Characterizations of trialities of type I_{id} in buildings of type D_4

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Abstract

A triality of type I_{id} in a building Δ of type D_4 is a type rotating automorphism of order 3 whose structure of fixed flags is the building of type G_2 related to Dickson's simple groups (in geometric term, this building is the split Cayley generalized hexagon over the field in question). Such a triality exists over any field and is unique up to conjugacy. In this paper, we present two characterizations of such trialities among all type rotating automorphisms (hence not necessarily of order 3). We prove that, if for a type rotating automorphism θ of Δ, no non-fixed line and its image are contained in adjacent chambers, and θ fixes at least one line, then θ is a triality of type I_{id} (here, lines are vertices of type 2, with Bourbaki labeling). Also, if a type rotating automorphism θ of Δ never maps a line to an opposite line, then it is also a triality of type I_{id}. We moreover show that this condition is equivalent with θ not mapping any chamber to an opposite one. The latter completes the programme for type rotating automorphisms of buildings of type D_4 of determining all domestic automorphisms of spherical buildings.

1 Introduction

A *domestic* automorphism of a spherical building is an automorphism which does not map any chamber to an opposite one. This notion arose from work of Abramenko & Brown [2], who proved that in (thick) non-spherical buildings, only the identity has bounded displacement on the set of chambers. In the spherical case, they proved that only the identity maps no flag to an opposite flag. This led Temmermans, Thas and the present author [8, 9, 10] to the more refined definition of J*-domestic automorphism*: this is an automorphism of a spherical building over the type set $I \supseteq J$, with J stable under the opposition map on types, mapping no flag of type J to an opposite flag. For $J = I$, we obtain domesticity as defined above. It is clear that J-domesticity for any (self-opposite) $J \subseteq I$ implies domesticity. Hence in order to understand all J-domestic automorphisms, one has to classify the domestic automorphisms. This was done for projective spaces in [8], for generalized quadrangles in [9], for type interchanging automorphisms of buildings of type E_6 in [14], and for generalized hexagons in work to be published by Parkinson and the author. Partial results for polar spaces are contained in [10]. In a lot of cases, it seems that domesticity is intimately related to the notion of geometric hyperplane in an appropriate point-line geometry arising from the building. For instance, in a projective space, the domestic dualities fix a hyperplane of the line-Grassmannian, namely a linear complex, also known as the set of lines fixed under a symplectic polarity. In a polar pace, an automorphism is line domestic if and only if it pointwise fixes a geometric hyperplane. Also, the domestic dualities of buildings of type E_6 fix pointwise a building of type F_4, the standard point set of which is a geometric hyperplane of the standard point set of the building of type E_6. In the situation of the present paper, this phenomenon will also show up again. But I do not know of an abstract reason for it.

We investigate domestic type rotating automorphisms of (thick) spherical buidings Δ of type D_4 and we show that there is only one conjugacy class of such automorphisms, namely, the trialities of type I_{id} (as defined by Tits in [11]), which fix a split Cayley hexagon. Now, it is well known (see Paragraph 5.2.2 of [11]) that the standard point set of this hexagon is the standard point set of a subbuilding of type B_3, and thus again constitutes a geometric hyperplane of the standard point set of Δ.

We will proceed by showing that a domestic automorphism maps no line to an opposite line, and then use this to prove that it is a triality of type I_{id}. But domesticity also implies that, if we view Δ as a hyperbolic quadric, no line and its image are contained in a common singular plane. We also show that a partial converse holds, in particular, that the fact that no non-fixed line and its image under a type rotating automorphism θ are contained in a plane, and some line is fixed, implies that θ is a triality of type I_{id}. We can rephrase these conditions with no reference to the hyperbolic quadric using the following observation. Two distinct lines of the quadric are contained in a common plane if and only if they are contained in respective chambers which are adjacent (these chambers contain the intersection point and the two maximal singular subspaces through the plane).

Geometrically, the split Cayley hexagons and the symplectic quadrangles are very similar objects, sharing a number of properties and characterizations, see Chapter 6 of [13]. The present paper adds one more such characterization to this list: they both arise from the unique type rotating domestic automorphism of their ambient building.

In fact, the same is true for all symplectic polar spaces. Note that the symplectic polar spaces and the split Cayley hexagons are *split* buildings whose diagram contains a multiple bond. There are two more classes of split buildings with a double bond in the diagram, namely the polar spaces (of type B_n) related to quadrics of maximal Witt index in even dimensional projective spaces, and the standard buildings of type F_4. The latter also arise from domestic polarities, in these cases in buildings of type E_6, and these polarities are (again) unique as domestic type rotating automorphisms. For the former (the polar spaces of type B_n), a characterization has yet to be proved, but there are "polarities" of buildings of type D_{n+1} producing them, and these polarities are domestic. For the moment it is not clear whether these are the only domestic dualities (for low ranks, $n \in \{2, 3\}$, it can be deduced from the results in [10] that they are unique). Hence it seems that the split spherical buildings with a double bond in the diagram can be generally characterized by domesticity of type rotating automorphisms of spherical buildings with only single bonds in the diagram. This would be a rather beautiful and remarkable synthetic characterization of such buildings.

For a relation between domesticity and the Freudenthal-Tits Magic Square, see the introduction of [14].

Finally, our result shows that the *Phan-geometry* of a type rotating automorhism of a building of type D_4 is empty if and only if it is a triality of type I_{id}. Phan-geometries (which are the geometries of chambers mapped on opposite ones by a certain automorphism) play a prominent role in Phan-Curtis-Tits theory, see [6]. Note that in this specific case, the *almost-Phan geometry* (roughly defined as the geometry induced on the chambers mapped "as far away as possible", but not opposite) is studied in [4], where its diagram is established and where it is proved that it is simply connected.

Remark 1.1 Although the general setting of the problem treated in this paper concerns spherical buildings, we are here only concerned about buildings of type D_4, which is a "classical" type. Hence we will always take the natural point-line geometry point of view to approach these buildings, i.e., we consider these buildings as non-thick polar spaces. Then the elements (vertices) of the building (viewed as a simplicial complex) are the singular subspaces except for the next-to-maximal ones (and the maximal ones represent two types of vertices, see below). In fact, in this way, we can identify the building with a hyperbolic quadric of Witt index 4, and we will do so explicitly below. Note that this is closer to the original approach [12] than to a more recent approach [15]. When we mention flags, vertices, chambers, then we mean the terminology of [12]; in the terminology of [15] these are all residues. But we will not use the terminology of [15] in the present paper.

2 Preliminaries and Main Results

Let Δ be a spherical building, i.e., a building with a finite Weyl group. For the definition of buildings, we refer to [1]. We always consider *thick* buildings, if not explicitly stated otherwise. Let $I = \{1, 2, 3, 4\}$ be the set of types of Δ (and we use Bourbaki labeling [3] for the nodes of the corresponding Coxeter diagram). As mentioned in the introduction, for a subset $J \subseteq I$, a *J-domestic* automorphism is an automorphism which does not map any flag of type J onto an opposite one. If $J = I$, then we simply talk about a *domestic* automorphism. If J is a singleton, then we sometimes replace the "J" in J-domestic by the name of the type of elements of type J. If J is not stable under the opposition relation, then every automorphism is automatically J-domestic, hence we assume from now on that J is fixed under the opposition relation. It is clear that in this case J-domesticity follows from J'-domesticity if $J' \subseteq J$. Hence the most general situation is that of domesticity. In the present paper, we concentrate on domestic type rotating automorphisms of buildings of type D_4. Our main result will classify all such automorphisms and show that, here, domesticity is equivalent to $\{2\}$-domesticity.

So let Δ be a building of type D_4. It can be identified with a hyperbolic quadric H in a projective space $\mathsf{PG}(7, \mathbb{K})$ of dimension 7 over some (commutative) field $\mathbb{K}$. Hence the elements of Δ are the *points*, the *lines*, and the two systems of maximal isotropic or singular subspaces of dimension 3 of H (the oriflamme complex of H). Each such system plays the same role as the point set, and hence there is the *principle of triality*, which just means that there exist type rotating automorphisms of Δ, inducing period 3 on the types. A *triality*, in the sense of Tits [11], is then a type rotating automorphism of order 3, the smallest possible order it can have. Note that sometimes a type rotating automorphism itself is already called a triality, but we will follow Tits's terminology here, and call a general type rotating automorphism a *tairality* (after a Welsh expression "tair" for three). Since lines are the elements of type 2 in Δ, we also speak about *line-domesticity* instead of $\{2\}$-domesticity, as explained in the previous paragraph.

Tits classified in [11] all trialities which admit at least one fixed flag of type $\{1, 3, 4\}$. The most interesting trialities are the ones of type I_σ, where σ is an automorphism of the underlying field $\mathbb{K}$. The fixed flags of such trialities define a building of type G_2, a so-called *generalized hexagon*. If $\sigma = \mathsf{id}$, the identity, then the standard point set of such hexagon is exactly the standard point set of a subbuilding of type B_3—in geometric terms, a parabolic quadric obtained from H by intersecting it with a non-tangent hyperplane of $\mathsf{PG}(7, \mathbb{K})$. The hexagons related to trialities of type I_{id} are called *split Cayley hexagons*, because they can be constructed directly using the split Cayley algebra over $\mathbb{K}$, see Schellekens [7].

We can now state our first main result.

Main Result 2.1 *A type rotating automorphism θ of a building Δ of type D_4 is a triality of type I_{id} if and only if θ is domestic if and only if θ is line-domestic.*

Note that two lines L, M in H can be in six distinct possible mutual positions:

1. L could be equal to M;
2. L could be concurrent with M and $L \cup M$ contained in a singular subspace of H;
3. L could be concurrent with L but some point of L is not collinear with some point of M;
4. $L \cup M$ could be contained in a singular subspace of H but not in a plane;
5. there could be a unique point on L collinear to all points of M (and then there is a unique point on M collinear with all points of L).
6. L and M are opposite, so each point of L is collinear to a unique point of M.

Viewing L and M as elements of Δ, the cases (3) and (4) are the same, since both say that L and M are incident with a unique common element of Δ. In case (5), we say that (L, M) is a *special pair*. Case (2) is equivalent to saying that L and M are distinct and contained in adjacent chambers.

Now we can state our second main result.

Main Result 2.2 *A type rotating automorphism θ of a building Δ of type D_4 is a triality of type I_{id} if and only if no non-fixed line and its image under θ are contained in adjacent chambers, and θ fixes some line.*

Concerning Main Result 2.2, it also seems to be a general phenomenon for the split buildings with a multiple bond in the diagram that a certain mutual position other than opposition of a vertex and its image of the corresponding type rotating automorphism of the ambient building with only single bonds in the diagram cannot occur, and that this automorphism is characterized by that property. For instance, [14] shows that symplectic polarities θ of buildings of type E_6 are characterized by the absence of pairs (p, p^θ), where p is a point and p^θ is a "symp" not opposite and not incident with p, and the fact that there is at least one pair (p, p^θ) with the point p incident with p^θ. A similar thing holds for symplectic polarities in projective space; we will show in a short appendix the following characterization.

Proposition 2.3 *Let θ be a duality of a projective space $\mathsf{PG}(n, \mathbb{K})$, $n \geq 3$, for some skew field $\mathbb{K}$. Then θ is a symplectic polarity if and only if for no line L the image L^θ intersects L in precisely one point, and there is some line L included in its image L^θ if and only if for no line L the image L^θ intersects L in precisely one point, and there is some absolute point.*

The restriction $n \geq 3$ stems from the fact that for $n = 2$, the hypotheses are self-contradictory (and there is indeed no symplectic polarity in a plane). Note also that the last criterion is the most general because if a line is included in its image, then every point of that line is absolute (a point being *absolute* if it is contained in its image).

3 Proof of Main Result 2.1

Let θ be a type rotating automorphism of a building Δ of spherical type D_4. We repeat that we call θ a *tairality* (see the introduction). If θ has order 3, then we call it a *triality*. Let H be the corresponding hyperbolic quadric in $\mathsf{PG}(7, \mathbb{K})$, for some well determined field $\mathbb{K}$. Then H can be viewed as a non-thick polar space of rank 4 containing two systems Σ_1 and Σ_2 of maximal singular subspaces (all of dimension 3; they are also called *generators*) with the property that two maximal singular subspaces intersect in a singular subspace of dimension 0 or 2 if and only if they belong to a different system. Each system of generators can be identified with one type of vertices of Δ. Without loss of generality, we assume that θ maps points to elements of Σ_1, maps Σ_1 to Σ_2 and maps Σ_2 to the point set of H. Notice that the planes of H correspond with flags of Δ containing an element of Σ_1 and one of Σ_2.

We will identify Δ with H. In particular, we will view each element of Δ as a set of points (the points incident with that element).

In order to prove Main Result 2.1, it suffices to prove that, if θ is a triality of type I_{id}, then it is line-domestic, and that, if θ is a domestic tairality, it is a triality of type I_{id}. We begin with the former.

Let θ be a triality of type I_{id}, and let L be a line of H. Then L contains at least one absolute point p (as the set of absolute points consists of a parabolic subquadric, see Paragraph 5.2.2 of [11]), and $L^\theta \subseteq p^\theta$. It follows that L and L^θ are not opposite, hence θ is line-domestic. Note that this has also been observed in [4] (Section 3) using a coordinate calculation.

So from now on we assume that θ is a domestic tairality, and we aim to show that it is a triality of type I_{id}. We first show that θ is line-domestic.

Lemma 3.1 *If θ is domestic, then it is line-domestic.*

Proof Suppose, for a contradiction, that some line L is mapped onto an opposite line L^θ. Let p be an arbitrary point on L. Then there is a unique member of Σ_1 incident with L not opposite p^θ, hence we can choose $U_1 \in \Sigma_1$ opposite p^θ. Likewise, there is a unique member of Σ_2 incident with L and not opposite U_1^θ, and there is a unique point q on L^θ not opposite p. Since there are at least three elements of Σ_2 incident with L, we can choose $U_2 \in \Sigma_2$ incident with L such that U_2 is opposite U_1^θ and such that $U_2^\theta \neq q$. Since we now have that p is opposite U_2^θ, that U_1 is opposite p^θ, and that U_2 is opposite U_1^θ, we conclude that $\{p, L, U_1, U_2\}$ is a chamber that is mapped onto an opposite chamber, a contradiction to the domesticity. □

We call a point p of H *absolute* if $p \in p^\theta$. Likewise, a member U of $\Sigma_1 \cup \Sigma_2$ is *absolute* if U^θ is incident with U, i.e., if $\{U, U^\theta\}$ is a flag of Δ, or, equivalently, if $U \cap U^\theta$ is a (singular) plane of H if $U \in \Sigma_1$, and $U^\theta \in U$ if $U \in \Sigma_2$. Note that $U \in \Sigma_1 \cup \Sigma_2$ being absolute is equivalent with U^θ being absolute.

Here is a fundamental observation.

Lemma 3.2 *Let L be a line such that (L, L^θ) is a special pair. Then the unique point p on L collinear with all points of L^θ is absolute.*

Proof Suppose, for a contradiction, that p is not absolute. Then p^θ does not share any point with L. Let π be the plane in p^θ all of whose points are collinear with p. Note that $L^\theta \subseteq \pi$. Let M^θ be a line in p^θ intersecting L^θ in a point x, and not contained in π. We choose x different from the unique point on L^θ collinear with all points of L. Then M is a line through p contained in a plane α together with L. By varying M^θ, we may assume that α does not meet L^θ. It follows that M and M^θ are not concurrent. Since p is not collinear to every point of M^θ, we see that M and M^θ are not contained in a common singular subspace. By domesticity of θ, the pair (M, M^θ) is special. Since p is not collinear to all points of M^θ, but it is collinear to x, it follows that x is collinear to all points of M. Since x is not collinear with all points of L, it is not collinear with all points of a line K in α through p, with $L \neq K \neq M$. But K^θ contains x and is not contained in π. Hence K and K^θ are opposite, a contradiction to the assumption that θ is a domestic tairality. Hence p is absolute. □

We now investigate properties of the set A of absolute points. Our first main aim in this respect is to show that θ has order 3 when restricted to A.

If p is an absolute point of θ, then we denote by π_p the intersection of p^θ with p^{θ^2}.

Lemma 3.3 *Let p be an absolute point of θ. Then all points of π_p are absolute.*

Proof Let x be an arbitrary point in π_p. Let L be any line in p^θ through x but not through p. Then L^θ is a line in p^{θ^2}, but not in p^θ. If $x \in L^\theta$, then, since $L^\theta \subseteq x^\theta$, clearly x is absolute. If $x \notin L^\theta$, then Lemma 3.2 implies that x is absolute. Hence every point x of $\pi := p^\theta \cap p^{\theta^2}$ is absolute. □

Lemma 3.4 *Let p be an absolute point of θ. Then $p \in p^{\theta^2}$.*

Proof Suppose by way of contradiction that $p \notin p^{\theta^2}$. Let x be an arbitrary point of π_p. By Lemma 3.3, we know that x is absolute. Then x^θ intersects π_p in a line L_x through x (since it intersects p^{θ^2} in a plane). It follows that $(px)^\theta = L_x$ is a line of π_p through x. But the map sending a line in p^θ through p to a line in π_p induces an isomorphism of projective planes; hence the mapping from π_p to π_p mapping the point x to the line L_x is the restriction to the points of a duality of π_p. But since $x \in L_x$, no flag can be mapped to an opposite, a contradiction to Theorem 3.1 of [8] and the fact that there are no symplectic polarities in a projective plane.

This contradiction proves the lemma. □

Lemma 3.5 *Let p be an absolute point of θ. Then every line in π_p through p is fixed under θ. Also, $p = p^{\theta^3}$.*

Proof Suppose, for a contradiction, that some line L through p in π_p is not fixed under θ. Since $p \in L$ and $L \subseteq p^\theta$, the image L^θ is contained in π_p. Put $z = L \cap L^\theta$. Let $y \in L \setminus \{p, z\}$. Since, by Lemma 3.3, y is absolute, $y \in y^\theta$. Since $y \in L$, we also have $L^\theta \subseteq y^\theta$. Hence $\pi_p \in y^\theta$, implying $p^\theta = y^\theta$, a contradiction. Hence L is fixed.

Now suppose $p \neq p^{\theta^3}$. Lemma 3.4 implies $p \in p^{\theta^2}$; hence p^θ contains p^{θ^3}. It follows that $p^{\theta^3} \in \pi_p$. The image of a line through p in π_p is a line through p^{θ^3} in π_p. Now, since $p \neq p^{\theta^3}$, there exists a line L through p in π_p not through p^{θ^3}, and so L cannot be fixed, a contradiction. We conclude that $p = p^{\theta^3}$. □

Remark 3.6 Lemma 3.5 implies that no non-fixed line L is mapped onto a line L^θ such that L and L^θ are contained in a plane. Indeed, suppose that L and L^θ are contained in some plane, and $L \neq L^\theta$. Let $x = L \cap L^\theta$, then $x \in L^\theta \subseteq x^\theta$ and so x is absolute. Then $L \subseteq x^{\theta^{-1}} = x^{\theta^2}$. Since no point of $x^\theta \setminus \pi_x$ is collinear with any point of $x^{\theta^2} \setminus \pi_x$, one of L, L^θ is contained in π_x and hence fixed, a contradiction.

Lemma 3.7 *Let p be an absolute point of θ. If q is an absolute point of θ collinear with p and not contained in π_p, then there is a unique point $r \in \pi_p$ such that $\{q, p\} \subseteq \pi_r$. In particular, all points of the line pq are absolute.*

Proof First we claim that q is not contained in $p^\theta \cup p^{\theta^2}$. Indeed, suppose $q \in p^\theta$. The line $(pq)^\theta$ is contained in p^{θ^2} but not in π_p. Then q^θ contains $(pq)^\theta$, and so, if q were absolute, q would be collinear with a point of $p^{\theta^2} \setminus \pi_p$. Hence q would be collinear with all points of p^{θ^2}, a contradiction. The same argument with θ substituted by θ^2 (and noting that $(p^{\theta^2})^{\theta^2} = p^\theta$) shows that $q \notin p^{\theta^2}$. Our claim is proved.

Hence there is a unique line L in π_p through p all of whose points are collinear with q, and there is a unique plane π containing q and L. Let U be the unique member of Σ_2 containing π. Since $L^\theta = L$, we see that the point $r := U^\theta$ is contained in L, and is absolute. Now, since q is contained in $U = r^{\theta^2}$, it must be contained in π_r by the our claim above. Hence $\pi = \pi_r$.

Clearly r is unique as $\pi_s = \pi_r$ for every other point $s \in \pi_p$ such that $\{p, s\} \subseteq \pi_s$. □

Lemma 3.8 *The set of absolute points is a geometric hyperplane of H.*

Proof We start by showing that every line L contains at least one absolute point. If $L^\theta \cap L \neq \emptyset$, then for any $x \in L \cap L^\theta$, the singular subspace x^θ contains L^θ, and hence x; so x is absolute and contained in L. If $L \cap L^\theta = \emptyset$, then, by Lemma 3.1, either (L, L^θ) is a special pair, in which case the assertion follows from Lemma 3.2, or L and L^θ are skew lines in a singular 3-space U. If $U \in \Sigma_1$, then, since $L^\theta \subseteq U$, we see that $U^{\theta^{-1}} \in L$ and so $U^{\theta^{-1}}$ is contained in U and is an absolute point on L. If $U \in \Sigma_2$, then U^θ is a point on L^θ, and hence $p := U^\theta$ is an absolute point and $L \subseteq p^{\theta^{-1}} = p^{\theta^2}$. Also, $L = (L^\theta)^{\theta^{-1}} \subseteq U^{\theta^{-1}}$ and the latter is by Lemma 3.5 equal to p^θ. Hence $L \subseteq \pi_p$ and so every point on L is absolute by Lemma 3.3.

Hence every line of H contains at least one absolute point. Lemmas 3.5 and 3.7 complete the proof. □

Lemma 3.9 *The tairality θ is a triality.*

Proof If p is absolute, then $p^{\theta^3} = p$ by Lemma 3.5. If p is not absolute, then we consider the 3-space p^θ. Since, by Lemma 3.8, the set of absolute points is a geometric hyperplane of H, it follows that there is a plane π in p^θ consisting of absolute points. Moreover, p^θ

is the unique member of Σ_1 containing π. But, by Lemma 3.5, π^{θ^3} (pointwise image) coincides with π, hence $(p^\theta)^{\theta^3} = p^\theta$, implying that $p^{\theta^3} = p$. Consequently θ has order 3 and is a triality. □

We can now finish the proof of Main Result 2.1. Since θ is a triality, and since every absolute point is incident with at least three fixed lines, it follows from the second part of Paragraph 5.2.9 of [11] that θ is a triality of type I_σ, where σ is a field automorphism of order 3. However, since for every absolute point p, every line through p inside π_p is fixed, the first part of Paragraph 5.2.9 of [11] implies that $\sigma = \mathsf{id}$.

The proof of Main Result 2.1 is complete.

4 Proof of Main Result 2.2

In this section, we let θ be a tairality of a building Δ of spherical type D_4 (which we again view as the hyperbolic quadric H just like in the previous section) with the property that, for every line L which is not fixed by θ, the lines L and L^θ are not contained in adjacent chambers of Δ, and such that there exists at least one fixed line. We must show that θ is a triality of type I_{id}. Hence, by Main Result 2.1, it suffices to show that θ maps no line to an opposite line. Note that, by Remark 3.6 and the fact that a triality of type I_{id} is domestic, we know that trialities of type I_{id} do not map any non-fixed line L to a line which is contained in a chamber adjacent to some chamber containing L.

The outline of the proof is as follows. We set A to be the the union of all fixed lines (viewed as sets of points of H). Our first goal is the show that A is a subspace, i.e., if $x, y \in A$ are collinear in H, then all points of the line xy belong to A. Then we show that every line meets A nontrivially. Finally, we show that every line containing a point of A is not mapped onto an opposite line.

We again assume that θ maps points to elements of Σ_1, and its inverse maps points to elements of Σ_2. Note that every element of A is an absolute point. In fact, θ/A has order 3:

Lemma 4.1 *If $x \in A$, then $x^{\theta^3} = x$.*

Proof Let L be a fixed line containing x. Then $L = L^\theta \subseteq x^\theta$, and similarly $L \subseteq x^{\theta^2}$. Replacing x by x^{θ^3} and θ by θ^{-1}, we see that $x^{\theta^3} \in x^\theta \cap x^{\theta^2} =: \pi_x$. If $x \neq x^{\theta^3}$, we consider a line $M \subseteq \pi_x$ through x but not through x^{θ^3}. Then $M^\theta \subseteq \pi_x$ and M^θ contains x^{θ^3}. Hence M and M^θ are distinct and contained in a common plane, a contradiction. □

We again introduce the notation π_x for $x^\theta \cap x^{\theta^2}$.

Lemma 4.2 *If $x \in A$, then all points of π_x are in A and every line in π_x through x is fixed under θ.*

Proof If L is a line through x in π_x, then L^θ is again a line through x in π_x. Our main assumption now implies that $L = L^\theta$. It follows that every point in π_x belongs to A. $\square$

Lemma 4.3 *The union A of all fixed lines is a non-empty subspace.*

Proof By our main assumption, A is non-empty. Now let $x, y \in A$ with L a line of H containing x and y. If L is fixed, then we are done. Now suppose that L is not fixed. We claim that $y \notin x^\theta$. Indeed, suppose the contrary. Then $x \in L \subseteq x^\theta$ and so $L^\theta \subseteq x^\theta \cap x^{\theta^2} = \pi_x$. Also $L^\theta = x^\theta \cap y^\theta \ni y$. Hence $y \in \pi_x$ and so L is fixed due to Lemma 4.2. Our claim is proved.

Similarly $x \notin y^\theta$. Hence L^θ, which is the intersection of x^θ with y^θ, does not contain x or y. Consider the 3-space S determined by the points x and y and the line L^θ. Since S intersects x^θ and y^θ in planes, it belongs to Σ_2. So S^θ is a point p. Since x and y belong to S, we see that p is contained in $x^\theta \cap y^\theta = L^\theta$. Since S intersects x^θ and y^θ in planes (and hence is incident with them as elements of the building Δ), we see that $p \in x^{\theta^2} \cap y^{\theta^2}$. Hence $p \in \pi_x \cap \pi_y$ and Lemma 4.2 implies that the lines px and py are fixed. Consequently, these lines belong to π_p, and so $L \subseteq \pi_p$. Lemma 4.2 implies that all points of L belong to A. $\square$

Lemma 4.4 *The only points of A in $x^\theta \cup x^{\theta^2}$, for a point $x \in A$, are the points of π_x.*

Proof Let $y \in x^\theta \setminus \pi_x$ and suppose $y \in A$. Let L be a fixed line through y. Then y^θ intersects x^{θ^2} in a plane, and contains L; hence L contains a point z of x^{θ^2}. Obviously, $z \in \pi_x$, and so L is contained in x^θ. Applying θ, we see that $L = L^\theta$ is contained in x^{θ^2}, hence in π_x, a contradiction since y is not contained in π_x. Similarly if $y \in x^{\theta^2} \setminus \pi_x$ and $y \in A$ (interchange θ and θ^{-1}). $\square$

Lemma 4.5 *Every line of H contains a point of A.*

Proof Choose a point $p \in A$ arbitrarily. We can identify the lines of H through p with the points of a quadric H_p in $\mathsf{PG}(5, \mathbb{K})$ isomorphic to the Klein quadric over the field $\mathbb{K}$. The points of A collinear with p induce a subset A_p of the points of H_p. The plane π_p corresponds with a line L in A_p. Each point $x \in \pi_p$ produces a plane π_x through p and this plane π_x corresponds with a line L_x entirely contained in A_p and meeting L in a point. Note that by Lemma 4.4 L_x and L are not contained in a plane of H_p. We can consider three points x_1, x_2, x_3 in π_p, with x_1 and x_2 on distinct lines through p, and x_3 on the line px_1. Then the lines L, L_{x_1}, L_{x_2} generate a 3-space of $\mathsf{PG}(5, \mathbb{K})$ intersecting H_p in a hyperbolic quadric Q, which is entirely contained in A_p, by Lemma 4.3. The subspace generated by Q and L_{x_3} either is a cone over Q entirely contained in A_p, or intersects H_p in a quadric Q' isomorphic to a parabolic quadric $Q(4, \mathbb{K})$ in $\mathsf{PG}(4, \mathbb{K})$. In the former case, A_p contains a plane through L, contradicting Lemma 4.4. In the latter case, the set A_p induces a subspace of $Q(4, \mathbb{K})$ containing a grid and an extra line through one of the points of the grid. Now, a subspace of a generalized quadrangle containing an ordinary quadrangle is a (full) subquadrangle, and so A_p induces in $Q(4, \mathbb{K})$ a thick (full) subquadrangle. But Proposition 5.9.4 of [13] says exactly that every thick full subquadrangle of $Q(4, \mathbb{K})$ coincides with $Q(4, \mathbb{K})$ itself. It follows that A_p contains Q'. Note that Q' is a geometric hyperplane of H_p.

Now let K be any line of H and let a be any point of A. Since A contains π_a, there is at least one point p of A collinear with all points of K. Now the plane determined by p and K defines a line in H_p. That line intersects the geometric hyperplane Q' in at least one point, and so there is a line through p contained in A and intersecting K. The lemma is proved. □

One could now go on and show that A is a geometric hyperplane consisting of all points of a parabolic subquadric of H, and then show that the fixed lines define a generalized hexagon. But we can end the proof quicker by noting that θ is domestic.

Indeed, let L be any line of H, and let x be a point in $L \cap A$. Then $L^\theta \subseteq x^\theta \ni x$. Hence the point x of L is collinear to all points of L^θ and so L cannot be opposite L^θ. Consequently θ is domestic and Main Result 2.2 now follows from Main Result 2.1.

Appendix: A characterization of symplectic polarities in projective spaces

Here we show Proposition 2.3. It suffice to show that the duality θ of the projective space $\mathsf{PG}(n, \mathbb{K})$, $n \geq 3$, maps every point x to a hyperplane x^θ containing x, as soon as θ

maps no line to a subspace intersecting the line in precisely one point, and some point is absolute. Then the result follows from Lemma 3.2 of [8].

So let, by way of contradiction, x be a point with $x \notin x^\theta$. Let p be an absolute point. If $p \in x^\theta$, then the line px is mapped onto $x^\theta \cap p^\theta$, which contains p but not x, a contradiction. Hence all points of x^θ are non-absolute. Let L be a line in x^θ. Since every hyperplane meets L, every point, and hence also p, is contained in some y^θ, with $y \in L$. Now the line py is mapped onto a subspace containing p but not y, a contradiction again. The proposition is proved.

It is an open problem whether we can strengthen Main Result 2.2 to the assumptions that no line is mapped onto a line in a common plane, and there exists at least one absolute point. In the finite case, this works (and the proof, which we omit, uses the fact that every duality of a finite projective plane has at least one absolute point, see [5]).

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