A non-classical unital of order four with many translations

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Abstract

We give a general construction for unitals of order q admitting an action of $\mathrm{SU}(2, q)$. The construction covers the classical hermitian unitals, Grüning's unitals in Hall planes and at least one unital of order four where the translation centers fill precisely one block. For the latter unital, we determine the full group of automorphisms and show that there are no group-preserving embeddings into (dual) translation planes of order 16.

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Introduction

Let $\mathbb{U} = (U, \mathcal{B})$ be a unital of order q, and let $\Gamma := \mathrm{Aut}(\mathbb{U})$. For each point $c \in U$ we consider the group $\Gamma_{[c]}$ of translations with center c, i.e., the set of all automorphisms of $\mathbb{U}$ fixing each block through c. We say that c is a translation center of $\mathbb{U}$ if $\Gamma_{[c]}$ is transitive on the set of points different from c on any block through c.

The main result of [9] states that the unital $\mathbb{U}$ is classical (i.e., isomorphic to the hermitian unital corresponding to the field extension $\mathbb{F}_{q^2}/\mathbb{F}_q$) if it has non-collinear translation centers. Unitals with precisely one translation center seem to exist in abundance (we indicate several quite different classes of examples in Section 5 below). If there are two translation centers c and c' then the orbit of c' under $\Gamma_{[c]}$ fills the complement of c in the block joining c with c'. We give an example of a unital (of order 4, see Section 1 below) where the translation centers fill just one block. As far as we know, this unital is the first (and up to now, the only) one with that property.

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1 A curious unital of order four

Let $\mathrm{Syl}_2(\mathrm{A}_5)$ be the set of all five Sylow 2-subgroups in the alternating group A_5, and let $S := \langle(01234)\rangle$. We consider the subsets $E_1 := \{\mathrm{id}, (023), (024), (123), (03421)\}$ and $E_2 := E_1^{(1243)} = \{\mathrm{id}, (041), (043), (124), (01342)\}$. For later reference, we abbreviate $\mathcal{E} := \{E_1, E_2\}$.

We construct an incidence structure $\mathbb{U}_{\mathcal{E}}$ with two kinds of points: elements of A_5 and elements of $\mathrm{Syl}_2(\mathrm{A}_5)$. The blocks are the following:

- cosets Tg for $T \in \mathrm{Syl}_2(\mathrm{A}_5)$, $g \in \mathrm{A}_5$,
- cosets Sg for $g \in \mathrm{A}_5$,
- sets E_jg for $g \in \mathrm{A}_5$ and $j \in \{1, 2\}$,
- a single block named $[\infty]$.

Note that the elements of $\mathrm{Syl}_2(\mathrm{A}_5)$ are used as (labels for) points and also as certain blocks. Incidence between points in A_5 and subsets is the obvious one. A point $T \in \mathrm{Syl}_2(\mathrm{A}_5)$ is incident with $[\infty]$, with each *left* coset gT ($= T^g g$), and with no other block.

In the present paper, we show that the incidence geometry $\mathbb{U}_{\mathcal{E}}$ is a non-classical unital of order 4, with the following properties: The action of A_5 by multiplication from the right on itself and on the blocks apart from $[\infty]$ is an action by automorphisms of $\mathbb{U}_{\mathcal{E}}$. Each $T \in \mathrm{Syl}_2(\mathrm{A}_5)$ acts by translations with center T. There are no other translations of $\mathbb{U}_{\mathcal{E}}$. The full automorphism group of $\mathbb{U}_{\mathcal{E}}$ is isomorphic to a semi-direct product of A_5 with a cyclic group of order four, where a generator of that cyclic group induces conjugation by (1243) on A_5.

2 A general construction

Motivated by the action (see 3.1 below) of $\mathrm{SU}(2, q)$ on the classical (hermitian) unital of order q, we study geometries as follows.

2.1 Lemma. *Let G be a group, let T be a subgroup such that conjugates T^g and T^h have trivial intersection unless they coincide (i.e., the conjugacy class T^G forms a T.I. set). Assume that there is a subgroup S and a collection $\mathcal{D}$ of subsets of G such that each set $D \in \mathcal{D}$ contains* 1, *and the following properties hold:*

(Q) *For each $D \in \mathcal{D}$, the map $(D \times D) \setminus \{(x, x) \mid x \in D\} \to G \colon (x, y) \mapsto xy^{-1}$ is injective. We abbreviate $D^* := \{xy^{-1} \mid x, y \in D, x \neq y\}$.*

(P) *The system consisting of $S \setminus \{1\}$, all conjugates of $T \setminus \{1\}$ and all sets D^* with $D \in \mathcal{D}$ forms a partition of $G \setminus \{1\}$.*

Then the incidence structure with point set G and block set

$$\mathcal{B}^\infty := \{Sg \mid g \in G\} \cup \{T^h g \mid h, g \in G\} \cup \{Dg \mid D \in \mathcal{D}, g \in G\}$$

is a linear space. Each involution of G is contained in $S \cup \bigcup_{g \in G} T^g$.

We consider each conjugate T^h as a point at infinity, call $[\infty] := \{T^h \mid h \in G\}$ the block at infinity (incident with each point at infinity, and no point in G), and extend the incidence relation in two different ways:

(a) *Make each conjugate T^h incident with each coset $T^{hg^{-1}}g = gT^h$ (and no other block in $\mathcal{B}^\infty$). This gives an incidence structure $\mathbb{U}_\mathcal{D} := \big(G \cup \{T^h \mid h \in G\}, \mathcal{B}^\infty \cup \{[\infty]\}, I\big)$.*

(b) *Make each conjugate T^h incident with each coset $T^h g$ (and no other block in $\mathcal{B}^\infty$). This gives an incidence structure $\mathbb{U}^\flat_\mathcal{D} := \big(G \cup \{T^h \mid h \in G\}, \mathcal{B}^\infty \cup \{[\infty]\}, I^\flat\big)$.*

Then both $\mathbb{U}_\mathcal{D}$ and $\mathbb{U}^\flat_\mathcal{D}$ are linear spaces, and the following hold.

1. *Via multiplication from the right on G and conjugation on the point row of $[\infty]$, the group G acts as a group of automorphisms on $\mathbb{U}_\mathcal{D}$.*
2. *On $\mathbb{U}^\flat_\mathcal{D}$ the group G also acts by automorphisms via multiplication from the right on G but trivially on the point row of $[\infty]$.*

Now let G be finite, and abbreviate $q := |T|$. Assume that $|G| = q^3 - q$, that there are $q+1$ conjugates of T, and that $|S| = q + 1 = |D|$ holds for each $D \in \mathcal{D}$. Note that we have $|\mathcal{D}| = q - 2$ in that case.

3. *Both $\mathbb{U}_\mathcal{D}$ and $\mathbb{U}^\flat_\mathcal{D}$ are $2 - (q^3 + 1, q + 1, 1)$ designs; i.e., unitals of order q.*
4. *On the unital $\mathbb{U}_\mathcal{D}$ each conjugate of T acts as a group of translations. Thus each point on the block $[\infty]$ is a translation center, and G is two-transitive on $[\infty]$.*
5. *On the unital $\mathbb{U}^\flat_\mathcal{D}$ the group G contains no translation except the trivial one.*

Proof. Assume first that some involution $s \in G$ lies outside $S \cup \bigcup_{g \in G} T^g$. Then the assumptions yield that s is of the form $s = xy^{-1}$ with $x, y \in D$ for some $D \in \mathcal{D}$. Then $xy^{-1} = s = s^{-1} = yx^{-1}$ contradicts assumption **(Q)**.

We consider blocks through 1 first. Cosets like $T^h g$ or Sg contain 1 if, and only if, they coincide with the subgroups T^h and S, respectively. The situation is different for Dg with $D \in \mathcal{D}$: here $1 \in Dg \iff g^{-1} \in D$. Thus Dg passes through 1 precisely if $Dg \subseteq D^* \cup \{1\}$. Now the partition required in condition **(P)** secures that each element in $G \smallsetminus \{1\}$ is joined to 1 by a unique block in $\mathcal{B}^\infty$. As G forms a transitive group of automorphisms of $(G, \mathcal{B}^\infty)$, that structure is a linear space.

The conditions imposed on the orders of G, T, S, and $|D|$ make it immediate that both $\mathbb{U}_\mathcal{D}$ and $\mathbb{U}^\flat_\mathcal{D}$ are $2 - (q^3 + 1, q + 1, 1)$ designs.

Each orbit of each conjugate of T is contained in a block, but these blocks are assigned points at infinity in different ways in the two incidence structures $\mathbb{U}_\mathcal{D}$ and $\mathbb{U}^\flat_\mathcal{D}$. For $(g, T^h) \in G \times [\infty]$ the unique joining block in $\mathbb{U}_\mathcal{D}$ is $T^{hg^{-1}}g$. Thus $\mathbb{U}_\mathcal{D}$ is a linear space. In $\mathbb{U}^\flat_\mathcal{D}$ the unique joining block for (g, T^h) is $T^h g$, and $\mathbb{U}^\flat_\mathcal{D}$ is a linear space, as well.

We note that the subgroup $T^h \leq G$ fixes each block through the point T^h in $\mathbb{U}_\mathcal{D}$ because $T^{hg^{-1}}g = gT^h$. Thus T^h is a group of translations with center T^h on $\mathbb{U}_\mathcal{D}$. As T^h

acts transitively on the set of points on the block T^h that are different from T^h (considered as a point), the point T^h is a translation center.

In $\mathbb{U}^\flat_{\mathcal{D}}$ the group T fixes $[\infty]$ and precisely $|\mathrm{N}_G(T)/T|$ blocks through each point $T^h \in [\infty]$, namely the blocks of the form $T^h g = gT^{hg}$ with $hg \in \mathrm{N}_G(T)$. As every element of G fixes each point at infinity, the group G contains no translations of the unital $\mathbb{U}^\flat_{\mathcal{D}}$ apart from the trivial one. □

2.2 Remarks. It may come as a surprise that the incidence relation for points at infinity is not determined by the affine part of $\mathbb{U}_{\mathcal{D}}$ (i.e., the linear space $(G, \mathcal{B}^\infty)$ together with the action of G on that geometry). We have at least one other possibility, namely the unital $\mathbb{U}^\flat_{\mathcal{D}}$ where G acts trivially on $[\infty]$, and there are no translations of the unital in G. Grüning's unital (see 5.5 below) is obtained by such a completion of $(G, \mathcal{B}^\infty)$.

3 Examples

3.1 Classical unitals. Let C be a field with an involutory automorphism $\kappa\colon x \mapsto \overline{x}$. On C^2 we consider the affine lines, the hermitian form $(x, y) \mapsto x\,\overline{x} - y\,\overline{y}$ and the subset $A := \{(x, y) \mid x\,\overline{x} - y\,\overline{y} = 1\}$. The special unitary group with respect to the given form is $\mathrm{SU}(2, C) = \left\{ \left(\begin{smallmatrix} x & y \\ \overline{y} & \overline{x} \end{smallmatrix}\right) \,\middle|\, x, y \in C, x\,\overline{x} - y\,\overline{y} = 1 \right\}$. It is known that $\mathrm{SU}(2, C) \cong \mathrm{SL}(2, R)$, where $R = \mathrm{Fix}(\kappa)$. Note that $\mathrm{SU}(2, C)$ acts regularly on A; we identify $(x, y) \in A$ with $\left(\begin{smallmatrix} x & y \\ \overline{y} & \overline{x} \end{smallmatrix}\right) \in \mathrm{SU}(2, C)$. We may even extend this to an identification of $C^2 \smallsetminus \{(x, y) \mid x\,\overline{x} = y\,\overline{y}\}$ with a subgroup of the group of similitudes of the hermitian form.

We describe the interesting intersections of A with lines through $(1, 0)$; in particular, all the blocks through $(1, 0)$:

1. The line $(1, 0) + C(0, 1)$ is a tangent to the unital, it meets the unital only in $(1, 0)$.
2. The (stabilizer of the) block induced by the line $(1, 0) + C(1, 0)$ is the subgroup $S = \{(x, 0) \mid x\,\overline{x} = 1\}$ of $\mathrm{SU}(2, C)$.
3. The (stabilizer of the) block induced by the line $(1, 0) + C(1, 1)$ is the subgroup $T_1 := \{(1 + x, x) \mid x + \overline{x} = 0\}$. For each $s \in \{c \in C \mid c\,\overline{c} = 1\}$ the line $(1, 0) + C(1, s)$ induces the block $T_s = \{(1 + x, xs) \mid x + \overline{x} = 0\}$ which is a conjugate of T_1; in fact, we have $(u^{-1}, 0)T_1(u, 0) = T_{\overline{u}/u}$.
4. For $t \in \{c \in C \mid c\,\overline{c} \notin \{0, 1\}\}$, the block induced by the line $(1, 0) + C(1, t)$ is $H_t := \{(1 + x, xt) \mid x + \overline{x} = (t\,\overline{t} - 1)x\,\overline{x}\}$. These subsets of $\mathrm{SU}(2, C)$ are not subgroups; the corresponding stabilizers are trivial.

   We note that $\widehat{H_t} := \{ab^{-1} \mid a, b \in H_t\}$ equals the union $\bigcup_{s \in C\colon s\,\overline{s} = 1} H_{st}$ of the orbit of H_t under conjugation by S. Mapping (x, y) to xy^{-1} gives a bijection from $(H_t \smallsetminus \{(1, 0)\})^2$ onto $\widehat{H_t}$.

If C is the finite field of order $q^2 = p^{2n}$ with $p = \operatorname{char} C$ then T_s is a Sylow p-subgroup of $\mathrm{SU}(2, C) = \mathrm{SU}(2, q)$, the group S is cyclic of order $q + 1$, and there are $q - 2$ orbits of blocks of type H_t under conjugation by S. We choose an arbitrary set $\mathcal{H}_q = \{H_1, \ldots, H_{q-2}\}$ of representatives for these orbits. Then $\mathbb{U}_{\mathcal{H}_q}$ is the *classical* (hermitian) unital.

Grüning's unitals (embedded in Hall planes and their duals, see [10] and 5.5 below) are obtained as $\mathbb{U}^\flat_{\mathcal{H}_q}$. For $q > 2$ the unitals $\mathbb{U}_{\mathcal{H}_q}$ and $\mathbb{U}^\flat_{\mathcal{H}_q}$ are not isomorphic because $\mathbb{U}^\flat_{\mathcal{H}_q}$ contains O'Nan configurations (see [10, 5.4]).

3.2 Example. There is only one isomorphism type of unitals of order 2 because such a unital is actually an affine plane of order 3. Our construction yields that unital from the group $S_3 \cong \mathrm{SL}(2,2)$, its Sylow 2-subgroups, and $S = \langle(012)\rangle$; the collection $\mathcal{H}_2$ is empty. The two possibilities for incidences on $[\infty]$ lead to isomorphic unitals $\mathbb{U}_{\mathcal{H}_2} \cong \mathbb{U}^\flat_{\mathcal{H}_2}$ but different actions of $\mathrm{SL}(2,2)$.

Unitals of order 3 exist in abundance, see [4]. However, there is only one candidate for D^*, and essentially only one choice for D:

3.3 Theorem. *Assume $q = 3$ and $G = \mathrm{SL}(2,3) \cong \mathrm{SU}(2,3)$, and let $\mathcal{D}$ be a collection as in 2.1. Then S is cyclic of order 4, and $\mathcal{D}$ consists of just one set D. Moreover, the pair (S, D) is unique, up to conjugation in $\mathrm{SL}(2,3)$. Thus we obtain two unitals, namely $\mathbb{U}_{\mathcal{H}_3}$ and $\mathbb{U}^\flat_{\mathcal{H}_3}$.*

Proof. Clearly we have $|\mathcal{D}| = q - 2 = 1$. We consider the element $\delta := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ of order 3, the element $\varphi_0 := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ of order 4, and the conjugates $\varphi_j := \delta^{-j}\varphi_0\delta^j$ for $j \in \{1,2\}$. The group $\mathrm{SL}(2,3)$ contains the elements

$$\begin{aligned}
&\mathrm{id} \text{ (of order 1)}, \ -\mathrm{id} \text{ (of order 2)},\\
&\delta, \delta^2, \varphi_j\delta = \delta\varphi_{j+1}, -\varphi_j\delta^2 = -\delta^2\varphi_{j-1} \text{ (of order 3)},\\
&\pm\varphi_j \text{ (of order 4), and}\\
&-\delta, -\delta^2, -\varphi_j\delta = -\delta\varphi_{j+1}, \varphi_j\delta^2 = \delta^2\varphi_{j-1} \text{ (of order 6)}.
\end{aligned}$$

Up to conjugation, we may assume that S is generated by φ_0. The point 1 is joined by blocks to the elements of the union $\mathcal{T} = \{\mathrm{id}, \delta, \delta^2\} \cup \{\varphi_j\delta, -\varphi_j\delta^2 \mid j \in \{0,1,2\}\}$ of Sylow 3-subgroups (these are the blocks that also meet the block at infinity), and joined by the block S to the elements of $S = \{\mathrm{id}, -\mathrm{id}, \varphi_0, -\varphi_0\}$.

Without loss, we assume $\varphi_1 \in D$. Then the conditions $S \cap (D \smallsetminus \{\varphi\})(-\varphi_1) \subseteq S \cap D^* = \emptyset$ and $S \cap (D \smallsetminus \{\varphi\})(-\varphi_1) \subseteq S \cap D^* = \emptyset$ yield $D = \{\mathrm{id}, \varphi_1, b, c\}$ with $\{b, c\} \subseteq \{-\delta, -\varphi_1\delta, -\varphi_2\delta, \varphi_2\delta^2\}$. For $b = -\varphi_1\delta$ we find $bc^{-1} \in S \cup S$ for each $c \in \{-\delta, -\varphi_2\delta, \varphi_2\delta^2\}$; i.e., for each remaining choice of c. Thus $-\varphi_1\delta \notin D$. Analogously, the choice $b = -\varphi_2\delta$ is excluded. The last remaining possibility is $D = \{\mathrm{id}, \varphi_1, -\delta, -\varphi_2\delta\}$. □

3.4 Example. The isomorphisms $\mathrm{SU}(2,4) \cong \mathrm{SL}(2,4) \cong A_5$ allow to give an alternative description for the classical unital of order 4 which will come handy if we compare $\mathbb{U}_{\mathcal{H}_4}$ and $\mathbb{U}_{\mathcal{E}}$. We use $\mathrm{Syl}_2(A_5)$, the subgroup $S = \langle(01234)\rangle$ and $\mathcal{C} = \{C_1, C_2\}$ with $C_1 := \{\mathrm{id}, (032), (134), (02134), (03214)\}$ and $C_2 := C_1^{(1243)}$. Then $\mathbb{U}_{\mathcal{C}} \cong \mathbb{U}_{\mathcal{H}_4}$ and $\mathbb{U}^\flat_{\mathcal{C}} \cong \mathbb{U}^\flat_{\mathcal{H}_4}$.

3.5 Example. In $G = A_5$, we take $\mathrm{Syl}_2(A_5)$, the group $S = \langle(01234)\rangle$ and the sets $E_1 := \{\mathrm{id}, (023), (024), (123), (03421)\}$, $E_2 := E_1^{(1243)} = \{\mathrm{id}, (023), (024), (123), (03421)\}$ as in

Section 1. Then $\mathcal{E} := \{E_1, E_2\}$ together with the Sylow 2-subgroups of G and $S = \langle(01234)\rangle$ satisfies the conditions in 2.1, so $\mathbb{U}_\mathcal{E}$ and $\mathbb{U}^\flat_\mathcal{E}$ are unitals.

We show the table for $\widehat{E_1}$ on the left, on the right there is the table for $\widehat{C_1}$ (leading to the classical unital, see 3.4); the entry in the row starting with δ on the left and the column starting with γ on top is the quotient $\delta\gamma^{-1}$:

	(032)	(042)	(132)	(01243)
(023)		(234)	(013)	(04312)
(024)	(243)		(01324)	(04123)
(123)	(031)	(04231)		(01432)
(03421)	(02134)	(03214)	(02341)	

	(023)	(143)	(04312)	(04123)
(032)		(01432)	(01243)	(124)
(134)	(02341)		(042)	(04231)
(02134)	(03421)	(024)		(031)
(03214)	(142)	(01324)	(013)	

4 Automorphisms and Embeddings

Recall that an O'Nan configuration consists of four blocks such that any two of them meet. The hermitian unitals do not contain such configurations (cp. [8, 2.2]), so the following observation secures that the unitals $\mathbb{U}_\mathcal{E}$ and $\mathbb{U}^\flat_\mathcal{E}$ are not classical.

In $\mathbb{U}_\mathcal{E}$ we have the O'Nan configuration

$$\begin{aligned} D_0 &= \{\mathrm{id}, (023), (024), (123), (03421)\}\,, \\ D_1(013) &= \{(013), (043), (04)(13), (01243), (03421)\}\,, \\ T &:= \{\mathrm{id}, (01)(34), (03)(14), (04)(13)\}\,, \\ (123)T &= \{(123), (01243), (03412), (04)(12)\}\,; \end{aligned}$$

note that the blocks T and $(123)T = T^{(132)}(123)$ share the point $T \in [\infty]$ in $\mathbb{U}_\mathcal{E}$.

In $\mathbb{U}^\flat_\mathcal{E}$ we have the O'Nan configuration

$$\begin{aligned} D_0 &= \{\mathrm{id}, (023), (024), (123), (03421)\}\,, \\ D_1(013) &= \{(013), (043), (04)(13), (01243), (03421)\}\,, \\ T &:= \{\mathrm{id}, (01)(34), (03)(14), (04)(13)\}\,, \\ T(024) &= \{(024), (01243), (03241), (13)(24)\}\,; \end{aligned}$$

now the blocks T and $T(024)$ share the point $T \in [\infty]$ in $\mathbb{U}^\flat_\mathcal{E}$.

The following result on the full automorphism group gives an alternative argument to see that $\mathbb{U}_\mathcal{E}$ is not classical.

4.1 Theorem. *The unital $\mathbb{U}_\mathcal{E}$ is not classical, and* $\mathrm{Aut}(\mathbb{U}_\mathcal{E}) \cong \mathrm{C}_4 \ltimes \mathrm{A}_5$.

Proof. Consider the stabilizer L of the block $[\infty]$ in $\mathrm{Aut}(\mathbb{U}_\mathcal{E})$. As the subgroup $\mathrm{A}_5 \leq \mathrm{Aut}(\mathbb{U}_\mathcal{E})$ is generated by the translations with centers on $[\infty]$, it is normal in L. Since A_5 acts transitively on the points not on $[\infty]$, the stabilizer L_id of the point id acts (via conjugation) faithfully by automorphisms of A_5. This means that L_id is a subgroup of S_5, acting by conjugation on A_5, cf. [16].

The block S is the only block through id with non-trivial stabilizer in A_5 and not meeting $[\infty]$. Thus L_id normalizes S, and is a subgroup of $\langle(1243), (01234)\rangle$. Conjugation with (01234) does not leave $\{\widehat{D_0}, \widehat{D_1}\}$ invariant, but $\sigma = (1243)$ does.

Analogously, we find that the stabilizer of a block in the classical unital $\mathbb{U}_{\mathcal{H}_4}$ is a semi-direct product $(C_4 \ltimes C_5) \ltimes A_5$. So the stabilizers of blocks are different, and $\mathbb{U}_{\mathcal{E}} \not\cong \mathbb{U}_{\mathcal{H}_4}$.

As the classical unitals are characterized by the fact that they have non-collinear translation centers (see [9]), the full group of automorphisms of $\mathbb{U}_{\mathcal{E}}$ coincides with the block stabilizer L. □

4.2 Corollary. *The unital $\mathbb{U}_{\mathcal{E}}$ is not isomorphic to a Buekenhout-Metz unital.*

Proof. Every Buekenhout-Metz unital of order q admits an automorphism group of order q^3 (see [7, Theorem 1] for the case of even q), but by 4.1 the order of $\mathrm{Aut}(\mathbb{U}_{\mathcal{E}})$ is not divisible by 4^3. □

We observe that $\mathrm{Aut}(\mathbb{U}_{\mathcal{E}})$ has index 5 in the stabilizer of the block $[\infty]$ in $\mathrm{Aut}(\mathbb{U}_{\mathcal{H}_4})$. The latter group is isomorphic to $\mathrm{P\Gamma U}(3, \mathbb{F}_4)$, and has order $2^8 \cdot 3 \cdot 5^2 \cdot 13 = 249\,600$. The block stabilizer has order $2^4 \cdot 3 \cdot 5^2 = 1\,200$.

We can repeat the arguments from the proof of 4.1 to conclude that the *normalizer* of A_5 in $\mathrm{Aut}(\mathbb{U}^\flat_{\mathcal{E}})$ is isomorphic to $C_4 \ltimes A_5$. From Grüning [10, 5.6] we know that $\mathrm{Aut}(\mathbb{U}^\flat_{\mathcal{E}})$ is isomorphic to a block stabilizer in the automorphism group of the classical unital. So $\mathrm{Aut}(\mathbb{U}^\flat_{\mathcal{E}}) \cong C_4 \ltimes A_5$.

4.3 Lemma. *Let $\mathbb{U}$ be a unital of order q embedded in a projective plane $\mathbb{P}$ of order q^2. If $\varphi \in \mathrm{Aut}(\mathbb{P})$ leaves $\mathbb{U}$ invariant and induces a translation of the unital $\mathbb{U}$ then φ is an elation of $\mathbb{P}$; the center is the unique fixed point x in $\mathbb{U}$, and the axis is the tangent to $\mathbb{U}$ at x in $\mathbb{P}$.* □

Computer based results

Using our knowledge of $\mathrm{Aut}(\mathbb{U}_{\mathcal{E}})$, we notice that $\mathbb{U}_{\mathcal{E}}$ does not occur in the collection of unitals of order 4 presented in [15].

According to the results of a computer-based search by Bamberg, Betten, Praeger, and Wassermann (see [3]), there are just two orbits of unitals in $\mathrm{PG}(2, 16)$, containing the hermitian unitals and Buekenhout-Metz unitals, respectively. Combining that result with 4.1 and 4.2, we obtain:

4.4 Theorem. *There are no embeddings of $\mathbb{U}_{\mathcal{E}}$ into the desarguesian plane of order* 16. □

4.5 Theorem. *The unital $\mathbb{U}_{\mathcal{E}}$ has no embedding into any (dual) translation plane of order* 16 *such that every translation of the unital extends to a collineation of the plane.*

Proof. Assume that such an embedding exists. Then the translations generate a non-solvable group isomorphic to $A_5 \cong \mathrm{SL}(2, 4)$.

The plane is not desarguesian by 4.4. Up to duality, it is then one of the planes listed in [12] (cf. also [6]). Inspection of the list yields that the involutions in that group (i.e., the collineations inducing the translations on the unital) are Baer involutions. This contradicts 4.3. □

4.6 Remarks. Among the known projective planes of order 16 (as found on Gordon Royle's homepage, see `http://staffhome.ecm.uwa.edu.au/~00013890/`), the ones admitting groups with orders divisible by 5 are the desarguesian one, the Hall plane, the Dempwolff plane, a plane called BBH2 in the list, and the duals of the last three planes. Apart from BBH2, all these planes are translation planes (up to duality), and treated in 4.5. Peter Müller (Würzburg) has checked that the automorphism group of BBH2 is solvable, using the generators for this group from the homepage of Eric Moorhouse (see http://www.uwyo.edu/moorhouse/pub/planes16/) and the computer algebra system Sage.

Therefore, there do not exist any embeddings of $\mathbb{U}_{\mathcal{E}}$ into any of the known projective planes of order 16 such that the translations extend.

5 Unitals with just one translation center, or none at all

There is an ample supply of unitals having precisely one translation center. We give various examples, starting with unitals in planes over twisted fields:

5.1 Definitions. Let q be a power of an odd prime, and let d be a divisor of q such that -1 is not in $\{x^{d-1} \mid x \in \mathbb{F}_q\}$. Then $\delta\colon x \mapsto x^d$ is an automorphism of $\mathbb{F}_q$; the additive map $\delta + \mathrm{id}\colon \mathbb{F}_q \to \mathbb{F}_q\colon x \mapsto x^d + x$ is injective, and thus also surjective.

The new multiplication $*$ defined[1] by $(a^d + a) * (b^d + b) := 2(a^d b + b^d a)$ yields a semifield $S_q^d := (\mathbb{F}_q, +, *)$. If $1 < d < q$ then S_q^d is not associative (cf. [5, 5.3.8]); but $S_q^1 = \mathbb{F}_q$. We will describe the affine translation plane as usual, using the sets $[m, t] := \{(x, m * x + t) \mid x \in S_q^d\}$ for the non-vertical lines[2].

For each $v \in \mathbb{F}_q^\times$ we find[3] that $\gamma_v\colon (x^d + x, y) \mapsto ((vx)^d + vx, v^{d+1} y)$ is a collineation of the (affine) plane over S_q^d; the line $[a^d + a, b]$ is mapped to the line $[(va)^d + (va), v^{d+1} b]$.

The projective plane $\mathbb{P}_{S_q^d}$ admits polarities: one of these is $\widehat{\mathrm{id}}$, interchanging (x, y) with $[x, -y]$. The absolute points form an oval $\mathcal{O} := \mathbb{U}(S_q^d, \mathrm{id})$. If q is a square then there exists a (unique) involution $\kappa\colon x \mapsto \overline{x}$ in $\mathrm{Aut}(\mathbb{F}_q)$. Since κ commutes with $\delta \in \mathrm{Aut}(\mathbb{F}_q)$ we find that κ is also an automorphism of S_q^d, and obtain a polarity $\hat{\kappa}$ interchanging (x, y) with $[\overline{x}, -\overline{y}]$. The absolute points form a unital $\mathbb{U}(S_q^d)$.

If $q = r^2$ is a square then $\mathbb{P}_{S_q^d}$ contains another unital, with rather surprising properties (see [2]): One combines translates of the polar oval $\mathcal{O}$, forming the point set $E_t := \{\infty\} \cup \bigcup_{s \in \mathbb{F}_r} (\mathcal{O} + (0, st))$ for some $t \in \mathbb{F}_q$ which is not a square. Then $\mathbb{E}_t := (E_t, \mathcal{B}_t)$ is a unital, where $\mathcal{B}_t$ consists of the traces of secants of E_t in $\mathbb{P}_{S_q^d}$. The unital $\mathbb{E}_t$ is invariant under the affine collineations of the form

$$\varphi_{u,s}\colon (S_q^d)^2 \to (S_q^d)^2\colon (x, y) \mapsto \left(x + u, y + u * x + \tfrac{1}{2}(u * u) + st\right)$$

[1] We have introduced the factor 2 in the definition in order to ensure that $1 \in \mathbb{F}_q$ is still the neutral element of the multiplication $*$ — without that factor (such as in [1]) this neutral element would be $1 + 1$.

[2] We deviate from the description in [1], [2] here. This implies changes in the formulae for certain collineations.

[3] See [1, Prop. 1]. Our description of the collineation differs from that in [1, Prop. 1] because we use a different description for the lines.

with $u \in \mathbb{F}_q$ and $s \in \mathbb{F}_r$. It is easy to see that $\{\varphi_{u,s} \mid u \in \mathbb{F}_q, s \in \mathbb{F}_r\}$ is an elementary abelian group acting sharply transitive on $E_t \setminus \{\infty\}$. There are further collineations leaving $\mathbb{E}_t$ invariant, namely the maps γ_v with $v \in \mathbb{F}_r^\times$.

5.2 Theorem. *The group $\Xi := \{\varphi_{0,s} \mid s \in \mathbb{F}_r\}$ consists of translations of the unital $\mathbb{E}_t$, and the group $\{\gamma_v \mid v \in \mathbb{F}_r^\times\}$ acts transitively on $\Xi \setminus \{\mathrm{id}\}$. Thus Ξ is a transitive group of translations of $\mathbb{E}_t$. The center ∞ is fixed by all automorphisms of $\mathbb{E}_t$, and $\mathbb{E}_t$ does not admit any translations apart from those in Ξ.*

Proof. The unital $\mathbb{E}_t$ is not classical because the classical unitals do not admit any abelian groups transitive on the complement of a point. If some automorphism of $\mathbb{E}_t$ would move ∞ then $\mathrm{Aut}(\mathbb{E}_t)$ would act two-transitively on $\mathbb{E}_t$, which is impossible on a non-classical unital by Kantor's result [13] (the Ree unitals also occurring in Kantor's list do not admit any translations at all, see [8, 1.8]). □

5.3 Examples. We conjecture that many polar unitals in semifield planes (among them $\mathbb{U}(S_q^d)$ as in 5.1 and 5.2) are examples of unitals with precisely one translation center. In fact, each of these polar unitals has at least on translation center (namely, its point at infinity); it only remains to prove that the unital is not classical.

See [11] for an explicit example where the full group of the unital is determined.

5.4 Examples. If the polar unital in a Coulter–Matthews plane of order r^2 (cf. [14, Sect. 6]) is not classical then it has precisely one translation center. In fact, the centralizer of the polarity acts on the unital with three orbits (cf. [14, 5.6]), of lengths 1, r^2, and $r^3 - r^2$, respectively. In particular, there is no invariant block, and the translation centers cannot form the point row of a block. Thus either every point is a translation center (and the unital is the classical one by [9]), or the fixed point of the centralizer of the polarity is the unique translation center. One knows that the unital is not classical in many cases, see [14, 6.8].

5.5 Example. For each prime power $q > 2$, Grüning [10] constructs a unital $U(q)$ of order q that is embedded both in the Hall plane of order q^2 and its dual. He also shows that every automorphism of that unital extends to an automorphism of the Hall plane.

In particular, there is a block $[\infty]$ invariant under every automorphism of $U(q)$ (see [10, 5.5]), and $\mathrm{SL}(2,q)$ acts faithfully on $U(q)$. The proof of [10, 5.6] actually shows that $U(q)$ is isomorphic to the unital $\mathbb{U}^\flat_{\mathcal{H}_q}$ constructed in 2.1, using the standard collection $\mathcal{H}_q$ in $G = \mathrm{SU}(2,q) \cong \mathrm{SL}(2,q)$. One can use the action of the group $\mathrm{Aut}(U(q))$ on the Hall plane (and 4.3) to see that it contains no translations of the unital, apart from the trivial one.

The phenomenon that G acts trivially on $[\infty]$ is due to the fact that the derivation set used to obtain the Hall plane from the desarguesian one is just our block $[\infty]$; the process of derivation then replaces a regulus with a transitive action of $\mathrm{SL}(2,q)$ by the opposite regulus, where $\mathrm{SL}(2,q)$ acts trivially. The confluent affine lines fixed by T are turned into Baer subplanes.

5.6 Example. Ree unitals do not admit any translations at all, see [8, 1.8].

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