A combinatorial characterization of the Lagrangian Grassmannian $\mathbf{LG}(3,6)(\mathbb{K})$

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Abstract

We provide a combinatorial characterization of $\mathbf{LG}(3,6)(\mathbb{K})$ using an axiom set which is the natural continuation of the Mazzocca-Melone set we used to characterize Severi varieties over arbitrary fields [10]. This fits within a large project aiming at constructing and characterizing the varieties related to the Freudenthal-Tits magic square.

1 Introduction

Classical varieties such as Veronese varieties, Segre varieties and Grassmann varieties are intensively studied in algebraic geometry, but are also important in combinatorial geometry, in particular in the area where groups and geometries meet and where the Tits buildings play a central role. However, in combinatorial geometry, the underlying field, if any, is arbitrary, and in this case a variety of tools from algebraic geometry can no longer be used. In 1984, Mazzocca & Melone suggested an axiom system for the Veronesean varieties over finite fields that was based on the very basic properties of these varieties as smooth complex varieties, but which can be phrased over any field (and they restricted to finite fields). The main property of such varieties responsible for allowing such an approach is the fact that they are the union of maximal quadratic varieties whose corresponding subspaces pairwise meet on the variety. This makes it possible to define the dimension of the variety via a condition on the tangent spaces to these quadrics—over an arbitrary field, the variety is just a set of points, whereas tangent spaces to quadrics

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are defined over any field. The success of such an approach is illustrated in [10], where the authors generalize Zak’s classification of complex Severi varieties [15] to their analogues over an arbitrary field, just using a straightforward extension of the axioms of Mazzocca & Melone. Another example is the recent characterization of the Veronese representation of projective planes over non-associative alternative division rings (Cayley-Dickson algebras) by Krauss [7]. Also his axioms are based on the Mazzocca-Melone approach.

The Mazzocca-Melone approach, however, was, up to now, only applied when it concerned geometries with point-line diameter 2, and then the first axiom says that every pair of points is contained in a quadric. The types of the geometries thus characterized mainly belong to the second row of the so-called Freudenthal-Tits magic square. The latter is an arrangement of sixteen Dynkin diagrams in a four-by-four square symmetric along the main diagonal. Chosen a field, the \( i \)-th column is parametrized by a (split or nonsplit, depending on the point of view) quadratic alternative algebra of dimension \( 2^{i-1} \), whereas the \( j \)-th row is parametrized by a Jordan algebra over a quadratic alternative algebra of dimension \( 2^{j-1} \). Each cell thus corresponds with an ordered pair of algebras, in a non symmetric way, and a general construction method of Tits [12] associates to this pair a Lie algebra of the type indicated by the magic square. To each Lie algebra in the square can be associated a variety, which turns out to be a point-line geometry of a building with corresponding Dynkin type. The geometries of the second row all have diameter at most 2 (these comprise projective planes, products of two projective planes, line Grassmannians of projective 5-space, and the exceptional \( E_6 \)-geometry). In the present paper, we start to apply this approach to geometries of larger diameter. The first natural choice is the Lagrangian Grassmannian \( \text{LG}(3,6)(\mathbb{K}) \), which is the geometry of totally singular planes of a symplectic space in 5-dimensional projective space \( \mathbb{P}^5(\mathbb{K}) \). Not coincidently, it corresponds to the first cell of the third row of the Freudenthal-Tits magic square. The first cell of the third row is intimately related to the first cell of the second row, which contains the ordinary quadratic Veronesean representations of projective planes. To prove our main result, we strenghten the Mazzocca-Melone approach to these geometries: we basically show that the third axiom can be deleted, if one assumes the right bound on the dimension of the ambient space (the third axiom expresses the dimension of the variety by means of the tangents). This assumption cannot further be weakened as there exist counterexamples for higher dimensions.

**Notation.** In this paper, we will use the following notation: the subspace spanned by a set \( S \) of points will be denoted by \( \langle S \rangle \). The finite field of \( q \) elements will be denoted by \( \mathbb{F}_q \). The \( n \)-dimensional affine (projective) space over the skew field \( \mathbb{K} \) will be denoted by \( \mathbb{A}^n(\mathbb{K}) \) (by \( \mathbb{P}^n(\mathbb{K}) \)).
2 Statement of the Main Results

Let us first recall the Mazzocca-Melone axioms for the quadratic Veronesean of the standard projective plane $\mathbb{P}^2(K)$ over any field $K$. First note that an oval $O$ in any projective plane is a set of points no three collinear and such that through every point $o \in O$ exactly one line does not intersect the set in two points. Examples are conics, if $K$ is commutative.

Let $X$ be a spanning point set of $\mathbb{P}^N(K)$, $N \in \mathbb{N} \cup \{\infty\}$, with $K$ any skew field, and let $\Xi$ be a collection of 2-spaces of $\mathbb{P}^N(K)$ containing at least two elements and such that for any $\xi \in \Xi$ the intersection $\xi \cap X =: X(\xi)$ is an oval in $\xi$ (and then, for $x \in X(\xi)$, we denote the tangent line at $x$ to $X(\xi)$ by $T_x(X(\xi))$ or sometimes simply by $T_x(\xi)$). Then $(X, \Xi)$ a called a Veronesean cap if (VC1), (VC2) and (VC3) below hold. It is called a pre-Veronesean cap if (VC1) and (VC2) hold.

(VC1) Any pair of points $x$ and $y$ of $X$ is contained in an element of $\Xi$, denoted by $[x, y]$ (its uniqueness follows straight from (VC2)).

(VC2) If $\xi_1, \xi_2 \in \Xi$, with $\xi_1 \neq \xi_2$, then $\xi_1 \cap \xi_2 \subset X$.

(VC3) If $x \in X$, then all tangent lines $T_x(\xi)$, $x \in \xi \in \Xi$, are contained in a plane.

It is proved in [9] that such a Veronesean cap is always the Veronesean representation of the standard projective plane over $K$, and $K$ is a field. Recall that the Veronesean representation of $\mathbb{P}^2(K)$ is the image $\mathcal{V}_2(K)$ of $K^3 \setminus \{(0,0,0)\}$ under the Veronesean map $(x, y, z) \mapsto (x^2, y^2, z^2, yz, zx, xy)$, where the latter is conceived as a point of $\mathbb{P}^5(K)$. Writing $(x^2, y^2, z^2, yz, zx, xy)$ as $(x, y, z)^T(x, y, z)$ (where $T$ means “transposed”) it is obvious that the points of the Veronesean representation of $\mathbb{P}^2(K)$ can be seen as the points corresponding to the rank 1 symmetric $(3 \times 3)$-matrices in the projective space corresponding to the vector space of all symmetric $(3 \times 3)$-matrices over $K$. In the proof, Axiom (VC3) seems to play an important, if not crucial, role. However, we will show below that, if $|K| > 2$ and $N \leq 5$, then we can delete Axiom (VC3)! This is our first Main Result.

**Main Result 1.** If $(X, \Xi)$ is a pre-Veronesean cap in $\mathbb{P}^N(K)$, $N \leq 5$, with $K$ any skew field distinct from $\mathbb{F}_2$, then $K$ is commutative, $(X, \Xi)$ is a Veronesean cap, and hence $X$ is projectively equivalent with $\mathcal{V}_2(K)$, the Veronesean representation of the standard projective plane over $K$.

We also classify the pre-Veronesean caps if $K \cong \mathbb{F}_2$, see Proposition 4.4; there is essentially one more example besides the Veronesean cap. Furthermore, we also provide a further
weakening of the axioms by allowing \( X \) to contain lines. For the motivation and precise statements, see Subsection 4.2.

Now we turn to the Lagrangian Grassmannian \( \text{LG}(3,6)(\mathbb{K}) \). As a point set, this is the set of points of \( \mathbb{P}^{19}(\mathbb{K}) \) on the plane Grassmannian of \( \mathbb{P}^{5}(\mathbb{K}) \), restricted to the planes totally isotropic with respect to a nondegenerate alternating bilinear form, which forces this point set into a 13-dimensional subspace \( \mathbb{P}^{13}(\mathbb{K}) \). As natural point-line geometry (lines are those from \( \mathbb{P}^{13}(\mathbb{K}) \) completely contained in \( \text{LG}(3,6)(\mathbb{K}) \)), \( \text{LG}(3,6)(\mathbb{K}) \) has diameter 3, but we want to leave the diameter open in the axioms (even infinite diameter will in principle be possible). Also, in the real case, \( \text{LG}(3,6)(\mathbb{R}) \) has dimension 6; in the finite case \( \text{LG}(3,6)(\mathbb{F}_q) \) has \((q^3 + 1)(q^2 + 1)(q + 1)\) points, confirming the 6-dimensionality. All this leads to the following definition (noting that quadrics only exist in projective spaces over fields, hence there is no point in starting from a skew field).

Let \( X \) be a spanning point set of \( \mathbb{P}^N(\mathbb{K}) \), \( N \in \mathbb{N} \cup \{\infty\} \), with \( \mathbb{K} \) any field, and let \( \Xi \) be a collection of at least two 4-spaces of \( \mathbb{P}^N(\mathbb{K}) \) (called the quadratic spaces) such that, for any \( \xi \in \Xi \), the intersection \( \xi \cap X := X(\xi) \) is a non-singular parabolic quadric \( Q(4,\mathbb{K}) \) (which we will call a symp, inspired by the theory of parapolar spaces, see [11]) in \( \xi \). For \( x \in X(\xi) \), we denote the tangent space at \( x \) to \( X(\xi) \) by \( T_x(X(\xi)) \) or sometimes simply by \( T_x(\xi) \). A line of \( \mathbb{P}^N(\mathbb{K}) \) all of whose points are contained in \( X \) is called a singular line, and the set of singular lines is denoted by \( S \). Also, we denote by \( \mathcal{G}(X) \) the geometry \((X, S)\) of points and singular lines, and with \( \Gamma(X) \) we denote the point graph of \( \mathcal{G}(X) \) (two points being adjacent if they are collinear in \( \mathcal{G}(X) \)). We call \((X, \Xi)\) a Lagrangian set if (LS1), (LS2) and (LS3) below hold.

(\text{LS1}) \( \mathcal{G}(X) \) is connected and any pair of points \( x \) and \( y \) of \( X \) such that the distance between \( x \) and \( y \) in \( \Gamma(X) \) is at most 2 is contained in at least one element of \( \Xi \), denoted by \([x,y]\) if unique.

(\text{LS2}) If \( \xi_1, \xi_2 \in \Xi \), with \( \xi_1 \neq \xi_2 \), then \( \xi_1 \cap \xi_2 \subset X \).

(\text{LS3}) If \( x \in X \), then all 3-spaces \( T_x(\xi), x \in \xi \in \Xi \), generate a subspace \( T_x \) of \( \mathbb{P}^N(\mathbb{K}) \) of dimension at most 6.

Our second Main Result says that \( \text{LG}(3,6)(\mathbb{K}) \) is the only Lagrangian set. More precisely:

\textbf{Main Result 2.} If \((X, \Xi)\) is a Lagrangian set in \( \mathbb{P}^N(\mathbb{K}) \), \( N \in \mathbb{N} \cup \{\infty\} \), then \( N = 13 \) and \( X \) is projectively equivalent to the Lagrangian Grassmannian \( \text{LG}(3,6)(\mathbb{K}) \).

The rest of the paper is devoted to proving Main Results 1 and 2. In the next section, we show that \( \text{LG}(3,6)(\mathbb{K}) \) is a Lagrangian set. Then, in Section 4 we show Main Result 1. In
Section 5 we show Main Result 2. This proof consists of two major parts. In the first part, we show that the diameter of $G(X)$ cannot be equal to 2. In the second part, we show that this implies that the diameter is equal to 3 and that $X$ is projectively equivalent to the Lagrangian Grassmannian $LG(3, 6)(\mathbb{K})$.

3 The Lagrangian Grassmannian $LG(3, 6)(\mathbb{K})$

In this section we give an explicit description of $LG(3, 6)(\mathbb{K})$ and show that it is a Lagrangian set. As already mentioned, $LG(3, 6)(\mathbb{K})$ is the plane Grassmannian of $\mathbb{P}^5(\mathbb{K})$ restricted to the planes totally isotropic with respect to a nondegenerate alternating form. As a geometry, consequently, it is isomorphic to the dual polar space denoted by $DW(5, \mathbb{K})$; the points are the planes of the symplectic polar space $W(5, \mathbb{K})$ and the lines correspond to the sets of planes of $W(5, \mathbb{K})$ containing a common line of $W(5, \mathbb{K})$. In this setting, a sym is the set of points corresponding to the planes of the polar space $W(5, \mathbb{K})$ containing a common point. It is naturally isomorphic to an orthogonal polar space of rank 2, the so-called orthogonal generalized quadrangle $Q(4, \mathbb{K})$ (which is a parabolic quadric; note that every parabolic quadric over a field of characteristic 2 admits a nucleus, which is a point through which no secant line passes). The following construction is taken from [2] (see also [4]).

We define certain types of points in $\mathbb{P}^{13}(\mathbb{K})$.

**Type I.** A point denoted by $[\infty]$ has coordinates

$$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1).$$

**Type II.** For $k \in \mathbb{K}$, a point denoted by $[k]$ has coordinates

$$(0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, k).$$

**Type III.** For $k, x \in \mathbb{K}$, a point denoted by $[x; k]$ has coordinates

$$(0, 0, 0, 0, 0, 0, 0, 0, x^2, 1, -x, 0, 0, k).$$

**Type IV.** For $k_1, k_2, x \in \mathbb{K}$, a point denoted by $[x; k_1, k_2]$ has coordinates

$$(0, 1, 0, 0, 0, 0, 0, 0, k_1, k_2, x, 0, 0, k_1 k_2 - x^2).$$
**Type V.** For \( k, x_1, x_2 \in \mathbb{K} \), a point denoted by \([x_1, x_2; k]\) has coordinates
\[
(0, 0, 0, 0, 0, 0, 1, x_1^2, x_2^2, -x_1x_2, x_2, x_1, k).
\]

**Type VI.** For \( k_1, k_2, x_1, x_2 \), a point denoted by \([x_1, x_2; k_1, k_2]\) has coordinates
\[
(0, x_2^2, 0, 1, 0, x_2, 0, k_1, k_2, k_1x_2^2, -x_1x_2, k_1x_2, x_1, k_1k_2 - x_2^2).
\]

**Type VII.** For \( k_1, k_2, x_1, x_2, x_3 \in \mathbb{K} \), a point denoted by \([x_1, x_2, x_3; k_1, k_2]\) has coordinates
\[
(0, x_3^2, 1, x_1^2, -x_1, -x_3x_1, x_3, k_1, k_2x_1^2 + k_1x_3^2 + x_2(x_1x_3) + (x_3x_1)x_2, k_2,
-x_3x_2 - k_1x_1, x_2, x_2x_1 + k_1x_3, k_1k_2 - x_2^2).
\]

**Type VIII.** For \( k_1, k_2, k_3, x_1, x_2, x_3 \in \mathbb{K} \), a point denoted by \([x_1, x_2, x_3; k_1, k_2, k_3]\) has coordinates
\[
(1, k_1, k_2, k_3, x_1, x_2, x_3, k_2k_3 - x_1^2, k_3k_1 - x_2^2, k_1k_2 - x_3^2, k_1k_1 - x_3x_2, k_2x_2 - x_3x_1, k_3x_3 - x_2x_1,
k_1k_2k_3 + 2x_1x_2x_3 - k_1x_1^2 - k_2x_2^2 - k_3x_3^2).
\]

The set \( X \) of all these points, together with the lines of \( \mathbb{P}^{13}(\mathbb{K}) \) contained in it is the dual polar space \(\text{DW}(5, \mathbb{K})\) and defines the Lagrangian Grassmannian variety \(\text{LG}(3, 6)(\mathbb{K})\). An example of a symp is given by all points of Type I, II, III and IV. These points all lie in the subspace \(U\) defined by \(X_1 = X_3 = X_4 = \cdots = X_8 = X_{12} = X_{13} = 0\) and their (other) coordinates satisfy the equation \(X_2X_{14} = X_9X_{10} - X_7^2\). Conversely, every point in \(U\) whose coordinates satisfy this equation lies on \(\text{LG}(3, 6)(\mathbb{K})\). Now it is shown in Corollary 1.2 (ii) of [6] that, if \(|\mathbb{K}| > 2\), this is the absolutely universal embedding of \(\text{DW}(5, \mathbb{K})\), i.e., every other (full) embedding arises as a quotient (i.e., a projection from a suitable subspace) from this one. If \(|\mathbb{K}| = 2\), then the universal embedding happens in a 14-dimensional projective space \(\mathbb{P}^{14}(\mathbb{F}_2)\) [1, 8], and the Lagrangian Grassmannian is a projection of it. Also, for arbitrary \(\mathbb{K}\), the absolutely universal embedding (and for \(\mathbb{K} \cong \mathbb{F}_2\) also the Lagrangian Grassmannian) is homogeneous, i.e., the group of collineations of the ambient projective space stabilizing the embedding induces the full group of automorphisms of the dual polar space. In particular, this group is transitive on the family of pairs of syms that intersect non-trivially, and also on the family of pairs of syms that have empty intersection; this group is also transitive on the set of points of the embedded dual polar space.

First we want to check that the intersection of a quadratic space with the point set \(X\) is a symp. We can take, by the transitivity properties mentioned in the previous
paragraph, the symp $\Sigma_1$ consisting of the points of Types I, II, III and IV. Put $n_U = \{1, 3, 4, \ldots, 8, 12, 13\}$ and note that $U = \langle \Sigma_1 \rangle$ is determined by the equations $X_i = 0$, for all $i \in n_U$. Since points of Type V, VI, VII and VIII have a nonzero coordinate in position 8, 4, 3 and 1, respectively, and all these numbers belong to $n_U$, we deduce that $X \cap U = \Sigma_1$, which completes the proof.

We now verify the axioms (LS1), (LS2) and (LS3).

Axiom (LS1) follows from the fact that $\text{DW}(5, K)$ is a strong parapolar space, see Example 2 of Section 13.4.2 in [11].

For (LS2), we introduce the following two syms:

- $\Sigma_2$ consists of the points of Type I, II, V (with $x_2 = 0$) and VI (with $x_2 = 0$). This symp spans the subspace $U_2$ with equations $X_1 = X_2 = X_3 = X_5 = X_6 = X_7 = X_{10} = X_{11} = X_{12} = 0$, which is indeed 4-dimensional. Clearly $U \cap U_1$ is the line spanned by $(0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1)$, which is a line contained in both syms (namely, the line consisting of all points of Types I and II).

- $\Sigma_3$ consists of the points of Type V (with $x_1 = x_2 = k = 0$), VI (with $x_1 = x_2 = k_1 = 0$), VII (with $x_2 = x_3 = k_2 = 0$) and VIII (with $x_2 = x_3 = k_1 = 0$). This symp spans the subspace $U_3$ with equations $X_2 = X_6 = X_7 = X_9 = X_{10} = \cdots = X_{14} = 0$, which is clearly disjoint from $U$.

The transitivity properties of the automorphism group of the Lagrangian Grassmannian variety mentioned before conclude the proof of (LS2).

Finally (LS3) follows by (1) of Theorem 1.3 of [2].

Let $x \in X$ be a point of the variety $\text{LG}(3, 6)(K)$. Then we denote by $\eta_x$ the subspace of $\mathbb{P}^{13}(K)$ generated by all points of $X$ collinear to $x$ in $\text{DW}(5, K)$, and by $\zeta_x$ the subspace of $\mathbb{P}^{13}(K)$ generated by all points of $X$ contained in a common symp with $x$ in $\text{DW}(5, K)$. Obviously we have $\eta_x \subseteq \zeta_x$, and it follows by (1) of Theorem 1.3 of [2] that $\dim \eta_x = 6$ and $\dim \zeta_x = 12$, for all $x \in X$.

**Lemma 3.1** Let $x \in X$. Then every 7-dimensional subspace $U$ containing $\eta_x$ and not contained in $\zeta_x$ contains a unique point $y \in X$ not in a common symp with $x$.

**Proof** By homogeneity, we may take $x = [\infty]$. It follows from the definition of $X$ in [2] (see also [4]) that $\eta_x$ is generated by the points of Types I, II, III and V, whereas
$\zeta_x$ is generated by the points of Types I, II, III, IV, V, VI and VII. It is easy to see that $\zeta_x$ has equation $X_1 = 0$. Hence an arbitrary point $z$ outside $\langle \zeta_x \rangle$ can be written as $(1, \ell_1, \ell_2, \ell_3, y_1, y_2, y_3, \ldots)$. Then one also calculates that $\eta_x$ has equations $X_1 = X_2 = \cdots = X_7 = 0$. Now there is a unique point $y$ of Type VIII in $X$ sharing the same initial seven coordinates with $z$, we see that $y \in \langle \eta_x, z \rangle$ and the lemma is proved.

We now want to show that no nontrivial projection of the Lagrangian Grassmannian is a Lagrangian set. We first need a lemma.

**Lemma 3.2** Let $Y$ be a point set of $\mathbb{P}^5(\mathbb{K})$ isomorphic to $V_2(\mathbb{K})$, and let $p$ be a point in $\mathbb{P}^5(\mathbb{K})$ not belonging to $Y$. Then there exists a point $y \in Y$ such that some plane containing a conic on $Y$ shares a point with $\langle p, y \rangle$ distinct from $y$.

**Proof** We identify $Y$ with the rank 1 symmetric $3 \times 3$-matrices over $\mathbb{K}$, up to a scalar nonzero multiple. Those of rank 2 correspond to points contained in a plane of one of the conics of $V_2(\mathbb{K})$, and those of rank 3 correspond to points not contained in any such plane. Clearly we may assume that $p$ corresponds to a rank 3 symmetric matrix $M$. Let $y \in Y$ correspond to the matrix with a 1 in one place somewhere on the diagonal and 0 elsewhere. Then clearly the line $\langle p, y \rangle$ contains a point $t$ corresponding to a rank $\leq 2$ matrix (in which case the assertion easily follows) if and only if the corresponding cofactor is nonzero. Hence we may assume that all diagonal cofactors vanish. If the characteristic of $\mathbb{K}$ is 2, then this implies that $M$ is singular, a contradiction.

If the characteristic of $\mathbb{K}$ is not 2, then we play the same game with the point $z \in Y$ corresponding with the rank 1 matrix all entries of which are 0 except for the entries in the north-west $2 \times 2$-square which are all 1. Since $M$ is nonsingular, an easy calculation implies that $\langle p, z \rangle$ must contain a point distinct from $z$ corresponding to a rank $\leq 2$ matrix.

**Proposition 3.3** No nontrivial projection of the Lagrangian Grassmannian is a Lagrangian set.

**Proof** Since (LS3) holds for the Lagrangian Grassmannian, it will hold for all its projections. Hence we have to find a contradiction against (LS1) or (LS2), and it suffices to do so for the projection $X'$ of $\text{LG}(3, 6)(\mathbb{K})$ from an arbitrary point $p$.

Suppose first that $p$ is contained in $\zeta_x$, for each $x \in X$. By [2] we have $\text{char}(\mathbb{K}) = 2$, $p$ is contained in a unique quadratic space and it is the nucleus of the corresponding symp. Hence the projection of this symp reduces to a 3-space, violating (LS1).
Hence we may assume that there is some \( x \in X \) with \( p \notin \zeta_x \). In that case the 7-space \( \langle p, \eta_x \rangle \) contains by Lemma 3.1 a point \( y \) of \( X \) outside \( \eta_x \) and at distance 3 from \( x \) in \( \Gamma(X) \). So, in the projection from \( p \) the subspace generated by \( \eta_x \) contains the projection of \( y \). Hence, our assertion boils down to showing that \( \eta_x \) cannot contain a point of \( X' \) at distance 3 from \( x \). Let, for a contradiction, \( q \) be such a point. Lemma 3.2 implies that there is some line \( L \subseteq X' \) through \( x \) such that the plane \( \langle q, L \rangle \) contains a line \( L' \neq L \) through \( x \) which is also contained in a symp through \( x \). Now, since some point \( u \) on \( L \) is at distance 2 from \( q \) in \( \Gamma(X) \), (LS2) yields \( \langle q, u \rangle \subseteq X' \) and so the distance from \( q \) to \( x \) is at most 2, a contradiction. \( \square \)

4 Proof of Main Result 1

4.1 Pre-Veronesean caps

Let \( X \) be a spanning point set of \( \mathbb{P}^N(\mathbb{K}) \), \( N \leq 5 \), with \( \mathbb{K} \) any skew field, which for the moment we allow to be isomorphic to \( \mathbb{F}_2 \), and let \( \Xi \) be a collection of at least two 2-spaces of \( \mathbb{P}^N(\mathbb{K}) \) such that for any \( \xi \in \Xi \) the intersection \( \xi \cap X \) is an oval in \( \xi \). Suppose that \((X, \Xi)\) satisfies (VC1) and (VC2) above, in other words, suppose \((X, \Xi)\) is a pre-Veronesean cap. With “oval”, we will in this section always refer to the intersection of \( X \) with a member of \( \Xi \). If \( \mathbb{K} \cong \mathbb{F}_2 \), then an oval has only three points \( x, y, z \) not on a common line. In this case, there is a unique line \( L \) in \( \langle x, y, z \rangle \) disjoint from \( \{x, y, z\} \), and the unique point in \( \langle x, y, z \rangle \) not on that line and not on the oval will be denoted by \( x + y + z \); it is usually called the nucleus of the oval. The points of \( L \) will be denoted by \( x + y, y + z, z + x \), where \( \{x, y, x + y\} \) is a line, etc.

**Lemma 4.1** Under the above assumptions, let \( \pi \in \Xi \) and let \( U \) be a subspace of \( \mathbb{P}^N(\mathbb{K}) \) complementary to \( \pi \). Then the projection of \( X \setminus X(\pi) \) from \( \pi \) onto \( U \) is injective.

**Proof** If \( x_1, x_2 \) are two points of \( X \setminus X(\pi) \) projected onto the same point, then \( \langle \pi, x_1, x_2 \rangle \) is a 3-space. Hence \([x_1, x_2] \) intersects \( \pi \) in a point of \( \langle x_1, x_2 \rangle \), which belongs to \( X \) by (VC2), contradicting the fact that \( X([x_1, x_2]) \) is an oval. \( \square \)

**Lemma 4.2** Under the above assumptions, we have \( N = 5 \).

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Suppose for a contradiction that \( N \leq 4 \).

First suppose \( K \cong \mathbb{F}_2 \). In this case, \( X \) contains an odd number of points. Indeed, if there are \( \ell \) ovals containing a point \( x \in X \), then (VC1) implies that there are \( 2\ell + 1 \) points in \( X \). Since there are at least 2 ovals, we have a point \( x \) and an oval \( C = \{x_1, x_2, x_3\} \not\ni x \). The ovals \( X([x_i, y_i]), i \in \{1, 2, 3\} \) are distinct, hence \( |X| \geq 7 \). On the other hand, Lemma 4.1 implies that there are at most 6 points in \( X \) (3 in the plane \( \pi \) and 3 projected onto the at most 1-dimensional subspace \( U \)), a contradiction.

Now suppose \( K \not\cong \mathbb{F}_2 \). Clearly (VC2) implies that \( N \geq 4 \), since \( |\Xi| \geq 2 \). Now suppose \( N = 4 \). Consider two intersecting ovals \( C, C_1 \), then the intersection \( x_1 \) of \( (C) \) and \( (C_1) \) belongs to \( X \) by (VC2). Let \( x_2 \in C \setminus \{x_1\} \). Let \( C_2 \) be an oval containing \( x_2 \) and some point \( y \in C_1 \setminus \{x_1\} \). We project \( (C_1 \cup C_2) \setminus \{x_1, x_2\} \) from \( (C) \) onto a line \( L \) skew to \( C \). It is clear that the images of both these sets comprise all points of the line \( L \), except one, say \( p_1 \) and \( p_2 \), respectively (then \( p_i, i = 1, 2 \) corresponds to the tangent line in \( x_i \) at \( C_i \)). Since \( |K| > 2 \), there is a point \( z \) on \( L \) in the image of both \( C_1 \setminus \{x_1, y\} \) and \( C_2 \setminus \{x_2, y\} \), contradicting Lemma 4.1.

**Lemma 4.3** Every pair of ovals intersects nontrivially.

**Proof** Suppose, by way of contradiction, that two ovals \( C \) and \( D \) do not meet. We consider the projection of \( X \setminus C \) from \( (C) \) onto \( (D) \), which is injective by Lemma 4.1.

Again, we first suppose that \( K \cong \mathbb{F}_2 \). Since \( (D) \) contains 7 points, and since \( |X| \) is odd, we have \( |X| \in \{7, 9\} \). Hence the geometry induced by the ovals on \( X \) is either a \( 2 \) - \( (7, 3, 1) \) design or a \( 2 \) - \( (9, 3, 1) \) design, which are both unique and isomorphic to \( \mathbb{P}^2(\mathbb{F}_2) \) and \( A^2(\mathbb{F}_3) \), respectively. In the former case, every pair of ovals intersects; in the latter case there exist three pairwise disjoint ovals \( C, D, E \). Since \( (C) \) and \( (D) \) are also disjoint (by (VC2)), every point \( z_i \) of \( E = \{z_1, z_2, z_3\} \) is contained in a unique line \( L_i \) that intersects both \( (C) \) and \( (D) \) nontrivially; put \( C = \{x_1, x_2, x_3\} \) and \( D = \{y_1, y_2, y_3\} \). Since the projection from \( (C) \) onto \( (D) \) is injective, we clearly have \( L_i \cap (D) \not\ni D \), for all \( i \in \{1, 2, 3\} \). Similarly \( L_i \cap (C) \not\ni C \), for all \( i \in \{1, 2, 3\} \). At most one of \( L_1, L_2, L_3 \) contains \( x_1 + x_2 + x_3 \) (by Lemma 4.1 projecting from \( (D) \)), and likewise at most one \( y_1 + y_2 + y_3 \). Hence there is at least one line, say \( L_1 \) containing a point \( x_2 + x_3 \) and a point \( y_2 + y_3 \) (without loss of generality). The oval \( X([x_1, z_1]) \) contains at most one of \( \{y_2, y_3\} \), hence we find an oval \( F \) containing, without loss of generality, the points \( z_1, y_2, x_3 \). Then \( F \) is contained in the 3-space \( \langle x_2, x_3, y_2, y_3 \rangle \) and so \( (F) \) and \( (x_2, y_3) \) meet nontrivially, implying by (VC2) that \( X([x_2, y_3]) \) contains three collinear points, a contradiction.

So we may assume that \( K \not\cong \mathbb{F}_2 \). Let \( x \in D \) be arbitrary. By the injectivity of the projection, and since \( |K| > 2 \), the projections of the planes generated by the conics.
containing $x$ and a point varying on $C$ are distinct lines through $x$. Consequently, there is a conic $E$ such that the projection $E'$ is not contained in the tangent line to $D$ at $x$. By injectivity, if $t$ is the projection of the tangent line to $E$ at $E \cap C$, then $E' \cup \{t\}$ is a full projective line, and $t \in D$. Let $u \in E \setminus (C \cup \{x\})$ be arbitrary. Since the projection is injective, the projection of $C_u := X([t, u])$ does not coincide with $\langle x, t \rangle$, and so the projection $C'_u$ of $C_u$ is an oval through $t$.

Now let $v$ be an arbitrary point of $C$ and let $C_v = X([t, v])$. Let $C'_v$ be the projection of $C_v$. Then, by injectivity, $C'_v$ is not contained in $\langle x, t \rangle$. For finite $\mathbb{K} \not\cong \mathbb{F}_2$, this is a contradiction, as there are precisely $|\mathbb{K}| + 1$ choices for $v$ and exactly as many lines in $\langle D \rangle$ through $p$. So we may assume that $\mathbb{K}$ is infinite. But then we consider two choices for $u$, say $u_1$ and $u_2$, and we can choose $v$ such that $C'_v$ is neither contained in the tangent line to $C'_{u_1}$ at $t$, nor in the tangent line to $C'_{u_2}$ at $t$. By injectivity of the projection, $C'_v$ is contained in a line minus two points (the latter are points in $C'_{u_1} \cup C'_{u_2}$, which are distinct, again by injectivity of the projection), a contradiction.

The proof of the lemma is complete. □

We can now finish the proof of Main Result 1. Since two distinct ovals always meet, the geometry of points of $X$ and ovals is a projective plane. Now Theorem 2.3 of [9] completes the proof.

We now briefly study the case $|\mathbb{K}| = 2$. In this case, we have 7 points in $\mathbb{P}^5(\mathbb{F}_2)$ and the geometry of ovals determines a projective plane of order 2 (a so-called Fano plane). Consider arbitrarily five points of $X$. In a Fano plane, every set of five points is the union of two lines, hence, by (VC2), the corresponding ovals generate a 4-space. Hence every set of five points in $X$ generates a 4-space. If every set of six points of $X$ generates a 5-space, then $X$ consists of a skeleton, and this is isomorphic to $\mathcal{V}_2(\mathbb{F}_2)$. Hence we may assume that there is a 6-subset of $X$ forming a skeleton in some 4-subspace $U$ of $\mathbb{P}^5(\mathbb{F}_2)$. Since the seventh point must lie outside $U$, and every point in the Fano plane plays the same role, this gives rise to a projectively unique situation, and the resulting point set will be called a disturbed Veronesean cap. Hence we have the following result.

**Proposition 4.4** Let $X$ be a pre-Veronesean cap in $\mathbb{P}^N(\mathbb{F}_2)$, $N \leq 5$, then it is either a Veronesean cap, and hence $X$ is projectively equivalent with $\mathcal{V}_2(\mathbb{F}_2)$, or it is a disturbed Veronesean cap.
4.2 Singular pre-Veronesean caps

Motivated by the proof of Main Result 2, we will now extend Main Result 1 in case the ambient space has dimension at most 5. We will weaken the hypotheses to again end up with a Veronesean cap in case \( K \not\sim F_2 \). For \( K \cong F_2 \), some more possibilities will turn up. The idea is to also allow degenerate conics, i.e., lines. However, we will only need to deal with the situation where the degenerate conic is a line with multiplicity 2 (and not a point, or a pair of distinct lines). These lines will be called singular lines, and, although a set containing lines is not a cap in the technical sense, we will call the new objects singular pre-Veronesean caps. This is harmless, as we will show that a singular pre-Veronesean cap is a Veronesean cap after all, at least when the underlying field is not the smallest field. In the latter case a few more possibilities occur, see below.

Let \( X \) be a point set of \( \mathbb{P}^5(K) \), with \( K \) any skew field, and let \( \Xi \) be a collection of 2-spaces (called the quadratic planes) of \( \mathbb{P}^5(K) \) containing at least two elements and such that for any \( \xi \in \Xi \) the intersection \( \xi \cap X =: X(\xi) \) is an oval in \( \xi \). Then \((X, \Xi)\) a called a singular pre-Veronesean cap if \((VC1')\) and \((VC2)\) below hold.

\((VC1')\) If \( x, y \in X \), then either all points of \( \langle x, y \rangle \) belong to \( X \), or there exists a unique member \([x,y]\) of \( \Xi \) containing both \( x \) and \( y \).

\((VC2)\) If \( \xi_1, \xi_2 \in \Xi \), with \( \xi_1 \not= \xi_2 \), then \( \xi_1 \cap \xi_2 \subset X \).

Clearly, every Veronesean cap is a singular pre-Veronesean cap. The converse is not true for \( K \cong F_2 \), and there are some counter examples.

**Example 1 (The projected Veronesean cap).** If we project one conic of the Veronesean cap \( V_2(F_2) \) from its nucleus onto a secant, then we obtain a singular pre-Veronesean cap, as one checks easily. If \( \{e_1, \ldots, e_6\} \) is a basis of \( \mathbb{P}^5(F_2) \), then such a set is projectively equivalent with \( \{e_1, e_2, e_3, e_4, e_5, e_6, e_4 + e_5\} \). The corresponding set of quadratic planes contains 6 elements, namely those corresponding with the conics \( \{e_1, e_2, e_4\}, \{e_2, e_3, e_5\}, \{e_3, e_4, e_5\}, \{e_1, e_5, e_6\}, \{e_2, e_6, e_4 + e_5\} \) and \( \{e_1, e_3, e_4 + e_5\} \).

**Example 2 (The biaffine singular cap).** Let \( \{e_1, \ldots, e_6\} \) again be a basis for \( \mathbb{P}^5(F_2) \) and let \( \Xi \) be the set of planes generated by the triples of points corresponding to the lines of a biaffine plane of order 3 (i.e., an affine plane with three points per line and one parallel class of lines removed, giving rise to parallel classes of points) with point set \( X := \{e_1, e_2, e_1 + e_2, e_3, e_4, e_3 + e_4, e_5, e_6, e_5 + e_6\} \), where the triples \( \{e_1, e_2, e_1 + e_2\}, \{e_3, e_4, e_3 + e_4\} \) and \( \{e_5, e_6, e_5 + e_6\} \) are point parallel classes. Then \((X, \Xi)\) is a singular pre-Veronesean cap.

In a series of lemmas, we will show the following classification.
Proposition 4.5 Every singular pre-Veronesean cap in $\mathbb{P}^5(\mathbb{K})$ is a Veronesean cap, except if $\mathbb{K} \cong \mathbb{F}_2$, in which case it could also be isomorphic to either a disturbed Veronesean cap, or a projected Veronesean cap, or a biaffine singular cap.

So let $(X, \Xi)$ be a singular pre-Veronesean cap, which we may assume to contain at least one singular line by Main Result 1. If all points of a certain subspace are contained in $X$, then we call that subspace singular. In the sequel, an oval is the intersection of $X$ with a member of $\Xi$. We start with proving a lemma similar to Lemma 4.1 now using $(VC1_1')$ and $(VC2)$ instead of $(VC1)$ and $(VC2)$.

Lemma 4.6 Let $\pi \in \Xi$ and let $U$ be a complementary subspace to $\pi$ in $U_p$. Then the projection from $\pi$ onto $U$ is injective when restricted to the points of $X \setminus \pi$ which are not on a singular line that intersects $\pi$.

Proof Suppose two points $x, y \in X_p \setminus \pi$ have the same image. Then $\langle x, y \rangle$ intersects $\pi$ and so, if $\langle x, y \rangle$ is not singular, then the conic through $x, y$ contains three collinear points, a contradiction. \[\Box\]

We now rule out singular subspaces of dimension at least 2. We denote by $k$ the dimension of $\langle X \rangle$. First we note that distinct maximal singular subspaces must be disjoint.

Lemma 4.7 Two singular subspaces $U$ and $V$ sharing a point $z$ generate a singular subspace.

Proof Since every point in the span $\langle U, V \rangle$ is contained in the span of two lines containing $z$, one in $U$ and one in $V$, it suffices to assume that both $U$ and $V$ are lines. Let $p$ be arbitrary in $\langle U, V \rangle \setminus \{z\}$ and assume $p \notin X$. Choose points $x_1, x_2 \in U \setminus \{z\}$ and $y_1, y_2 \in V \setminus \{z\}$ such that $p = \langle x_1, y_1 \rangle \cap \langle x_2, y_2 \rangle$. Since $p \notin X$, Axiom $(VC1_1')$ implies that $[x_1, y_1]$ and $[x_2, y_2]$ are well defined. But Axiom $(VC2)$ implies $p \in X$, a contradiction. Hence $p \in X$ and so $\langle U, V \rangle$ is a singular plane. \[\Box\]

Lemma 4.8 There are no singular planes in $X$.

Proof Let $U$ be a singular subspace of dimension $\ell \geq 2$ in $X$ and assume that $\ell$ is maximal with this property. Since $X$ contains at least one plane $\pi$ such that $\pi \cap X_p$ is a conic, we see that $\ell \leq k - 2 \leq 3$. If $\ell = 3$, then $k = 5$, and we can consider a point $x \in X$ outside $U$. For any $u \in U$, the line $\langle u, x \rangle$ is non-singular, as Lemma 4.7 would otherwise
lead to a singular subspace of dimension 4. Pick two distinct points \( u, v \in U \). So we have an oval \( C \subseteq X \) through \( x \) and \( u \), and for each point \( y \) of \( C \setminus \{ u \} \), we have an oval \( C_y \) containing \( v \) and \( y \). Let \( y_1, y_2 \) be two distinct points of \( C \setminus \{ u \} \). An arbitrary 4-space \( W \) through \( U \) not containing the tangent lines at \( v \) to the ovals \( C_{y_1} \) and \( C_{y_2} \), respectively, intersects \( C_{y_i} \) in a point \( z_i, i = 1, 2 \). The line \( \langle z_1, z_2 \rangle \) intersects \( U \) and so is singular, a contradiction to Lemma 4.7.

Next suppose \( \ell = 2 \). If \( k = 4 \), then we argue similarly as above and obtain a contradiction. So we may assume that \( k = 5 \). Let \( \pi \) be a plane in \( \mathbb{P}^5(\mathbb{K}) \) skew to \( U \). If \( \pi \in \Xi \), then, by Lemma 4.6, the projection from \( \pi \) onto \( U \) is injective, even on \( X \setminus \pi \), which leads to \( X_p = U \cup X_p(\pi) \), a contradiction as is easily seen. Hence all ovals intersect \( U \) nontrivially. Now, the projection of \( X \setminus U \) from \( U \) onto \( \pi \) is also injective, as the line joining two points with same image must meet \( U \) and hence is singular, a contradiction to Lemma 4.7. If \( \mathbb{K} \cong \mathbb{F}_2 \), then considering all conics joining a point off \( U \) with a point of \( U \), we obtain \( 7 + 1 \) points of \( X \) off \( U \), contradicting the injectivity. So suppose \( \mathbb{K} \not\cong \mathbb{F}_2 \). Now let \( C_1, C_2 \) be two conics intersecting \( U \) in the same point \( u \). The projection onto \( \pi \) of \( C_1 \setminus \{ u \} \) and \( C_2 \setminus \{ u \} \) are two affine lines (an affine line is the point set of a line, except for one point) \( L_1 \setminus \{ c_1 \}, L_2 \setminus \{ c_2 \} \), respectively, where \( L_1, L_2 \) are lines of \( \pi \) and \( c_i \) is a point of \( L_i, i = 1, 2 \). Suppose \( c_1 \neq c_2 \). By injectivity, we may assume \( c_1 \in L_2 \setminus \{ c_2 \} \). Take an arbitrary point \( c_1' \) of \( L_1 \setminus \{ c_1 \} \). The conic defined by the inverse images of \( c_1, c_1' \) in \( X \) intersects \( U \) and projects into \( L_1 \), contradicting the injectivity (since that conic is certainly different from both \( C_1, C_2 \)).

Hence all conics in \( X \) through the same point \( u \) of \( U \) project onto affine lines of \( \pi \) sharing the same point \( p_u \). For different \( u \), the points \( p_u \) are also different as otherwise, by injectivity of the projection, we find two conics through a common point of \( X \setminus U \) intersecting in all points but the ones in \( U \), a contradiction. This now implies that two different conics containing a (possibly different) point of \( U \) meet in a unique point of \( X \). We now choose a line \( L \subseteq U \) and project \( X \setminus L \) from \( L \) onto some skew 3-space \( \Sigma \). Let \( u_i, i = 1, 2, 3 \), be three distinct points on \( L \). The conics through these points project onto three families of lines such that lines from different families intersect in a unique point. Considering two families, we see that these lie either on a hyperbolic quadric, and the third family cannot exist (\( |\mathbb{K}| > 2 \)), or in a plane. In the latter case, we easily see that all points of \( X_p \setminus U \) are contained in a 4-space together with \( L \), a contradiction considering a conic through some point of \( U \setminus L \) (and once again using \( |\mathbb{K}| > 2 \)).

So we now know that \( X \) does not contain planes. Before we start a detailed analysis when there are singular lines, we note two easy properties.
Lemma 4.9 No point $x$ of any singular line is contained in the span of two other singular lines. Also, no oval $C$ misses at least two singular lines $L_1, L_2$.

Proof Note that by Lemma 4.7 singular lines do not intersect each other. But the transversal through $x$—i.e., the line through $x$ intersecting the two singular lines—must be a singular line by (LS1), a contradiction. For the second assertion, let $x \in \langle C \rangle \cap \langle L_1, L_2 \rangle$ (our assumption implies $x \notin L_1 \cup L_2$) and consider the unique transversal to $L_1, L_2$ containing $x$. Then Axiom (LS2) implies that $x \in X$ and the transversal is singular, a contradiction. □

We first treat the case where $X$ spans a 4-space. From now one, we will frequently have to make a distinction between $|K| = 2$ en the rest (a few times also $|K| = 3$ requires special arguments). Note that $|X|$ is odd if $|K| = 2$ (if there are $\ell$ ovals through a point $x \in X$, then there are either $2\ell + 1$ points—if there is no singular line through $x$—or $2\ell + 3$—otherwise). Also, if $|K| = 2$, the geometry induced by the singular lines and the ovals on $X$ is a 2-design, which is the Fano plane if $|X| = 7$, and the affine plane of order 3 if $|X| = 9$.

Lemma 4.10 We have $k = \dim(X) = 5$.

Proof Since there are at least two ovals, Axiom (VC2) implies $k \geq 4$. Hence for a contradiction, we assume $k = 4$. We claim that there are at most two singular lines. Indeed, suppose $L_1, L_2, L_3$ are three different singular lines. Notice that they are disjoint by Lemma 4.7. The 3-space $\langle L_1, L_2, \rangle$ intersects $L_3$ in at least a point, contradicting Lemma 4.9. The claim is proved.

Now suppose that there are precisely two singular lines $L_1, L_2$. Let $x_i \in L_i, i = 1, 2$ and consider the projection of $X \setminus [x_1, x_2]$ from $[x_1, x_2]$ onto some skew line $L$. Clearly, $\langle L_1, L_2, [x_1, x_2] \rangle$ is 4-dimensional, so we can choose $L$ to contain a point $y_i$ of $L_i, i = 1, 2$.

- If $|K| = 2$, then the injectivity of the projection on $X \setminus ([x_1, x_2] \cup L_1 \cup L_2)$ implies $6 \leq |X| \leq 8$. Hence $|X| = 7$, contradicting the fact that in a Fano plane every two lines meet (and $L_1$ and $L_2$ are disjoint).

- If $|K| > 2$, we consider three conics through $y_1$ intersecting $L_2 \setminus \{x_2\}$ nontrivially. These project onto three affine lines in $L$ containing $y_1, y_2$. Hence there is at least one point $L \setminus \{y_1, y_2\}$ covered twice. This contradicts the injectivity of the projection on $X \setminus ([x_1, x_2] \cup L_1 \cup L_2)$.
Now suppose that there is a unique singular line \( L \). If some conic \( C \) is disjoint from \( L \), then projecting \( X \setminus C \) from \( \langle C \rangle \) onto \( L \) implies that \( X = C \cup L \), an easy contradiction. Hence every conic intersects \( L \) and the geometry of conics and \( L \) is a projective plane (as in a 4-space every pair of planes intersects). We project \( X \setminus C \) from \( \langle C \rangle \) onto some disjoint line \( M \), which we may assume to contain a point \( x \in L \setminus C \).

- If \(|K| > 2\), then we may consider three conics through \( y \), which all project onto some affine line in \( M \) containing \( x \); this again implies that two distinct points have the same image giving rise to an extra singular line, a contradiction.

- If \(|K| = 2\), then \( |X_p| = 7 \) and we can coordinatize as follows: the points on \( L \) are \( e_1, e_2, e_1 + e_2 \). Let \( x_1, x_2, x_3 \) three arbitrary other points of \( X \). The planes \([x_1, x_2]\) and \([x_2, x_3]\) generate the 4-space, hence we may assume that they are \( e_3, e_4, e_5 \), with \( \{e_1, e_2, e_3, e_4, e_5\} \) a basis. There are two projectively inequivalent choices for the last point, namely \( e_2 + e_3 + e_4 + e_5 \) and \( e_3 + e_4 + e_5 \). In the former case, the plane \([e_1, e_3 + e_4 + e_5]\), which we may assume to contain without loss of generality \( e_3 \), contains \( e_4 + e_5 \), a contradiction. In the latter case we may assume that the conic planes through \( e_1 \) are \( \langle e_1, e_3, e_4 \rangle \) and \( \langle e_1, e_5, e_3 + e_4 + e_5 \rangle \), which both contain \( e_3 + e_4 \), a contradiction.

So from now one we may assume that \( k = 5 \). We first treat the case where there are at least three singular lines.

**Lemma 4.11** There are at most three singular lines, and in case there are three of them, \(|K| = 2\) and \((X, \Xi)\) is a biaffine singular cap.

**Proof** Suppose for a contradiction that there are at least four singular lines \( L_1, L_2, L_3, L_4 \). Then the 3-spaces \( \langle L_1, L_2 \rangle \) and \( \langle L_3, L_4 \rangle \) have a line \( K \) in common, with \( K \cap X = \emptyset \). Each point \( a \in K \) belongs to a transversal to \( L_1, L_2 \), and to a transversal to \( L_3, L_4 \). Axiom (LS2) now implies that these two transversals span a quadratic plane. We conclude that every point \( a \in K \) is contained in a unique such quadratic plane, and so each point \( x \in L_1 \cup L_2 \cup L_3 \cup L_4 \) is contained in a unique oval \( C_x \) which intersects each \( L_i \), \( i \in \{1, 2, 3, 4\} \), nontrivially. This already implies \(|K| > 2\). Consider two such ovals \( C_x \) and \( C_y \), with \( x, y \in L_1 \) and project \( X \setminus C_x \) from \( C_x \) onto \( \langle C_y \rangle \). Let \( a_i \) be the projection of \( L_i, i = 2, 3, 4 \).

- Suppose first that \(|K| > 3\). Let \( u, v \in C_x \setminus \{x\} \) be arbitrary and consider the projections \( L_u \) and \( L_v \) of the ovals \( X([u, y]) \) and \( X([v, y]) \), respectively. If \( L_u = L_v \),
then at least three points on \( L_u \) are the image of at least two points of \( X([u,y]) \cup X_p([v,y]) \), and at most one of these is contained in \( C_y \). So there are at least two singular lines intersecting \( C_x \) and projected off \( C_y \). It follows that \( C_y \) misses at least two singular lines, a contradiction. Hence \( L_u \neq L_v \), and since there are at least four choices for \( u \in C_x \setminus \{x\} \), there is a choice such that \( L_u \) misses \( \{a_1, a_2, a_3\} \), and hence \( X([u,y]) \) misses at least two of \( \{L_2, L_3, L_4\} \) (it intersects at most one of these in the plane \( \langle C_x \rangle \)).

Now let \(|K| = 3\). Then an oval through \( x \) and a point of \( C_y \setminus \{y\} \) gives rise to a point \( z \in X \) not contained in \( L_1 \cup L_2 \cup L_3 \cup L_4 \). Every oval through \( z \) must have precisely three points in common with \( L_1 \cup L_2 \cup L_3 \cup L_4 \), which has 16 points, a contradiction as 16 is not divisible by 3.

Now assume that there are exactly three singular lines \( L_1, L_2, L_3 \). Suppose for a contradiction that some point \( x \in X \) is not contained in \( L_1 \cup L_2 \cup L_3 \). Then \( \langle L_1, L_2 \rangle \) shares a point \( y \) with \( \langle x, L_3 \rangle \). As above, this implies that \( x \) and the unique transversal to \( L_1, L_2 \) through \( y \) span a quadratic plane. We conclude that every point outside \( L_1, L_2, L_3 \) is contained in an oval intersecting each of \( L_1, L_2, L_3 \) (and so \(|K| > 2\)). Let \( C \) be such an oval and project \( X \setminus C \) from \( \langle C \rangle \) onto some disjoint plane \( \pi \). The projection of \( L_i \) is some point \( a_i, i = 1, 2, 3 \). Let \( z \in X \) be a point not contained in \( L_1 \cup L_2 \cup L_3 \cup C \), which we may assume to be contained in \( \pi \).

Suppose \(|K| > 3\). The conics through \( z \) and a point of \( C \) project into distinct lines of \( \pi \) through \( z \), because, if not, then by Lemma 4.6, there are at least three points on such projection with inverse image consisting of at least three points, contradicting the fact that \( a_1, a_2, a_3 \) are the only such points, and they are not contained in one line. Hence at least one such line misses \( a_1, a_2 \) and \( a_3 \), and so can only meet one of \( L_1, L_2, L_3 \) (namely, in a point of \( C \)), a contradiction to the second assertion of Lemma 4.9.

Suppose \(|K| = 3\). Since \( x \) is contained in exactly one conic meeting each of \( L_1, L_2, L_3 \), and every other conic through \( x \) meets exactly two of \( L_1, L_2, L_3 \), an easy count implies that there are \( 1 + 9/2 \) conics through \( x \), a contradiction.

Consequently \(|K| = 2\) and we have exactly nine points and nine ovals, as is easily checked. These ovals form a biaffine plane; if we add the singular lines, we have an affine plane of order 3. The uniqueness of this structure in easily proved. \( \square \)

**Lemma 4.12** The set \( X \) cannot contain exactly two singular lines.
Proof Suppose for a contradiction that there are exactly two singular lines \( L_1, L_2 \).

- If \(|K| > 3\), then choose a conic \( C \) containing a point \( x_i \in L_i \) and project \( X \setminus C \) from \( C \) onto some disjoint plane \( \pi \). Let \( a_i \) be the projection of \( L_i, i = 1, 2 \). Since \( k = 5 \), there is some point \( x_3 \in X \), which we may assume to be in \( \pi \), such that \( \langle a_1, a_2, x_3 \rangle = \pi \). As in the previous proof, no two conics through \( x_3 \) and a point of \( C \setminus \{ x_1, x_2 \} \) project into the same line. Hence we can find such an oval whose projection misses \( a_1 \) and \( a_2 \) and hence which does not contain a point of \( L_1 \cup L_2 \), a contradiction to Lemma 4.9.

- Now suppose \(|K| = 3\). If some point \( x \in X_p \setminus (L_1 \cup L_2) \) is only contained in ovals which meet \( L_1 \cup L_2 \) in two points, then all points are contained in \( \langle L_1, L_2, x \rangle \), contradicting \( k = 5 \).

Hence each point is contained in at least one oval intersecting \( L_1 \cup L_2 \) in exactly one point. Suppose the point \( x \in X \setminus (L_1 \cup L_2) \) is contained in \( t \) ovals intersecting \( L_1 \cup L_2 \) in exactly two points; then it is contained \( 8 - 2t \) ovals intersecting \( L_1 \cup L_2 \) in exactly one point. Hence \(|X| = 1 + 3(8 - t)\) and it follows that \( t \) is constant. If \( y \in L_1 \cup L_2 \), then exactly four ovals through \( y \) intersect \( L_1 \cup L_2 \) in two points, leaving \( 9 - 3t \) points. Hence there are \( 3 - t \) ovals through \( y \) intersecting \( L_1 \cup L_2 \) in just \( y \). So in total there are \( 24 - 8t \) ovals intersecting \( L_1 \cup L_2 \) in just one point. On the other hand, there are \( 17 - 3t \) points in \( X \setminus (L_1 \cup L_2) \), each in \( 8 - 2t \) ovals intersecting \( L_1 \cup L_2 \) in just one point. Equating the two expressions obtained for this number, we obtain \( 136 - 58t + 6t^2 = 72 - 24t \), implying \( 32 - 17t + 3t^2 = 0 \), a contradiction.

- Now suppose \(|K| = 2\). A similar count as in the case \(|K| = 3\) implies (with similar definition for \( t \)) that \( \frac{(7-2t)(6-2t)}{2} = 18 - 6t \), so \( t = \frac{1}{2}, \) a contradiction. \( \square \)

The next lemma concludes the proof of Proposition 4.5.

**Lemma 4.13** If the set \( X \) contains a unique singular line \( L \), then \(|K| = 2\) and \((X, \Xi)\) is a projected Veronesean cap.

**Proof**

- Suppose first \(|K| > 2\). We claim that every two ovals that intersect \( L \), intersect mutually. Indeed, let \( C, D \) be two ovals intersecting \( L \) in \( x, y \), respectively. Suppose \( C \) and \( D \) are disjoint. The projection from \( \langle C \rangle \) onto \( \langle D \rangle \) of \( X \setminus (C \cup L) \) is injective.
Hence there are at least two ovals $E_1, E_2$ through $y$ meeting $C$ and projected onto affine lines $A_1, A_2$, respectively, whose projective extensions $M_1, M_2$, respectively, are not tangent to $D$ at $x$. Injectivity implies that $M_i \setminus A_i \in D, i = 1, 2$. If there were a second oval $D'$ through $y$ disjoint from $C$, then its projection would be an oval through $y$ tangent to both $M_1, M_2$, a contradiction. Hence the points of $X \setminus (C \cup L)$ are projected onto the union $U$ of a set of affine lines through $y$ and the oval $D$. Now consider an oval through a point of $L \setminus \{x, y\}$ and some point of $C \setminus \{x\}$. Its projection is an affine line $T$ through $y$, contained in $U$. Since no affine line can be contained in $D$, $T$ is contained in the projective extension $M$ of some projection $A$ of an oval through $y$ and some point of $C$. If $|K| > 3$, then $|(A \cap T) \setminus \{y\}| \geq 2$, contradicting injectivity. If $|K| = 3$, then there are 16 points in total, hence five ovals through a point $z$ of $X \setminus L$. Consequently there is an oval $E$ through $z$ disjoint from $L$. The projection of $X \setminus E$ from $\langle E \rangle$ onto a plane $\pi$ skew to $\langle E \rangle$ containing $L$ is injective. The four ovals through $x$ intersecting $E$ project into four distinct lines of $\pi$. But these lines should also differ from $L$, contradicting the fact that we have only four lines through $x$ in $\pi$.

Hence all conics that intersect $L$ meet mutually. Projection from $L$ onto some disjoint 3-space yields a system of $|K| + 1$ families of $|K|$ lines generating 3-space such that each pair of lines from different families intersect non-trivially. This is only possible for $|K| = 2$.

- Hence let $K \cong \mathbb{F}_2$. If $|X| = 7$, then the four points of $X$ off $L$ are projectively unique; indeed, if $L = \{e_1, e_2, e_1 + e_2\}$, then the other points are $e_3, e_4, e_5, e_6$, where $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ is a basis. The conics and $L$ form a Fano plane. If $|X| > 7$, then there is an oval disjoint from $L$, and hence projection from such an oval onto a plane containing $L$ is injective, implying $|X| \leq 10$. Since $|X|$ is odd, we have $|X| = 9$ and the ovals and $L$ form an affine plane of order 3. Hence there are two disjoint ovals that are also disjoint from $L$. If $\{e_1, \ldots, e_6\}$ is a basis as above, we may assume that the two ovals are $C_1 = \{e_1, e_2, e_3\}$ and $C_2 = \{e_4, e_5, e_6\}$. Since projection from $\langle C_1 \rangle$ onto $\langle C_2 \rangle$ is injective on $X \setminus C_1$ and vice versa, and there is only one line disjoint from $C_i$ in $\langle C_i \rangle$, $i = 1, 2$, we deduce that $L$ is contained in $\langle e_1 + e_2, e_2 + e_3, e_3 + e_1, e_4 + e_5, e_5 + e_6, e_6 + e_4 \rangle$. Hence without loss of generality we may take $L = \{e_1 + e_2 + e_4 + e_5, e_2 + e_3 + e_5 + e_6, e_3 + e_1 + e_6 + e_4\}$. The plane $[e_1, e_4]$ does not contain $e_1 + e_2 + e_4 + e_5$ as it would also contain $e_2 + e_5$, which does not belong to $X_p$, but is also contained in $[e_2, e_5]$. Likewise $[e_1, e_4]$ does not contain $e_3 + e_1 + e_6 + e_4$. Hence it must contain $e_2 + e_3 + e_5 + e_6$. Likewise $[e_2, e_5]$ contains $e_3 + e_1 + e_6 + e_4$. But then $[e_1, e_4]$ and $[e_2, e_5]$ share the point $e_1 + e_2 + \cdots + e_6$, which does not belong to $X$, contradicting (VC2). □
5 Proof of Main Result 2

5.1 General Properties

In this section, \((X, \Xi)\) is a Lagrangian set in \(\mathbb{P}^N(\mathbb{K})\), with \(\mathbb{K}\) a field and \(N\) possibly infinite. We denote by \(\mathcal{G}(X)\) the corresponding geometry of points and singular lines, and by \(\Gamma(X)\) we denote the point graph of \(\mathcal{G}(X)\) (which is the graph with point set \(X\) and adjacency is collinearity). The diameter of \(\Gamma(X)\) is by definition the diameter of \((X, \Xi)\).

The distance between two points \(x, y \in X\) in \(\Gamma(X)\) is denoted by \(\delta(x, y)\). Two points of \(X\) on a singular line will be called \(X\)-collinear. The elements of \(\Xi\) are called the quadratic spaces. Subspaces of \(\mathbb{P}^N(\mathbb{K})\) consisting entirely of points of \(X\) are called singular.

The following is exactly the Quadrangle Lemma of [10], proved there for similar objects, although having diameter 2. We give a proof for completeness’ sake.

**Lemma 5.1 (The Quadrangle Lemma)** Let \(L_1, L_2, L_3, L_4\) be four (not necessarily pair-wise distinct) singular lines such that \(L_i\) and \(L_{i+1}\) share a (not necessarily unique) point \(p_i, i = 1, 2, 3, 4 \mod 4\), and suppose that \(p_1\) and \(p_3\) are not \(X\)-collinear. Then \(L_1, L_2, L_3, L_4\) are contained in a unique common symp.

**Proof** Since \(\langle p_1, p_3 \rangle\) is not singular, we can pick a point \(p \in \langle p_1, p_3 \rangle\) which does not belong to \(X\). Since \(p_1\) and \(p_3\) are \(X\)-collinear with \(p_2\), we have \(\delta(p_1, p_3) = 2\). Hence, by (LS1), there is a unique quadratic space \(\xi\) containing \(p_1\) and \(p_3\). We choose two arbitrary distinct lines \(M_1, M_2\) through \(p\) inside the plane \(\langle L_1, L_2 \rangle\) not containing \(p_2\). Denote \(M_i \cap L_j = \{p_{ij}\}, \{i, j\} \subseteq \{1, 2\}\), then \(\delta(p_{i1}, p_{i2}) = 2, i = 1, 2\). By (LS1) there is a quadratic space \(\xi\) containing \(p_{i1}\) and \(p_{i2}\), \(i = 1, 2\). If \(\xi_1 \neq \xi_2\), then (LS2) implies that \(p\), which is contained in \(\xi_1 \cap \xi_2\), belongs to \(X\), a contradiction. Hence \(\xi_1 = \xi_2 = \xi\) and contains \(L_1, L_2\). We conclude \(\xi\) contains \(L_1, L_2\), and similarly also \(L_3, L_4\). \(\square\)

Now let \(p \in X\) be arbitrary. Let \(U_p\) be a hyperplane of \(T_x\) not containing \(p\) and define \(X_p\) to be the set of points obtained by intersecting \(U_p\) with all singular lines of \(X\) through \(p\). Let \(\Xi_p\) be the set of subspaces of \(U\) obtained by intersecting \(U\) with all tangent spaces at \(p\) to the syms of \((X, \Xi)\) through \(p\). The pair \((X_p, \Xi_p)\) is called the residue of \((X, \Xi)\) in \(p\).

We denote the dimension of \(U_p\) by \(k\). Note \(k \leq 5\).

We have the following result.

**Lemma 5.2** For every \(p \in X\), the residue \((X_p, \Xi_p)\) is a singular pre-Veronesean cap.
Proof Clearly, for any $\xi \in \Xi_p$, we have $X_p \cap \xi$ is a conic. Also, clearly (VC2) is inherited from $(X, \Xi)$. Now suppose $x, y \in X_p$. Assume first that some point of $\langle x, y, p \rangle$ does not belong to $X$. Then there are two points on $\langle x, p \rangle \cup \langle y, p \rangle$ which are not $X$-collinear and the Quadrangle Lemma implies that a unique quadratic space $\xi$ contains $\langle x, p \rangle \cup \langle y, p \rangle = X \cap \langle x, y, p \rangle$. In this case $T_p(\xi) \cap X_p$ is a conic. Assume now that all points of $\langle x, y, p \rangle$ belong to $X$. Then all points of $\langle x, y \rangle$ belong to $X_p$. This shows (VC1').

Since $\Xi$ contains at least two elements, it follows from the connectivity and (LS1) that there is at least one symp $\xi$ through $p$. Now let $x \in X \setminus \xi$. Let $(p, p_1, p_2, p_3, \ldots, x)$ be a minimal path connecting $p$ and $x$ in $\Gamma(X)$. If $p_2 \not\in \xi$, then $X([p, p_2])$ is a second symp through $p$. So suppose $p_2 \in \xi$. Then $p_2 \neq x$ and $p_3$ exists. But now we find a point $y$ outside $\xi$ in $X([p_1, p_3])$ collinear with $p_1$, and so $X([p, y])$ is a symp distinct from $\xi$ and containing $p$. Hence $|\Xi_p| \geq 2$ and the lemma is proved. □

The previous lemma motivates the following terminology. For $p \in X$, if $(X_p, \Xi_p)$ is a Veronesean cap, we call $p$ a straight point. All points are straight as soon as $K \ncong \mathbb{F}_2$. If $K \cong \mathbb{F}_2$, then we also have almost straight points (when $(X_p, \Xi_p)$ is a projected Veronesean), 1-singular points (when $(X_p, \Xi_p)$ contains exactly one singular line) and 3-singular points (when $(X_p, \Xi_p)$ contains exactly three singular lines).

5.2 Lagrangian sets of diameter 2

We now suppose that $\Gamma(X)$ has diameter 2 and prove the following lemma.

Lemma 5.3 If $(X, \Xi)$ has diameter 2, and $x \in X$ is not a 3-singular point, then every point at distance 2 from $x$ in $\Gamma(X)$ is a 3-singular point.

Proof Let $y \in X$ be at distance 2 from $x$ and let $L$ be any singular line of $X$ through $y$ not contained in $[x, y]$. Choose a point $z \in L \setminus \{y\}$. Since $x$ is not 3-singular, every pair of ovals in $(X_x, \Xi_x)$ intersects and hence the symp $X([x, y])$ and $X([x, z])$ intersect in a line $M$ containing $x$. Consequently, looking in $X([x, z])$, there is a point $x'$ of $M$ collinear with $z$. Clearly $x' \neq y$, hence the Quadrangle Lemma implies that $y$ and $x'$ are $X$-collinear, and hence $\langle z, y, x' \rangle$ is a singular plane.

Hence every point of $(X_y, \Xi_y)$ not on the oval corresponding to $[x, y]$ is contained in a singular line. This implies, by Proposition 4.5, that $y$ is a 3-singular point. □

We are now ready to prove the nonexistence of Lagrangian sets of diameter 2.
Proposition 5.4 Lagrangian sets of diameter 2 do not exist.

Proof Since, if $K \not\cong F_2$, every point of $X$ is straight, the assertion follows in that case directly from Lemma 5.3.

Now suppose $K \cong F_2$. Suppose first that some point $p$ is not 3-singular. Then we can select two singular lines $L_1, L_2$ through $p$ which are not contained in a singular plane. It follows that, if $x_i \in L_i$, $i = 1, 2$, are points distinct from $p$, none of the points $x_1, x_2$ is 3-singular. But by our choice, we have $\delta(x_1, x_2) = 2$. This contradicts Lemma 5.3.

Hence we may assume that all points of $X$ are 3-singular. Select an arbitrary point $p$; there are 9 symps through $p$ giving rise to exactly 72 points of $X$ at distance 2 from $p$. There are 9 singular lines through $p$ giving rise to exactly 18 points $X$-collinear with $p$. Together with $p$, this amounts to $91 = |X|$ points. A double count of the pairs $(x, \xi) \in X \times \Xi$ with $x \in \xi$ yields $91 \times 9 = |\Xi| \times 15$, a contradiction. □

Remark 5.5 In fact, in the proof of Proposition 5.4 we did not use Axiom (LS3) explicitly anymore; the facts that in every residue every pair of conics intersects nontrivially and that there are no singular planes, or that $|K| = 2$ and that either every residue has seven points with at most one singular line, or nine points with exactly three singular lines, suffice.

5.3 Lagrangian sets of diameter at least 3

From now on we may assume that the diameter of $\Gamma(X)$ is either unbounded or at least 3. We first aim at showing that the diameter is always equal to 3. Along the way, this will also prove that there are no singular planes.

Lemma 5.6 If $\pi$ is a singular plane, then every point $p \in X$ not contained in $\pi$ is $X$-collinear with exactly one point of $\pi$.

Proof Suppose for a contradiction that no point of $\pi$ is $X$-collinear with $p$. Then, by connectivity, we may assume that there is a point $x \in \pi$ with $\delta(p, x) = 2$. Note that, by Proposition 4.5, every symp through $x$ intersects $\pi$ in a line. Applied to $X([p, x])$, this yields a line $L \in \pi \cap [p, x]$. Inside the symp $X([p, x])$, there is a point $y \in L$ that is $X$-collinear with $p$, contradicting our hypothesis. Hence there is always at least one point of $\pi$ collinear with $p \in X \setminus \pi$.
If at least two points \( x_1, x_2 \in \pi \) are collinear with \( p \), then \( \pi' = \langle x_1, x_2, p \rangle \) is singular and Lemma 4.7 applied to the residue at \( x_1 \) leads to a singular plane in that residue, a contradiction to Lemma 4.8.

\[ \square \]

**Lemma 5.7** There are no singular planes in \( X \).

**Proof** This follows from Proposition 4.5 if \( \mathbb{K} \not\cong \mathbb{F}_2 \); so suppose \( \mathbb{K} \cong \mathbb{F}_2 \). For a contradiction, suppose there is a singular plane, and hence \( X \) contains some 1-singular point or 3-singular point. If all points are either 1-singular or 3-singular, then every point is contained in a singular plane and consequently, using Lemma 5.6, the diameter of \( \Gamma(X) \) equals 2, a contradiction.

Hence we may assume that some point \( p \) is (almost) straight. Noting that (almost) straight points are never \( X \)-collinear with 3-singular points, connectivity leads to the existence of a 1-singular point \( q \). Since \( q \) is contained in a singular plane, it is at distance \( \leq 2 \) from any other point of \( X \), and hence every other point is contained in a symp together with \( q \). Since there are 6 symps through \( q \), there are exactly 48 points at distance 2 from \( q \); since there are 7 singular lines through \( q \), there are 14 points \( X \)-collinear with \( q \). Hence \(|X| = 63\). But a similar count yields already \( 1 + 14 + 56 = 71 \) points at distance 0, 1, 2 from \( p \), a contradiction.

\[ \square \]

**Lemma 5.8** The graph \( \Gamma(X) \) has diameter 3.

**Proof** Suppose for a contradiction that \( x_1, x_2, x_3, x_4, x_5 \) are five points of \( X \) with \( \delta(x_i, x_j) = |i - j|, i, j \in \{1, 2, 3, 4, 5\} \). The symps \( X([x_1, x_3]) \) and \( X([x_3, x_5]) \) intersect in a line \( L \). It follows that there are points \( z_1, z_5 \) on \( L \) which are \( X \)-collinear to \( x_1, x_5 \), respectively. This leads to a path \( (x_1, z_1, z_5, x_5) \) or \( (x_1, z_1, x_5) \) of length 3 or 2 joining \( x_1 \) to \( x_5 \) (depending on whether \( z_1 \neq z_5 \) or \( z_1 = z_5 \)), a contradiction.

We can now determine the isomorphism class of the geometry of points and singular lines of \( X \).

**Lemma 5.9** If \( \mathcal{L} \) denotes the set of singular lines of \( X \), then \( (X, \mathcal{L}) \) is the dual polar space associated to the building of absolute and relative type \( \mathcal{C}_3 \) over the field \( \mathbb{K} \); in other words, \( X \) can be viewed as the set of totally isotropic planes with respect to a symplectic polarity in \( \mathbb{P}^5(\mathbb{K}) \), and the singular lines correspond to the planes intersecting in a common totally isotropic line with respect to that polarity.
Proof Define a geometry $\mathcal{G}$ over the type set $\{1, 2, 3\}$ where the points of $X$ are the elements of type 3, the singular lines in $X$ are the elements of type 2, and the symps in $X$ are the elements of type 1. Incidence is symmetrized containment. From the previous it follows that this is a geometry of type $C_3$. Moreover, properties (LL) and (O) of [13], p. 543, required for the geometry to correspond to a building, are in our setting equivalent to the requirement that if two lines are both contained in two distinct quads $S_1$ and $S_2$, then they coincide, which trivially holds. Hence the geometry corresponds to a building, and since the residue of the elements of type 1 are precisely the symps, hence orthogonal quadrangles $Q(4, \mathbb{K})$, we see that $\mathcal{G}$ is the geometry of the totally isotropic subspaces of a symplectic polarity in $P^5(\mathbb{K})$. Consequently, $(X, \mathcal{L})$ is the corresponding dual polar space $\text{DW}(5, \mathbb{K})$. □

If $|\mathbb{K}| > 2$, then by [5] and [6], $\text{LG}(3, 6)(\mathbb{K})$ is the absolute universal embedding of $(X, \mathcal{L})$, and Proposition 3.3 completes the proof of Main Result 2.

Finally suppose $|\mathbb{K}| = 2$.

Let $Y$ be the point set of the universal embedding of $\text{DW}(5, \mathbb{F}_2)$. By [14], $Y$ spans a 14-dimensional space $\mathbb{P}^{14}(\mathbb{F}_2)$ and the stabilizer of $Y$ in $\text{PGL}_{15}(2)$ induces the full group of automorphisms of $\text{DW}(5, \mathbb{F}_2)$; in particular, it is transitive on the points of $Y$. For $y \in Y$, denote by $\eta_y$ the subspace of $\mathbb{P}^{14}(\mathbb{F}_2)$ generated by all points of $Y$ collinear with $y$ in $\text{DW}(5, \mathbb{F}_2)$. We now show that $\mathbb{P}^{14}(\mathbb{F}_2)$ is generated by $\langle \eta_y, \eta_z \rangle$, for $y, z \in Y$ at distance 3 from each other.

Let $x$ be an arbitrary point of $Y$. If $x$ is at distance at most 2 from one of $y$ and $z$, then we claim it is contained in $\langle \eta_y, \eta_z \rangle$. Indeed, suppose $x$ is at distance 2 from $y$. Let $S$ be the symph through $x$ and $y$; then there is a unique point $t \in S$ collinear with $z$. Since $t$ cannot be collinear with $y$, $S$ is generated by $t$ and the points of $S$ in $\eta_y$. Hence $S \subseteq \langle \eta_y, \eta_z \rangle$ and the claim follows.

So we may assume that $x$ has distance 3 from both $y$ and $z$. In the polar space $W(5, \mathbb{F}_2)$, the points $x, y, z$ correspond to mutually disjoint planes $\pi_x, \pi_y, \pi_z$. We claim that there is a plane $\pi$ intersecting $\pi_x$ in a line and $\pi_y \cup \pi_z$ in a single point. Indeed, clearly no plane intersecting $\pi_x$ in a line can meet $\pi_y \cup \pi_z$ in more than two points. Suppose now, for a contradiction, that each plane $\pi_L$ which intersects $\pi_x$ in a line $L$ and $\pi_y$ in a point $y_L$ (there are precisely 7 such planes) intersects $\pi_z$ in a point $z_L$. If $y_L = y_M$, for $L, M$ lines of $\pi_x$, then $y_L$ is collinear with all points of $\pi_x$, a contradiction. Since the planes $\pi_x, \pi_y, \pi_z$ are disjoint, one deduces that the mapping $\pi_y \to \pi_z : y_L \mapsto z_L$ induces a collineation, and so also the mapping $\pi_y \to \pi_x : y_L \mapsto L \cap \langle y_L, z_L \rangle$ is a collineation. Now the projection mapping $\pi_x \to \pi_y : L \mapsto y_L$ is a duality; hence the mapping $\pi_x \to \pi_z : L \mapsto L \cap \langle y_L, z_L \rangle$ is a duality, every point of which is incident with its image. It is easy to see that this is a
contradiction. This proves our claim. So there are planes $\alpha_y$ and $\alpha_z$ intersecting $\pi_x$ in a common line $L$, intersecting $\pi_y$ and $\pi_z$, respectively, in some point, and disjoint from $\pi_z$ and $\pi_y$, respectively.

Now, this implies that the line $L'$ in $Y$ corresponding to the line $L$ of $W(5, \mathbb{F}_2)$ contains the point $x$, and the points at distance 2 from $y$ and $z$ on $L'$ are distinct. Hence $x \in \langle \eta_y, \eta_z \rangle$ by our first claim, and we have shown $\langle \eta_y, \eta_z \rangle$ is the whole space. By transitivity of the automorphism group on $Y$, we either have $\dim \eta_z = \dim \eta_y = 6$, or $\dim \eta_z = \dim \eta_y = 7$. In the former case, $\dim \langle \eta_y, \eta_z \rangle \leq 13$, a contradiction. Hence $\dim \eta_y = 7$, for all $y \in Y$. Since $LG(3,6)(\mathbb{F}_2)$ is isomorphic to the projection of $Y$ from a point $c \notin Y$, Axiom (LS3) yields that $c$ is contained in $\eta_y$, for all $y \in Y$. Choosing coordinates in $\eta_y$ appropriately, we may assume $y = (1, 0, 0, 0, 0, 0, 0, 1, 0, \ldots, 0)$, and the other points of $Y \cap \eta_y$ are $(0, \ldots, 0, 1, 0, \ldots, 0)$ and $(1, 0, \ldots, 0, 1, 0, \ldots, 0)$ (the 1 is twice in the $i$th position), $i = 2, \ldots, 8$. The point $c$ consequently has coordinates either $(1, 1, \ldots, 1)$ or $(0, 1, \ldots, 1)$. Without loss of generality, we may assume the former.

Now suppose for a contradiction that $X$ does not arise from $Y$ by projection from $c$. Then it must arise from $Y$ by projection from a subspace $C$ that intersects $\eta_y$ in a unique point $y_C$, for every $y \in Y$. Since (the projection of) $y$ is either a straight or an almost straight point, the point $y_C$ either has coordinates $(0, 1, \ldots, 1)$ (in case of a straight point), or we may assume without loss of generality that $y_C$ has coordinates $(0, 0, 1, \ldots, 1)$. In both cases, the projection of $c$ coincides with the projection of a point of $Y \cap \eta_y$, namely, $y$ and $(1, 1, 0, \ldots, 0)$, respectively. Since this holds for all $y \in Y$, it implies that the projection from $C$ is not injective on $Y$, a contradiction.

Hence $X$ arises from $Y$ by projection from $c$, and we obtain $LG(3,6)(\mathbb{F}_2)$. The proof of Main Result 2 is complete.

References


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