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Abstract

Among all 2-($q^3+1, q+1, 1$)-designs, we characterize the hermitian unitals by the existence of sufficiently many translations. In arbitrary 2-($q^3+1, q+1, 1$)-designs, each group of translations with given center acts semi-regularly on the set of points different from the center.
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The classical examples of unitals (i.e., 2-($q^3+1, q+1, 1$)-designs) are the hermitian unitals, which are embedded in projective planes (as absolute points and non-absolute lines of a polarity). Here we study abstract unitals, without requiring embeddings in projective planes, and we characterize the classical examples by the existence of sufficiently many translations:

Main Theorem. *Let $\mathbb{U}$ be a unital of order q with the following property: for every point c of $\mathbb{U}$ there exist q translations of $\mathbb{U}$ with center c. Then $\mathbb{U}$ is isomorphic to the hermitian unital of order q.*

The translations of the hermitian unital of order q generate the unitary group $\mathrm{PSU}(3, q)$. Translations are defined in Section 1 where we also prove the semi-regularity of translation groups (Theorem 1.3). This implies that every unital of order q has at most q translations with given center. In particular, our assumption that there are q translations with given center c is equivalent to the transitivity of the translation group $\Gamma_{[c]}$ on $L \setminus \{c\}$ for one (and then each) line L through c.

For the conclusion of our Main Theorem it suffices to assume that there are three non-collinear centers with translation groups of order q, see 1.6 below. There exist examples of non-hermitian unitals with at least one center such that the corresponding translation group is transitive, among them unitals arising from polarities in semifield planes but also in planes that are not translation planes (see [12, Theorem 5.6]).

The transitivity of translation groups with given center is a "Moufang" condition in the spirit of Tits; see [27, Addenda, p. 274]. This condition makes sense also for other 2-designs, like affine planes; note that in affine planes, "translations" as defined in 1.1 below are often called "dilatations with affine fixed point" or "homologies". A finite or infinite affine plane is desarguesian (hence coordinatized by a skew field) if for each center the corresponding "translation" group is transitive in the sense of 1.5 below, see [19, 3.2.27]. The traditional Moufang condition for projective planes is slightly different and requires both a center and a dual center (axis).

1 Translations

Let $\mathbb{U} = (U, \mathcal{L})$ be a unital of order $q > 1$, i. e., a finite linear space with $q^3 + 1$ points and $q + 1$ points per line; then there are q^2 lines per point. In other words, $\mathbb{U}$ is a 2-$(q^3 + 1, q + 1, 1)$-design. We note that q need not be a prime power; e.g., Mathon [17] and Bagchi-Bagchi [3] have constructed a unital of order 6.

The line joining $x, y \in U$ will simply be denoted by xy, and $\mathcal{L}_x$ is the set of all lines through x. The group of all automorphisms will be abbreviated by $\Gamma := \mathrm{Aut}(U, \mathcal{L})$.

1.1 Definition. An automorphism of $(U, \mathcal{L})$ is called a *translation* of $(U, \mathcal{L})$ with center c if it fixes each line through c. The set of all translations with center c is denoted by $\Gamma_{[c]}$.

J. Tits [29] has noted that every translation of a classical (hermitian) unital over a commutative field extends to an automorphism of the ambient projective plane, and used that fact in order to determine the full group of automorphisms of such a unital. In particular, it follows that every translation centralizes the polarity defining the hermitian unital, and Γ consists of automorphisms of the group generated by all translations. This result has been extended to various cases of non-commutative fields in [25, 4.2], [24], and to alternative fields in [4] and [13, 7.1].

1.2 Lemma. *Assume that $(P, \mathcal{L})$ is a linear space. Consider a finite subset $S \subset P$ with a point $c \in S$ and a point $\infty \in P \setminus S$ with the following properties:*

- *Not all points of S are collinear.*
- *If a line joins two points in S but does not pass through c then it meets S in precisely $q + 1$ points, where $q \geq 1$ is a fixed number.*
- *There are at most q^2 lines through ∞.*
- *If a line through ∞ meets S in more than one point then that line is ∞c.*

Then every line joining c with another point of S contains precisely one point of $S \setminus \{c\}$.

Proof. Let $x \in S \setminus \{c\}$ be chosen such that the number t of points of S on the line xc is maximal. As S is not contained in a line, there exists a line B passing through x and containing q further points of S.

Aiming at a contradiction, we assume $t > 2$ and pick $y \in (xc \cap S) \setminus \{x, c\}$. The lines joining y with the q points in $(B \cap S) \setminus \{x\}$ contain q^2 points of S apart from y. Thus $q^2 + t \leq |S|$.

Let S' denote the set of points in S but not on the line ∞c. Joining each point of S' to ∞ gives an injective map into a set with at most $q^2 - 1$ lines because the line ∞c is avoided, and $|S| \leq |S'| + t \leq q^2 - 1 + t$ follows. This is a contradiction. □

1.3 Theorem. *Let $(U, \mathcal{L})$ be a unital of order q, and let c be some point in U. Then $\Gamma_{[c]}$ acts semi-regularly on $U \setminus \{c\}$, and semi-regularly on the line set $\mathcal{L} \setminus \mathcal{L}_c$.*

Proof. First of all, we note that a translation fixing $M \in \mathcal{L} \setminus \mathcal{L}_c$ fixes each point on M. In particular, semi-regularity on the set $\mathcal{L} \setminus \mathcal{L}_c$ is clear if semi-regularity on $U \setminus \{c\}$ is established.

Aiming at a contradiction, we assume that there exists $\vartheta \in \Gamma_{[c]} \setminus \{\mathrm{id}\}$ such that ϑ fixes a point $x \neq c$. Without loss, we may then pass to a suitable power of ϑ such that the order of ϑ is a prime p. If p divides q then ϑ fixes at least one more point on cx, and at least one line in

$\mathcal{L}_a \smallsetminus \{ca\}$ for each fixed point $a \neq c$. If p does not divide q then each member of $\mathcal{L}_c$ contains a point fixed by ϑ, and ϑ fixes the joining lines of these points. If ϑ would fix precisely one point apart from c on each member of $\mathcal{L}_c$ then joining one of these points to all the others yields a set of fixed lines. Each of these contains q points apart from the chosen one. Thus q divides $q^2 - 1$. This is impossible, and we infer that at least one line through c contains more than two fixed points.

Let S be the set of fixed points of ϑ. As ϑ is not the identity, we find a point $\infty \in U \smallsetminus S$. A line through ∞ but not through c cannot meet S in more than one point because otherwise every point on that line would be fixed by ϑ. Thus the set S and the point ∞ satisfy the conditions imposed in 1.2, contradicting the fact that there exists a line through c with more than two fixed points on it. □

1.4 Lemma. *Let $(U, \mathcal{L})$ be a unital of order q, and assume that $S \subset U$ is a proper subset closed under taking all the points on each line joining two points in S. Then S is contained in a line; thus either S is empty, or consists of a single point, or of the points on a single line.*

Proof. Aiming at a contradiction, we assume that S contains a triangle. Joining the points on one side of that triangle to the vertex not on that side we see that there are at least $(q+1)q+1$ points in S. Joining with any point $z \in U \smallsetminus S$ we obtain an injective map from S into the set of lines through z but this set of lines has only q^2 members. □

1.5 Definition. Abusing notation, we say that the group $\Gamma_{[c]}$ of translations with center c is *transitive* if it is transitive on $L \smallsetminus \{c\}$ for some line L through c.

Transitivity on $L \smallsetminus \{c\}$ is equivalent to $|\Gamma_{[c]}| = q$ because $\Gamma_{[c]}$ acts semi-regularly on $U \smallsetminus \{c\}$ by 1.3. Thus a transitive group $\Gamma_{[c]}$ of translations is transitive on $K \smallsetminus \{c\}$ for each line K through c.

1.6 Proposition. *If a, b, c are three non-collinear points in a unital $(U, \mathcal{L})$ such that $\Gamma_{[x]}$ is transitive for each $x \in \{a, b, c\}$ then the group T generated by $\Gamma_{[a]} \cup \Gamma_{[b]} \cup \Gamma_{[c]}$ is transitive on U, and contains $\Gamma_{[u]}$ for each $u \in U$. In particular, the translation groups form a conjugacy class in T (and in Γ), each $\Gamma_{[u]}$ is transitive, and T is a normal subgroup of Γ.*

Proof. Let S be the orbit of a under T. Then T contains $\Gamma_{[u]}$ for each $u \in S$, in fact $\Gamma_{[u]}$ is a conjugate of $\Gamma_{[a]}$ under some element of T. This implies that $\Gamma_{[u]}$ is also transitive, and S contains all points on any line joining two points of u. Now 1.4 yields $S = U$. □

1.7 Proposition. *Let c be a point in a unital $(U, \mathcal{L})$. Assume that $\Gamma_{[c]}$ is transitive, let $L \in \mathcal{L}_c$, and let $x \in L \smallsetminus \{c\}$.*

1. *The stabilizer $\mathrm{C}_\Gamma(\Gamma_{[c]})_x$ in the centralizer $\mathrm{C}_\Gamma(\Gamma_{[c]})$ acts trivially on L.*
2. *The kernel $\Gamma_{[L]}$ of the action of the stabilizer Γ_L on the line L centralizes $\Gamma_{[c]}$.*
3. *Assume that $\Gamma_{[u]}$ is transitive for each $u \in U$. Then $\Gamma_{[L]}$ acts semi-regularly on $U \smallsetminus L$.*

Proof. If we fix $x \neq c$ and centralize $\Gamma_{[c]}$ we clearly fix each point in the orbit of x under $\Gamma_{[c]}$, and this orbit is $L \smallsetminus \{c\}$. If γ fixes each point on L then γ normalizes $\Gamma_{[c]}$. For $\tau \in \Gamma_{[c]}$ we evaluate $\vartheta := \gamma\tau\gamma^{-1} \in \Gamma_{[c]}$ at $y \in L \smallsetminus \{c\}$ as $y^\vartheta = y^{\vartheta\gamma} = (y^\gamma)^\tau = y^\tau$. Semi-regularity (see 1.3) of $\Gamma_{[c]}$ yields $\vartheta = \tau$, and γ centralizes $\Gamma_{[c]}$.

In the situation of assertion 3, assume that $\gamma \in \Gamma_{[L]}$ fixes a point y outside L. As every point is the center of a transitive translation group, the first two assertions yield that the set S of fixed points of γ is closed under taking all the points on lines joining two points in S. Now 1.4 yields $\gamma = \mathrm{id}$. □

1.8 Lemma. *Let* $\mathrm{RU}(q)$ *be a Ree unital, i.e., the unital associated with the Ree group* $\mathrm{R}(q) = {}^2G_2(q)$ *for* $q = 3^r$ *with odd* r*. Then each translation of* $\mathrm{RU}(q)$ *is trivial.*

Proof. In [8] it has been shown that the full group $\mathrm{Aut}(\mathrm{RU}(q))$ of automorphisms of the Ree unital is the semi-direct product $\mathrm{Aut}(\mathrm{R}(q)) \cong \mathrm{Aut}(\mathbb{F}_q) \ltimes \mathrm{R}(q)$. For an explicit description of the Ree group one needs the *Tits automorphism* θ of $\mathbb{F}_q$ with the property that θ^2 is the Frobenius automorphism $x \mapsto x^3$.

By 1.3 the group $\Gamma_{[c]}$ of translations acts semi-regularly on $L \setminus \{c\}$ for any line L through a point c. Therefore, there are no elements of prime order different from 3 in $\Gamma_{[c]}$, and $\Gamma_{[c]}$ is contained in a Sylow 3-subgroup of $\mathrm{Aut}(\mathbb{F}_q) \ltimes \mathrm{R}(q)$.

We use Lüneburg's description [15] of $\mathrm{RU}(q)$: the points are the Sylow 3-subgroups of $\mathrm{R}(q)$, the lines are the involutions in $\mathrm{R}(q)$, and a point is incident with a line if it is normalized by the latter. Now we use the standard description of a point stabilizer in $\mathrm{R}(q)$ (due to Tits [28], cf. [30, 7.7.7]): the Sylow 3-subgroup is $S = \mathbb{F}_q^3$ with multiplication $(a,b,c) * (x,y,z) = (a+x, b+y+ax^\theta, c+z+ay-bx-ax^{\theta+1})$, and its normalizer in $\mathrm{R}(q)$ is the semi-direct product of S and the multiplicative group of $\mathbb{F}_q$ where $h \in \mathbb{F}_q \setminus \{0\}$ acts on S via $(x,y,z) \mapsto (hx, h^{\theta+1}y, h^{\theta+2}z)$. In particular, the element $-1 \in \mathbb{F}_q$ induces an involution j in the normalizer of S acting via $(x,y,z) \mapsto (-x, y, -z)$. Thus j is one of the lines through the (unique) point fixed by S (which is, in Lüneburg's model, just S itself). The centralizer $\mathrm{C}_S(j)$ has order q, so the conjugacy class j^S has q^2 elements, and this conjugacy class is the set of all lines through S. It is easy to see that j^S generates $\langle j \rangle S$.

The Sylow 3-subgroup of the stabilizer of the point S is the product AS of S with a group A isomorphic to a Sylow 3-subgroup of $\mathrm{Aut}(\mathbb{F}_q)$. Now $\Gamma_{[S]} = \mathrm{C}_{AS}(j^S) \le \mathrm{C}_{AS}(Z) = S$ where $Z = \{0\}^2 \times \mathbb{F}_q$ is the center of S; in fact, the group A acts faithfully on Z. Finally, it remains to note that $\Gamma_{[S]} \le \mathrm{C}_Z(j)$ is trivial. □

2 Unitals with many translations

There is, up to isomorphism, just one unital of order 2, namely the affine plane of order 3, and the translations of the unital are homologies of the affine plane. These involutions generate a group of order 18 with trivial center, and the stabilizer of a point in that group fixes all lines through the point. This shows that the assumption $q > 2$ is in fact necessary in the results of this section.

2.1 Proposition. *If* $\mathbb{U} = (U, \mathcal{L})$ *is a unital of order* $q > 2$ *such that each translation group is transitive, then the stabilizer* Γ_c *of a point* c *does not fix any line through* c*.*

Proof. Assume, to the contrary, that Γ_c fixes some line L through c. Pick any point y outside L and consider the set S of all points on lines joining y with a point on L. Then S has $(q+1)q+1 = q^2+q+1$ points. We claim that the lines joining two points of S carry either $q+1$ points of S (this clearly happens if the line passes through y or is in the orbit of L under $\Gamma_{[y]}$) or q points of S (we are going to show that this happens in all other cases).

Consider $u \in cy \setminus \{c, y\}$ and $b \in L \setminus \{c\}$. Let $\tau \in \Gamma_{[y]}$ be defined by $c^\tau = u$. For any $v \in by \setminus \{b^\tau\}$ take $\alpha \in \Gamma_{[y]}$ with $b^\alpha = v$; then $c^\alpha \neq u$ and there exists $\beta \in \Gamma_{[u]}$ with $(c^\alpha)^\beta = c$. Now $\alpha\beta$ fixes c and thus fixes L by our assumption. Thus $v^\beta = b^{\alpha\beta} \in L$, and we have proved that

uv meets L, for each $v \in by \setminus \{b^\tau\}$. This means that q lines through u meet both L and by. For any $a \in L \setminus \{b, c\}$ we also have q lines through u that meet both L and ay. Among the lines joining u with points on by we therefore have at most one that does not meet ay. That line is easily found: apply $\vartheta \in \Gamma_{[c]}$ with $y^\vartheta = u$ to ay. Thus there are two lines through u that are completely contained in S (namely, the lines L^τ and cy) and all other lines joining u to points of S carry precisely q points of S.

Applying translations with centers on L we see that Γ_x fixes L whenever $x \in L$. Repeating our arguments above with x replacing c we see that through each point of $S \setminus \{y\}$ there are precisely two lines that are completely contained in S and q lines each containing q points of S.

Let k be the number of lines with q points in S. Counting the incident point-line pairs in two ways, we find $(q^2 + q)2 + (q + 1) + (q^2 + q)q = (q + 1)(2q + 1) + qk$ and thus $k = q^2 + q$. These k lines cover at most $|S| + k = 2q^2 + 2q + 1$ points of U. Our assumption $q > 2$ now yields $q^3 + 1 > 2q^2 + 2q + 1$, and there does exist a point z not on any of these k lines. Joining with z gives an injective map from S into the set of lines through z. This contradicts the fact that there are only $q^2 < |S|$ lines through z. □

2.2 Corollary. *If $\mathbb{U}$ is a unital of order $q > 2$ such that each translation group is transitive, then the group generated by all translations acts primitively on the set of points.*

Proof. The translation groups of any two points on a line generate a two-transitive group on that line. Thus every block of imprimitivity is closed under taking all the points on each line joining two elements of the block. If there were a nontrivial block B of imprimitivity then 1.4 thus yields that B is a line, and the stabilizer of any point in that line would also fix B. This is impossible by 2.1. □

2.3 Proposition. *Let L be a line in a unital $\mathbb{U} = (U, \mathcal{L})$ of order $q > 2$ such that $\Gamma_{[x]}$ is transitive for each $x \in L$. If the group G induced on L by $\hat{G} = \langle \Gamma_{[x]} \mid x \in L \rangle$ is sharply two-transitive then $q + 1$ is a power of some prime p which divides the order of the center $Z := Z(\hat{G})$.*

Proof. From 1.7.2 we know that the kernel $K := \hat{G}_{[L]}$ of the action of $\hat{G}$ on L is contained in Z. The sharply two-transitive group G has a regular normal subgroup N which is elementary abelian (compare [21, 7.3.1]), hence $q + 1 = p^m$ for some prime p. Let $\hat{N}$ be a Sylow p-subgroup of $\hat{G}$. Since $\hat{N}$ is characteristic in the normal subgroup $K\hat{N}$ of G we infer that $\hat{N}$ is normal in $\hat{G}$, whence $\Gamma_{[c]}\hat{N}$ is a subgroup of $\hat{G}$ for $c \in L$. Now $\hat{N}$ is transitive on L, and $\{g\Gamma_{[c]}g^{-1} \mid g \in \hat{N}\} = \{\Gamma_{[x]} \mid x \in L\}$ yields $\hat{G} = \Gamma_{[c]}\hat{N}$.

Aiming for a contradiction we assume now that p does not divide $|Z|$. Then $K \cap \hat{N}$ is trivial, hence $|\hat{N}| = |N| = q + 1$ and $|\hat{G}| = |\Gamma_{[c]}\hat{N}| = q(q + 1) = |G|$. Thus $\hat{G} = G$ and every element of $\hat{N} = N$ is the product of two elements of stabilizers of points on L. Since these stabilizers are just the translation groups, we infer that N acts semi-regularly on $U \setminus L$. As the complement $G \setminus N$ is covered by translation groups, we find that G acts semi-regularly on $U \setminus L$.

If q is even then $p \geq 3$ and $|\mathcal{L}| = q^4 - q^3 + q^2 \equiv 3 \pmod{p}$ yields that N fixes at least 2 lines in $\mathcal{L} \setminus \{L\}$. We choose z on such a line. Semi-regularity implies that N acts transitively on that line. For $y \in z^G \setminus z^N$ the (unique) element $\tau \in G$ with $z^\tau = y$ belongs to $G \setminus N = \bigcup_{x \in L} \Gamma_{[x]} \setminus \{1\}$. Thus there exists $c \in L$ with $\tau \in \Gamma_{[c]}$, and the line zy contains the q points of the orbit $z^{\Gamma_{[c]}} \subset z^G$. We obtain that every line meeting z^G in more than one point meets in q or $q + 1$ points, and every line meeting z^G in q points is completely contained in $z^G \cup L$. Since $q > 2$ we have

$|U \smallsetminus L| = q^3 - q > q^2 + q = |G|$. Thus there exists a point $w \in U \smallsetminus (L \cup z^G)$. Joining points of z^G with w yields an injective map, whence $q^2 + q = |z^G| \le q^2$ which is absurd.

Thus q is odd. Then $p = 2$ and $q + 1$ is divisible by 4. We have $q \equiv -1 \pmod 4$ and $|\mathcal{L}| \equiv 3 \pmod 4$. If N fixes a line different from L we obtain a contradiction as in the previous paragraph. Thus N has some orbit $\{M, M'\}$ of length 2 in $\mathcal{L} \smallsetminus \{L\}$. Note that M and M' do not meet L. We choose $z \in M$. Then z^N is distributed evenly onto M and M', and each of these lines contains $(q + 1)/2$ points from z^N.

As before, each line joining z with a point in $z^G \smallsetminus z^N$ is invariant under a translation and contains q points of z^G. Thus the joining lines in z^G either contain q points of z^G and meet L, or contain $(q + 1)/2$ points of z^G and are orbits under a subgroup of index 2 in N, or contain just two points (this happens if we join points on $M \cap z^G$ with points on $M' \cap z^G$).

Pick a point $m \in M \smallsetminus z^G$, then $|m^G| = q^2 + q$. The intersection $M \cap m^G$ is the orbit of m under the stabilizer of M in N, and has $(q + 1)/2$ points.

Let $x \in m^G \smallsetminus M$ and assume (by way of contradiction) that the line zx contains a second point x' of m^G. The line $zx = x'x$ does not meet L because it is not contained in $z^G \cup L$. Repeating the argument above for x instead of z we see that $x' \in x^N$. Thus there exists an involution in the elementary abelian group N interchanging x and x'. The joining line is fixed by that involution. As the latter does not fix z, it will interchange M and M', and x' is the intersection point of M' and zx. For any point $y \in m^G \smallsetminus (M \cup M')$ we obtain that $zy \cap m^G = \{y\}$. This gives $|m^G \smallsetminus (M \cup M')| = q^2 + q - (q + 1) = q^2 - 1$ lines through z with precisely one point in m^G. Together with the line M, this covers the whole pencil $\mathcal{L}_z$.

Now we count the points in m^G by joining them with z: there are $(q+1)/2$ points on M, and $q^2 - 1$ points on the other lines. This gives $q^2 + q = q^2 - 1 + (q + 1)/2$ and then the contradiction $q + 1 = (q + 1)/2$. □

3 Moufang sets induced on lines

Recall that a *Moufang set* is a set X together with a collection of groups $(R_x)_{x \in X}$ of permutations of X such that each R_x fixes x and acts regularly (i. e., sharply transitively) on $X \smallsetminus \{x\}$, and such that $\{R_y \mid y \in X\}$ is invariant under conjugation by each R_x. The permutation group $G := \langle R_x \mid x \in X \rangle$ is called the *little projective group* of the Moufang set; the groups R_x are called *root groups*.

3.1 Proposition. *Let $\mathbb{U}$ be a unital with a line L such that $\Gamma_{[x]}$ is transitive for each $x \in L$. Then the following is true.*

1. *$(L, (\Gamma_{[x]}|_L)_{x \in L})$ is a Moufang set.*
2. *The kernel $\hat{G}_{[L]}$ of the action of $\hat{G} := \langle \Gamma_{[x]} \mid x \in L \rangle$ on L is the center of $\hat{G}$.*
3. *If the little projective group G induced by $\hat{G}$ on L is perfect, then the commutator series of $\hat{G}$ terminates at a perfect group $\tilde{G}$ such that $\tilde{G}/\mathrm{Z}(\tilde{G}) \cong G$, and $\mathrm{Z}(\tilde{G})$ is a subgroup of the Schur multiplier of G.*

Proof. Clearly $\hat{G}$ acts two-transitively on L. The set $\{\Gamma_{[x]} \mid x \in L\}$ is invariant under conjugation, and $\Gamma_{[x]}$ acts regularly on $L \smallsetminus \{x\}$ by 1.3 and our transitivity assumption. Thus we have a Moufang set. By 1.7.2 the kernel of the action is contained in $\bigcap_{x \in L} \mathrm{C}_{\hat{G}}(\Gamma_{[x]})$ which is the center of $\hat{G}$. As $\mathrm{C}_{\hat{G}}(\Gamma_{[x]})$ fixes the unique point x fixed by $\Gamma_{[x]}$, the reverse inclusion follows.

If G is perfect then every member of the commutator series of $\hat{G}$ is mapped surjectively onto G. As $\hat{G}$ is finite, the series terminates at a group $\tilde{G}$ with the required properties. □

All finite Moufang sets are explicitly known:

3.2 Theorem. *The little projective group of a finite Moufang set is either sharply two-transitive, or it is permutation isomorphic to one of the following two-transitive permutation groups of degree $q+1$: $\mathrm{PSL}(2,q)$ with a prime power $q > 3$, $\mathrm{PSU}(3,f)$ with a prime power $q = f^3 \geq 3^3$, a Suzuki group $\mathrm{Sz}(2^s) = {}^2B_2(2^s)$ with $q = 2^{2s} \geq 2^6$, or a Ree group $\mathrm{R}(3^r) = {}^2G_2(3^r)$ with $q = 3^{3r}$, where r and s are positive odd integers.*

This was proved (in the context of split BN-pairs of rank one) by Suzuki [26] and Shult [22] for even q, and by Hering, Kantor and Seitz [9] for odd q; these papers rely on deep results on finite groups, but not on the classification of all finite simple groups. See also Peterfalvi [18]. Note that $\mathrm{PSL}(2,2) \cong \mathrm{AGL}(1,3)$, $\mathrm{PSL}(2,3) \cong \mathrm{A}_4 \cong \mathrm{AGL}(1,4)$, $\mathrm{PSU}(3,2) \cong \mathrm{ASL}(2,3)$ and $\mathrm{Sz}(2) \cong \mathrm{AGL}(1,5)$ are sharply two-transitive. The smallest Ree group $\mathrm{R}(3) \cong \mathrm{P\Gamma L}(2,8)$ is almost simple, but not simple.

3.3 Lemma. *Consider two finite Moufang sets with isomorphic root groups, and assume that the corresponding little projective groups G_1 and G_2 are not permutation isomorphic. Then G_1 and G_2 are both sharply two-transitive, or G_1 and G_2 are permutation isomorphic to $\mathrm{AGL}(1,2^m)$ and $\mathrm{PSL}(2,2^m-1)$ in some order and 2^m-1 is a Mersenne prime with $2^m-1 \geq 7$.*

The examples where G_1 and G_2 are sharply two-transitive and not isomorphic originate from nonisomorphic nearfields with isomorphic multiplicative groups; compare [16, 7.4] and note that each Dickson pair determines the multiplicative group of a Dickson nearfield up to abstract isomorphism.

Proof. Assume that G_1 is not sharply two-transitive. Then G_1 is one of the almost simple permutation groups of degree $q+1$ listed in 3.2, where the prime power q is the order of the root groups. The root groups of $\mathrm{PSL}(2,q)$ are elementary abelian. The root groups of $\mathrm{PSU}(3,f)$ have order f^3 and prime exponent, and their centers have order f. The root groups of $\mathrm{Sz}(2^s)$ are Suzuki 2-groups of order 2^{2s} with centers of order 2^s. The root groups of $\mathrm{R}(3^r)$ have order 3^{3r} and exponent 9, and their centers have order 3^r. We observe that the root groups of these almost simple groups are mutually not isomorphic.

Thus G_2 is sharply two-transitive by 3.2. Hence $q+1$ is a prime power, and the root groups of G_2 are fixed-point-free automorphism groups of order q of the regular normal subgroup of G_2; compare [21, 7.3.1]. Thus the root groups are cyclic or generalized quaternion groups, see [10, V.8.12], cf. [21, 10.5.5]. Comparison with the root groups of G_1 shows that the root groups are cyclic, in fact $G_1 \cong \mathrm{PSL}(2,q)$ for some prime $q > 3$, and $G_2 \cong \mathrm{AGL}(1,q+1)$. Moreover, the prime power $q+1$ is even, hence a power of 2, and q is a Mersenne prime with $q \geq 7$. □

We collect some explicit information on $\mathrm{SL}(2,F)$.

3.4 Transvections and elations. Let F be a field. In $\mathrm{SL}(2,F)$ we call an element a transvection if its characteristic polynomial is $(X-1)^2$. An element of $\mathrm{PSL}(2,F)$ is called an elation if it is induced by a transvection of $\mathrm{SL}(2,F)$. The collection of groups of elations with common fixed point forms a Moufang set on the projective line over F such that the little projective group is $\mathrm{PSL}(2,F)$.

3.5 Lemma. *Let F be a field. Every element of* $\mathrm{SL}(2,F) \setminus \{-\mathrm{id}\}$ *is a product of at most two transvections, and every element of* $\mathrm{PSL}(2,F)$ *is the product of at most two elations.*

Proof. Since $\mathrm{SL}(2,F)$ acts two-transitively on the set of one-dimensional subspaces of F^2 every product of two transvections in $\mathrm{SL}(2,F)$ is conjugate to

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1+ab & a \\ b & 1 \end{pmatrix}$$

with $a, b \in F$. Non-central elements of $\mathrm{SL}(2,F)$ are conjugates if, and only if, they have the same trace. As $2+ab$ runs over all elements of F, every non-central element of $\mathrm{SL}(2,F)$ is (conjugate to) a product of at most two transvections. The assertion about elations follows immediately. □

3.6 Lemma. *Assume that* $\mathrm{SL}(2,t)$ *acts (perhaps not faithfully) on a unital* $(U,\mathcal{L})$ *of order q in such a way that some line L of the unital is fixed, every non-trivial transvection induces a non-trivial translation of the unital, and non-commuting transvections induce translations with different centers. Then the stabilizer* $\mathrm{SL}(2,t)_x$ *of any point $x \in U \setminus L$ is contained in the center of* $\mathrm{SL}(2,t)$*. If $t = q$ then* $\mathrm{SL}(2,t)$ *induces a semi-regular group on $U \setminus L$.*

Proof. Let H denote the group induced by $\mathrm{SL}(2,t)$ on U. By 3.5 every non-central element of H is the product of at most two translations (induced by transvections) with different centers on L. Such an element cannot fix any point outside L. If $q = t$ then the translation groups are transitive, and the center of H acts semi-regularly by 1.7.1 and 1.7.3. □

4 Proof of the Main Theorem

The following elementary observation can be traced back to Gleason; see [5, pp. 190 f].

4.1 Lemma. *Let p be a prime, and let H be a group acting on a finite set X. Assume that for every $x \in X$ there exists in H an element of order p which fixes x but no other element of X. Then H is transitive on X.*

Proof. If H is not transitive then X is a disjoint union of two non-empty H-invariant subsets A and B. Pick $a \in A$ and $b \in B$. The orbits of the stabilizer H_a in B have lengths divisible by p, and the orbits of H_b in $B \setminus \{b\}$ also have lengths divisible by p. This is impossible. □

The next result will help to exclude a "mixed case" below.

4.2 Proposition. *Let H be a primitive rank 3 permutation group of degree q^3+1 where q is a prime. Then $q \le 5$.*

Proof. By the O'Nan–Scott theorem, H is of grid type, or an affine group, or almost simple, see [14] or [31, Section 2.6]. In the grid case, $q^3 + 1 = n^2$ for some integer n, hence $q^3 = (n-1)(n+1)$. As $n - 1 > 1$ the prime q divides both factors, and their difference, whence $q = 2$. If H is an affine group and $q > 2$, then the even degree $q^3 + 1 = (q+1)(q^2 - q + 1)$ is a prime power, hence a power of 2; thus the odd factor $q^2 - q + 1$ has to be 1, which is absurd.

It remains to deal with almost simple groups H. These permutation groups have been classified by Bannai, Kantor and Liebler, and Liebeck and Saxl; the completeness of the list depends on the classification of finite simple groups. Below we require only the knowledge of the (sub)degrees, as listed in [6]. We denote the subdegrees of H by $1, k, \ell$, thus $q^3+1 = 1+k+\ell$ and $k+\ell = q^3$.

(i) If $H \le S_n$ acts on 2-sets with $n \ge 5$, then $q^3+1 = \binom{n}{2}$, hence $2q^3 = (n-2)(n+1)$. Since $n-2 > 2$, the prime q divides both factors, and their difference, whence $q = 3$ (and $n = 8$).

(ii) Now let H act on (singular) lines, or on singular points, or on singular 4-spaces, or on the points of an E_6-building. Then k and ℓ are both divisible by some prime power $Q > 1$, and $k + l > Q^3$, compare the list in [6]. From $k+\ell = q^3$ we infer that Q divides q^3. Hence the prime q divides Q, and $k+\ell = q^3 \le Q^3$, which is a contradiction.

(iii) Let H act on an orbit of non-singular points (over the field with 2 or 3 elements). Then $k+\ell = q^3$ is of the form $(2^{n-1}-e)(2^n+e)$ or $\frac{1}{2}(3^{n-1}-e)(3^n+2e)$ or $\frac{1}{2}(3^n-e)(3^n+2e)$ or $\frac{1}{3}(2^{n-1}+(-1)^n)(2^n-3(-1)^n)$ with $n \ge 3$ and $e = \pm 1$. Again q divides both factors in brackets, hence each integer linear combination of these factors. Thus q divides $3e$ or $5e$, whence $q \le 5$.

(iv) If H belongs to one of the two series acting on an orbit of quadratic forms, then $k+\ell = q^3$ is of the form $(4^n-e)(2\cdot 4^{n-1}+e)$ or $(8^n-e)(4\cdot 8^{n-1}+e)$ with $e = \pm 1$. As in case (iii) we infer that $q \le 5$.

(v) Now finitely many cases remain, compare the list in [6] or [20]. In each of these cases, k and ℓ have a common prime divisor $r \le 17$ such that $k+\ell \ne r^3$ (which implies $r = q$, a contradiction), except if $\{k, \ell\} = \{25, 100\}$ and $q = 5$ (here H is A_{10} or S_{10}, acting on partitions of type 5, 5, with degree $\frac{1}{2}\binom{10}{5} = 5^3+1$). □

Proof of the Main Theorem. If $q = 2$, then $\mathbb{U}$ is a 2-$(9,3,1)$-design, i.e. an affine plane of order 3, which is uniquely determined and isomorphic to the hermitian unital of order 2. Now let $q > 2$.

Fix a point c of $\mathbb{U} = (U, \mathcal{L})$. For every line L through c we define below a "Gleason" prime $g(L)$ coprime to q such that the stabilizer $\Gamma_{c,L}$ *contains an element $\varphi = \varphi_{c,L}$ of order $g(L)$ fixing no point in $U \setminus L$*. Recall that L carries the structure of a Moufang set and that the center of $\hat{G} = \langle \Gamma_{[x]} \mid x \in L \rangle$ is the kernel of the action of $\hat{G}$ on L, see 3.1.2. We denote by G^L the group induced by $\hat{G}$ on L, and by $\tilde{G}^L$ the final term of the commutator series of $\hat{G}$. If G^L is simple then the perfect group $\tilde{G}^L$ is a quotient of the universal cover of G^L modulo some subgroup of the Schur multiplier of G^L. By 3.2 we have one of the following 6 cases.

(1) G^L is sharply two-transitive. By 2.3, $q+1$ is a power of some prime p and the center of $\hat{G}$ contains an element φ of order p. We put $g(L) = p$. By 1.7.1 and 1.7.3, the automorphism φ fixes all points on L and no point in $U \setminus L$.

(2) $G^L \cong \mathrm{PSL}(2, q)$ with $q > 3$. Let $g(L)$ be the largest prime divisor of $q-1$.

For $q \notin \{4, 9\}$ the group $\tilde{G}^L$ is isomorphic to $\mathrm{SL}(2, q)$ or $\mathrm{PSL}(2, q)$ because $\mathrm{SL}(2, q)$ has trivial Schur multiplier, see [10, 25.7] or [23]. For $q = 4$ there is the further possibility $\tilde{G}^L \cong \mathrm{SL}(2, 5)$, see [10, 25.7] or [2, 33.15], which is a double cover of $\mathrm{SL}(2, 4) \cong A_5 \cong \mathrm{PSL}(2, 5)$, cf. [31, 3.3.5, p. 51], and contains only one involution. However, this possibility is ruled out by the elementary abelian subgroup $\Gamma_{[c]}$ of order 4 in $\tilde{G}^L$.

The Schur multiplier of $\mathrm{PSL}(2, 9) \cong A_6$ is cyclic of order 6, see [2, 33.15] or [31, 2.7.3], hence for $q = 9$ the group $\tilde{G}^L$ has a center Z of order $|Z| \in \{1, 2, 3, 6\}$. The translation group $\Gamma_{[c]}$ is (elementary) abelian of order 3^2 and has trivial intersection with Z by 1.7.1, hence the Sylow 3-subgroups of $\tilde{G}^L$ are abelian. However, the Sylow 3-subgroups of the universal cover are not abelian, cf. [31, 2.7.3, p. 30] or [10, V § 25, p. 647]. This excludes the cases where $|Z| \in \{3, 6\}$.

Hence $\tilde{G}^L$ is isomorphic to $\mathrm{SL}(2,q)$ or to $\mathrm{PSL}(2,q)$ in all cases. Thus 3.6 entails that $\tilde{G}^L$ acts semi-regularly on $U \setminus L$. For $d \in L \setminus \{c\}$ the stabilizer $\tilde{G}^L_{c,d}$ has order $q-1$ if $\tilde{G}^L \cong \mathrm{SL}(2,q)$ and $(q-1)/2$ otherwise, and the largest prime divisor of $q-1$ also divides $(q-1)/2$ for odd $q > 3$.

(3) $G^L \cong \mathrm{PSU}(3,f)$ with $f = \sqrt[3]{q} \geq 3$. Let $g(L)$ be the largest prime divisor of $f-1$.

Here $\mathrm{SU}(3,f)$ is the universal cover of $\mathrm{PSU}(3,f)$, see [7, Thm. 2]. Thus the order of the center Z of $\tilde{G}^L$ divides $\gcd(3,f+1) = \mathrm{Z}(\mathrm{SU}(3,f))$. Now $\mathrm{SU}(3,f)$ has a subgroup isomorphic to $\mathrm{SU}(2,f) \cong \mathrm{SL}(2,f)$, contained in the stabilizer of a line of the hermitian unital of order f, and intersecting $\mathrm{Z}(\mathrm{SU}(3,f))$ trivially. Let $\Delta \cong \mathrm{SL}(2,f)$ denote the corresponding subgroup of $\hat{G}^L$. Each transvection of Δ belongs to a Sylow subgroup of order f^3 in $\tilde{G}^L$ (note that f is coprime to $|Z|$) and is therefore a translation of $\mathbb{U}$. Now 3.6 yields that Δ is semi-regular on $U \setminus L$. As in case (2), for $d \in L \setminus \{c\}$ the order of Δ_c is divisible by $g(L)$ unless $f = 3$.

For $f = 3$ we have $Z = \{1\}$ and $|\tilde{G}^L_c| = |G^L_c| = 2 \cdot 3^3$. All involutions in $\mathrm{SU}(3,3) = \mathrm{PSU}(3,3)$ are conjugate and are products of two elements of root groups: with respect to the form $\overline{x_1}\,x_3 + \overline{x_2}\,x_2 + \overline{x_3}\,x_1$, the matrices $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ belong to root groups, and their product is an involution. Thus each involution in $\tilde{G}^L_c$ is a product of two translations of $\mathbb{U}$ (with different centers), hence semiregular on $U \setminus L$.

(4) $G^L \cong \mathrm{Sz}(2^s)$ with $q = 2^{2s} \geq 2^6$. Let $g(L)$ be the largest prime divisor of $2^s - 1$. The two-point stabilizers in $\mathrm{Sz}(2^s)$ have order $2^s - 1$. We choose $d \in L \setminus \{c\}$ and show below that every element φ of odd order in $\tilde{G}^L_{c,d}$ acts semi-regularly on $U \setminus L$.

Every element of odd order in $\mathrm{Sz}(2^s)$ is strongly real, i.e., a product of two involutions, see [16, 24.7, 24.6]. If $s > 3$ then $\tilde{G}^L \cong G^L$ because the Schur multiplier is trivial (see [1], cf. [2, 4.2.4]). In that case, the involutions are translations (with different centers) and their product acts semi-regularly on $U \setminus L$. For $s = 3$ the Schur multiplier is elementary abelian of order 4, see [1], cf. [2, 4.2.4]. If $\varphi \in \tilde{G}^L_{c,d}$ has odd order then there exist involutory translations ϑ, τ and an element $\zeta \in \mathrm{Z}(\tilde{G}^L)$ such that $\varphi = \vartheta\tau\zeta$. Then $\zeta^2 = 1$ and $\varphi^2 = \vartheta\tau\vartheta\tau\zeta^2 = \vartheta\vartheta^\tau\zeta^2 = \vartheta\vartheta^\tau$ is the product of two translations (with different centers). Hence $\langle\varphi\rangle = \langle\varphi^2\rangle$ acts semi-regularly on $U \setminus L$.

(5) $G^L \cong \mathrm{R}(3^r)$ with $q = 3^{3r}$ and $r > 1$. Let $g(L)$ be the largest prime divisor of $(3^r - 1)/2$. Since $r > 1$ the Ree group $\mathrm{R}(3^r)$ is simple and has trivial Schur multiplier (see [1]), hence $\tilde{G}^L \cong G^L$. This group has a subgroup $\Delta \cong \mathrm{PSL}(2,3^r)$, namely the unique subgroup of index 2 of the stabilizer of a line in the Ree unital of order 3^r, see [15, p. 257]. Each elation in Δ belongs to a Sylow 3-subgroup of G^L and is therefore a translation of $\mathbb{U}$. Now 3.6 yields that Δ is semi-regular on $U \setminus L$. Moreover, $|\Delta_c| = 3^r(3^r-1)/2$ is divisible by $g(L)$.

(6) $G^L \cong \mathrm{R}(3) \cong \mathrm{P\Gamma L}(2,8)$ and $q = 3^3$. Let $g(L) = 2$.

The final term Δ of the commutator series of $\hat{G} = \langle \Gamma_{[x]} \,|\, x \in L \rangle$ is a cover of $\mathrm{PSL}(2,8) \cong \mathrm{SL}(2,8)$, which has no proper cover (see case (2)). Hence $\Delta \cong \mathrm{SL}(2,8)$ and $|\Delta_c| = 2 \cdot 3^2$. The translation group $\Gamma_{[c]}$ is a semidirect product $\Gamma_{[c]} = \langle\tau\rangle \rtimes \langle\alpha\rangle$ where τ has order 9 and α has order 3, with $\alpha\tau = \tau^4\alpha$. If $\alpha \in \Delta$, then α and $\tau^{-1}\alpha\tau = \tau^3\alpha$ together generate an elementary abelian subgroup of order 9 in $\Delta \cong \mathrm{SL}(2,8)$, which has cyclic Sylow 3-subgroups. This contradiction shows that $\alpha \notin \Delta$. Thus $\langle\alpha\rangle\Delta$ induces all of G^L on L, and $\langle\alpha\rangle\Delta \cong \mathrm{P\Gamma L}(2,8)$.

The involutions in $\Delta \cong \mathrm{SL}(2,8)$ are conjugate and can be written as quotients of elements of order 9: let $\mathbb{F}_8 = \mathbb{F}_2(w)$ with $w^3 = w+1$; then $\mu = \begin{pmatrix} 0 & 1 \\ 1 & w \end{pmatrix}$ and $\nu = \begin{pmatrix} w^2 & w \\ 1 & w \end{pmatrix}$ have order 9 and their quotient $\mu\nu^{-1} = \begin{pmatrix} 1 & w^2 \\ 0 & 1 \end{pmatrix}$ has order 2. Now $\mu\zeta$ is a translation of $\mathbb{U}$ for a suitable element ζ in the center of $\hat{G}$. All elements of order 9 in Δ are conjugate in $\langle\alpha\rangle\Delta \cong \mathrm{P\Gamma L}(2,8)$, hence also $\nu\zeta$

is a translation of $\mathbb{U}$, and $\mu\nu^{-1} = (\mu\zeta)(\nu\zeta)^{-1}$. Thus each involution $\varphi \in \Delta_c$ is a product of two translations of $\mathbb{U}$ of order 9 (with different centers), whence φ is semiregular on $U \setminus L$.

This concludes our definition of $g(L)$ and $\varphi_{c,L}$. Note that the prime $g(L)$ depends only on the isomorphism type of G^L. Thus $g(L) = g(L^\gamma)$ for every $\gamma \in \Gamma_c$, and Γ_c acts on each line set $g^{-1}(r) = \{L \in \mathcal{L}_c \mid g(L) = r\}$ where r is a prime. Moreover, $\varphi_{c,L}$ fixes no line $M \in \mathcal{L}_c \setminus \{L\}$, otherwise it would fix a point of $M \setminus \{c\} \subseteq U \setminus L$ since $g(L)$ is coprime to $q = |M \setminus \{c\}|$. By Gleason's Lemma 4.1, the stabilizer Γ_c is transitive on each line set $g^{-1}(r)$.

If the mapping $g\colon \mathcal{L}_c \to \mathbb{N}$ is not constant, then $\mathbb{U}$ contains lines L and M such that G^L and G^M are not isomorphic. However, the corresponding root groups are conjugate in Γ (because they are translation groups and Γ is transitive on U). From 3.3 and our definition of g we infer that $q \geq 7$ is a Mersenne prime and G^L is sharply two-transitive or isomorphic to $\mathrm{PSL}(2,q)$; this holds for every line L through c. Thus g assumes only two values, and Γ_c has two orbits on the pencil $\mathcal{L}_c$ and two orbits on $U \setminus \{c\}$. With 2.2 we obtain that Γ is a primitive permutation group of rank 3 on U, and 4.2 gives $q \leq 5$, which is a contradiction.

Thus g is constant, and Γ_c is transitive on the pencil $\mathcal{L}_c$. Hence Γ is transitive on $\mathcal{L}$ and two-transitive on U. According to Kantor [11], $\mathbb{U}$ is the unital associated with $\mathrm{PSU}(3,q)$, i.e. the hermitian unital of order q, or the unital associated with $\mathrm{R}(3^r) = {}^2G_2(3^r)$, i.e. a Ree unital (this result of Kantor relies on the classification of finite simple groups). However, a Ree unital does not admit any nontrivial translation by 1.8.

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