\Pi_{i+1}$  is not contained in the space  $\Gamma_i$  generated by  $\Pi_1, \Pi_2, \dots, \Pi_i$ . Also, denote  $\Gamma_{n+1} := \langle \Gamma_n, \Pi_{n+1} \rangle$ . We claim that  $\Gamma_{n+1}$  is equal to  $\text{PG}(d, q)$ .

Indeed, if not, we can find  $\Pi_{n+2}, \dots, \Pi_t \in \Sigma$ ,  $t \geq n+2$ , such that  $\Pi_{j+1}$  is not contained in the space  $\Gamma_j$  generated by  $\Gamma_n, \Pi_{n+1}, \Pi_{n+2}, \dots, \Pi_j$ ,  $t-1 \geq j \geq n+1$ , and such that  $\langle \Gamma_{t-1}, \Pi_t \rangle = \text{PG}(d, q)$ . Put  $s_i = \dim(\Gamma_{i-1} \cap \Pi_i) + 1$  for  $2 \leq i \leq t$ . Then  $s_2 = 0$  and  $s_i \leq m$  for  $i \in \{3, \dots, t\}$ . Also,  $d+1 = t(m+1) - \sum_{i=2}^t s_i$  and by assumption  $e := t - (n+1) > 0$ .

Let  $x \in \Pi_1$  be arbitrary, let  $\tilde{\Pi}$  be the element of  $\tilde{\Sigma}$  containing  $x$ , and put  $\{x_i\} = \tilde{\Pi} \cap \Pi_i$ , for  $i = 1, 2, \dots, t$ . Then  $x_1 = x$  and  $\langle x_1, \dots, x_i \rangle \subseteq \Gamma_i$ ,  $i = 1, 2, \dots, t$ . If for  $i \geq 2$ ,  $x_i$  is not contained in  $\Gamma_{i-1}$ , then  $\dim\langle x_1, \dots, x_i \rangle = 1 + \dim\langle x_1, \dots, x_{i-1} \rangle$ . As  $\dim\langle x_1, x_2 \rangle = 1$  and  $\dim\langle x_1, \dots, x_t \rangle \leq n$ , it follows that at least  $e$  indices  $i \in \{3, \dots, t\}$  satisfy  $x_i \in \Gamma_{i-1} \cap \Pi_i$ . As this holds for every point  $x \in \Pi_1$ , and as the elements of  $\tilde{\Sigma}$  are disjoint, it follows that  $e|\Pi_1| \leq \sum_{i=2}^t g_{s_i}$  (with above notation). Now  $s_i \leq m$ , for all  $i$ , and  $\sum_{i=2}^t s_i \leq e(m+1)$ , so that the number theoretic result above implies

$$e \frac{q^{m+1} - 1}{q - 1} = e|\Pi_1| \leq \sum_{i=2}^t g_{s_i} \leq \frac{e(m+1)}{m} g_m = e \frac{(m+1)(q^m - 1)}{m(q-1)},$$

implying  $m(q-1)q^m \leq q^m - 1$ , clearly a contradiction in view of  $m \geq 1$  and  $q \geq 2$ . Hence  $\Gamma_{n+1}$  is equal to  $\text{PG}(d, q)$

A dimension argument now implies that  $\Pi_{i+1}$  is disjoint from  $\Gamma_i$ , for all  $i = 1, 2, \dots, n$ . In particular, by the arbitrary choices of the  $\Pi_{i+1}$  outside  $\Gamma_i$ , we see that, whenever some member  $\Pi' \in \Sigma$  meets the space generated by an arbitrary subset of  $\Sigma$  in at least one point, it is contained in it. We also deduce that  $d = mn + m + n$ . Let  $\tilde{\Pi}$  be an arbitrary element of  $\tilde{\Sigma}$ . Let  $d_i$  be the dimension of the intersection of  $\tilde{\Pi}$  with  $\Gamma_i$ ,  $i = 1, 2, \dots, n+1$ . Then, since  $\tilde{\Pi}$  meets  $\Pi_j$  in a point, and  $\Pi_j$  is disjoint from  $\Gamma_{j-1}$ , for all  $j \in \{2, 3, \dots, n+1\}$ , we see that  $0 = d_1 < d_2 < \dots < d_n < d_{n+1} = n$ . Hence  $d_i = i - 1$ . In particular each element of  $\tilde{\Sigma}$  intersects  $\Gamma_2$  in a line, and so the  $q+1$  elements of  $\Sigma$  in  $\Gamma_2$  form a family of generators of a Segre variety  $\mathfrak{S}_{m,1}(q)$ .

Now let  $\tilde{\Pi} \in \tilde{\Sigma}$  be arbitrary and let  $\{x_1, x_2, \dots, x_{n+1}\}$  be a generating set of points of  $\tilde{\Pi}$ . We can now choose spaces  $\Pi_i \in \Sigma$  such that  $\Pi_i$  contains  $x_i$ ,  $i = 1, 2, \dots, n+1$ . Since every member of  $\Sigma$  meets  $\tilde{\Pi}$  in a point, it is contained in the span of all  $\Pi_i$ ,  $i = 1, 2, \dots, n+1$ . Our previous paragraph implies that this span is  $\text{PG}(d, q)$ , with  $d = mn + m + n$ . Hence, for every  $i = 1, 2, \dots, n+1$ , we have that  $\Pi_i$  is disjoint from the subspace generated by the other  $\Pi_j$ 's.

Now let  $\Pi, \Pi', \Pi''$  be three members of  $\Sigma$  meeting  $\tilde{\Pi}$  in collinear points  $y, y', y''$ , respectively. Hence these three subspaces are contained in a subspace of dimension  $2m+1$ . If  $\Pi, \Pi', \Pi''$  would meet another member of  $\tilde{\Sigma}$  in non-collinear points, then they would generate, by a previous paragraph, a subspace of dimension  $3m+2$ , a contradiction. Hence  $\Pi, \Pi', \Pi''$  meet every other member of  $\tilde{\Sigma}$  in three collinear points. This is half of Herzer's linearity condition [1].

Hence, for two arbitrary members  $\tilde{\Pi}_1$  and  $\tilde{\Pi}_2$  of  $\tilde{\Sigma}$ , the mapping  $\tilde{\sigma} : \tilde{\Pi}_1 \rightarrow \tilde{\Pi}_2 : x_1 \mapsto x_2$ , where  $x_1$  and  $x_2$  are contained in a common member of  $\Sigma$ , is a collineation.

By interchanging the roles of  $\Sigma$  and  $\tilde{\Sigma}$ , we also have the other half of Herzer's linearity condition, and so, for two arbitrary members  $\Pi_1$  and  $\Pi_2$  of  $\Sigma$ , the mapping  $\sigma : \Pi_1 \rightarrow \Pi_2 : x_1 \mapsto x_2$ , where  $x_1$  and  $x_2$  are contained in a common member of  $\tilde{\Sigma}$ , is a collineation.

Now our First Main Result follows from Teorema 2.8 of [1], see also [3] and [6] (in the language of *product spaces*).

## 4 Proof of the Second Main Result

First we note that, by the numbers, the set of points covered by  $\Sigma$  coincides with that covered by  $\tilde{\Sigma}$ , which means that each point that lies on a member of  $\Sigma$  is contained in a unique member of  $\tilde{\Sigma}$ .

Now we pick independent elements  $\Pi_1, \Pi_2, \dots, \Pi_{n+1} \in \Sigma$  of  $\Delta$ . The subspace of  $\text{PG}(d, q)$  generated by the elements  $\Pi_1, \Pi_2, \dots, \Pi_i$  is denoted by  $\Gamma_i$ ,  $i = 1, 2, \dots, n+1$ , and the subspace of  $\Delta$  generated by these elements is denoted by  $\Sigma_{i-1}$ . As for any two elements  $\xi_1, \xi_2 \in \Sigma$  the  $q+1$  points of the line  $\xi_1\xi_2$  of  $\Delta$  are contained in the  $(2m+1)$ -dimensional subspace  $\langle \xi_1, \xi_2 \rangle$  of  $\text{PG}(d, q)$ , it follows that any  $i$  independent points of  $\Sigma_{i-1}$  generate the same subspace  $\Gamma_i$  of  $\text{PG}(d, q)$ . Clearly  $\Gamma_{n+1} = \text{PG}(d, q)$ .

Now we embed  $\text{PG}(d, q)$  in a  $\text{PG}(mn+m+n, q)$  and we construct an  $\mathfrak{S}_{m,n}(q)$  and a  $\text{PG}(mn+m+n-d-1, q)$  in  $\text{PG}(mn+m+n, q)$  skew to  $\text{PG}(d, q)$  such that the set of points covered by the Segre pair  $(\Sigma, \tilde{\Sigma})$  is the projection of  $\mathfrak{S}_{m,n}(q)$  from  $\text{PG}(mn+m+n-d-1, q)$  onto  $\text{PG}(d, q)$ .

Put  $m_i = \dim(\Gamma_i)$ ,  $i = 2, 3, \dots, n+1$ . Then  $m_i \leq mi + (i-1)$ . Let  $\Gamma'_i$  be a projective space of dimension  $mi + (i-1)$  containing  $\Gamma_i$ , with  $\Gamma'_i \cap \text{PG}(d, q) = \Gamma_i$ . Assume that  $\Phi_i$  is a subspace of  $\Gamma'_i$  skew to  $\Gamma_i$  with  $\langle \Phi_i, \Gamma_i \rangle = \Gamma'_i$ , that  $\mathfrak{S}_{m,i-1}(q)$  is a Segre variety in  $\Gamma'_i$  skew to  $\Phi_i$  and containing  $\Pi_1$  and  $\Pi_2$ , and that the set  $\Sigma_{i-1}$  is the projection of the family of generators of dimension  $m$  (a family of generators if  $m = i-1$ ) of  $\mathfrak{S}_{m,i-1}(q)$  from  $\Phi_i$  onto  $\Gamma_i$ . We have  $\dim(\Phi_i) = mi + i - 2 - m_i$ , so for  $i = 2$  this means  $\dim(\Phi_i) = \dim(\Phi_2) = -1$ . Also  $|\Sigma_{i-1}| = (q^i - 1)/(q - 1)$ .

Put  $\xi_{i+1} = \Pi_{i+1} \cap \Gamma_i$ ,  $i = 2, \dots, n$ . Then  $\dim(\xi_{i+1}) = m + m_i - m_{i+1} =: u_{i+1}$ . Choose an arbitrary subspace  $\eta_{i+1}$  in  $\Pi_{i+1}$  complementary to  $\xi_{i+1}$  and choose a  $u_{i+1}$ -dimensional subspace  $\xi'_{i+1}$  skew to  $\langle \Gamma'_i, \text{PG}(d, q) \rangle$ . Put  $\Pi'_{i+1} = \langle \xi'_{i+1}, \eta_{i+1} \rangle$  and  $\Gamma'_{i+1} = \langle \Gamma'_i, \Pi'_{i+1} \rangle$ . In the  $(2u_{i+1} + 1)$ -dimensional space  $\langle \xi_{i+1}, \xi'_{i+1} \rangle$  we choose a subspace  $\tilde{\Phi}_{i+1}$  of dimension  $u_{i+1}$ , skew to both  $\xi_{i+1}$  and  $\xi'_{i+1}$ . The projection operator from  $\langle \Phi_i, \tilde{\Phi}_{i+1} \rangle =: \Phi_{i+1}$  onto  $\Gamma_{i+1}$  is denoted by  $\delta_{i+1}$ . Then  $(\xi'_{i+1})^{\delta_{i+1}} = \xi_{i+1}$ ,  $(\Pi'_{i+1})^{\delta_{i+1}} = \Pi_{i+1}$  and  $\mathfrak{S}_{m,i-1}(q)^{\delta_{i+1}}$  is the union of the elements of  $\Sigma_{i-1}$ . We will now "lift" the  $(q^{i+1} - 1)/(q - 1)$  elements of  $\Sigma_i$  so that the lifted set becomes the family of  $m$ -dimensional generators (a family of generators if  $m = i$ ) of a  $\mathfrak{S}_{m,i}(q)$  and these  $(q^{i+1} - 1)/(q - 1)$  generators of dimension  $m$  are projected back by  $\delta_{i+1}$  to the elements of  $\Sigma_i$ . We will define this lifting pointwise, and we denote the corresponding operator with  $\lambda_{i+1}$ . By definition, each point  $x$  of  $\text{PG}(d, q)$  on some

element of  $\Sigma_{i-1}$  is lifted onto the unique point  $y$  in the union of all subspaces contained in  $\mathfrak{S}_{m,i-1}(q)$  such that  $y^{\delta_{i+1}} = x$ . Hence we can put  $x^{\lambda_{i+1}} = y$ .

The lifted elements of  $\Sigma_{i-1}$  are the generators of dimension  $m$  of  $\mathfrak{S}_{m,i-1}(q)$  (the generators belonging to one family if  $m = i - 1$ ). For a subset  $\Omega$  of  $\Sigma$ , we denote by  $S_\Omega$  the set of elements of  $\Sigma$  generated in  $\Delta$  by  $\Omega$ .

Let  $\bar{x}$  be any point of  $\bar{\Pi} \in \Sigma_{i-1}$ , and let  $\Pi$  be any element of  $S_{\{\bar{\Pi}, \Pi_{i+1}\}}$ . Further let  $L$  be the line containing  $\bar{x}$  and intersecting  $\bar{\Pi}$  and  $\Pi_{i+1}$ , and put  $L \cap \Pi = \{x\}$ . Consider the subspace

$$\langle \bar{x}^{\lambda_{i+1}}, \Pi'_{i+1} \rangle \cap \langle x, \Phi_{i+1} \rangle =: x^{\lambda_{i+1}}.$$

Since  $\langle \Phi_{i+1}, \Pi'_{i+1} \rangle = \langle \Pi_{i+1}, \Pi'_{i+1}, \Phi_i \rangle$ , and  $x$  is contained in  $\langle \bar{x}, \Pi_{i+1} \rangle$ , the subspace generated by  $\bar{x}$ ,  $\Pi'_{i+1}$ ,  $x$  and  $\Phi_{i+1}$  coincides with the subspace generated by  $\Phi_i$ ,  $\bar{x}$ ,  $\Pi_{i+1}$  and  $\Pi'_{i+1}$ . The latter subspace has dimension

$$\dim(\Phi_i) + \dim\langle \bar{x}, \Pi_{i+1}, \Pi'_{i+1} \rangle + 1,$$

so

$$mi + i - 2 - m_i + m + u_{i+1} + 3 = mi + i - m_i + m + m + m_i - m_{i+1} + 1,$$

which simplifies to  $mi + i + 2m - m_{i+1} + 1$ . As  $\langle \bar{x}, \Pi'_{i+1}, x, \Phi_{i+1} \rangle = \langle \bar{x}^{\lambda_{i+1}}, \Pi'_{i+1}, x, \Phi_{i+1} \rangle$ , the dimension of  $x^{\lambda_{i+1}}$  is equal to

$$m + 1 + m(i + 1) + i - 1 - m_{i+1} + 1 - (mi + i + 2m - m_{i+1} + 1) = 0.$$

Hence  $x^{\lambda_{i+1}}$  is a point. But clearly, the definition of  $x^{\lambda_{i+1}}$  immediately implies that  $(x^{\lambda_{i+1}})^{\delta_{i+1}} = x$ .

Hence the set  $\Pi^{\lambda_{i+1}} = \Pi'$  contains  $(q^{m+1} - 1)/(q - 1)$  distinct points and is contained in  $\langle \bar{\Pi}^{\lambda_{i+1}}, \Pi'_{i+1} \rangle \cap \langle \Pi, \Phi_{i+1} \rangle = \zeta$ . As  $\Pi$  and  $\bar{\Pi}$  are skew or coincide, the space  $\zeta$  has dimension at most  $m$ . It follows that  $\Pi' = \zeta$ , hence is an  $m$ -dimensional subspace of  $\langle \Gamma'_{i+1}, \Phi_{i+1} \rangle = \langle \Gamma_{i+1}, \Phi_{i+1} \rangle$ . The space  $\langle \Gamma_{i+1}, \Phi_{i+1} \rangle$  has dimension

$$m_{i+1} + m(i + 1) + i - 1 - m_{i+1} + 1 = mi + m + i.$$

We put  $\langle \Gamma_{i+1}, \Phi_{i+1} \rangle = \text{PG}(mi + m + i, q)$ . Also,  $\text{PG}(mi + m + i, q) = \langle \mathfrak{S}_{m,i-1}(q), \Pi'_{i+1} \rangle$ . Hence we obtain a set  $\Sigma'_i$  of  $(q^{i+1} - 1)/(q - 1)$  mutually skew subspaces  $\Pi'$  of dimension  $m$  generating  $\text{PG}(mi + m + i, q)$ .

The  $q + 1$  elements of  $S_{\langle \bar{\Pi}, \Pi_{i+1} \rangle} =: S$  are lifted by  $\lambda_{i+1}$  to  $q + 1$   $m$ -dimensional subspaces of the  $(2m + 1)$ -dimensional space  $\langle \bar{\Pi}^{\lambda_{i+1}}, \Pi'_{i+1} \rangle$ ; the set of these  $q + 1$  subspaces of  $\text{PG}(mi + m + i, q)$  is denoted by  $S'$ . Each line intersecting the  $q + 1$  elements of  $S$  is lifted to a line intersecting the  $q + 1$  elements of  $S'$ . Hence the union of the elements of  $S'$  is a Segre variety  $\mathfrak{S}_{m,1}(q)$ .

If  $\alpha_1, \alpha_2$  are any two elements of  $\Sigma_{i-1}$ , then we know that the elements of  $S_{\langle \alpha_1, \alpha_2 \rangle}$  are lifted to the elements of the family (a family if  $m = 1$ ) of  $m$ -dimensional generators of

some  $\mathfrak{S}_{m,1}(q)$ . The lines intersecting all elements of  $S_{\langle\alpha_1,\alpha_2\rangle}$  are lifted to the elements of the family (a family if  $m = 1$ ) of 1-dimensional generators of  $\mathfrak{S}_{m,1}(q)$ .

Now put  $i = 2, 3, \dots, n$  and put  $\Sigma'_n = \Sigma'$ . Let  $\tilde{\Pi} \in \tilde{\Sigma}$  and let  $L$  be any line of  $\tilde{\Sigma}$ . The elements of  $\Sigma$  intersecting  $L$  constitute a family of generators of some  $\mathfrak{S}_{m,1}(q)$ . If  $L$  intersects  $\Pi_{n+1}$ , then  $L^{\lambda_{n+1}}$  is a line. Assume now that  $L \cap \Pi_{n+1} = \emptyset$ . Let  $\Pi_{n+1} \cap \tilde{\Pi} = \{x\}$  and  $L = \{y_1, y_2, \dots, y_{q+1}\}$ . Further, let  $\gamma_i$  be the element of  $\Sigma$  containing  $y_i$ , let  $\beta_i$  be the common element of  $S_{\langle\Pi_{n+1}, \gamma_i\rangle}$  and  $\Sigma_{n-1}$ , and let  $\beta_i \cap xy_i = \{z_i\}$ , with  $i = 1, 2, \dots, q + 1$ . Then the line  $\{z_1, z_2, \dots, z_{q+1}\} = M$  is lifted by  $\lambda_{n+1}$  to a line of  $\text{PG}(mn + m + n, q)$ . It follows that the plane  $\langle x, L \rangle = \alpha$  is lifted to a plane  $\alpha'$  of  $\text{PG}(mn + m + n, q)$ . If  $L^{\lambda_{n+1}}$  were not a line, then all lines  $(xy_i)^{\lambda_{n+1}}$  would be projected from  $\Phi_{n+1}$  onto a common line of  $\text{PG}(d, q)$ , clearly a contradiction. Hence  $L^{\lambda_{n+1}}$  is a line. Consequently  $\tilde{\Pi}^{\delta_{n+1}}$  is an  $n$ -dimensional subspace of  $\text{PG}(mn + m + n, q)$  intersecting each element of  $\Sigma'$  in a point. By our First Main Result the union of the elements of  $\Sigma'$  is an  $\mathfrak{S}_{m,n}(q)$  whose families of generators are  $\Sigma'$  and the set  $\tilde{\Sigma}'$  of the elements  $\tilde{\Pi}^{\delta_{n+1}}$ . This proves our Second Main Result.

## 5 Proof of the Third Main Result

We denote by  $\mathfrak{M}$  the set of maximal subspaces entirely contained in  $X$ .

Step 1: *If two intersecting lines  $L, M$  entirely contained in  $X$  do not belong to a common hyperbolic space, then the entire plane  $\langle L, M \rangle$  belongs to  $X$ .*

Take any point  $x$  in  $\langle L, M \rangle$ ,  $x \notin L \cup M$ , and choose two distinct lines  $K_1, K_2$  through  $x$  in the plane  $\langle L, M \rangle$  and not containing the intersection  $L \cap M$ . A hyperbolic space  $\xi_i$  containing  $K_i$ ,  $i = 1, 2$ , does not contain  $\langle L, M \rangle$  by assumption, hence  $\xi_1 \neq \xi_2$  and we conclude with (S2) that  $x \in \xi_1 \cap \xi_2 \subseteq X$ .

Some consequences:

Step 1': *If a triangle of lines is contained in  $X$ , then so is the entire plane spanned by these three lines.*

A *quadrangle* (in  $X$ ) is a set of four lines  $L_1, L_2, L_3, L_4$  in  $X$  such that  $L_i$  and  $L_{i+1}$  meet for all  $i$ , subscripts modulo 4, and  $\{L_1, L_3\}$  and  $\{L_2, L_4\}$  are two sets of skew lines. Also, none of the planes  $\langle L_i, L_{i+1} \rangle$  is allowed to be contained in  $X$ .

Step 1'': *Any quadrangle of lines  $L_1, L_2, L_3, L_4$  is contained in a hyperbolic space.*

If not, then, with the notation above, the lines  $L_1, L_2$  and the lines  $L_3, L_4$  are contained in hyperbolic spaces (use Step 1), which must by assumption be different. But then the intersection of these two hyperbolic spaces is contained in  $X$ , and we obtain a triangle of lines, contradicting the definition of quadrangle.

Step 2: *If two planes  $\Pi$  and  $\Pi'$  are entirely contained in  $X$  and meet in a line  $K$ , then the solid  $\langle \Pi, \Pi' \rangle$  is entirely contained in  $X$ .*

Suppose by way of contradiction that some point  $x \in \langle \Pi, \Pi' \rangle$  is not contained in  $X$ . Let  $\alpha$  be a plane through  $x$  intersecting  $\Pi$  and  $\Pi'$  in distinct lines  $L$  and  $L'$ , respectively. By Step 1, the lines  $L$  and  $L'$  are contained in a hyperbolic space  $\xi$ . Note that  $\xi$  does not contain  $K$ . Considering a second similar plane  $\beta$  through  $x$ , we see that  $x$  is contained in the intersection of two distinct hyperbolic spaces, a contradiction.

It now follows easily that:

Step 2': *Two members of  $\mathfrak{M}$  meet in at most one point.*

Step 2'': *If a set of four lines  $L_1, L_2, L_3, L_4$  is such that  $L_i$  and  $L_{i+1}$  meet for all  $i$ , subscripts modulo 4, and  $\{L_1, L_3\}$  and  $\{L_2, L_4\}$  are two sets of skew lines, then either this is a quadrangle, or the solid generated by  $L_1, L_2, L_3, L_4$  is entirely contained in  $X$ .*

Step 3: *If three members of  $\mathfrak{M}$  meet pairwise in a point, then they all have a point in common.*

Indeed, if not, then the plane spanned by the three respective intersection points is contained in  $X$ , by Step 1, which contradicts Step 2'.

Step 4: *Let  $x \in X$  and let  $L$  be a line contained in  $X$ , such that no point on  $L$  is collinear in  $X$  with  $x$ . Then  $T(x, L) \cap X$  is the union of a plane  $\alpha$  and a line  $R$ . Moreover, the line  $R$  is contained in every hyperbolic space containing  $x$  and a point of  $L$ , and there is a point  $r \in R$  collinear in  $X$  with every point of  $L$ .*

Let  $\xi, \xi'$  be two hyperbolic spaces containing  $x$  and a point  $y, y'$ , respectively, of  $L$ . Let  $H, H'$  be the respective corresponding hypos. If  $\xi = \xi'$ , then  $L \subseteq H$  and so  $L$  contains a point collinear in  $X$  with  $x$ , a contradiction. Suppose the generators through  $x$  of  $H$  and  $H'$  are four distinct lines, say  $G$  and  $\overline{G}$  of  $H$  and  $G'$  and  $\overline{G}'$  of  $H'$ . Then no three of these generators are contained in a plane, and the line  $\langle G, \overline{G} \rangle \cap \langle G', \overline{G}' \rangle \subseteq \xi \cap \xi'$  is contained in  $X$ , a contradiction, since this line cannot belong to either  $\xi$  or  $\xi'$ . Hence  $H$  and  $H'$  have exactly one generator  $R$  in common. Let  $r$  and  $r'$  be the points on  $R$  collinear in  $X$  with  $y$  and  $y'$ , respectively. If  $r \neq r'$ , then, since  $x$  is not collinear to any point on  $L$ , the lines  $L, yr, R, y'r'$  cannot form a quadrangle in  $X$ . Step 2'' implies that  $R$  and  $L$  are contained in a solid belonging to  $X$ , again a contradiction. We conclude that  $r = r'$ . Also,  $\langle r, L \rangle$  is contained in  $X$ .

Now, since  $x$  is not collinear in  $X$  with any point on  $L$ , there is no plane containing  $R$  and a point of  $L$  which is entirely contained in  $X$ . Consequently, by Step 1, there is a hyperbolic space through  $x$  and any line  $rz$ , with  $z \in L$ . It follows that  $R$  is a generator of every hypo in  $X$  containing  $x$  and a point  $z$  of  $L$ .

Now let  $M, M', M''$  be three generators distinct from  $R$  belonging to three respective hypos  $H, H', H''$  through  $x$  and three respective points on  $L$ , and suppose that  $M, M', M''$  are not contained in a plane. If, on the one hand,  $M, M'$  were not contained in a hyperbolic space, then the line  $\langle M, M' \rangle \cap \langle H'' \rangle$  would belong to  $X$  and hence to  $H''$ . Hence  $\langle M, M' \rangle \cap \langle H'' \rangle = R$ . But then  $\langle H \rangle \cap \langle H' \rangle$  is a plane, clearly a contradiction. If, on the other hand,  $M, M'$  were contained in a hyperbolic space, then exactly the same conclusion would hold, a contradiction again.

Step 5: *Let  $x \in X$  and let  $M \in \mathfrak{M}$  with  $x \notin M$ . Then  $x$  is collinear with a unique point of  $M$ .*

By Step 2',  $x$  is collinear with at most one point of  $M$ . If  $x$  is not collinear with any point of  $M$ , then we can pick any line  $L$  inside  $M$  and apply Step 4. We thus obtain a point  $r$  collinear in  $X$  with  $x$  and with every point of  $L$ . By maximality of  $M$ , the point  $r$  belongs to  $M$ , contradicting the fact that  $x$  is not collinear in  $X$  with any point of  $M$ .

Step 6: *Let  $x \in X$  and let  $M \in \mathfrak{M}$  with  $x \notin M$ . Then  $x$  belongs to at most one member of  $\mathfrak{M}$  disjoint from  $M$ .*

Suppose, by way of contradiction, that  $M_1, M_2$  are two members of  $\mathfrak{M}$  containing  $x$  and disjoint from  $M$ . Let  $u$  be the unique point of  $M$  collinear in  $X$  with  $x$ . Choose an arbitrary point  $y$  in  $M$  different from  $u$ . Let  $z_i$  be the unique point of  $M_i$  collinear in  $X$  with  $y$ ,  $i = 1, 2$ . Then  $z_1 \neq x \neq z_2 \neq z_1$ . Since  $x$  is not collinear in  $X$  with  $y$ , the lines  $xz_i$  and  $yz_i$ ,  $i = 1, 2$ , are contained in a hyperbolic space, and so are the lines  $xu$  and  $yu$ . Since at least two of these three hyperbolic spaces must be different, this contradicts (S2).

Step 7: *"Being disjoint" is an equivalence relation in  $\mathfrak{M}$  giving rise to exactly two equivalence classes. Every point is contained in exactly one member of each equivalence class.*

The first part of the first assertion follows directly from Step 6. Suppose that there are at least three equivalence classes and let  $M, M', M''$  be three respective members of them. By Step 3 they all meet in a common point  $x$ . Let  $L$  be a generator not through  $x$  of some hypo in  $X$  through  $x$ . Then  $L$  is contained in a unique member  $N$  of  $\mathfrak{M}$  (by definition of maximal subspace). Clearly  $N$  does not contain  $x$ , and so by Step 3,  $N$  meets at most one of  $M, M', M''$ . But then this contradicts Step 6.

Since every point  $x$  is contained in at least one hypo  $H$  in  $X$ , the point  $x$  is contained in at least two members of  $\mathfrak{M}$  (defined by the two respective generators of  $H$  through  $x$ ). Since all members of  $\mathfrak{M}$  through  $x$  belong to different equivalence classes, they are at most two of them, and hence exactly two.

Step 8: *Let  $M, M' \in \mathfrak{M}$  be distinct and belong to the same equivalence class. Then the map  $\rho : M \rightarrow M' : x \mapsto x'$ , where  $x'$  is collinear in  $X$  with  $x$  (i.e.,  $x$  and  $x'$  belong to a common member of  $\mathfrak{M}$ ), is an isomorphism of projective spaces.*

By Step 5, the map  $\rho$  is bijective. We now show that  $\rho$  and its inverse preserve collinearity. Let three distinct collinear points  $x_1, x_2, x_3$  of  $M$  belong to the line  $L$  and let  $x'_i$  belong to  $M'$  and be collinear with  $x_i$ ,  $i = 1, 2, 3$ . Since  $x_i$  and  $x'_j$  are not collinear in  $X$  for  $i, j \in \{1, 2, 3\}$  and  $i \neq j$ , we deduce by Step 1'' that there is a unique hyperbolic space  $\xi_{i,j}$  through  $x_i, x_j, x'_i, x'_j$ ,  $i = 1, 2, 3$  and  $i \neq j$ . These spaces must coincide as otherwise the intersection of two of them, say  $\xi_{1,2} \cap \xi_{1,3}$ , belongs to  $X$  but contains  $x_2$  and  $x'_1$ . Hence the lines  $x'_1x'_2$  and  $x'_1x'_3$  coincide and hence  $x'_1, x'_2, x'_3$  are collinear. Similarly, the inverse of  $\rho$  preserves collinearity.

Using Teorema 2.8 of [1] again (or [3] and [6]), we conclude that  $X$  is the projection of a Segre variety.

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### Address of the authors

Department of Mathematics, Ghent University,  
Krijgslaan 281-S22, B-9000 Ghent, BELGIUM  
{jat,hvm}@cage.ugent.be