Projective planes over quadratic 2-dimensional algebras

J. Schillewaert H. Van Maldeghem

Abstract

The split version of the Freudenthal-Tits magic square stems from Lie theory and constructs a Lie algebra starting from two split composition algebras [5, 20, 21]. The geometries appearing in the second row are Severi varieties [24]. We provide an easy uniform axiomatization of these geometries and related ones, over an arbitrary field. In particular we investigate the entry $A_2 \times A_2$ in the magic square, characterizing Hermitian Veronese varieties, Segre varieties and embeddings of Hjelmslev planes of level 2 over the dual numbers. In fact this amounts to a common characterization of "projective planes over quadratic 2-dimensional algebras", in casu the split and non-split Galois extensions, the inseparable extensions of degree 2 in characteristic 2 and the dual numbers.

1 Introduction, Notation and Main Result

1.1 Mazzocca-Melone axioms and $\mathcal{C}$-Veronesean sets

In this paper we present a far-going generalization of the Mazzocca-Melone approach to quadric Veronese varieties. In this introduction, we describe the formal situation, and mention some history. The compelling motivation for our approach (why exactly generalizing in the way we do) is explained in the final section of the paper in order not to interfere with the mathematical flow of the paper. The reader might want to read the final section first. It puts our result in the broader perspective of the Freudenthal-Tits Magic Square [5, 20, 21], certain alternative algebras, and representations of a class of spherical buildings (containing those having exceptional type E_6) in projective space. Let us just mention here that our intention is not just "generalizing", but rather a new geometric approach to the aforementioned magic square.

The Mazzocca-Melone axioms for Veronese varieties have proved to be of fundamental importance for the theory of Veronese varieties. Mazzocca and Melone [10] were the first ones to prove such a characterization (for quadric Veronese varieties of finite projective planes), and the same axioms, with only some minor changes depending on the context, were used by others to characterize finite quadric Veronese varieties of projective spaces [17], quadric Veronese varieties in general [13], finite Hermitian Veronese varieties [18], and Hermitian Veronese varieties in general [12]. In this paper, we introduce a further minor change in these axioms to include more varieties over arbitrary fields.

Let us start by briefly recalling the "classical" Mazzocca-Melone axioms, in their simplest form, namely, for the quadric Veronesean variety X of a projective plane $\mathsf{PG}(2,\mathbb{K})$ in the 5-dimensional projective space $\mathsf{PG}(5,\mathbb{K})$ (see [13]). One hypothesizes a family Π of planes each member of which intersects X in a conic (or, more generally, an *oval*, see below). The set X corresponds to the point set of $\mathsf{PG}(2,\mathbb{K})$, whereas the conics correspond to the lines of $\mathsf{PG}(2,\mathbb{K})$. The pair (X,Π) satisfies three properties. The first is that every pair of points in X is contained in a unique member of Π; the second is that every pair of planes in Π intersects inside X; the third is that the tangent lines at any point $x \in X$ to the conics obtained from Π and going through x are contained in a plane only depending on x. The axioms for a Hermitian Veronesean are the same, except that Π is replaced by a family of 3-spaces each member of which intersects the point set X in a quadric of Witt index 1. One possible way to go on would be to replace 3-spaces with n-spaces (see [8] for such an approach). Another generalization is to consider other classes of quadrics in 3-space. That is exactly what we will do. The achievement is then that the corresponding varieties precisely correspond in a uniform and explicit way to projective planes over 2-dimensional unital algebras.

So our first aim is to write down a satisfactory new version of the classical Mazzocca-Melone axioms that capture exactly the varieties we are aiming at. Two features have to be taken into account. The first one is that the quadrics no longer have Witt index 1; hence the set X will no longer be a *cap*, i.e., a set not containing 3 points on a line), which implies that more than one quadric can contain two different points. The second one is the quadrics no longer need to be nondegenerate, in which case the singular points do not belong to the set X.

We now discuss how to deal with the first feature, and we do this by considering the case of Segrean Veronesean sets; here the quadrics have Witt index 2 and are nondegenerate.

A Segre variety of type (2,2) is the image $\mathcal{S}_{2,2}$ of the direct product $\mathsf{PG}(2,\mathbb{K}) \times \mathsf{PG}(2,\mathbb{K})$ of the point sets of two isomorphic projective planes over a commutative field $\mathbb{K}$ under the mapping

$$\begin{aligned}\sigma:\ & \mathsf{PG}(2,\mathbb{K})\times\mathsf{PG}(2,\mathbb{K})\to\mathsf{PG}(8,\mathbb{K}):\\ & ((x,y,z),(x',y',z'))\mapsto(xx',xy',xz',yx',yy',yz',zx',zy',zz').\end{aligned}$$

(In general, for the definition of a Segre variety $\mathcal{S}_{n,m}$ of type (m,n), $m,n\geq 1$, one considers the direct product of the point sets of a projective m-space with a projective n-space and the obvious generalization of the mapping above.)

One observes that the image of the direct product of two lines inside these planes is a hyperbolic quadric in some 3-dimensional subspace of $\mathsf{PG}(8,\mathbb{K})$. Clearly, every pair of points in the image is contained in at least one such hyperbolic quadric, but considering a fixed line in one of the planes, we see that distinct hyperbolic quadrics contain the same line, and hence the same pairs of distinct points. Hence the first Mazzocca-Melone axiom has to be modified in this way. As for the other two axioms, we note that the automorphism group of $\mathcal{S}_{2,2}$ is transitive on the points, and on the hyperbolic quadrics (indeed, if a linear collineation acts on the first component with matrix A, and one on the second component with matrix B, then the Kronecker product $A\otimes B$ acts in a natural way on $\mathcal{S}_{2,2}$). Thus we can choose the coordinates conveniently to verify that (1) the intersection of two arbitrary 3-spaces each containing distinct such hyperbolic quadrics is a subset of $\mathcal{S}_{2,2}$, and (2) all generators of all hyperbolic quadrics through a fixed point are contained in a 4-space. Let us record in some more detail the resulting axiomatization.

A *hypo* H in a 3-dimensional projective space Σ (over $\mathbb{K}$) is the set of points of Σ on some hyperbolic quadric. For every point $x\in H$, there is a unique plane π through x intersecting O in two intersecting lines both of which contain x. The plane π contains all lines through x that meet H in only x and is called the *tangent plane* at x to H and denoted $T_x(H)$. The lines contained in a hypo are called *generators* of the hypo.

Let X be a spanning point set of $\mathrm{PG}(N,\mathbb{K})$, $N>3$ (possibly infinite), and let Ξ be a set of 3-dimensional projective subspaces of $\mathrm{PG}(N,\mathbb{K})$, called the *hyperbolic spaces* of X, such that, for any $\xi\in\Xi$, the intersection $\xi\cap X$ is a hypo $X(\xi)$ in ξ (and then, for $x\in X(\xi)$, we sometimes denote $T_x(X(\xi))$ simply by $T_x(\xi)$). We call X a *Segrean Veronesean set* if the following properties hold :

(S1) Any two points x and y in X lie in at least one element of Ξ, which is denoted by $[x,y]$ if it is unique.

(S2) If $\xi_1,\xi_2\in\Xi$, with $\xi_1\neq\xi_2$, then $\xi_1\cap\xi_2\subset X$.

(S3) If $x\in X$, then the planes $T_x(\xi)$, with $x\in\xi\in\Xi$, are contained in a common 4-dimensional subspace of $\mathrm{PG}(N,\mathbb{K})$, which we denote by $T(x)$.

Note that (S1) immediately implies that the cardinality of Ξ is at least 2, since $N > 3$ and X spans $\mathrm{PG}(N, \mathbb{K})$. This rules out the trivial case of X being a hyperbolic quadric in 3-space.

Now about the second feature, namely, the possibility to allow degenerate quadrics. Then Axioms (S1) and (S3) can remain the same, but it is clear that axiom (S2) has to be reformulated in a more general context. Indeed, we do not want the singular points of the degenerate quadric to belong to the variety. To make our point clearer, we will introduce the new axioms for a general class of "hypersurfaces".

Let $n \geq 1$ be a natural number. Let us call a point set $S \subsetneq \mathsf{PG}(n, \mathbb{K})$, $\mathbb{K}$ any skew field, a *hypersurface* if (1) for any point $x \in S$ the union $T_x(S)$ of the set of lines through x that either are contained in S or intersect S precisely in $\{x\}$ either forms a hyperplane of $\mathsf{PG}(n, \mathbb{K})$ (and then x is called a *regular point* of S), or is the whole point set of $\mathsf{PG}(n, \mathbb{K})$ (and then x is called a *singular point* of S), and (2) the set of regular points spans $\mathsf{PG}(n, \mathbb{K})$. Let $\mathcal{C}$ be a class of hypersurfaces of $\mathsf{PG}(n, \mathbb{K})$. Let there be given a set X of points spanning of some projective space $\mathrm{PG}(N, \mathbb{K})$, with $N > n$ and $\mathbb{K}$ still any skew field, and let Ξ be a collection of subspaces of $\mathrm{PG}(N, \mathbb{K})$ of common dimension n (the "$\mathcal{C}$-subspaces"), such that, for any $\xi \in \Xi$, the intersection $\xi \cap X$ is the set $X(\xi)$ of regular points of some hypersurface $\overline{X(\xi)}$ of ξ belonging to the class $\mathcal{C}$ (and then, for $x \in \overline{X(\xi)}$, we sometimes denote $T_x(X(\xi))$ simply by $T_x(\xi)$). Put $\overline{X}$ equal to the union of all $\overline{X(\xi)}$, $\xi \in \Xi$. We call X a $\mathcal{C}$-*Veronesean set* if the following properties hold :

(V1) Any two points $x, y \in X$ lie in at least one element of Ξ, denoted by $[x, y]$, if unique.

(V2) If $\xi_1, \xi_2 \in \Xi$, with $\xi_1 \neq \xi_2$, then $\xi_1 \cap \xi_2 \subset \overline{X(\xi_1)} \cap \overline{X(\xi_2)}$, and $\xi_1 \cap \xi_2 \cap (\overline{X} \setminus X)$ is contained in some subspaces of $\xi_1 \cap \xi_2$ of codimension 1.

(V3) If $x \in X$, then each of the subspaces $T_x(\xi)$, with $x \in \xi \in \Xi$, is contained in a $(2n-2)$-dimensional subspace of $\mathrm{PG}(N, \mathbb{K})$, denoted by $T(x)$.

Axiom (V1) is self-explanatory in the frame of projective planes over rings (every pair of points is contained in a least one line). Axiom (V2) expresses two facts: (1) two members of Ξ intersect in the same way as the corresponding hypersurfaces do, up to the singular points; (2) they intersect in the minimal possible dimension. The motivation here is to exclude certain projections of the varieties that we want to characterize. Finally, Axiom (V3) expresses the 4-dimensionality of the variety over $\mathbb{K}$, and hence bounds the dimension N. Indeed, as an algebraic variety, the tangent space has the same dimension as the variety, and it is the dimension of the tangent space that is bounded in (V3) in an abstract way. Deletion of (V3) would result in many more examples, such as the varieties corresponding to projective spaces over 2-dimensional unital algebras, even infinite dimensional ones, and many suitable projections. Also Veronesean representations of other linear spaces, such as Hermitian curves, in 6- and 7-dimensional projective space and related to triality (see [4]) satisfy (V1) and (V2). Finally, free constructions not related to any variety satisfy (V1) and (V2) but not (V3), and such objects are considered unclassifiable.

The original Mazzocca-Melone axioms amount to the case $n = 2$ and $\mathcal{C}$ the class of irreducible finite plane conics. Mazzocca & Melone characterized finite quadric Veronese varieties over fields of odd order [10]. This was later generalized by Hirschfeld & Thas to include all finite fields [7]. Thas & Van Maldeghem [16] then considered the case where $\mathcal{C}$ is the class of finite plane ovals (and proved each of these ovals must automatically be a conic; an *oval* is a set of points of a projective plane no three on a line and through each of its point a unique line intersecting the oval in one point), whereas Schillewaert & Van Maldeghem [13] classified $\mathcal{C}$-Veronesean sets for $\mathcal{C}$ the class of all ovals of any (finite or infinite) projective plane.

Cooperstein, Thas & Van Maldeghem [3] characterized finite Hermitian Veronese varieties as the $\mathcal{C}$-Veronesean sets, where $\mathcal{C}$ is the class of elliptic quadrics of finite 3-space. Thas & Van Maldeghem [18] generalized this to the class of ovoids of finite 3-space, whereas Schillewaert & Van Maldeghem [12] proved the same characterization for Hermitian Veronese varieties over any field, as $\mathcal{C}$-Veronesean sets, where $\mathcal{C}$ is the class of all ovoids of $\mathsf{PG}(3, \mathbb{K})$, with $\mathbb{K}$ any skew field.

In the present paper, we want to consider the case $n = 3$ and $\mathcal{C}$-Veronesean sets for $\mathcal{C}$ either the class of ruled quadrics of $\mathsf{PG}(3, \mathbb{K})$, with $\mathbb{K}$ an arbitrary field, or the class of oval cones in $\mathsf{PG}(3, \mathbb{K})$, with $\mathbb{K}$ any skew field. In the former case, we also speak of a *Segrean Veronesean set* (see above), whereas in the latter case we speak of a *Hjelmslevian Veronesean set*.

Let X be a Hermitian, Segrean or Hjelmslevian Veronesean set. We define the geometry $\mathcal{G}(X)$ as follows. The points are the elements of X, the lines are the members of Ξ and incidence is given by containment. Note that, for X a Hermitian Veronesean set, the geometry $\mathcal{G}(X)$ is a projective plane over a quadratic extension of $\mathbb{K}$, see [12]. But for the Segrean and Hjelmslevian cases, $\mathcal{G}(X)$ will be a ring geometry. The corresponding ring will be a 2-dimensional unital algebra over a field. So let us have a look at such geometries and algebras in the next subsection.

1.2 Planes over some 2-dimensional algebras

Let $\mathbb{K}$ be a (commutative) field and let V be a 2-dimensional unital algebra over $\mathbb{K}$. Let V_0 be the set of zero-divisors of V and put $V^* = V \setminus V_0$.

We have the following classification result. Note that all algebras are quadratic, i.e., every element of the algebra satisfies a quadratic equation with coefficients in $\mathbb{K}$.

Proposition 1.1 (Bourbaki [1], III $2 Proposition 3, p. 441) *Let V be a unital $\mathbb{K}$-algebra of dimension two. Then precisely one of the following holds.*

(a) *V is étale, i.e., $V/\mathbb{K}$ is either a separable quadratic field extension or $V = \mathbb{K} \oplus \mathbb{K}$ splits.*

(b) *$V = \mathbb{K}[\epsilon]; \epsilon^2 = 0$; is the algebra of dual numbers.*

(c) *$\mathbb{K}$ has characteristic 2, and $V/\mathbb{K}$ is a (purely) inseparable field extension (of exponent one).*

We define the following geometry $\mathcal{G}(V)$. The points are the classes of triples $V^*(x, y, z)$, $x, y, z \in V$, such that, if $v \in V$ and $v \times (x, y, z) = (0, 0, 0)$, then $v = 0$. The lines are the classes of triples $V^*[a, b, c]$, $a, b, c \in V$, such that, if $v \in V$ and $v \times [a, b, c] = [0, 0, 0]$, then $v = 0$. A point $V^*(x, y, z)$ is incident with a line $V^*[a, b, c]$ if $a \times x + b \times y + c \times z = 0$. We will usually omit the multiplication sign "$\times$" in the sequel.

A 3×3 matrix with entries in V will be said to *have rank* 1 if for every two rows R_1, R_2, there are elements $a, b \in V$ such that $aR_1 + bR_2$ is the zero-row, and such that no nonzero element c of V exists such that $ac = bc = 0$. Below we will only use this notion for "Hermitian matrices", where it is clear that one can substitute "row" by "column" in this definition without changing the meaning.

There are always three ways to represent $\mathcal{G}(V)$ in an 8-dimensional projective space over $\mathbb{K}$, and sometimes there is a fourth.

We denote the unique identity element in V with respect to $\times$ by $\mathbf{1}$. Proposition 1.1 implies that there exists a unique (linear) automorphism σ of V of order 2 in Case (a), of order at most 2 in Case (b), and of order 1 in Case (c), fixing $\mathbf{1}$ and such that both $v + v^\sigma$ and $v \times v^\sigma$ belong to $\mathbb{K} \cdot \mathbf{1}$, for all $v \in V$. We will henceforth identify $\mathbb{K} \cdot \mathbf{1}$ with $\mathbb{K}$.

1. Let V and σ be as above. The set of points of $\mathcal{G}(V)$ is in bijective correspondence with the set of rank 1 Hermitian 3×3 matrices over V (i.e., diagonal elements belong to $\mathbb{K} \cdot \mathbf{1}$ and corresponding symmetric elements are images under σ), up to a scalar multiple, by mapping a point $V^*(x, y, z)$ onto the matrix $(x\ y\ z)^t(x\ y\ z)^\sigma$, where t denotes transposition. Viewing the set of all Hermitian 3×3 matrices over V as a 9-dimensional vector space over $\mathbb{K}$, we see that this defines a point set X in $\mathsf{PG}(8, \mathbb{K})$. It is easy to see that the lines of the geometry correspond to images of rank 1 Hermitian 2×2 matrices over V, and these in turn define quadrics in $\mathsf{PG}(3, \mathbb{K})$. This hints to the fact that X is indeed a $\mathcal{C}$-Veronesean set, where $\mathcal{C}$ is either the class of all hyperbolic quadrics, or the class of all quadratic cones, or an isomorphism class of elliptic quadrics. For ease of reference, we call the set X here defined a *V-set defined by matrices.*

2. Consider the free module $\mathcal{W}$ of rank 3 over V, and reconsider $\mathcal{W}$ as a 6-dimensional vector space over $\mathbb{K}$. The set of points of $\mathcal{G}(V)$ defines a family of one-dimensional subspaces over V of $\mathcal{W}$, and this corresponds to a set of 2-dimensional subspaces over $\mathbb{K}$. This yields a set $\mathcal{L}$ of lines in the projective 5-space over $\mathbb{K}$. If we take the line Grassmannian of $\mathcal{L}$, then we end up in $\mathsf{PG}(14, \mathbb{K})$. But actually, one can prove that the image X of $\mathcal{L}$ under the line Grassmannian spans a subspace $\mathsf{PG}(8, \mathbb{K})$ of dimension 8. We will refer to X as a *V-set defined by reduction.*

   To see the lines, one must consider certain 2-dimensional submodules of $\mathcal{W}$ over V, which correspond to certain 4-dimensional subspaces of $\mathcal{W}$ over $\mathbb{K}$. Projectively, we have a line set in a projective 3-space, and the Grassmannian then lies on the Klein quadric; in fact we obtain a quadric in projective 3-space (again either an elliptic quadric, a ruled quadric, or a quadratic cone).

3. There is another construction leading to the same set of lines $\mathcal{L}$. Indeed, it follows from Proposition 1.1 that the algebra V is a matrix algebra, a subalgebra of the full 2×2 algebra over $\mathbb{K}$. Now the set of lines in the projective 5-space obtained above can also be obtained by juxtaposition of three arbitrary 2×2 matrices (corresponding to triples satisfying the same restrictions as in the definition of points of $\mathcal{G}(V)$) of the corresponding matrix algebra, and then taking the joins of the points represented by the rows of the 6×2 matrices thus obtained. The set X obtained by taking the image under the line Grassmannian will be referred to as a *V-set defined by juxtaposition.*

4. We provide a parameter representation of $\mathcal{G}(V)$. Let U be the vector space $\mathbb{K} \times \mathbb{K} \times \mathbb{K} \times V \times V \times V$ and identify $\mathsf{PG}(8, \mathbb{K})$ with $\mathsf{PG}(U)$. Then we let the point $V^*(x, y, z)$

correspond to the point in $\mathsf{PG}(8,\mathbb{K})$ having coordinates $(x^\sigma x, y^\sigma y, z^\sigma z, x^\sigma y, y^\sigma z, z^\sigma x)$ and one checks easily that this is independent of the chosen representative. We call this correspondence the *Veronese correspondence*. Then, by definition, the set of images of points of $\mathcal{G}(V)$ under the Veronese correspondence is X. The lines of $\mathcal{G}(X)$ are the images under the Veronese correspondence of the lines of $\mathcal{G}(V)$. Since each line involves two free parameters over V, hence four over $\mathbb{K}$, every line will be contained in a projective 3-subspace, and hence here we also see that this hints to a $\mathcal{C}$-Veronesean set with $\mathcal{C}$ the a set of quadrics in projective 3-space. We call the set X thus constructed a *V-set defined by parametrization.*

1.3 Main Result

Our main result connects the abstractly defined ring geometries with the notion of $\mathcal{C}$-Veronesean set as follows.

Main Result. *Let $\mathbb{K}$ be any field. Let V be a 2-dimensional unital algebra over $\mathbb{K}$. Then the V-set defined by matrices is isomorphic to the V-set defined by reduction, to the V-set defined by juxtaposition, and to the V-set defined by parametrization (which allows to briefly talk about V-sets). Also, such a V-set is either a Hermitian Veronesean, or a Segrean Veronesean, or a Hjelmslevian Veronesean set X, and $\mathcal{G}(V)$ is isomorphic to $\mathcal{G}(X)$. Conversely, every Hermitian, Segrean or Hjelmslevian Veronesean set in $\mathsf{PG}(N,\mathbb{K})$, $N \geq 8$, can be obtained from a V-set over $\mathbb{K}$ and hence is unique, up to projectivity (and then $N = 8$). Also, a Segrean Veronesean set is projectively equivalent to a Segre variety of type $(2,2)$ and the geometry $\mathcal{G}(X)$ for X a Hjelmslevian Veronesean set is a projective Hjelmslev plane of level 2 over the dual numbers over $\mathbb{K}$.*

For the definition of projective Hjelmslev plane of level 2, we refer to Proposition 3.13. We can also classify Segrean and Hjelmslevian Veronesean sets with $4 \leq N \leq 8$, but for that we refer to the exact statements in the next sections.

We are going to prove the Main Result in small pieces. The main work is done in the next two sections, where we first classify Segrean Veronesean sets, and then prove that Hjelmslevian Veronesean sets are projectively unique. We then briefly pause and mention some properties of the ring geometries $\mathcal{G}(V)$. At last, we show that each V-set is an appropriate $\mathcal{C}$-Veronesean set, establishing the Main Result. In the final section we put our investigations in a broader perspective, thus motivating the results of the present paper.

Remark 1.2 In fact, it is easy to check that V is a ring of stable rank 2, and that the geometry $\mathcal{G}(V)$ coincides with the projective plane over V, as defined by Veldkamp in [23]. We will make use of the results in [23] in Section 5

2 A characterization of Segrean Veronesean Sets

In this section, our main goal is to characterize *Segre varieties of type* $(2,2)$. More precisely, we prove

Theorem 2.1 *The point set X of a Segrean Veronesean set of index 2 is the point set of a Segre variety $\mathcal{S}_{1,2}$ (and then $N=5$), $\mathcal{S}_{1,3}$ (and then $N=7$) or $\mathcal{S}_{2,2}$ (and then $N=8$).*

From now on, we let X be a set in $\mathsf{PG}(N,\mathbb{K})$, $N \geq 4$ and possibly infinite, and Ξ a family of 3-spaces of $\mathsf{PG}(N,\mathbb{K})$ such that every member of Ξ intersects X in a hypo, and satisfying (S1) and (S2). Note that we want X to contain at least two hypos, which implies by (S2) that we can assume $N \geq 5$. We will denote the (projective) span of a set S of points of $\mathsf{PG}(N,\mathbb{K})$ by $\langle S\rangle$.

Our first aim is to show that X contains a plane. Call a line of $\mathsf{PG}(N,\mathbb{K})$ entirely contained in X a *singular* line. Similarly for *singular plane.*

Lemma 2.2 (1) *Every pair of intersecting singular lines not contained in a singular plane is contained in a unique hypo.*

(2) *Every quadrangle of singular lines such that no two consecutive sides are contained in a singular plane, is contained in a unique hypo, determined by any pair of intersecting lines of that quadrangle.*

Proof Suppose by way of contradiction that the intersecting singular lines L_1, L_2 are not contained in a hypo. Let x be any point of the plane π spanned by L_1 and L_2. Then we can choose two lines M_1, M_2 through x meeting both L_1 and L_2 in distinct points. By (S1), the line M_i, $i=1,2$, is contained in a hyperbolic space ξ_i. If $\xi_1=\xi_2$, then it would contain the plane generated by M_1 and M_2, and therefore also $L_1 \cup L_2$, contradicting our hypothesis. Hence $\xi_1 \neq \xi_2$ and so $x \in \xi_1 \cap \xi_2$ belongs to X by (S2). Consequently π is a singular plane, a contradiction.

Hence L_1, L_2 are contained in a hypo. This hypo is unique as otherwise the plane $\langle L_1, L_2\rangle$ would be singular as it would belong to the intersection of two distinct hyperbolic spaces.

The second assertion follows from considering the hypos through two opposite pairs of intersecting sides of the quadrangle, and applying the first part of the lemma, noting that opposite points of the quadrangle do not lie on a common singular line. □

An easy consequence is that a non-singular plane does not contain at least three singular lines.

Next we want to show that the singular lines through a point x span a space of dimension at least 4. When we later assume (S3), this space will then necessarily have to coincide with T_x.

Lemma 2.3 *If X does not contain singular planes, and $N \geq 5$, then through every point $x \in X$, there exist at least four singular lines. Also, four arbitrary singular lines through x span a 4-space. Moreover, given four singular lines L_1, L_2, L_3, L_4 through x, the hyperbolic spaces containing L_1, L_2, and L_3, L_4, respectively, meet only in $\{x\}$, which implies $N \geq 6$.*

Proof By considering an arbitrary hypo H through x, which exists by (S1) and the fact that X generates $\mathsf{PG}(N, \mathbb{K})$, $N \geq 5$, we see that there are at least two singular lines through x. We can now choose a point $y_1 \in X \setminus \langle H \rangle$ and consider a hypo H_1 containing x and y_1. Axiom (S2) implies that $H \cap H_1$ does not contain the two singular lines of H through x. Hence there is at least one more singular line through x. If no more singular lines through x existed, Lemma 2.2 would imply that there are exactly three hypos containing x.

Now choose a point $y \in H$ not on a singular line with x. Interchanging the roles of x and y in the previous paragraph, we can choose a singular line L through y not contained in H. If $|\mathbb{K}| \geq 3$, we want to show we get at least four hypos, which yields a contradiction. In this case, we can consider 2 points z_1, z_2 different from y and not in H_1. If there is more than one hypo through x and z_1 or more than one hypo through x and z_2 we are done. Otherwise, if $[x, z_1] = [x, z_2]$ then y would be collinear with X by (S2), a contradiction, so $[x, z_1] \neq [x, z_2]$ and we are done.

Now suppose $|\mathbb{K}| = 2$. If z is a point on a singular line with x, then the number of singular lines distinct from $\langle x, z \rangle$ through z is equal to the number of singular lines distinct from $\langle x, z \rangle$ through x (indeed, a bijection is given by "being contained in the same hypo with $\langle x, z \rangle$", and use Lemma 2.2). By connectivity, we obtain that through every point there are a constant number of singular lines. If this constant is equal to 3, then one counts $|X| = 19$, and a double count of the pairs (point,hypo), where the point is in the hypo, results in $19 \times 3 = n \times 9$, where n is the total number of hypos. Clearly a contradiction.

So we have at least four singular lines L_1, L_2, L_3, L_4 through the point $x \in X$. Now (S2) implies that the hyperbolic spaces (which really are hyperbolic spaces by Lemma 2.2 and the assumption that there are no singular planes) containing L_1, L_2, and L_3, L_4, respectively, meet only in $\{x\}$, since otherwise we would obtain more than two lines through x in a hypo. Consequently $N \geq 6$.

The lemma is proved. $\square$

Now we fix a hypo H. We want to study the projection of $X \setminus H$ from $\langle H \rangle$ onto some $(N-4)$-dimensional subspace F. In order to do so, we first prove some additional lemmas.

Lemma 2.4 *Suppose X does not contain singular planes. Let H be a hypo, and let L_1 and L_2 be two distinct singular lines of X meeting H in the points x_1, x_2, respectively. Then the subspace generated by H, L_1, L_2 is 5-dimensional.*

Proof If $x_1 = x_2$, then this follows from Lemma 2.3. Now suppose $x_1 \neq x_2$, and assume that the subspace generated by H, L_1, L_2 is 4-dimensional. Then the 3-space $\langle L_1, L_2 \rangle$ (which is a 3-space by (S2) and Lemma 2.2) intersects $\langle H \rangle$ in a plane π. Then π contains a point y not on H and not on the line $\langle x_1, x_2 \rangle$. It follows that y lies on a line M meeting both L_1 and L_2 in points, say z_1, z_2, respectively, not in $\langle H \rangle$. Since $y \in [z_1, z_2] \cap \langle H \rangle$, and $[z_1, z_2] \neq \langle H \rangle$, this contradicts (S2). $\square$

The next lemma is the last one before we invoke (S3). It basically says that $N \geq 7$.

Lemma 2.5 *Suppose X does not contain singular planes. Let H be a hypo and L a singular line on H. Let H_1 and H_2 be two different hypos distinct from H containing L. Then $W := \langle H, H_1, H_2 \rangle$ has dimension 7.*

Proof Since H, H_1, H_2 all contain L, we already have $\dim W \leq 7$. Suppose now, by way of contradiction, that $\dim W \leq 6$. Let M_i, $i = 1, 2$, be a singular line on H_i disjoint from L. Since $W_i := \langle H, H_i \rangle$ has dimension 5, because of (S2) and since there are no singular planes, for $i = 1, 2$, and since $W = \langle W_1, W_2 \rangle$, we see that $W_1 \cap W_2$ has dimension at least 4, hence 4 or 5. If it has dimension 5, then we choose a point m_1 on M_1 and we set $U = \langle H, m_1 \rangle$. If it has dimension 4, then we put $U = W_1 \cap W_2$. In both cases, U has codimension 1 in $\langle W_i \rangle$, for $i \in \{1, 2\}$. Since M_i, $i = 1, 2$, does not meet $\langle H \rangle$, U intersects M_i in exactly one point, which we may denote m_i. If we denote the generator of H_i through m_i distinct from M_i by R_i, then R_i intersects L, and hence H. Lemma 2.4

implies that $R_1 = R_2$. But then Lemma 2.2 tells us that $H_1 = H_2$, since both contain L and $R_1 = R_2$. This contradiction concludes the proof of the lemma. $\square$

We can now prove the following two important lemmas.

Lemma 2.6 *Suppose X does not contain singular planes, and assume that (X, Ξ) satisfies* (S3). *Let H be a hypo and L a singular line on H. Let x_1 and x_2 be two distinct points on L. Then the 5-spaces $U_1 = \langle H, T_{x_1}\rangle$ and $U_2 = \langle H, T_{x_2}\rangle$ meet in $\langle H\rangle$, so $\dim\langle H, T_{x_1}, T_{x_2}\rangle = 7$.*

Proof Let L_1 and L_1' be two singular lines through x_1 not in H. By Lemma 2.3, T_{x_1} is generated by L_1, L_1' and the two singular lines of H passing through x_1. The unique hypos G_1 and G_1' containing L, L_1 and L, L_1', respectively, contain some lines L_2 and L_2', respectively, through x_2 distinct from L. Clearly $\langle H, G_1, G_1'\rangle = \langle U_1, U_2\rangle$, which is, by Lemma 2.5, 7-dimensional. It follows that $U_1 \cap U_2$ is 3-dimensional, and hence coincides with $\langle H\rangle$. $\square$

A natural property to investigate is whether the tangent space T_x at some point $x \in X$ contains points of X not on a singular line with x. The following lemma will imply that this is not the case.

Lemma 2.7 *Suppose X does not contain singular planes, and assume that (X, Ξ) satisfies* (S3). *Let $x \in X$ be arbitrary, and let H be an arbitrary hypo containing x. Then all points of $\langle H, T_x\rangle$ in X either lie on H, or are on a singular line together with x. In particular, all points of $T_x \cap X$ are contained in a singular line with x.*

Proof Suppose by way of contradiction that some point $y \notin H$, for which $\langle x, y\rangle$ is not a singular line, is contained in $\langle H, T_x\rangle$. Let H' be the unique hypo through x and y. Then $H' \subseteq \langle H, T_x\rangle$, and so, since the latter is 5-dimensional, $\langle H'\rangle \cap \langle H\rangle$ is 1-dimensional. This implies that H' and H share some singular line L through x, and hence there is a singular line M containing y and a point z of L. By assumption, $z \neq x$. But the space $\langle H, T_x, M\rangle$ is 5-dimensional, hence, since $M \subseteq T_z$ and M is not in $\langle H\rangle$, this implies that the dimension of $\langle H, T_x, T_z\rangle$ is at most 6, contradicting Lemma 2.6. $\square$

Lemma 2.8 *Suppose X does not contain singular planes, and assume that (X, Ξ) satisfies* (S3). *Then $N = 8$.*

Proof We first show that $N > 7$. Indeed, suppose $N = 7$. Fix an arbitrary hypo H and two distinct points x_1, x_2 on a singular line L of H. Choose two singular lines L_1, L_1' through x_1 outside $\langle H\rangle$. Let G_1 be the hypo through L_1, L_1'. Then $\langle H, G_1\rangle$ is 6-dimensional by Lemma 2.3. Let S be a 3-space skew to $\langle H\rangle$. Then the projection of $G_1 \setminus (L_1 \cup L_1')$ from $\langle H\rangle$ onto S is an affine part π^* of some plane π of S. The projection of T_{x_2} is a line T_2 in S. Noting that the line $\pi \setminus \pi^*$ is the projection of T_{x_1}, it follows from Lemma 2.4 that T_2 intersects π in a point z' of π^*. Let z be the point of G_1 projected onto z'. Then z lies in the 5- space $\langle H, T_{x_2}\rangle$, contradicting Lemma 2.7.

Now we show that $N \leq 8$. Indeed, suppose by way of contradiction that $N > 8$. Let H again be an arbitrary hypo, and let x_1, x_2, L be as above. Since $W := \langle H, T_{x_1}, T_{x_2}\rangle$ is 7-dimensional, by Lemma 2.6, we can choose two points $y, z \in X$ such that the line $\langle y, z\rangle$ is skew to W. Then any hyperbolic space containing y and z has at most a line in common with W, implying that we find a singular line R skew to W. We claim that there can be at most one singular line R_i connecting a point of R with a point of M_i, where M_i is the singular line of H through x_i distinct from L, for $i = 1, 2$. Indeed, if there were at least two such lines, then Lemma 2.2 would imply that R and M_i lie in some hypo together, and so some point of R would be contained in T_{x_i}, a contradiction to the choice of R. Hence we can find a point v on R not on any singular line meeting $M_1 \cup M_2$ nontrivially. This implies that $X([v, x_1])$ and $X([v, x_2])$ meet H in x_1 and x_2 respectively. Consequently, T_v meets T_{x_i} in a line, for $i = 1, 2$, and these lines are skew as they do not meet $\langle H\rangle \supseteq T_{x_1} \cap T_{x_2}$. Hence T_v intersects W in a 3-space W_v. But T_v also contains R, and our choice of R implies that $\langle W_v, R\rangle$ is 5-dimensional, contradicting Axiom (S3). $\square$

We can now show that there is at least one singular plane.

Lemma 2.9 *Assume that* (X, Ξ) *satisfies* (S3)*. Then there exists a singular plane in* X*.*

Proof Suppose on the contrary that there is no singular plane in X. Let H be an arbitrary hypo, and, as above, let x_1, x_2 be two points on H on a common singular line L. Let L_1, L_1' be two distinct singular lines through x_1, not inside H, which exist by Lemma 2.3. Let H_1 be the hypo through L_1 and L and let L_2 be the singular line of H_1 not in H and passing through x_2. Finally, let L_2'' be an arbitrary singular line through x_2, distinct from L_2 and not in H. We denote the hypo through L_1 and L_1' by G_1. By Lemma 2.3, we have $\langle G_1\rangle \cap \langle H\rangle = \{x\}$. By Lemma 2.8, we have $N = 8$ and so we can choose a 4-dimensional subspace F skew to $\langle H\rangle$. With "projection", we mean in this proof the projection from $\langle H\rangle$ onto F. As in the proof of Lemma 2.8, the projection of $\langle G_1 \setminus (L_1 \cup L_1')\rangle$ is a set of points of an affine plane π_1^* in F, with projective completion π_1.

Likewise the projection of the hypo G_2 through L_2, L_2'', except for L_2 and L_2', is an affine plane π_2^* in F, with projective completion π_2. The projective planes π_1 and π_2 meet by a dimension argument. Also, by Lemma 2.6, the intersection $(\pi_1 \setminus \pi_1^*) \cap (\pi_2 \setminus \pi_2^*)$ is empty.

Suppose now that a point of π_1 coincides with a point at infinity of π_2 (so belonging to $\pi_2 \setminus \pi_2^*$). This means that a point z of G_1 is contained in $\langle H, T_{x_2}\rangle$, contradicting Lemma 2.7.

Hence, by symmetry, the affine planes π_1^* and π_2^* meet in a unique (affine) point and so we have points z_1 in G_1 and z_2 in G_2 lying in a common 4-space S with H. We now prove that $z_1 = z_2$. To that aim, suppose $z_1 \neq z_2$. Let G be a hypo through z_1, z_2. Considering $G \cap \langle H\rangle$ and (S2), we see that $\langle z_1, z_2\rangle$ is a singular line hitting H in some point u. By possibly interchanging the roles of x_1 and x_2, we may assume that $\langle u, x_2\rangle$ is not a singular line ($u \notin L$ because otherwise L belongs to G_1 since by Lemma 2.2, there is a (unique) hypo through z_1, x_1, L). Note that z_1 does not belong to T_{x_2} (as z does not belong to the projection $\pi_2 \setminus \pi_2^*$ of T_{x_2}). So we can consider the unique hyperbolic space $[z_1, x_2]$ and the corresponding hypo G_2'. Suppose that G_2' contains a singular line of H. Then we can find a quadrangle of singular lines containing z_1, u and two singular lines of H. By Lemma 2.2, this implies that z_1 belongs to H, a contradiction. Hence the two singular lines of G_2' through x_2 do not lie in $\langle H\rangle$ and consequently, the projection from $\langle H\rangle$ of $[z_1, x_2] \setminus \{x_2\}$ coincides with the plane π_2. Now, the arguments in the previous paragraph imply that π_1 and π_2 span F, and this implies that the singular lines in G_1 and G_2' through z_1 span a 4-dimensional space which must necessarily coincide with T_{z_1}, and which is projected onto F. Consequently T_{z_1} is disjoint from $\langle H\rangle$, contradicting $u \in T_{z_1}$.

Hence we have shown that $z_1 = z_2$. Now let M_i be the singular line in G_i meeting L_i, $i = 1, 2$. Note that $M_1 \neq M_2$ as otherwise the quadrangle M_1, L_1, L, L_2 and Lemma 2.2 imply that $G_1 = G_2$. Remember that L_1, L_2 are contained in the hypo H_1, and so there is a singular line intersecting L_1 and containing the intersection point of L_2 and M_2. This gives us again a quadrangle of singular lines (it cannot degenerate to a triangle, as this contradicts the hypothesis that we do not have singular planes). Again, Lemma 2.2 implies that z_1 belongs to H_1, a contradiction.

This contradiction finally implies that a singular plane exists. $\square$

Next we show that if there is a *singular* 3*-space* in X, we obtain $\mathcal{S}_{1,3}$.

Lemma 2.10 *Assume that* (X, Ξ) *satisfies* (S3). *If there is a* 3*-space entirely contained in* X*, then* X *is* $\mathcal{S}_{1,3}$.

Proof Suppose that some 3-space S is entirely contained in X. Let x be any point of S and let H be a hypo through x. Then H cannot contain two lines of S. If none of the singular lines L_1, L_2 of H through x belongs to S, then (S2) implies that the intersection of the plane $\langle L_1, L_2\rangle$ with S, which is a line by (S3), belongs to H, a contradiction. Hence exactly one of L_1, L_2 belongs to S.

In fact, there is a unique singular line L_x through x not in S. Indeed, in the previous paragraph we showed that there was at least one. If there were at least two, then the plane generated by these would contain a third one (in S), and so this would be a singular plane π by Lemma 2.2. If some point p in T_x did not belong to X, then we find two intersecting lines L_S and L_π in S and π, respectively, such that $p \in \langle L_S, L_\pi\rangle$. As in the proof of Lemma 2.2, this implies that $L_S \cup L_\pi$ is contained in a unique hypo H_L. We now consider another pair of intersecting lines M_S, M_π, in S and π, respectively, such that $L_S \cap L_\pi \neq M_S \cap M_\pi$, and such that $p \in \langle M_S, M_\pi\rangle$. Then M_S and M_π are contained in a unique hypo $H_M \neq H_L$, implying that p, which belongs to $\langle H_L\rangle \cap \langle H_M\rangle$, belongs to X after all. Hence we see that this would imply that all lines of T_x are singular, a contradiction (this would imply that there are no hypos through x).

Considering a hypo through any point $y \in X \setminus S$ and any point $x \in S$, we see that every point y of X not in S is on a (necessarily unique) singular line that intersects X.

Now let y_1, y_2 be two points of $X \setminus S$, and let x_1, x_2 be the unique points of S on a singular line with y_1, y_2, respectively. Suppose that $\langle y_1, y_2\rangle$ is not a singular line. Since the hypo containing y_1, x_2 automatically contains the singular line through x_2 not contained in S, it also contains y_2 and hence it coincides with the hypo through y_1, y_2. We have shown that all hypos have a unique line in S. This immediately implies that there is no hypo containing two intersecting singular lines not meeting S. The same argument as in the proof of Lemma 2.2 implies that all singular lines through a point $y \in X \setminus S$ distinct from the one intersecting S in a point, form a singular subspace S_y, which is thus defined by any of its points. It follows that, using the hypos through a point of S and a point of S_y, there is a (bijective) collineation between S and S_y given by "lying on a common singular line", and hence S_y has dimension 3. Also, all points of X are on lines intersecting both S and S_y (we can see this also by letting S_y play the role of S). Since S and S_y generate a space of dimension seven, Theorem 3 of [25] implies that we obtain $S(1, 3)$. $\square$

From now on we assume that X does not contain a singular 3-space so that X is not a Segre variety of type $(1, 3)$.

Our next goal is to single out $\mathcal{S}_{1,2}$ and to show that otherwise every point of a singular plane belongs to a second singular plane. We first show that, if X is not $\mathcal{S}_{1,2}$, then there must be a lot of other singular lines through a point of such singular plane besides the ones in that singular plane.

Lemma 2.11 *Assume that* (X, Ξ) *satisfies* (S3). *Let* π *be a singular plane and let* x *be a point in* π. *Then either there are at least three singular lines through* x *not contained in* π, *or* $N = 5$ *and* X *is projectively equivalent to* $\mathcal{S}_{1,2}$.

Proof Considering an arbitrary hypo through x we see that there must be at least one singular line through x not in π. Suppose now that there is only one such, namely L_x. Suppose that there are at least two singular lines L_y, L'_y not in π but through some other point $y \in \pi$. Then there must be hypos H containing L_y and $\langle x, y \rangle$ and H' containing L'_y and $\langle x, y \rangle$. These must automatically contain L_x, but the intersection of the corresponding hyperbolic spaces is then a plane, a contradiction. Hence through every point of π there is a unique singular line not contained in π. As in the proof of Lemma 2.13, we deduce that X is contained in the space generated by two disjoint planes. Similarly to the last part of the proof of Lemma 2.13, we obtain the Segre variety $\mathcal{S}(1, 2)$.

Now suppose that there are exactly two lines L_x, L'_x through x not in P. Then there is a hypo H_x containing these.

Consider three points x_1, x_2, x_3 on a line M in π not through x, and a point t on L_x. Consider the hypos H_1, H_2 and H_3 through L_x and x_1, x_2, x_3, respectively. Denote the lines through t different from L_x and contained in these hypos by M_1, M_2 and M_3 respectively. The points x_1, x_2 and x_3 are collinear with points t_1, t_2 and t_3 on M_1, M_2 and M_3 respectively.

Suppose first that M_1, M_2 and M_3 are not coplanar. Now consider the hypos H'_1, H'_2 and H'_3 determined by M and t_1, t_2 and t_3, respectively. Since t_1, t_2 and t_3 are not collinear we obtain three different lines L_1, L_2 and L_3 through x_2 outside of π. Now looking at hypos determined by L_i and x yields three different lines through x, a contradiction.

Hence M_1 and M_2 are contained in a singular plane. Now choose a point z on L'_x and a point z' on M_1, but not on L_x. If $\langle z, z' \rangle$ were singular, then the quadrangle $L_x, L'_x, \langle z, z' \rangle, M_1$ would show, using Lemma 2.2(2), that M_1 belongs to H_x, a contradiction. Let H_z be the hypo through z and z'. Let L_z be a singular line of H_z through z. Suppose that L_z is contained in H_x. Let z'' be the unique point on L_z such that $\langle z', z'' \rangle$ is singular, and let z''' be the unique point on L_z such that $\langle t, z''' \rangle$ is singular. Considering the quadrangle $L_z, \langle t, z''' \rangle, M_1, \langle z', z'' \rangle$, it follows from Lemma 2.2(2) that M_1 belongs to H_x, a contradiction. Hence no singular line through z in H_z belongs to H_x. But now interchanging the roles of L_x and L'_x, and of t and z, we arrive at a contradiction as in the previous paragraph.

The lemma is proved. □

From now one, we assume that through any point x of any singular plane π, there are at least three singular lines not contained in π. We can now show the penultimate step.

Lemma 2.12 *Assume that* (X, Ξ) *satisfies* (S3). *Let* π *be a singular plane and* $x \in \pi$. *Then there is a second singular plane through* x. *The union of both planes contains all singular lines through* x.

Proof By the previous lemma there are at least three singular lines through x not in π, name them L_1, L_2, L_3. If they are contained in a plane, then this plane is singular. If they are not contained in a plane, then the 3-space they generate contains a line L_4 of π (using (S3)). If no pair of $\{L_1, L_2, L_3\}$ is contained in a singular plane, then the planes $\langle L_1, L_2 \rangle$ and $\langle L_3, L_4 \rangle$ are distinct and hence meet in line L_5. Lemma 2.2(1) implies that L_5 is a singular line, and therefore, $\langle L_3, L_4 \rangle$ is singular after all.

So we always have at least two singular planes. If another singular line through x existed (not contained in either singular plane), then it would be contained in a plane together with a line of each singular plane. This now leads to a singular 4-space, a contradiction. □

It follows from the previous lemmas that through every point of the two singular planes through x there are precisely two singular planes. By connectivity, this is true for all points of X.

Lemma 2.13 *Assume that* (X, Ξ) *satisfies* (S3) *and that there is a point* $x \in X$ *contained in two singular planes. Then* $N \geq 8$.

Proof Let L_1, L_2, L_3, L_4 be singular lines through x such that $\langle L_1, L_2 \rangle$ and $\langle L_3, L_4 \rangle$ are the two singular planes through x. Let H be the hypo containing L_1, L_3 and let y be a point in the hypo G defined by L_2, L_4, not on a singular line with x. Let M_1, M_2 be the singular lines of G through y. Suppose for a contradiction that y is on a singular line M with some point z of H. Then $z$does not belong to the singular planes through x as otherwise there is a singular plane through y intersecting a singular plane through x in a line. Now, the singular line $\langle y, z \rangle$ is contained in a singular plane with one of M_1, M_2 and this leads to the contradiction that z is collinear with at least two points of one of the singular plans through x. Hence no point of H is on a singular line with y.

Suppose now some point u is contained in $\langle H \rangle \cap T(y)$. Then there exist singular lines N_1, N_2 through y such that $u \in \langle N_1, N_2 \rangle$. But then Axiom (S2) ensures that $u \in X$, a contradiction to the conclusion in the previous paragraph. Hence $N \geq \dim\langle H\rangle + \dim T(y) + 1 = 8$. $\square$

Now we can finish the proof of Theorem 2.1.

Consider two disjoint singular planes. By considering appropriate hypos, we see that every singular plane intersecting one of them, intersects the other. We can use this to see that, starting from two intersecting singular planes π, π', the set X is the disjoint union of all singular planes meeting π and likewise for those meeting π'. Moreover, as in the proof of Lemma 2.13, the hypos define collineations between disjoint planes and this is enough to conclude with Proposition 1.2 of [25] that $N \leq 8$, hence $N = 8$ and Theorem 3 of [25] implies that we have a Segre variety of type $(2, 2)$.

3 Hjelmslevian Veronesean Sets

In this section, we consider the Mazzocca-Melone axioms for the class of "tubes". Let us rewrite the general axioms (V1), (V2) and (V3) for this special case.

A *tube* C (as a short synonym for "cylinder") in a 3-dimensional projective space Σ (over $\mathbb{K}$) is the set of points of Σ on some nondegenerate cone with base a plane oval O, except for the vertex v (v will also be called the *vertex* of the tube, although it does never belong to the tube). For every point $x \in C$, there is a unique plane π through x intersecting C in the unique generator which contains x. The plane π contains all lines through x that meet C in only x and is called the *tangent plane* at x to C and denoted $T_x(C)$.

Let X be a spanning point set of $\mathrm{PG}(N, \mathbb{K})$, $N > 3$, and let Ξ be a collection of 3-dimensional projective subspaces of $\mathrm{PG}(N, \mathbb{K})$, called the *cylindric spaces* of X, such that, for any $\xi \in \Xi$, the intersection $\xi \cap X$ is a tube $X(\xi)$ in ξ (and then, for $x \in X(\xi)$, we sometimes denote $T_x(X(\xi))$ simply by $T_x(\xi)$). Conversely, for a tube C, we denote $\Xi(C)$ the unique member of Ξ containing C. We denote by Y the set of all vertices and we obviously have $X \cap Y = \emptyset$. We call (X, Ξ), or briefly X, a *Hjelmslevian Veronesean set (of index* 2*)* if the following properties hold :

(H1) Any two points x and y lie in at least one element of Ξ, which we denote by $[x, y]$ if it is unique.

(H2) If $\xi_1, \xi_2 \in \Pi$, with $\xi_1 \neq \xi_2$, then $\xi_1 \cap \xi_2 \subset X \cup Y$, and $\xi_1 \cap \xi_2 \cap Y$ is contained in a codimension 1 subspace of $\xi_1 \cap \xi_2$. If $Y \cap \xi_1 \cap \xi_2 \neq \emptyset$, then $Y \cap \xi_1 \neq \emptyset \neq Y \cap \xi_2$.

(H3) For each $x \in X$, all planes $T_x([x,y])$, $y \in X \setminus \{x\}$, are contained in a fixed 4-dimensional subspace of $\mathrm{PG}(N, \mathbb{K})$, denoted by T_x.

We again have that (H1) implies that $|\Xi| > 1$.

The rest of this section is devoted to prove the following theorem.

Theorem 3.1 *If (X, Ξ) is a Hjelmeslevian Veronesean set in $\mathsf{PG}(N, \mathbb{K})$, $N \geq 4$ (possibly infinite), then either $N = 6$ or $N = 8$. In either case, the point set X and the set of cylindric spaces Ξ are projectively unique.*

A *singular line* L is a generator of a tube; if $|\mathbb{K}| > 2$, then L is obviously characterized by saying that it contains at least three points of X, and then all of its points belong to X, except for one, which belongs to Y, and which we denote by $y(L)$. The set $L \setminus y(L)$ is sometimes referred to as a *singular affine line.*

The case $|\mathbb{K}| = 2$ will be treated separately at the end. So, for most of the results below, we assume $|\mathbb{K}| > 2$, although for the moment, we allow $|\mathbb{K}| = 2$.

Lemma 3.2 *Let C, C' be two tubes with common vertex. Then C and C' share a unique generator.*

Proof By Axiom (H2), the intersection $\Xi(C) \cap \Xi(C')$ is either a generator of both C and C' (in which case there is nothing to prove), or it is a point t of $X \cup Y$. In our assumptions, t is the vertex, and so belongs to Y. But this contradicts (H2) as a codimension 1 subspace of a point is the empty set. $\square$

Lemma 3.3 *Let C, C' be two tubes and suppose $|\Xi(C) \cap \Xi(C')| > 1$. Then $C \cap C'$ is a singular affine line. In particular, C and C' have the same vertex.*

Proof From (H2) it follows that $\Xi(C) \cap \Xi(C')$ is a generator L of both C and C'. It suffices to show that the vertices of C and C' coincide. Let t and t' be these respective vertices, and assume $t \neq t'$. Then both t and t' belong to $X \cap Y$, a contradiction. $\square$

A plane π in $\mathsf{PG}(N, \mathbb{K})$ will be called *singular* if it contains a unique line L all of whose points belong to Y, and all other points of π belong to X. The line L will be referred to as the *radical line* of π.

Lemma 3.4 *Let L and L' be two singular lines intersecting in a point $y \in Y$. Then either L and L' belong to a unique common tube, or $\langle L, L' \rangle$ is singular.*

Proof Suppose L and L' do not belong to a unique common tube. Take a point p in the plane π spanned by L and L' arbitrarily, but not on $L \cup L'$. We consider two lines M, M' in π through p not incident with y. These lines intersect L and L', respectively, in the points $x_{LM}, x_{LM'}$ and $x_{L'M}$ and $x_{L'M'}$, respectively, using self-explaining notation. Arbitrary tubes C and C' through $x_{LM}, x_{L'M}$ and $x_{LM'}, x_{L',M'}$, respectively, satisfy $p \in \Xi(C) \cap \Xi(C')$. Now note that, if $C = C'$, then C contains L and L', a contradiction. Hence $C \neq C'$. By (H2), p belongs to $X \cup Y$. It is impossible that p belongs to X for all legal choices of p, since this would imply that there are lines all of whose points are contained in X. Hence we may assume that p belongs to Y (if $|\mathbb{K}| = 2$, then this is automatic, and we have a singular plane; so from now on we may assume $|\mathbb{K}| > 2$). Then all other points of $M \cup M'$ belong to X. The line $N = \langle p, y \rangle$ contains two elements of Y; hence we claim it can contain at most one element of X. Indeed, if it contained two points of X, then it would be contained in some member of Ξ, which contains only one element of Y, by Lemmas 3.2 and 3.3. Suppose now that N contains a unique point $x \in X$. Now note that every point z of $\langle L, L' \rangle \setminus N$ belongs to X, as the line $\langle p, z \rangle$ contains at least two points of X, namely the intersections with L and L'. So any line through x in $\langle L, L' \rangle$ only contains points of X, a contradiction. Hence $\langle L, L' \rangle$ is a singular plane. $\square$

Lemma 3.5 *Let C be a tube and $c \in C$. Then either $N = 6$ and $X \cup Y$ is projectively equivalent to a cone with vertex some point t and base a quadric Veronese variety of $\mathsf{PG}(2, \mathbb{K})$, with $Y = \{t\}$, or there exists some tube C' intersecting C in the singleton $\{c\}$.*

Proof Suppose that all tubes whose intersection with C contains c have a generator in common with C. Let $x \in X$ be arbitrary. Then any tube through x and c must have the same vertex t as C, by Lemma 3.3. Hence $\langle x, t \rangle$ is a singular line. Now we project X from t onto some hyperplane H not containing x and denote the projection operator by ρ. We define the following geometry $\mathcal{G}$. The point set is $\rho(X)$, and the line set is the family of projections of tubes with vertex t. In H, the points of $\mathcal{G}$ are projective points, and the lines of $\mathcal{G}$ are planar ovals. By Lemma 3.2, every pair of distinct lines of $\mathcal{G}$ has a unique point in common. Now let L and L' be two singular lines through t. Suppose that $\langle L, L' \rangle$ is singular. Let C' be any tube containing L. Choose a singular line M in $\langle L, L,' \rangle$ not though t and a tube D through M. Let z be a point of D not on M. Since by the beginning of our proof, $K := \langle t, z \rangle$ is a singular line, we have, by Lemma 3.4, that either

$\langle K, L\rangle$ is a singular plane, or K, L are contained in a unique tube D'. In the latter case, $\Xi(D) \cap \Xi(D')$ violates (H2), because it contains the two points z and $z' := L \cap M$ not on a singular line. In the former case, the line $\langle z, z'\rangle$ is singular, again a contradiction.

So we conclude that $\langle L, L'\rangle$ is not singular, and so, by Lemma 3.4, there is a unique tube containing both of them. This implies that every pair of points of $\mathcal{G}$ is contained in a unique line of $\mathcal{G}$. Hence $\mathcal{G}$ is a projective plane (since lines have at least three points). Since the 3-spaces of two different tubes intersect in a line, we deduce that the two planes generated by the two ovals corresponding to two lines of $\mathcal{G}$ meet in a unique point, which implies that the dimension of H is at least 4. We claim that H has dimension 5. Indeed, by the second main result of [13], the dimension of H is at most 5. So assume that H has dimension 4. Let D be any oval corresponding to a line of $\mathcal{G}$, and let $x \in D$. Let D_1, D_2 be two distinct ovals corresponding to lines of $\mathcal{G}$ containing x, and distinct from D. Let ζ be a 3-dimensional space containing D and not containing the tangent lines at x of D_1 and D_2 (ζ exists since there are at least three 3-spaces through the plane $\langle D\rangle$). Then ζ intersects D_1 and D_2 in distinct points x_1, x_2, respectively. There is a unique oval D' corresponding to a line of $\mathcal{G}$ containing x_1, x_2, and since x_1, x_2 belongs to ζ, the planes $\langle D\rangle$ and $\langle D'\rangle$ intersect in a point of $\langle x_1, x_2\rangle$, which is a point of $\mathcal{G}$ by Axiom (H3). But then D' contains three points on a projective line, a contradiction. Our claim is proved.

Now, again by the second main result of [13], $\mathcal{G}$ is isomorphic to the quadric Veronese variety of the projective plane $\mathsf{PG}(3, \mathbb{K})$. Adding t, we see that $N = 6$. $\square$

From now on we assume $|\mathbb{K}| > 2$ and we may also assume that X is not a point over a quadratic Veronese variety so that the second conclusion of Lemma 3.5 holds.

Lemma 3.6 *Two singular lines L, L' intersecting in a point $x \in X$ generate a singular plane.*

Proof Let t and t' be the members of Y on L, L', respectively. Then every point z on $\langle t, t'\rangle$ except for t and t' themselves is the intersection of two lines meeting $(L \cup L') \setminus \{t, t'\}$ in two points. Since $L \cup L'$ cannot be contained in a tube (since that tube would have two vertices t and t'), these two lines are contained in distinct tubes. Hence $z \in X \cup Y$. But at most one point on $\langle t, t'\rangle$ can belong to X, as otherwise a tube through $\langle t, t'\rangle$ would have at least two vertices. So, there exists a point $z_0 \in Y$ on $\langle t, t'\rangle$ contained in two singular lines of $\langle L, L'\rangle$. Noting that the plane $\langle L, L'\rangle$ contains already at least four distinct singular lines, the assertion follows from Lemma 3.4. $\square$

Lemma 3.7 *Let $x \in X$ be arbitrary. Then the set of points of X on a common singular line with x is either an affine plane, or an affine 3-space.*

Proof Let V_x be the set of points of X on a common singular line with x. Lemma 3.5 tells us that there are at least two singular lines L, L' through x. Hence, by Lemma 3.6, the affine plane consisting of the points of X in $\langle L, L' \rangle$ belongs to V_x. If there is some other singular line L'' through x not contained in $\langle L, L' \rangle$, then a repeated use of Lemma 3.6 shows that the set of points of the 3-space $\langle L, L', L'' \rangle$ belonging to X (and on a singular line with x) is an affine 3-space. A similar argument shows that, if there was still another singular line not contained in $\langle L, L', L'' \rangle$, then an affine part of T_x would be contained in V_x, but this contradicts the fact that T_x contains planes which intersect X in just an affine line.

The proof is complete. $\square$

We will call an affine 3-space as in the statement of Lemma 3.7 a *singular* 3*-space*.

Before stating the next proposition, we need a definition. Let $k \leq l; n = k + l + 1$ and take complementary subspaces Π, Π' in $\mathsf{PG}(n, \mathbb{K})$ of dimensions k and l respectively. Now choose normal rational curves C and C' in Π and Π' respectively and a bijection $\phi : C' \to C$ between them which preserves the cross-ratio. Then we define a *normal rational scroll* $\mathfrak{S}_{k,l} = \cup_{p \in C'} \langle p, p^{\phi} \rangle$. From now on we call $\mathfrak{S}_{1,2}$ a *normal rational cubic scroll.*

Proposition 3.8 *Let $y \in Y$ and let X_y be the set of all members x of X such that $\langle x, y \rangle$ is a singular line. Then $\langle X_y \rangle$ is a* 5*-dimensional subspace. Also, $X_y \cup \{y\}$ is a cone with vertex y on a normal rational cubic scroll. In particular, all tubes arise from quadratic cones. Also, for every $x \in X$, the set of points of X on a common singular line with x is an affine plane.*

Proof Let C be a tube with vertex y. Pick a generator L on C, and a singular plane π through L, which exists by Lemma 3.7. Let G be any generator of C distinct from L, and let M be any singular line in π through y distinct from L. Suppose $\langle G, M \rangle$ is singular. Take any point $t \neq y$ in $\langle G, M \rangle$, with $t \in Y$. Let G' be a line through t meeting G in a point x' of X, and let M' be a line through t meeting L in a point x'' of X. If $G' \cup M'$ is contained in a cylindric space, then the intersection with $\Xi(C)$, which contains $\langle x', x'' \rangle$, violates (H2), noticing that x' and x'' are not contained in a common generator of C. Hence, by Lemma 3.4, the plane $\langle G', M' \rangle$ is singular. But that would mean that all points of $\langle x', x'' \rangle$ belong to $X \cup Y$, also a contradiction (to $X(\Xi(C)) = C$).

Hence G and M define a unique tube C_{GM}. Now we fix two generators G, G' on C distinct from L. We denote by U an $(N-3)$-space skew to the plane $\alpha := \langle G, G' \rangle$, and we let ρ be the projection operator from α onto U. With "projection", we always mean ρ in the rest of this proof, except for the last paragraph.

The projection of C_{GM}, for any line M in π through y distinct from L, is an affine part of a line L_M; we denote the projection of the tangent plane to C_{GM} at any point of $G \cap X$ by p_M (it is the point at infinity of L_M with regard to the affine part alluded to above). Likewise, the projection of $C_{G'M}$ is a line L'_M minus a point p'_M. All these lines have a point in common with the projection of π, which is a line L_π. As two tubes never belong to the same 4-space due to (H2), we see that L_π differs from any such L_M and any such L'_M, and no L_M coincides with any $L'_{M'}$. Moreover, intersecting L_π defines a bijection from $\{L_M : M \text{ a line in } \pi \text{ through } y \text{ distinct from } L\}$ into $L_\pi \setminus \{\rho(L)\}$. All this now easily implies that all L_M and all L'_M above are contained in a unique plane β.

Now suppose that L is contained in a singular 3-space S. One easily checks that $\rho(S)$ cannot coincide with β. Consider a line N through y in S projecting outside β. As above, there is a unique tube C_{GN} containing G and N. This tube has to intersect all tubes $C_{G'M}$, with M a line in π distinct from L and incident with y, in distinct generators. Since ρ is injective on the set of generators of C_{GN} distinct from G, this easily implies that the projection of C_{GN} is contained in β, and hence $\rho(N)$ belongs to β, contradicting our choice.

We have shown that singular 3-spaces do not exist. Hence Lemma 3.7 implies that every singular line is contained in a unique singular plane. Denote the singular planes through G and G' by π_G and $\pi_{G'}$, respectively.

Now let P be any singular line through y, $P \notin \{G, G'\}$. Then P cannot be contained in both π_G and $\pi_{G'}$; suppose it is not contained in π_G. Then P and G are contained in a unique tube C_{PG}, and the arguments in the paragraph preceding the previous one show that the projection of this tube is contained in β. Hence $\langle X_y \rangle$ is a 5-dimensional subspace, as it coincides with $\langle \alpha, \beta \rangle$. But we can say more. Let L_{PG} be the projection of $C_{PG} \setminus G$. Then L_{PG} and L_π have a point z in common. Taking inverse images, we see that we have two cases. First case: there is a generator of C_{PG}, which we may take to be P, such that the 3-space $\langle \alpha, P \rangle$ contains a line M of π through y. If $\langle \alpha, P \rangle \in \Xi$, then $P = L = M$ and so $P \subseteq \pi$. If $\langle \alpha, P \rangle \notin \Xi$, then $M \neq L$. If $P \neq M$, then $P \not\subseteq \pi$ and so P and M determine a unique tube, and (H2) leads to a contradiction. Hence, in this first case, we have shown that P belongs to π.

Second case: the tangent plane to C_{PG} at G belongs to a 3-space Σ containing G, G' and some line M of π through y. Then again (H2) implies that C_{PG} and $C_{G'M}$ share a line in that tangent plane, which must necessarily be G, and so $M = L$. But then $\Sigma = \Xi(C)$ and $\Xi(C_{PG}) \cap \Xi(C)$ is a plane, contradicting (H2) once again. So the second case cannot occur.

All in all, we have shown that every tube through G intersects π in an affine line. Varying L, we conclude that every tube with vertex y intersects every singular plane containing y in a singular affine line. In other words, the geometric structure $\mathcal{G}_y$ with point set the set of singular lines through y and line set the set of tubes and singular planes through y is a dual affine plane.

Now let A be a 4-space in $\langle \alpha, \beta \rangle$ not containing y. We now project X_y from y onto A. This yields a generalized Veronesean embedding of the projective completion of $\mathcal{G}_y$ in $\langle \alpha, \beta \rangle$, provided we let y play the role of the unique point at infinity. The Main Result—General Version of Section 5 of [19] implies that the projection of X_y from y onto A is a normal rational cubic scroll. Hence, since X_y consists of the union of lines through y, the intersection of X_y with A is itself a normal rational cubic scroll, finalizing the proof of the proposition. $\square$

For the record, we explicitly write down a particular thing we proved in the course of the previous proof.

Lemma 3.9 *For $y \in Y$, the geometric structure $\mathcal{G}_y$ with point set the set of singular lines through y and line set the set of tubes and singular planes through y is a dual affine plane.* $\square$

There are a few interesting corollaries of Proposition 3.8, the first of which does not require a proof since it follows immediately from the fact that for every $x \in X$, the set of points of X on a common singular line with x is an affine plane.

Corollary 3.10 *Every singular line L is contained in a unique singular plane, containing all the singular lines that intersect L in a point of X.* $\square$

Corollary 3.11 *The set Y is the point set of a plane π_Y of $\mathsf{PG}(N, \mathbb{K})$.*

Proof Let $y \in Y$ be arbitrary and let C be a tube with vertex y. Then, by Proposition 3.8, there is a unique singular plane through every generator of C, and the union of the radical lines of all these planes is a plane π_y.

Now consider a second point $y' \in Y$, and let $x \in X$ be such that $\langle x, y' \rangle$ is a singular line. We may suppose $y' \notin \pi_y$ and consequently $\langle x, y \rangle$ is not singular. If $c \in C$, then any tube C' containing x and c has a vertex t belonging to π_y (as t must belong to the radical line of the singular plane through c). We clearly have $y' \in \pi_t$. Considering a tube through a point of $C \setminus \langle c, y \rangle$ and $C' \setminus \langle c, t \rangle$, we see that some point of $\pi_y \setminus \langle y, t \rangle$ belongs to π_t. Since also $y \in \pi_t$, this implies that $\pi_y = \pi_t$. But now $y' \in \pi_t = \pi_y$, a contradiction, and so $y' \in \pi_y$ after all.

The proof of the corollary is complete. □

We keep π_Y of the previous corollary as standard notation.

Corollary 3.12 *Every line L of π_Y is contained in a unique singular plane. The union of all these planes is precisely $X \cup Y$.*

Proof Let $y \in L$. Since, by Proposition 3.8, the projection of $\pi_Y \setminus \{y\}$ is a line (corresponding to the "vertex line" of the normal rational cubic scroll), the line L is the radical line of some singular plane α. If L were the radical line of some second singular plane α', then we can consider a line M in α and a line M' in α', with $M \cap M' = \{t\}$, $t \in L$. The plane $\langle M, M' \rangle$ is not singular, as this would lead to a singular 3-space $\langle \alpha, \alpha' \rangle$. Hence there is a tube $C_{MM'}$ containing M and M'. Now Proposition 3.8 implies that the projection of $\alpha \setminus \{t\}$ and $\alpha' \setminus \{t\}$ from t are disjoint lines on a normal rational cubic scroll, hence $\alpha \cap \alpha' = \{t\}$, a contradiction.

By Lemmas 3.5 and 3.6, every point $x \in X$ is contained in some singular plane, which, by Corollary 3.11, meets π_Y in some line. □

We can now determine the geometric structure of X.

Proposition 3.13 *Let T be the set of all tubes. Then (X, T) is a* projective Hjelmslev plane of level 2. *More exactly this means the following: the map $\chi : X \to \pi_Y^*$, where π_Y^* is the plane dual to π_Y, sending the point $x \in X$ to the radical line of the unique singular plane through x, is an epimorphism of (X, T) onto π_Y^* enjoying the following properties.*

(Hj1) *Two points of X are always joined by at least one member of T; they are joined by a unique member of T if and only if their images under χ are distinct.*

(Hj2) *Two members of T always intersect in at least one point; they intersect in a unique element of X if and only if their images under χ are distinct.*

(Hj3) *The inverse image under* χ *of a point, endowed with the intersections with non-disjoint tubes, is an affine plane.*

(Hj4) *The set of tubes contained in the inverse image under* χ *of a line, endowed with all mutual intersections, is an affine plane.*

Proof Surjectivity of χ follows from Corollary 3.12. Now consider the image $\chi(C)$ of an arbitrary tube C. Trivially, the radical lines of all singular planes containing points of C contain the vertex t of C. By Proposition 3.8, all lines in π_Y through t can be obtained this way. Hence χ is an epimorphism. Note that the image under χ of a tube is simply its vertex.

We now show (Hj1). By definition, two elements of X are contained in a tube. From Proposition 3.8 we deduce that this tube is not unique if and only if the two elements are contained in a singular line. By the last assertion of Proposition 3.8 and Corollary 3.10, this happens if and only if the two elements are contained in a unique singular plane, hence if and only if their images under χ coincide.

Now we show (Hj2). Let C, C' be two tubes. If they have the same vertex (hence their images under χ coincide), then they share a singular affine line (Lemma 3.2). Suppose now that they have distinct vertices t and t', respectively. Let $\pi_{tt'}$ be the unique singular plane containing t, t' (see Corollary 3.12). Then Lemma 3.9 implies that C and C' intersect $\pi_{tt'}$ in singular lines L and L', respectively. Since both lines are contained in a common plane, they intersect in a point (and one easily verifies that the intersection point belongs to X, and not to Y!).

Given a point x, the affine plane in (Hj3) is the affine plane of Corollary 3.10 arising from the unique singular plane containing x by removing its radical line.

Finally, (Hj4) follows immediately from Lemma 3.9, by dualizing. □

We say that a tube C and a point $x \in X$ are *neighbors* if $\chi(x)$ is incident with $\chi(C)$. We denote by $C(X)$ all neighbors of C.

Now we prove the following major step.

Proposition 3.14 *The set* X *contains a quadratic Veronese variety* $\mathcal{V}$ *whose conics are conics on tubes and whose points are in canonical bijection with the singular planes. The* 5*-space generated by* $\mathcal{V}$ *is skew to* π_Y*, and* $N = 8$*.*

Proof Let us fix a tube C (with vertex y), and consider a subspace F complementary to $\Xi(C)$. We consider the projection ρ of $X \setminus C$ from $\Xi(C)$ into F. We first claim that ρ is injective on $X \setminus C(X)$. Indeed, suppose two points $x_1, x_2 \in X \setminus C(X)$ are projected onto the same point $a \in F$. Then the 5-space $\langle C, a\rangle$ contains x_1, x_2, and hence the line $\langle x_1, x_2\rangle$, which is contained in any member of Ξ through x_1, x_2, must intersect $\Xi(C)$ in a point u of $X \cup Y$. If $u \in X$, then $\langle x_1, x_2\rangle$ is singular and x_1, x_2 are neighbors of C, a contradiction. Hence $u \in Y$, which implies, by Proposition 3.8, that x_1, x_2 belong to a singular plane containing a generator of C, leading to the same contradiction. Our claim follows.

Now consider two tubes C_1, C_2, both intersecting C in unique points, and such that $|C_1 \cap C_2| = 1$. Applying χ, we see that $x := C_1 \cap C_2$ is no neighbor of C. Let $i \in \{1, 2\}$ and put $C \cap C_i = x_i$. Since $\langle C, C_i\rangle$ is 6-dimensional, the projection of $C_i \setminus \{x_i\}$ from $\Xi(C)$ is isomorphic to the projection of $C_i \setminus \{x_i\}$ from x_i (onto a suitable plane), and hence it is an affine plane α_i with one "point at infinity" p_i^∞ added (the projection of the generator through x_i). By injectivity of ρ, $\alpha_1 \cap \alpha_2 = \{\rho(x)\}$ and so $\langle \alpha_1, \alpha_2\rangle$ is a 4-space inside F. This implies $N \geq 8$. The line at infinity L_i^∞ corresponds to the tangent plane to C_i at x_i. Note that, by (H3), all tubes intersecting C in just x_i will be projected onto planes containing L_i^∞. Considering the projections of all tubes connecting x_1 with a point of C_2 not in $C(X)$, we see that $X \setminus C(X)$ is projected bijectively into the affine 4-space A in $\langle \alpha_1 \alpha_2\rangle$ obtained by deleting the 3-space $A^\infty := \langle L_1^\infty, L_2^\infty\rangle$. It follows that $N = 8$.

In α_1, we see that the lines through p_1^∞ are the projections of the generators of C_1; all other lines distinct from L_1^∞ are projections of conics on C_1 through x_1. Noting that every pair of points is contained in a tube and every tube containing points of $X \setminus X(C)$ intersects C in a unique point, we see that every line of A is the projection of either a conic on some tube (and the conic intersects C nontrivially), or a generator of some tube intersecting C in a unique point. The latter case happens if and only if the point at infinity of the line is contained in the projection of $\Pi_Y \setminus \{y\}$, which is a line L^∞ connecting p_1^∞ and p_2^∞.

We take a line L in A^∞ skew to L^∞, and meeting the non-intersecting lines L_1^∞ and L_2^∞, say in the points z_1 and z_2, respectively. The set B_i of affine points on $\langle x, z_i$, $i \in \{1, 2\}$, is the projection of a "pointed" conic. Hence, translated to the Hjelmslev plane (X, T), it corresponds to a set of points mapped bijectively under χ to an affine line B_i^* of π_Y^*, where the point at infinity of that affine line is incident with $\chi(C)$. Let z be any point on L, then a similar property holds for the set B of affine points of $\langle x, z\rangle$; denote the corresponding affine line of π_Y^* with B^*. Suppose $B^* = B_i^*$ for some $i \in \{1, 2\}$. Then there are points $u \in B$ and $u_i \in B_i$, distinct from z corresponding to the same point of π_Y^*. Hence the corresponding points of X lie on a singular line, and so, by the above, the line $\langle u, u_i\rangle$ meets L^∞, contradicting the fact that L is skew to L^∞. By varying the point x in $\alpha := \langle x, L\rangle \setminus L$, we see that α corresponds to a set α' of points of (X, T) mapped bijectively under χ to an affine part of π_Y^*, namely all points of π_Y^* except those corresponding to the image of C. Also, α' defines an affine subplane of (X, T). We now claim that we can extend this to a projective subplane in a unique way.

Indeed, if we interchange the roles of C and C_1, then the conics corresponding to those lines of α intersecting $\langle x, z_1\rangle$, project again onto affine lines. It is easy to see, since $|\mathbb{K}| > 2$, see also Lemma 3.1 of [19], that all these affine lines are contained in a unique affine plane α_1. Also, if two lines of α intersect $\langle x, z_1\rangle$ in the same point, then it is obvious that the corresponding lines in α_1 are parallel, and so the line $\langle x, z_1\rangle$ of α corresponds to the line at infinity of α_1. The plane α_1 also contains an affine line L_O that corresponds to a conic O on C and which corresponds to the line at infinity of α. It follows that O extends α' to a projective plane, except possibly in the point that corresponds with the point at infinity of L_O (meaning that we do not know whether the unique point of O that does not correspond to any affine point of L_O is incident with all lines of π' that neighbor that point). But the same reasoning with now C_2 in the role of C shows that O really extends α' to a projective plane, and the claim is proved.

Let us denote the point set of this projective plane by P.

Now, the conic O lies in the 6-space $\langle C_1, \alpha_1\rangle$. The latter does not contain y. So, all points of α which are the projection of a point of X that corresponds with an affine point of α_1 are contained in the two 6-spaces $\langle C, \alpha\rangle$ and $\langle C_1, \alpha_1\rangle$, which intersect in a 5-space (since $y \in \langle C, \alpha\rangle$; the intersection cannot have dimension < 5 as O spans a plane, hence a codimension 1 subspace of $\langle C\rangle$). Consequently, we see that P spans a 5-space, and so it defines a Veronesean embedding of $\mathsf{PG}(2, \mathbb{K})$. By [13], P defines a quadric Veronese variety $\mathcal{V}$.

By the properties of α' mentioned above, it is now also clear that χ defines a bijection between P and the point set of π_Y^*. Also, as y does not belong to $\langle\mathcal{V}\rangle$, and C can be regarded as an arbitrary conic of $\mathcal{V}$, we see that $\langle\mathcal{V}\rangle$ and π_Y are disjoint. $\square$

This proposition hints at the following construction of X. Consider the plane π_Y, and consider the quadric Veronese variety $\mathcal{V}$ in a complementary 5-space. These objects are projectively unique. As we saw, there is a collineation, which we denote by χ, from $\mathcal{V}$ to π_Y^* such that for each point $p \in \mathcal{V}$, the point set $\langle p, \chi(p)\rangle \setminus \chi(p)$ is contained in X, and all points of X arise in this way. In order to show that X is projectively unique, it remains to nail down the collineation χ. Since all linear collineations are projectively equivalent, it suffices to show that χ is linear, i.e., χ preserves the cross-ratio. This will be established in our last proposition.

Corollary 3.15 *Let p_1, p_2, p_3, p_4 be four points on a common conic of $\mathcal{V}$ corresponding to the singular planes $\pi_{L_1}, \pi_{L_2}, \pi_{L_3}, \pi_{L_4}$, respectively. Then L_1, L_2, L_3, L_4 are concurrent lines of π_Y and the cross-ratios $(p_1, p_2; p_3, p_4)$ and $(L_1, L_2; L_3, L_4)$ are equal.*

Proof The fact that L_1, L_2, L_3, L_4 are concurrent lines of π_Y follows from the fact that χ defines a collineation. Now let y be the common point of the L_i, $i = 1, 2, 3, 4$. The points p_1, p_2, p_3, p_4 lie on singular lines together with y. If we consider the projection of all points of X that lie on a singular line together with y, then, according to Proposition 3.8, we obtain a normal rational cubic scroll, and p_1, p_2, p_3, p_4 can be seen as four points on the base conic, while the projections of L_1, L_2, L_3, L_4 are the four corresponding points on the vertex line. Just by the very definition of a normal rational cubic scroll, the assertion follows. $\square$

Now we treat the case $|\mathbb{K}| = 2$. Let $x \in X$. Since T_x contains at most 7 planes sharing pairwise at most a line, we see that there are at most 7 different tubes through x. This implies that $N \leq 4 + 7 = 11$, hence N is finite. Let $n = |X| \in \mathbb{N}$ be the number of points. Notice that not all tubes contain x (indeed, consider two tubes C_1, C_2 through x, then no tube through the points $x_i \in C_i$, $i = 1, 2$, where x_i is not collinear to x on C_i, contains x). Let C be a tube not through x. Since C has three generators, there are at least three tubes connecting x with C. So the number n_x of tubes through x satisfies $3 \leq n_x \leq 7$, and it follows from the previous that, if $n_x = 3$, then the three tubes through x all meet in a fixed singular affine line. Let $x_0 \in X$ be such that there are two tubes through x_0 only meeting in x (x_0 exists by Lemma 3.5). Then $n_{x_0} \geq 4$. Let, for $x \in X$, g_x be the number of generators through x, then one calculates that $|X| = 4n_x + g_x + 1$, where obviously $1 \leq g_x \leq n_x$. We also have $g_{x_0} \geq 2$ by assumption. Hence $|X| \geq 19$.

For $x \in X$, let T'_x be the 3-space obtained from T_x by factoring out x. Then every tube C through x gives rise to a point-line flag in T'_x, where the point corresponds to the singular line of C through x, and the line to the tangent plane to C at x. It follows from (H2) that the set of all tubes through x yields a set of such flags with the properties that, if two lines of different flags meet, then they must meet in their point (and therefore they have their point in common). It follows for instance that $g_x \leq 5$ since T'_x contains g_x disjoint lines.

Now we consider the different possibilities for g_x. If $g_x = 1$, then $n \in \{22, 26, 30\}$. Note that, if $g_x = 2$, then looking in T'_x, the only possibilities are $n_x \in \{2, 3, 4, 5, 6\}$, hence if $g_x = 2$, then $n \in \{19, 23, 27\}$. If $g_x = 3$, then likewise $n_x \in \{3, 4, 5, 6\}$, hence $n \in \{20, 24, 28\}$. If $n_x = 4$, then $g_x = 4$ and $n = 21$. Finally, if $g_x = 5$, then $n_x = 5$ and $n = 26$.

Since all these possibilities yield different possibilities for n, except for $n = 26$, we see that either n_x is constant, or there exist $x \in X$ with $(g_x, n_x) = (1, 6)$ and $y \in X$ with $(g_y, n_y) = (5, 5)$. Suppose first the latter happens, and let x' be the other point on the unique singular line through x.

The five points $u_i, i = 1..5$ on a common singular line with y also have $g_{u_i} = 5$. Let t be the vertex of the tubes through x. Suppose there is some point $w \in X \setminus \{x, x'\}$ with $g_w = 1$. Then all tubes through w also have vertex t, and so the singular lines of these tubes coincide with the singular lines of the tubes through x. hence all points off the tube $X[x, w]$ are contained in singular lines that are contained in at least two tubes; hence $g_z = 1$ for all such points z. There are now not enough points u left with $g_u = 5$. Hence x, x' are unique with $g_x = 1$. Let y' be any point different from x, x', y, but contained in the tube $X[x, y]$. Let C' be any tube through y not containing x. Then there are unique tubes through y' and the five respective points of $C' \setminus \{y\}$. Since these tubes can have at most one point in common with C', these must all be different. But then one of these must coincide with $X[y, y']$, since there are only 5 tubes through y'. That is clearly a contradiction.

We conclude that $g_x = g_y$ and $n_x = n_y$ for all $x, y \in X$. By Lemma 3.5 we have $g_x > 1$. Moreover this implies that the number of cones $\frac{nn_x}{6}$ has to be an integer. This excludes $(g_x, n_x, n) \in \{(2, 4, 19); (2, 5, 23); (3, 4, 20); (5, 5, 26)\}$.

Consequently $(g_x, n_x, n) \in \{(2, 6, 27); (3, 5, 24); (3, 6, 28); (4, 4, 21)\}$.

If $g_x = n_x = 4$, then $\mathcal{G}(X)$ is a linear space. For any point x, there is a line L, of size 6, not incident with x. hence the number of lines through x joining a point of L is at least 6, implying $n_x \geq 6$, a contradiction. This rules out the case $(4, 4, 21)$.

Now suppose $g_x = 2; n_x = 6, n = 27$. Since $n_x > g_x$ in this case, there is some singular line L contained in at least two tubes C_1, C_2. Let L_1 be a singular line of C_1 with $L_1 \neq L$ and let L_2 be a singular line of C_2 with $L_2 \neq L$. Suppose $L_i = \{t, x_i, y_i\}$, $i = 1, 2$. Suppose the tube $X[x_1, x_2]$ does not contain L_1. Then neither $X[y_1, y_2]$ contains L_1, and, since X does not contain three points on a line, the intersection $\langle x_1, x_2 \rangle \cap \langle y_1, y_2 \rangle$ is the common vertex of the tubes $X[x_1, x_2]$ and $X[y_1, y_2]$, and so x_1 and x_2 are on a singular line. Likewise x_1 and y_2 are on a singular line. This implies $g_{x_1} \geq 3$, a contradiction. Hence the singular lines through t form a linear space with line sizes 3 defined by the tubes through t. Such a space has at least seven elements, and each singular line is contained in at least three tubes. Hence we have shown that, if a singular line is contained in at least two tubes, it is contained in at least three. In T'_x, this implies that we have two sets of point-line flags, each set with common point, and one set contains at least three members. Then it is obvious that this set contains at most four elements and the other set contains at most two elements, hence exactly one! Each point x is contained in a unique singular lines that is contained in at least two tubes, with common vertex t. The inverse of the application $x \mapsto t$ defines a partition of X, each partition set consisting of $2(2(n_x - 1) + 1) = 4n_x - 2$ points. Hence there are $\frac{4n_x+3}{4n_x-2} = \frac{19}{14}$ partition classes, which is not an integer, excluding the case $(2, 6, 27)$.

Hence $g_x = 3$. Note that the structure of point-line flags in T'_x implies that every singular line is contained in either one or two tubes. Let $L = \{t, x, y\}$ be a singular line, with $t \in Y$, and such that there are exactly two cylindric spaces through L. Let L, L_1, M_1 and L, L_2, M_2 be the singular lines of the corresponding tubes. Then L_1 cannot be contained in tubes together with L_2 and M_2, as in that case L_1 would be contained in at least three tubes. As in the previous paragraph, we deduce that, if $x_1 \in L_1 \cap X$, and $x_2 \in L_2 \cap X$, and $L_1 \cup L_2$ is not contained in a cylindric space, then $\langle x_1, x_2 \rangle$ is a singular line. Since there are only three singular lines through x_1, we deduce that we may assume that L_1, L_2 are contained in a cylindric space, and so are M_1, M_2, but L_1 and M_2 are not contained in a cylindric space, and neither are L_2, M_1. By symmetry, it follows that there is a unique additional singular line L' through t contained in the cylindric spaces determines by L_1, L_2 and by M_1, M_2. Hence the set of singular lines through a fixed point of Y covers 12 points of X. The intersection of two such 12-point sets contains at most 4 points (such as those of $L_1 \cup M_2$). If $n = 24$, then every point is contained in a unique singular line which is contained in a unique cylindric space. No singular lines of such unique cylindric space is contained in a second cylindric space, as otherwise we can apply our argument above and obtain a 12-set. So those cylindric spaces partition X, and there are exactly 4 of them. But every other tube intersects each of these cylindric spaces in at most one point, contradicting the fact that a tube contains 6 points of X.

Hence $n = 28$. Now with the above, it is not difficult to see that any 12-set as defined before forms the set of points neighboring a line in a level 2 Hjelmslev plane over the dual numbers of size 4.

The result now follows as in the general case and the proof of Theorem 3.1 is complete.

4 Equivalence of V-sets

In this section, we consider the constructions of the various V-sets given in Subsection 1.2, and show some equivalence amongst them. The merit of this section is that all proofs are uniform, whereas many objects hidden in de V-sets for the different special cases of V are treated separately in the literature. Most proofs are not deep, and we will content ourselves by mentioning the main ideas.

First we introduce our uniform point of view. To that aim, we note that any 2-dimensional unital algebra over a field $\mathbb{K}$ is a matrix algebra over $\mathbb{K}$. We now make this more explicit. It will allow us to treat all cases in the same way.

If V is a quadratic field extension, say with respect to the quadratic polynomial x^2-tx+n, then V can be thought of to consist of the matrices

$$\begin{bmatrix} x & y \\ -ny & x+ty \end{bmatrix}, x, y \in \mathbb{K}.$$

If V is the direct product of $\mathbb{K}$ with itself, then obviously we can identify V with the matrix algebra of the diagonal matrices. However, in order to allow for a uniform treatment, we identify the algebra with the algebra of matrices

$$\begin{bmatrix} x & y \\ 0 & x+y \end{bmatrix}, x, y \in \mathbb{K}.$$

If V is the ring of dual numbers of $\mathbb{K}$, then it can be thought of to consist of the matrices

$$\begin{bmatrix} x & y \\ 0 & x \end{bmatrix}, x, y \in \mathbb{K}.$$

In each case, the automorphism σ is given by assigning to each matrix A its adjugate A^{adj}, i.e.,

$$\sigma : V \to V : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

for all appropriate a, b, c, d such that the given matrix belongs to V.

Now we observe that the second and the third of the three above algebras can be described in exactly the same way as the first one by putting (t, n) equal to $(1, 0)$ and $(0, 0)$, respectively. In fact, isomorphic descriptions are obtained if one now varies (t, n) over all possible values in $\mathbb{K} \times \mathbb{K}$. If the quadratic polynomial $x^2 - tx + n$ has no solutions, exactly one solution, or two distinct solutions in $\mathbb{K}$, then we we obtain a matrix algebra isomorphic to first, second or third case above, respectively.

So from now on, we assume that we have been given the pair $(n, t) \in \mathbb{K} \times \mathbb{K}$ arbitrarily, and we consider the corresponding matrix algebra V. But we emphasize that there are other matrix representations of these algebras not captured by assigning values to (t, n). Also for these representations, the results below hold, with a similar proof.

We write each element of V as $x\mathfrak{R} + y\mathfrak{I}$, with

$$\mathfrak{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \mathfrak{I} = \begin{bmatrix} 0 & 1 \\ -n & t \end{bmatrix}.$$

Although $\mathfrak{R}$ is the identity matrix, we insist on keeping the notation $\mathfrak{R}$ for it. The advantage is that all results below will also hold for $\mathfrak{R}$ different from the identity matrix, hence for different representations of the algebra V. Hence all results below hold for arbitrary 2×2 matrix representations of V.

For an arbitrary element A of V, write $x(A)$ and $y(A)$ for the unique elements of $\mathbb{K}$ such that $A = x(A)\mathfrak{R} + y(A)\mathfrak{I}$. This is a canonical way to write V as a 2-dimensional vector space over $\mathbb{K}$.

The following lemma is easily verified by direct computation.

Lemma 4.1 *For every matrix $M \in V$, we have* $M = \begin{bmatrix} x(\mathfrak{R}M) & y(\mathfrak{R}M) \\ x(\mathfrak{I}M) & y(\mathfrak{I}M) \end{bmatrix}$. $\square$

And also the next lemma is straightforward and requires no explicit proof.

Lemma 4.2 *Let $M, N \in V$, then the six determinants corresponding to the six 2×2 matrices obtained by deleting two columns in the 2×4 matrix constructed from M and N by juxtaposition, equal, up to sign and as a multiset, the four entries of the matrix $N^{\mathrm{adj}}M$ and the two determinants* $\det M$ *and* $\det N$. $\square$

With this set-up, the verification of the equivalence of the various V-sets becomes rather easy. In the sequel, we fix the field $\mathbb{K}$.

First equivalence: *The V-set defined by reduction is projectively equivalent with the V-set defined by juxtaposition.*

Indeed, writing any element of V as $x\mathfrak{R} + y\mathfrak{I}$, and denoting this as (x, y), an arbitrary point of $\mathcal{G}(V)$ is a triple $T = ((x, y), (x', y'), (x'', y''))$ of elements of V, up to a scalar factor in V^*, such that no common multiple of (x, y), (x', y') and (x'', y'') using a nonzero factor in V is zero. The 1-space over V defined by T is given by all multiples over V of T, and viewed as vector space over $\mathbb{K}$, the 2-space over $\mathbb{K}$ defined by T is hence generated by $\mathfrak{R}T$ and $\mathfrak{I}T$. Lemma 4.1 implies that this space is hence generated by the two points obtained by juxtaposition of the matrices of $(x, y), (x', y')$ and (x'', y'') and taking the points with coordinates in $\mathbb{K}$ given by the respective rows in this juxtaposition.

Second equivalence: *The V-set defined by matrices is projectively equivalent with the V-set defined by juxtaposition.*

Indeed, the coordinates in $\mathsf{PG}(8, \mathbb{K})$ of the point corresponding with the point $V^*(M, N, L)$, where M, N, L are elements of the matrix algebra V, are given by the nine values

$$\det M, \det N, \det L, x(M^{\mathrm{adj}}N), y(M^{\mathrm{adj}}N), x(N^{\mathrm{adj}}L), y(N^{\mathrm{adj}}L), x(L^{\mathrm{adj}}M), y(L^{\mathrm{adj}}M).$$

Now Lemma 4.2 implies that the line Grassmannian coordinates in $\mathsf{PG}(14,\mathbb{K})$ of the line of $\mathsf{PG}(5,\mathbb{K})$ determined by the points whose coordinates correspond with the two rows of the 2×6 matrix obtained by juxtaposing M,N,L, are given by the three determinants $\det M,\det N,\det L$, and by the twelve entries of the matrices $M^{\mathrm{adj}}N, N^{\mathrm{adj}}L$ and $L^{\mathrm{adj}}M$. But the entries in the first rows of these coincide with $x(M^{\mathrm{adj}}N)$, $y(M^{\mathrm{adj}}N)$, $x(N^{\mathrm{adj}}L)$, $y(N^{\mathrm{adj}}L$, $x(L^{\mathrm{adj}}M)$ and $y(L^{\mathrm{adj}}M)$, whereas the entries in the second rows are simple linear combinations of the other entries. This determines a projective linear transformation between these two V-sets.

Third equivalence: *The V-set defined by matrices is projectively equivalent with the V-set defined by juxtaposition.*

Indeed, this is clearly equivalent with showing that the set S_1 of rank 1 Hermitian 3×3 matrices over V coincides with the set S_2 of all scalar multiples of matrices $(M\ N\ L)^t(M\ N\ L)^{\mathrm{adj}}$, for $V^*(M,N,L)$ a point of $\mathcal{G}(V)$. Obviously, $S_2\subseteq S_1$. In order to show the converse, let

$$H=\begin{bmatrix} k_1 & R_3 & R_2^{\mathrm{adj}} \\ R_3^{\mathrm{adj}} & k_2 & R_1 \\ R_2 & R_1^{\mathrm{adj}} & k_3 \end{bmatrix},$$

with $k_1,k_2,k_3\in\mathbb{K}$ and $R_1,R_2,R_3\in V$ be a Hermitian matrix of rank 1. We show the assertion for $(k_1,k_2,k_3)\neq(0,0,0)$, leaving the easier case $k_1=k_2=k_3=0$ to the reader (note that this case only occurs for V the direct product of $\mathbb{K}$ with itself). We may assume $k_1\neq 0$ and hence $k_1=0$. Now we note that a Hermitian matrix of rank 1 with a 1 in the position of the first row and first column is completely determined by the first row and the first column (this is easily verified with standard arguments). Since both H and $(1,R_3^{\mathrm{adj}},R_2)^t(1,R_3^{\mathrm{adj}},R_2)^{\mathrm{adj}}$ have rank 1, have the same first row and first column, and both have a 1 on the position of the first row and first column, they coincide and the assertion is proved.

Preparing the verification of the Mazzocca-Melone axioms, we here show that all V-sets we considered span an 8-dimensional space. By isomorphism, it is enough to do so for the V-sets defined by matrices.

Proposition 4.3 *The V-set defined by matrices spans an 8-dimensional projective space.*

Proof We consider the nine Hermitian matrices $(M\ N\ L)^t(M\ N\ L)^{\mathrm{adj}}$ with (M,N,L) equal to $(\mathfrak{R},0,0)$, $(0,\mathfrak{R},0)$, $(0,0,\mathfrak{R})$, $(0,\mathfrak{R}\mathfrak{R})$, $(\mathfrak{R},0,\mathfrak{R})$, $(\mathfrak{R},\mathfrak{R},0)$, $(0,\mathfrak{R},\mathfrak{I})$, $(\mathfrak{I},0,\mathfrak{R})$, $(\mathfrak{R},\mathfrak{I},0)$, respectively. Then it is easily checked that these are $\mathbb{K}$-linearly independent. $\square$

5 Every V-set is a $\mathcal{C}$-Veronesean set

In this section we show the converses of Theorems 2.1 and 3.1 for $N = 8$, i.e., we show that a Segrean and Hjelmslevian Veronesean set for $N = 8$ really exists. In order to do so, it suffices to check the axioms (V1), (V2), (V3) for one of the equivalent V-sets of the previous section—and we will do it for V-sets defined by matrices since the $\mathcal{C}$-subspaces are most conveniently described—and show that the set of $\mathcal{C}$-subspaces is unique. It is convenient to use transitivity properties of the V-set to reduce the calculations. We will achieve this by first establishing these transitivity properties for $\mathcal{G}(V)$, and then show that the action of the group responsible for these properties extends to the V-set.

5.1 Transitivity properties for $\mathcal{G}(V)$

In this section we essentially gather some (transitivity and neighboring) properties of the geometry $\mathcal{G}(V)$, for V a 2-dimensional unital algebra over some commutative field $\mathbb{K}$, from [23] (see also [14] in case the characteristic of $\mathbb{K}$ is not equal to 2 or 3).

The transitivity properties we need require the notions of "neighboring relations". Hence we first define these.

Definition 5.1 Two points $V^*(x, y, z)$ and $V^*(u, v, w)$ of $\mathcal{G}(V)$ are called *neighboring*, denoted $V^*(x, y, z) \sim V^*(u, v, w)$, if there exist elements $k, \ell \in V$ such that $k(x, y, z) = \ell(u, v, w)$. Similarly for lines. Also, a point $V^*(x, y, z)$ and a line $V^*[a, b, c]$ are *neighboring*, denoted $V^*(x, y, z) \sim V^*[a, b, c]$, if $ax + by + cz \in V_0$.

We note that, if two distinct points $V^*(x, y, z)$ and $V^*(u, v, w)$ of $\mathcal{G}(V)$ are neighboring, and $k, \ell \in V$ are such that $k(x, y, z) = \ell(u, v, w)$, then both k and ℓ are zero divisors and $k \in \mathbb{K}\ell$. Indeed, if k is invertible, then clearly the points are equal; if k and ℓ would be linearly independent over $\mathbb{K}$, then we know $k^2 \in \mathbb{K}k$ and $k\ell = 0$, hence $k^2(x, y, z) = (0, 0, 0)$ without k^2 being 0, a contradiction.

In order to provide a better understanding of these relations we list the following properties for the neighboring relations. Consider two lines $L_1 := V^*[a_1, b_1, c_1]$ and $L_2 := V^*[a_2, b_2, c_2]$. We set

$$A := \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \qquad B := \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \qquad C := \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

In the next Proposition, (i) is easy to show (but since we do not need it, we omit the proof), (ii) is 2.6 of [23] and (iii) is 2.7 of [23].

Proposition 5.2 *(i) With the above notation, the lines L_1 and L_2 are not neighboring if and only if $V^*(A, B, C)$ is a point of $\mathcal{G}(V)$.*

(ii) A point $P := V^*(x, y, z)$ neighbors a line $L := V^*(a, b, c)$ if and only if P neighbors some point of L if and only if L neighbors some line through P.

(iii) Two lines are neighboring if and only if they intersect in at least two common points. $\square$

Now we define the objects on which a certain group will act transitively.

Definition 5.3 A triple of points (P_1, P_2, P_3) in $\mathcal{G}(V)$ is called a *proper ordered triangle* if P_1, P_2, P_3 are pairwise non-neighboring and the unique lines determined by P_1, P_2, by P_2, P_3 and by P_3, P_1, are pairwise non-neighboring.

Definition 5.4 *A* proper ordered quadrangle (P_1, P_2, P_3, P_4) *of points is a quadruple for which every ordered subtriple is a proper ordered triangle.*

We now come to defining the group of collineations that we will consider acting on $\mathcal{G}(V)$.

Let M be any 3×3 matrix with entries in V such that the usual determinant is an invertible element in V. Then the transposed of the inverse exists, and we denote it by M^*. The set of all these matrices forms a group $\mathsf{GL}_3(V)$.The action is defined in a standard way: the point $V^*(x, y, z)$ is mapped onto the point $V^*(u, v, w)$, where $(u\ v\ w) = (x\ y\ z)M$. We denote $(u, v, w) = (x, y, z)^M$.

Standard arguments (which we will not repeat here) show that the above action is well defined, i.e., the image of a point is a triple that represents a point. Moreover, every member of $\mathsf{GL}_3(V)$ maps lines to lines (also this only requires a standard argument, which we omit). Now *(ii)* and *(iii)* of Proposition 5.2 imply that $\mathsf{GL}_3(V)$ respects the neighbor relations.

A special class of members of $\mathsf{GL}_3(V)$ are, in the language of [23], the "central transvections", and the "dilatations". These are conjugate to

$$\begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

respectively, where $\alpha \in V$ arbitrary and $\beta \in V^*$. Hence, Corollary 4.25 of [23] implies the following theorem.

Theorem 5.5 *The group $\mathsf{GL}_3(V)$ acts sharply transitively on the family of all proper ordered quadrangles.* $\square$

Actually, Corollary 4.25 of [23] only asserts that $\mathsf{GL}_3(V)$ acts transitively on the family of proper ordered quadruples. The sharp transitivity (which we will not need in the sequel) follows from a standard argument (the only member of $\mathsf{GL}_3(V)$ fixing the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$—which form a proper ordered quadruple—is the identity).

This now immediately implies the following transitivity properties. A *flag* is an incident point-line pair.

Corollary 5.6 *The group $\mathsf{GL}_3(V)$ acts transitively on the family of all flags of $\mathcal{G}(V)$, on the family of all triples (P_1, L_1, P_2), where P_1, P_2 are non-neighboring points on the line L_1, on the family of all quadruples (P_1, L_1, P_2, L_2), where P_1, P_2 are non-neighboring points on the line L_1, and L_1, L_2 are non-neighboring lines through the point P_2, and on the family of proper ordered triangles.* $\square$

Note that the proof of Corollary 5.6 is not entirely trivial; one must show for instance show that every flag is contained in a proper ordered quadruple, and similarly for the other objects. As this is rather straightforward, we omit the proof. However, to emphasize that the intuition is not the usual one, we mention that, in general, the group $\mathsf{GL}_3(V)$ *does not* act transitively on the set of pairs (P, L), where P is a point and L is a line neighboring P.

5.2 Action of $\mathsf{GL}_3(V)$ on the V-set defined by matrices

We now show that the action of the group $\mathsf{GL}_3(V)$ on $\mathcal{G}(V)$ extends to the V-set X defined by matrices.

Lemma 5.7 *Let X be the V-set defined by matrices in $\mathsf{PG}(8, \mathbb{K})$. Then there is a group $G \leq \mathsf{GL}_9(\mathbb{K})$ acting on X, and an isomorphism $\theta : G \to \mathsf{GL}_3(V)$ such that for $g \in G$, and for $V^*(x, y, z)$ a point of $\mathcal{G}(V)$,*

$$(V^*(x, y, z))^{\theta(g)} \text{ corresponds to } ((x\ y\ z)^t (x\ y\ z)^\sigma)^g,$$

Proof This follows from the fact that, if $M \in \mathsf{GL}_3(V)$ and A is a Hermitian matrix over $V = \mathbb{K}\mathfrak{R} + \mathbb{K}\mathfrak{I}$ (the coefficient of $\mathfrak{R}$ will be referred to as the *real* part, the one of

$\mathfrak{I}$ as the *imaginary* part) then the real and imaginary part of each entry of M^tAM^θ is a linear combination of the real and imaginary parts of the entries of A, with as coefficients quadratic expressions in the real and imaginary parts of the entries of M, also involving the elements $t, n \in \mathbb{K}$ for which $\mathfrak{I}^2 - t\mathfrak{I} + n = 0$. The latter uses the fact that V is a quadratic algebra over $\mathbb{K}$. $\square$

5.3 V-sets defined by matrices satisfy the Mazzocca-Melone axioms

We now check that the V-sets defined by matrices satisfy the Mazzocca-Melone axioms.

Lemma 5.8 *The V-sets defined by matrices are $\mathcal{C}$-Veronesean sets, when endowed with the set Ξ of 3-spaces obtained by considering the spans of the lines of $\mathcal{G}(V)$, and where $\mathcal{C}$ is the class of elliptic quadrics, of quadratic cones with removed vertex, or of hyperbolic quadrics according to whether V_0 is trivial, a 1-dimensional subspace of V, or the union of two 1-dimensional subspaces of V.*

Proof Recall that the coordinates $(X_0, \cdots, X_8)$ of the point in $\mathsf{PG}(8, \mathbb{K})$ corresponding with the point $V^*(M, N, L)$, with $M, N, L \in V$, where V is viewed as matrix algebra, are

$$(\det M, \det N, \det L, x(M^{\mathrm{adj}}N), y(M^{\mathrm{adj}}N), x(N^{\mathrm{adj}}L), y(N^{\mathrm{adj}}L), x(L^{\mathrm{adj}}M), y(L^{\mathrm{adj}}M)).$$

We first show the claim that the points on an arbitrary line of $\mathcal{G}(V)$ span a 3-space which intersects X in a point set lying on a quadric (corresponding to the class $\mathcal{C}$). Indeed, by Corollary 5.6, we may consider the line $V^*[0, 0, 1] = V^*[0, 0, \mathfrak{R}]$, where this is easily checked, denote the quadric by H.

Now we show that every regular point P of H is a point of the line $V^*[0, 0, \mathfrak{R}]$, and no singular point is.

Suppose P has coordinates $(x_0, x_1, 0, x_3, x_4, 0, 0, 0, 0)$. First assume that $x_0 \neq 0$; then we may assume $x_0 = 1$. Then one easily calculates that P corresponds to the point $V^*(\mathfrak{R}, x_3\mathfrak{R} + x_4\mathfrak{I}, 0)$. Similarly if $x_1 \neq 0$, then P corresponds to a point of $V^*[0, 0, \mathfrak{R}]$. If H is elliptic, then there are no more points; if H is a cone, then the only point of H with $x_0 = x_1 = 0$ is the vertex and we claim this does not belong to $V^*[0, 0, \mathfrak{R}]$. Indeed, by Corollary 5.6 it would follow that every point of H is singular, a contradiction. Finally, if H is hyperbolic, then there are precisely two points with $x_0 = x_1 = 0$, and if one or both of them would not belong to $V^*[0,0,\mathfrak{R}]$, then the collineation group stabilizing the set of points H belonging to $V^*[0,0,\mathfrak{R}]$ could not act transitively, contradicting Corollary 5.6.

Hence we have shown that the intersection of X with the space generated by the points on $V^*[0,0,\mathfrak{R}]$ is a member of $\mathcal{C}$. Now Property (V1) is obvious.

By transitivity we can prove property (V3) by checking it at $P = (1,0,0,0,0,0,0,0,0)$, which is an easy calculation.

Finally, we show property (V2) and first the non-neighboring situation. By Corollary 5.6 we may take the lines $V^*[\mathfrak{R},0,0]$ and $V^*[0,0,\mathfrak{R}]$. The corresponding subspaces in $\mathsf{PG}(8,\mathbb{K})$ meet only in the point which corresponds to the point $V^*(0,\mathfrak{R},0)$ of $\mathcal{G}(V)$.

Now we consider the neighboring situation. By Corollary 5.6 and Definition 5.1, we may consider the lines $V^*[0,0,\mathfrak{R}]$ and $V^*[0,kM,\mathfrak{R}+\ell M]$, where $M \in V_0 \setminus \{0\}$. Put $M = a\mathfrak{R} + b\mathfrak{I}$, with $a, b \in \mathbb{K}$. One calculates that the points $V^*(0,\mathfrak{R}+\ell M,-kM)$ and $(1,\mathfrak{R}+\ell M,-kM)$ belong to $V^*[0,kM,\mathfrak{R}+\ell M]$, and the last four coordinates of these points in $\mathsf{PG}(8,\mathbb{K})$ are $(-ka,-kb,0,0)$ and $(-ka,-kb,kx+kyt,-ky)$, respectively. This implies, in view of the fact that the last four coordinates of every point of $V^*[0,0,\mathfrak{R}]$ are 0, that the set of points of $\mathsf{PG}(8,\mathbb{K})$ corresponding with the set of points of $\mathcal{G}(V)$ on the union of the lines $V^*[0,0,\mathfrak{R}]$ and $V^*[0,kM,\mathfrak{R}+\ell M]$ span a subspace of dimension at least 5. But since these lines meet in at least 2 points, this dimension is exactly 5 and the corresponding 3-spaces meet in a line T. Now, the intersection points of these two lines can be generically written as $V^*(m\mathfrak{R},M',0)$, with $m \in \mathbb{K}$ and where $MM' = 0$, $M' \neq 0$ (we overlook at most one point, and that happens precisely when M and M are linearly independent; then we have the additional point $V^*(M,M',0)$). Writing the coordinates of such generic point in $\mathsf{PG}(8,\mathbb{K})$, we see that m appears linearly, so that we miss at most one point of T. This shows Property (V2). $\square$

To finish the proof of the Main Result, it now suffices to show the following uniqueness result.

Proposition 5.9 *Every (abstract) ovoid, tube and hypo contained in a Hermitian, Hjelmslevian and Segrean Veronesean set, respectively, is contained in an elliptic, cylindric and hyperbolic space, respectively.*

Proof If the groundfield has order at least 3, then there is a uniform approach. Let X be a Hermitian, Hjelmslevian or Segrean Veronesean set with Ξ the set of corresponding elliptic, cylindric and hyperbolic spaces, respectively. Let Q be an ovoid, tube or hypo contained in X, and not contained in a member of Ξ. Let $x, y \in X$ be two points such that the projective line spanned by x and y is not entirely contained in Q (and then x and y are the only two points of X on that line). Such points exist since otherwise one easily sees that the 3-space generated by Q is entirely contained in X, a contradiction. Let H be the unique elliptic quadric, tube or hypo, respectively, contained in a member ξ of Ξ and containing both x and y. Choose an oval O in Q through x and y. If O does not belong to ξ, then we can select two points x', y' in O distinct from x, y, and a member ξ' of Ξ containing x', y'. Since $\xi' \neq \xi$, the axioms imply $\xi' \cap \xi \subseteq X$, a contradiction (for the Hjelmslevian case, it really follows from Lemma 3.3).

Hence $O \subseteq \xi$ and so $O \subseteq H$. But we do the same for a second oval through x, y and obtain $Q \subseteq \xi$.

Now we consider the cases of Hermitian and Segrean Veronesean sets over the field $\mathbb{F}_2$. In the Segrean case, it is easily seen that a hypothetical hypo Q not contained in a member of Ξ shares two generators with a hypo H contained in a member of Ξ. Now taking two appropriate points of Q outside H and considering a hyperbolic space through them leads to a contradiction, as above. In the Hermitian case, the same contradiction arises when a hypothetical ovoid Q in X not contained in a member of Ξ intersects an elliptic space in an ovoid; hence we may assume that it intersects some elliptic space ξ in just two points x, y. But the projection from ξ of $X \setminus \xi$ is injective and contained in an affine space of dimension 4, see [12], and the three points of Q distinct from x, y would project on a line; a contradiction.

For the Hjelmslevian Veronesean set X over $\mathbb{F}_2$, we note that it is easily seen that 4 points in general position can never lie in a common plane. Suppose we have 4 points of X in a common plane, then three of them must lie on singular lines through a common vertex t. One now easily checks that the space generated by all points on a singular line through t (a 12-set in the terminology of the last part of Section 3) does not contain more points than those 12 points; hence the fourth point must also be in that set. It now follows easily that a given tube must be contained in a cylindric space.

The proposition, and hence our Main Result, is completely proved. □

6 Further motivation

In this section we put our results in a broader perspective, at the same time motivating why we look at these generalizations of Hermitian Veronese varieties, and explaining why we stated the Axioms (V1), (V2) and (V3) in such generality.

Consider an alternative composition division algebra $\mathbb{A}$, and suppose $\mathbb{A}$ is either 4- or 8-dimensional over its center (hence we have a quaternion or octonion algebra). Then there is a unique projective plane $\mathbb{P}_2(\mathbb{A})$ associated with $\mathbb{A}$, and this projective plane corresponds to a real form of an algebraic group of absolute type either A_5 or E_6. The (standard) Veronese varieties associated to these planes arise from a standard representation of the spherical buildings of type A_5 and E_6, respectively, in a 15- and 27-dimensional vector space, respectively, by suitably restricting coordinates. These Veronese varieties can also be constructed directly as certain Hermitian matrices with entries in $\mathbb{A}$. Now, if we take for $\mathbb{A}$ the split version of the algebra, then we obtain in this way the aforementioned representations of the respective spherical buildings.

So, we may conclude that there is a uniform construction of all Veronese varieties corresponding to composition division algebras, and certain representations of spherical buildings. Our ultimate aim is to characterize all these structures in a uniform way by the axioms (V1), (V2) and (V3). Moreover, by allowing not only split and non-split algebras, but also certain other quadratic variants V, for which the associated geometry $\mathcal{G}(V)$ as defined in Section 1 has the properties that any two points lie on at least one common line and any pair of lines has at least one point in common, we hope to include some less standard varieties, such as the Hjelmslev planes of the present paper, which are still connected to Tits-buildings. Indeed, the Hjelmslev planes of level 2 are precisely the structures emerging from the set of vertices at distance 2 from a given vertex in an affine building of type $\widetilde{\mathsf{A}}_2$, see [6, 22].

The geometries referred to in the previous paragraph correspond to the second row of the Freudenthal-Tits magic square (over any field). In particular, our characterization should ultimately include a characterization of the standard module for groups of type E_6 over any field. It would be rather remarkable that this intricate geometric structure allows to be caught by a rather simple and short list of axioms.

By intersecting the models of the $\mathcal{C}$-Veronesean sets corresponding to the second row of the Freudenthal-Tits magic square with an appropriate hyperplane (codimension 1 space), we obtain models for the geometries and groups in the first row of this square. A second aim of ours is to find adequate axioms for these representations. These axioms should have the same spirit as those considered in the present paper. This would yield common characterizations of certain representations of buildings of types A_2 (the analogues of those considered in [15] for arbitrary fields), C_3 (the line Grassmannian representation) and F_4 (the ordinary 25-dimensional module).

Of course, further perspectives include the third and the final row of the Freudenthal-Tits magic square, where, amongst others, non-embeddable polar spaces and buildings of types E_7 and E_8 appear. The former, although already apparent in Tits' original approach, were formally proved to be in the Magic Square by Mühlherr in [11]. This indicates that a geometric approach, as started here is worthwhile to pursue.

Finally we note that [2, 9] contain related approaches to the magic square, describing the algebraic geometry and representation theory associated to the Freudenthal-Tits magic square (hence restricted to certain fields, for instance not over finite fields).

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Address of the authors
Jeroen Schillewaert
Department of Mathematics, Imperial College
South Kensington Campus, London SW7-2AZ, UK
jschillewaert@gmail.com
Hendrik Van Maldeghem
Department of Mathematics, Ghent University,
Krijgslaan 281, S22, B-9000 Ghent, BELGIUM
hvm@cage.ugent.be