Finite affine planes in projective spaces

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Abstract

We classify all representations of an arbitrary affine plane $\mathcal{A}$ of order q in a projective space $\mathsf{PG}(d,q)$ such that lines of $\mathcal{A}$ correspond with affine lines and/or plane q-arcs and such that for each plane q-arc which corresponds to a line L of $\mathcal{A}$ the plane of $\mathsf{PG}(d,q)$ spanned by the q-arc does not contain the image of any point off L of $\mathcal{A}$.

1 Introduction and Main Results

The quadric Veronesean surface $\mathcal{V}_2^4$ over a finite field is a well known example of a representation of the finite projective plane $\mathsf{PG}(2,q)$ in a projective space $\mathsf{PG}(5,q)$ where the points of $\mathsf{PG}(2,q)$ are some points of $\mathsf{PG}(5,q)$, and the lines of $\mathsf{PG}(2,q)$ are certain conics. This also induces a representation of the affine plane $\mathsf{AG}(2,q)$ into $\mathsf{PG}(5,q)$. This time, the lines of $\mathsf{AG}(2,q)$ correspond to plane q-arcs in $\mathsf{PG}(5,q)$

Another well known representation of $\mathsf{AG}(2,q)$ using plane q-arcs as lines is an ovoid in $\mathsf{PG}(3,q)$ from which one point p is removed, and the q-arcs arise as intersections of the ovoid with the non-tangent planes containing the point p. Instead of an ovoid, one can also use a hyperbolic quadric, or an oval cone, with appropriate point sets removed. But in these cases, some lines of $\mathsf{AG}(2,q)$ correspond with "affine lines" in $\mathsf{PG}(3,q)$.

The above representations all have the common property that, for each line L of $\mathsf{AG}(2,q)$, the subspace of $\mathsf{PG}(d,q)$ ($d=3$ or 5) spanned by the points of the corresponding affine line or q-arc, does not contain any point of $\mathsf{AG}(2,q)$ off L. In the present paper, we take this as an axiom if the line L corresponds to a plane q-arc and classify all representations of any affine plane of order q in $\mathsf{PG}(d,q)$, $d \geq 3$, with this property and for which lines correspond with affine lines and/or plane q-arcs of $\mathsf{PG}(d,q)$, for large enough q. The above well known examples of course appear, along with some new representations. Note that the case $d = 2$ is trivial.

More exactly, we will prove the following theorem; in order to distinguish the lines of the affine plane from the lines of the ambient projective space, we call the former "blocks".

Main Result—Affine Planes. *Let $\mathcal{A} = (\mathcal{P}, \mathcal{B})$ be an affine plane of order q with point set $\mathcal{P}$ and set of blocks $\mathcal{B} \subseteq 2^{\mathcal{P}}$. Assume that*

- (i) *the set $\mathcal{P}$ is a point set of $\mathsf{PG}(d,q)$, $d \geq 3$ and $q \geq 7$, with $\langle \mathcal{P} \rangle = \mathsf{PG}(d,q)$;*
- (ii) *the elements of $\mathcal{B}$ are either affine lines or plane q-arcs of $\mathsf{PG}(d,q)$;*
- (iii) *if $L \in \mathcal{B}$ is a plane q-arc, then $\langle L \rangle \cap \mathcal{P} = L$.*

Then exactly one of the following possibilities occurs.

- (a) $d = 5$. *Here, the set $\mathcal{P}$ is the Veronesean surface $\mathcal{V}_2^4$ where a given conic C is removed. The elements of $\mathcal{B}$ are the sets $D \cap \mathcal{P}$, with D any conic on $\mathcal{V}_2^4$ distinct from C.*
- (b) $d = 3$. *Exactly one of the following cases occurs.*
  - $(b1)$ *The set $\mathcal{P}$ is an ovoid $\mathcal{O}$ of $\mathsf{PG}(3,q)$ where one point $p \in \mathcal{O}$ is removed. The elements of $\mathcal{B}$ are the sets $D \setminus \{p\}$, with D any oval on $\mathcal{O}$ through p.*
  - $(b2)$ *The set $\mathcal{P}$ is a cone $\mathcal{K}$ with vertex x and base an oval O, where we remove one generator L. There is a point p on L such that every q-arc on $\mathcal{P}$ arising from a plane intersection of $\mathcal{K}$ with a plane through p is a block of $\mathcal{A}$, and the other blocks are the affine lines which are the intersections of $\mathcal{P}$ with the generators of $\mathcal{K}$ distinct from L.*
  - $(b3)$ *The set $\mathcal{P}$ is a hyperbolic quadric $\mathcal{H}$ with two intersecting generators L, M removed, $L \cap M = \{p\}$. The blocks are the q-arcs arising from plane intersections of $\mathcal{H}$ with planes through p not tangent to $\mathcal{H}$, and the intersections of $\mathcal{P}$ with the generators of $\mathcal{H}$ distinct from L and M.*
  - $(b4)$ *The prime power q is even and the set $\mathcal{P}$ is a cone $\mathcal{K}$ with vertex p and base an oval O, where we remove O. The blocks are the q-arcs arising from plane intersections of $\mathcal{K}$ with planes not containing O through the nucleus of O and not passing through p, and the intersections of $\mathcal{P}$ with the generators of $\mathcal{K}$.*

(c) $d = 4$. *Exactly one of the following cases occurs.*

(c1) *Let* $\mathcal{S}$ *be a rational normal cubic scroll with directrix line* V *and let* W *be a given generator. Then* $\mathcal{P} = \mathcal{S} \setminus (V \cup W)$*, every affine line which is the intersection of* $\mathcal{P}$ *with a generator of* $\mathcal{S}$ *distinct from* W *belongs to* $\mathcal{B}$*, and the other members of* $\mathcal{B}$ *are the plane* q*-arcs arising from the intersection of* $\mathcal{P}$ *with the non-singular conics on* $\mathcal{S}$*.*

(c2) *Let* π *be a conic plane of the Veronesean surface* $\mathcal{V}_2^4$ *in* $\mathsf{PG}(5,q)$*, set* $C = \pi \cap \mathcal{V}_2^4$*, and let* $p \in \pi \setminus C$*, with* p *not the nucleus of* C *if* q *is even. Let* $\mathsf{PG}(4,q)$ *be a hyperplane of* $\mathsf{PG}(5,q)$ *not containing* p*. Then* $\mathcal{P}$ *is the projection of* $\mathcal{V}_2^4 \setminus C$ *from* p *into* $\mathsf{PG}(4,q)$*. The set* $\mathcal{B}$ *does not contain any affine line and the plane* q*-arcs in* $\mathcal{B}$ *are the projections of* $E \setminus C$*, with* E *ranging over the conics of* $\mathcal{V}_2^4$ *distinct from* C*.*

For q *even, this defines a unique model; for* q *odd, this defines exactly two projectively nonequivalent models, and the two cases correspond to whether* p *is an interior or exterior point for* C*.*

(c3) *Here,* q *is even, and coordinates can be chosen such that the set* $\mathcal{P}$ *consists of all points with coordinates* $(x_0, x_1, x_2, x_3, x_4)$*, where*

$$\begin{cases} x_0 &= a^\sigma, \\ x_1 &= b^\sigma, \\ x_2 &= c^\sigma, \\ x_3 &= ac^{\sigma-1}, \\ x_4 &= bc^{\sigma-1}, \end{cases}$$

with $a, b, c \in \mathsf{GF}(q)$*,* $c \neq 0$*, and* σ *an automorphism which generates* $\mathrm{Aut}\mathsf{GF}(q)$*. The set* $\mathcal{B}$ *does not contain any affine line, and the* q^2+q *plane* q*-arcs of* $\mathcal{B}$ *are all plane* q*-arcs lying in* $\mathcal{P}$*.*

For $\sigma = 2$*, the affine plane* $\mathcal{A}$ *can also be described as in* (c2) *above, but with* p *being the nucleus of* C*.*

(c4) *Here,* q *is even, and coordinates can be chosen such that the set* $\mathcal{P}$ *consists of all points with coordinates* $(x_0, x_1, x_2, x_3, x_4)$*, where*

$$\begin{cases} x_0 &= a^\sigma, \\ x_1 &= b^\sigma, \\ x_2 &= c^\sigma, \\ x_3 &= ac^{\sigma-1}, \\ x_4 &= bc^{\sigma-1}, \end{cases}$$

with $a, b, c \in \mathsf{GF}(q)$*,* $b \neq 0$*, and* σ *an automorphism which generates* $\mathrm{Aut}\mathsf{GF}(q)$ *(the expression* $0^{\sigma-1}$ *equals* 0*). The set* $\mathcal{B}$ *contains exactly one affine line, which is the affine line with equations*

$$\begin{cases} X_2 = X_3 = X_4 & = & 0, \\ X_1 & \neq & 0, \end{cases}$$

which corresponds to $c = 0$*, and the* $q^2 + q - 1$ *plane* q*-arcs of* $\mathcal{B}$ *are all plane* q*-arcs lying in* $\mathcal{P}$*.*

For $\sigma = 2$*, the affine plane* $\mathcal{A}$ *can also be described as follows. Let* $\mathcal{V}_2^4$ *be the Veronesean surface in* $\mathsf{PG}(5, q)$*,* q *even, and let* $\mathcal{N}$ *be the nucleus plane of* $\mathcal{V}_2^4$*. Let* $p \in \mathcal{N}$ *and let* $\mathsf{PG}(4, q)$ *be a hyperplane of* $\mathsf{PG}(5, q)$ *not containing* p*. Let* C *be the conic on* $\mathcal{V}_2^4$ *with nucleus* p *and let* D *be some conic on* $\mathcal{V}_2^4$ *distinct from* C*. Then* $\mathcal{P}$ *is the projection of* $\mathcal{V}_2^4 \setminus D$ *from* p *into* $\mathsf{PG}(4, q)$*. The projection of* $C \setminus D$ *is the unique affine line of* $\mathcal{B}$*. The other elements of* $\mathcal{B}$ *are the projections of* $E \setminus D$*, with* E *ranging over all conics of* $\mathcal{V}_2^4$ *distinct from* C *and* D*.*

In particular, $d \leq 5$ *and* $\mathcal{A}$ *is always isomorphic to* $\mathsf{AG}(2, q)$*.*

For all the above models, except possibly for $(c2)$, $(c3)$ and $(c4)$, it is clear that Condition (iii) is satisfied. For $(c3)$ and $(c4)$, we will show (iii) in our proof; for $(c2)$, we only have to show that, for any conic $D \neq C$ of $\mathcal{V}_2^4$, the 3-space Σ generated by p and D does not contain any point y of $\mathcal{V}_2^4 \setminus (D \cup C)$. Choosing two points $z, u \in C \setminus D$ such that p, z, u are on a line of $\mathsf{PG}(4, q)$, we see that the conics through z and y, and through u and y, are contained in the 4-space spanned by Σ and C, contradicting the fact that every three conics mutually intersecting in three distinct points span the whole space $\mathsf{PG}(5, q)$.

Remark 1.1 For each of these models the set $\mathcal{P}$ is contained in some affine space $\mathsf{AG}(d, q) \subseteq \mathsf{PG}(d, q)$.

We mention the following special cases of the Main Result. Let $\mathcal{A} = (\mathcal{P}, \mathcal{B})$ be as in the Main Result—Affine Planes.

1. If $\mathcal{B}$ does not contain affine lines, then $\mathcal{A}$ is as in (a), $(b1)$, $(c2)$ or $(c3)$.
2. If every plane q-arc of $\mathcal{B}$ is contained in a conic, then $\mathcal{A}$ is one of (a), $(b1)$ where $\mathcal{O}$ is an elliptic quadric, $(b2)$ where the oval O is a conic, $(b3)$, $(b4)$ where the oval O is a conic, $(c1)$, $(c2)$, $(c3)$ with $\sigma = 2$, or $(c4)$ with $\sigma = 2$.

3. If $\mathcal{B}$ does not contain affine lines and if every plane q-arc of $\mathcal{B}$ is contained in a conic, then $\mathcal{A}$ is one of (a), $(b1)$ where $\mathcal{O}$ is an elliptic quadric, $(c2)$, or $(c3)$ with $\sigma = 2$.

Another corollary concerns representations of projective planes.

Main Result—Projective Planes. *Let $\Pi = (\mathcal{P}, \mathcal{B})$ be a projective plane of order q with point set $\mathcal{P}$ and set of blocks $\mathcal{B} \subseteq 2^{\mathcal{P}}$. Assume that*

(i) the set $\mathcal{P}$ is a point set of $\mathsf{PG}(d,q)$, $d \geq 3$ and $q \geq 7$, with $\langle \mathcal{P} \rangle = \mathsf{PG}(d,q)$;

(ii) the elements of $\mathcal{B}$ are either lines or ovals of $\mathsf{PG}(d,q)$;

(iii) if $L \in \mathcal{B}$ is an oval, then $\langle L \rangle \cap \mathcal{P} = L$.

Then exactly one of the following possibilities occurs.

(A) $d = 5$. Here, the set $\mathcal{P}$ is the Veronesean surface $\mathcal{V}_2^4$ and the elements of $\mathcal{B}$ are the conics on $\mathcal{V}_2^4$.

(B) $d = 3$ and q is even. The set $\mathcal{P}$ is a cone $\mathcal{K}$ with vertex p and base an oval O. The blocks are the ovals on $\mathcal{K}$ which are intersections of $\mathcal{K}$ with planes through the nucleus of O and not passing through p, together with the generators of $\mathcal{K}$.

(C) $d = 4$ and q is even. The coordinates can be chosen such that the set $\mathcal{P}$ consists of all points with coordinates $(x_0, x_1, x_2, x_3, x_4)$, where

$$\begin{cases} x_0 &= a^\sigma, \\ x_1 &= b^\sigma, \\ x_2 &= c^\sigma, \\ x_3 &= ac^{\sigma-1}, \\ x_4 &= bc^{\sigma-1}, \end{cases}$$

with $a, b, c \in \mathsf{GF}(q)$, $(a,b,c) \neq (0,0,0)$, and σ an automorphism which generates $\mathrm{Aut}\mathsf{GF}(q)$. The set $\mathcal{B}$ contains exactly one line, which has equations $X_2 = X_3 = X_4 = 0$ and which corresponds to $c = 0$. The other members of $\mathcal{B}$ are translation ovals and coincide with the ovals lying on $\mathcal{P}$.

For $\sigma = 2$, the projective plane Π can also be described as follows. Let $\mathcal{V}_2^4$ be the Veronesean surface in $\mathsf{PG}(5,q)$, q even, and let $\mathcal{N}$ be the nucleus plane of $\mathcal{V}_2^4$. Let $p \in \mathcal{N}$ and let $\mathsf{PG}(4,q)$ be a hyperplane of $\mathsf{PG}(5,q)$ not containing p. Then $\mathcal{P}$ is the projection of $\mathcal{V}_2^4$ from p into $\mathsf{PG}(4,q)$. If C is the unique conic on $\mathcal{V}_2^4$ with nucleus p, then the projection of C is the unique line of $\mathcal{B}$. The other elements of $\mathcal{B}$ are the projections of D, with D ranging over all conics of $\mathcal{V}_2^4$ distinct from C.

As for the case of affine planes, we mention the following special cases. Let $\Pi = (\mathcal{P}, \mathcal{B})$ be as in the Main Result—Projective Planes.

1. If $\mathcal{B}$ does not contain lines, then Π is as in (A).
2. If every oval of $\mathcal{B}$ is a conic, then Π is one of (A), (B) where O is a conic, or (C) with $\sigma = 2$.

We now explain the motivation of our research, and mention an immediate application afterwards, but not related to the original motivation.

Our result embeds in the theory of *translation generalized quadrangles of order* (q^n, q^{2n}), for some prime power q and some natural number n. This is equivalent with the theory of generalized ovoids via a construction similar to the André-Bose-Bruck construction linking translation planes to certain spreads of projective spaces. For details on translation generalized quadrangles we refer to the monograph of Thas, Thas and Van Maldeghem [5].

A generalized ovoid $O = O(n, 2n, q)$ consists of a set of $q^{2n}+1$ subspaces of dimension $n-1$ of $\mathsf{PG}(4n-1, q)$ such that every three such subspaces generate a subspace of dimension $3n-1$, and such that for every element π of O, there is a unique subspace τ of dimension $3n-1$ which does not meet any other member of O. Assume that the generalized quadrangle $\mathcal{S}$ defining O contains a regular line M not containing the point (∞) of $\mathcal{S}$ and let M be concurrent with the line $\pi \in O$ of $\mathcal{S}$. An arbitrary $3n$-space $\mathsf{PG}(3n, q)$ containing τ intersects the union of all elements of O except π in a set $\mathcal{P}$ of q^{2n} points. One can show that every pair of distinct elements of $\mathcal{P}$ is contained in a conic contained in $\mathcal{P} \cup \pi$ (and each such conic meets π in a unique point). If $\mathcal{B}$ is the family of all q-arcs obtained by deleting from the previous conics the points belonging to π, then $(\mathcal{P}, \mathcal{B})$ is an affine space, and every plane of this space is embedded as in our Main Result—Affine Planes. More exactly, by the third special case, there are only four possibilities. This should be a major step in the investigation of such generalized ovoids, more in particular, to prove the conjecture that O is *good* at π. An immediate consequence would be a purely geometric proof of the classification of finite Moufang quadrangles (this is the missing part in the monograph of Payne and Thas [4]).

But our main results are also of independent interest. Indeed, up to now it was not known whether all embeddings of finite affine planes in projective 4-space $\mathsf{PG}(4, q)$, where lines are represented by plane q-arcs, are covered by model (a). Example $(c3)$ shows that this is not the case if q is even. For q odd, it is true for those models in $\mathsf{PG}(4, q)$ that additionally satisfy Assumption (iii) of our Main Result—Affine Planes. Also, it was not known whether the examples in $(b2)$, with q even and O not a conic, admit coverings. Our theorem shows that they indeed admit a covering as soon as the base plane q-arc is a translation oval where the common point of the oval and the axis is removed, see Case $(c3)$.

2 Proof of the Main Results

Throughout this section, we let $\mathcal{A} = (\mathcal{P}, \mathcal{B})$ be an affine plane of order q, with point set $\mathcal{P}$ spanning $\mathsf{PG}(d, q)$ and such that all members of $\mathcal{B}$ are either affine lines or plane q-arcs of $\mathsf{PG}(d, q)$.

We will denote the subspace of $\mathsf{PG}(d, q)$ spanned by sets $S_1, S_2, \ldots$ of points by $\langle S_1, S_2, \ldots \rangle$.

In this section we also assume Condition (iii), i.e., for each q-arc $L \in \mathcal{B}$ we have $\langle L \rangle \cap \mathcal{P} = L$. Before we start the case distinction with respect to the dimension d, we prove a general lemma.

Lemma 2.1 *If the block $L \in \mathcal{B}$ is an affine line of $\mathsf{PG}(d, q)$, then $\langle L \rangle \cap \mathcal{P} = L$.*

Proof. Let $\langle L \rangle \setminus L = \{x\}$ and assume, by way of contradiction, that $x \in \mathcal{P}$. Consider a block M through x not parallel to L. Then M is necessarily a plane q-arc. But this contradicts Condition (iii). □

2.1 Case $d \geq 5$

We assume that $q \geq 7$.

Let L_1, L_2, L_3 be three blocks of $\mathcal{A}$ pairwise intersecting in a distinct point. Let P_1, P_2, P_3 be these three distinct points. It is clear that every point of $\mathcal{A}$, distinct from P_1, P_2, P_3, is contained in a block that intersects $L_1 \cup L_2 \cup L_3$ in three distinct points. Hence $\mathsf{PG}(d, q)$ is generated by $L_1 \cup L_2 \cup L_3$. If L_1 is an affine line, then $d \leq 4$, a contradiction. Since $d \geq 5$, none of L_1, L_2, L_3 is an affine line. In particualr, since $\langle L_1, L_2, L_3 \rangle = \mathsf{PG}(d, q)$, it follows that $d = 5$. Hence all blocks of $\mathcal{A}$ are plane q-arcs. Note also that with a similar argument, every triple of distinct blocks generates $\mathsf{PG}(5, q)$.

Now let x and y be distinct points of $\mathcal{A}$, and let $L \in \mathcal{B}$ be the unique block containing x, y. Let $M_1, \ldots, M_q$ be the blocks of $\mathcal{A}$ containing x but not y, and let $N_1, \ldots, N_q$ be the blocks of $\mathcal{A}$ containing y but not x. We may choose this notation in such a way that M_i and N_i are parallel blocks in $\mathcal{A}$, for all $i \in \{1, 2, \ldots, q\}$. Let $\mathsf{PG}(3, q)$ be a 3-space of $\mathsf{PG}(5, q)$ skew to $\langle x, y\rangle$, and project $\mathcal{P} \setminus \{x, y\}$ from $\langle x, y\rangle$ onto $\mathsf{PG}(3, q)$. The projections of all points of $L \setminus \{x, y\}$ coincide with some point ℓ; the projection of the set $M_i \setminus \{x\}$ is some set M'_i, $i \in \{1, 2, \ldots, q\}$, and the projection of $N_i \setminus \{y\}$ is denoted by N'_i, $i \in \{1, 2, \ldots, q\}$. As $\mathcal{P}$ generates $\mathsf{PG}(5, q)$, our previous paragraph implies that every pair of blocks of $\mathcal{A}$ generates a 4-space of $\mathsf{PG}(5, q)$, and so the projection of $\mathcal{P} \setminus L$ is injective. Also, the lines $\langle M'_1\rangle, \langle M'_2\rangle, \ldots, \langle M'_q\rangle$ are pairwise skew, and so are the lines $\langle N'_1\rangle, \langle N'_2\rangle, \ldots, \langle N'_q\rangle$. Also, the point ℓ does not belong to $\langle M'_1\rangle, \ldots, \langle M'_q\rangle, \langle N'_1\rangle, \ldots, \langle N'_q\rangle$.

Let $\mathcal{H}$ be the hyperbolic quadric containing the generators $\langle M'_{q-2}\rangle, \langle M'_{q-1}\rangle, \langle M'_q\rangle$ and $\langle N'_1\rangle, \langle N'_2\rangle, \langle N'_3\rangle$. Then $\langle M'_4\rangle$ and $\langle N'_4\rangle$ are also generators of $\mathcal{H}$ (as these lines intersect three mutually skew generators; remember $q \geq 7$), and by the same token now every line $\langle M'_i\rangle$ and $\langle N'_i\rangle$ is a generator of $\mathcal{H}$, $i \in \{1, 2, \ldots, q\}$. It is clear that the $\langle M'_i\rangle$ belong to one system of generators, say $\mathcal{M}$, and the $\langle N'_i\rangle$ to the other system, say $\mathcal{N}$, $i \in \{1, 2 \ldots, q\}$. Put $\{u_i\} = \langle M'_i\rangle \cap \langle N'_i\rangle$, $i = 1, 2 \ldots, q$. The point of $\langle M'_i\rangle$ not contained in $\langle N'_1\rangle \cup \langle N'_2\rangle \cup \cdots \cup \langle N'_q\rangle$ is denoted by v_i and the point of $\langle N'_i\rangle$ not on $\langle M'_1\rangle \cup \langle M'_2\rangle \cup \cdots \cup \langle M'_q\rangle$ is denoted by w_i, $i = 1, 2, \ldots, q$. The points $v_1, \ldots, v_q$ belong to a common generator $V \in \mathcal{N}$ of $\mathcal{H}$ and the points $w_1, \ldots, w_q$ belong to a common generator $W \in \mathcal{M}$. Let $\widetilde{\ell}$ be the intersection of V and W.

Let $U \neq L$ be a block of $\mathcal{A}$ parallel to L. The projection U' of U belongs to $\mathcal{H}$, so is contained in a conic. By injectivity of the projection, U belongs to a conic. Varying L, it follows that every block of $\mathcal{A}$ belongs to a conic. Now let D be a block of $\mathcal{A}$ not parallel to L and containing neither x nor y. The projection D' of D contains at least $q - 1 \geq 6$ points of $\mathcal{H}$, contains ℓ, and is contained in a conic $\widetilde{D}$. Hence $\widetilde{D} \subseteq \mathcal{H}$ and so $\ell \in \mathcal{H}$. It follows that $\widetilde{\ell} = \ell$. Assume without loss of generality that D and M_1 are parallel in $\mathcal{A}$. Then the intersection $\widetilde{D} \cap M'_1$ must be u_1 and so $\widetilde{D} = D' \cup \{u_1\}$. Consequently, the projections of all blocks of $\mathcal{A}$ parallel to M_i and N_i, but distinct from M_i and N_i, extend to conics on $\mathcal{H}$ by adding the point u_i, $i = 1, 2, \ldots, q$; the projections of all blocks parallel to L, but distinct from L, extend to conics on $\mathcal{H}$ by adding the point ℓ.

Let D and E be parallel to M_1, N_1, with $|\{D, E, N_1, M_1\}| = 4$. The conics which extend D and E are denoted by $\overline{D}$ and $\overline{E}$, respectively; let $\overline{D} = D \cup \{d\}$ and $\overline{E} = E \cup \{e\}$. Clearly d and e are projected into u_1. Assume, by way of contradiction, that $d \neq e$. Then the line $\langle d, e\rangle$ intersects the line $\langle x, y\rangle$. Varying x, y in $L \setminus (D \cup E)$, we see that $\langle d, e\rangle$ must intersect at least three non-concurrent lines of $\langle L\rangle$, implying that $\langle d, e\rangle \subseteq \langle L\rangle$. Hence $\langle L\rangle$ and $\langle D\rangle$ have a line in common, and so do $\langle L\rangle$ and $\langle E\rangle$. Consequently $\langle L, D, E\rangle$ is at most 4-dimensional, a contradiction. Hence $d = e$. We conclude that, if $\{C_1, C_2, \ldots, C_q\}$ is a parallel class of blocks in $\mathcal{A}$, then there is a unique point $c \in \mathsf{PG}(5, q)$ extending all the q-arcs $C_1, C_2, \ldots, C_q$ to conics $\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_q$.

Let us denote the points extending the q-arcs of $\mathcal{B}$ to conics by $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_q, \overline{u}_{q+1}$, subscripts chosen in such a way that their projections from $\langle x, y\rangle$ onto $\mathsf{PG}(3, q)$ are the points $u_1, u_2, \ldots, u_q, \ell$, respectively. Consider the q blocks of $\mathcal{A}$ parallel to M_1. With these q-arcs correspond in $\mathsf{PG}(3, q)$ the lines $\langle M_1'\rangle$, $\langle N_1'\rangle$ and $q-2$ non-singular conics on $\mathcal{H}$; the latter all contain the points ℓ and u_1. Now the union of the q planes of $\mathsf{PG}(3, q)$ through ℓ and u_1 corresponding to these $q-2$ nonsingular conics, and to the singular conics $M_1' \cup V$ and $N_1' \cup W$ contains all points of $\mathcal{H}$ except for $u_2, u_3, \ldots, u_q$. Hence the $(q+1)$th plane through ℓ and u_1 contains $u_2, u_3, \ldots, u_q$ and so $C' = \{\ell, u_1, u_2, \ldots, u_q\}$ is a nonsingular conic, and it is the projection of $C = \{\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_{q+1}\}$. Hence C is contained in the 4-space $\xi_{x,y} = \langle L, C'\rangle$. We choose a point $z \in \mathcal{P}$, $z \notin L$, and we interchange the roles of y and z to see that C is similarly contained in another 4-space $\xi_{x,z}$, and similarly also in $\xi_{y,z}$. Suppose two of these 4-spaces coincide, say $\xi_{x,y} = \xi_{x,z}$. Then an arbitrary block T not through x and parallel neither to the block L nor to the block R containing x and z, has two points in common with the blocks L and R; furthermore, the conic corresponding to T also intersects C in a point. Hence $\mathcal{P} \subseteq \xi_{x,y}$, a contradiction. If $\xi_{x,y} \cap \xi_{x,z} = \xi_{x,z} \cap \xi_{y,z} = \xi_{y,z} \cap \xi_{x,y}$, then x, y, z and C are contained in a common 3-space Σ. Then Σ contains the conics corresponding to the blocks joining x and y, joining x and z, and joining y and z, a contradiction. We conclude that $\xi_{x,y} \cap \xi_{y,z} \cap \xi_{x,z}$ is a plane and so C is a planar set. Since it projects on a conic C', it is itself a conic. Now we add C to $\mathcal{P}$ and obtain an embedding of the projective closure of $\mathcal{A}$ in $\mathsf{PG}(5, q)$ where lines are conics. By Thas and Van Maldeghem [6], $\mathcal{P} \cup C$ is projectively equivalent to the Veronese surface $\mathcal{V}_2^4$. This yields part (a) of the Main Result—Affine Planes, and (A) of the Main Result—Projective Planes.

2.2 Case $d = 4$

We continue to assume $q \geq 7$.

2.2.1 Generalities

We first establish the possibilities of configurations of the affine lines in $\mathcal{B}$. If there is no affine line, then this configuration is empty. Hence we may suppose that there is at least one affine line, say L. If there was a second affine line L' not parallel to L as a block of $\mathcal{A}$, then we consider a third block K intersecting both L and L', but not in $L \cap L'$. It is clear that, since $q \geq 4$, all points of $\mathcal{A}$ are contained in $\langle L, L', K\rangle$, which is at most

3-dimensional, a contradiction. Hence every block which is an affine line is parallel to L. A similar argument also implies that, if L' is a block which is parallel to L, and if L' is an affine line, then $\langle L, L' \rangle$ is 3-dimensional.

For each block K not parallel to L there is a unique point m_K extending K to an oval, if q is odd, and there are two points m_K and n_K extending K to a hyperoval, if q is even. We choose a plane π in $\mathsf{PG}(4, q)$ skew to $\langle L \rangle$ and we project $\mathcal{P} \setminus L$ from $\langle L \rangle$ into π. The points of each block $K \in \mathcal{B} \setminus \{L\}$ which has nontrivial intersection x_K with L are projected into a line L_K minus two points a_K, b_K in π. We choose the notation in such a way that, for q odd, b_K is the projection of the tangent line at x_K to the completion $K \cup \{m_K\}$ of K, and a_K is the projection of m_K; for q even we assume that a_K corresponds to m_K and b_K to n_K. Denote the set of points of $\mathcal{A}$ not on L by A.

Suppose now, by way of contradiction, that some point $x \in A \setminus K$ is projected into L_K. Then all points of A except possibly those of the blocks of $\mathcal{A}$ through x parallel to L or to K, or those on the block joining x and $L \cap K$, are contained in the inverse image of L_K under the projection. Since there are at least two lines through x completely contained in the 3-space $\langle L, L_K \rangle$, it is now easy to see that this forces all points of $\mathcal{A}$ to be contained in $\langle L, L_K \rangle$, a contradiction.

Hence, we deduce that the projection is injective, and that no image point coincides with a point a_K or b_K, for any $K \in \mathcal{B} \setminus \{L\}$.

Consider the q blocks distinct from L through some point p of L. The projective lines corresponding with their projections into π meet in points of type a_K and b_K. It is easy to see that $q \geq 4$ forces these projective lines to meet in a common point; denote it by x_p. The same thing holds for the q blocks of a fixed parallel class C not containing L, and we denote the corresponding point by x_C. It follows easily that the points x_t, for t varying in L, all lie on the unique line T through x_p which is not the image of one of the q aforementioned blocks through p (indeed, T must contain x_r, for r a point of L, $r \neq p$ since any block through r and p, respectively, different from L must meet $q - 1$ blocks through p and r, respectively, distinct from L). Since there are exactly q points like this, there remains one point on that line, and we denote it by x_∞. A similar argument implies that the q points x_C corresponding to the q parallel classes not containing L are contained in a second line through x_∞. The projections of the blocks contained in the parallel class of L are the equivalence classes of points with respect to the relation of not lying on the projection of a block intersecting L. But clearly, these equivalence classes correspond to the lines in π through x_∞ distinct from the two particular lines containing the points x_p and x_C. Hence the blocks parallel to L which are q-arcs are all completed to an oval by adding the point $\ell = \langle L \rangle \setminus L$.

Hence, if there is a second block L' which is an affine line (necessarily parallel to L), then, if $\ell' = \langle L' \rangle \setminus L'$, every block parallel to L and which is a q-arc, is completed to a hyperoval by adding ℓ and ℓ' to it. It now follows easily that either there is a unique affine line in $\mathcal{B}$, or there are precisely two (parallel) affine lines in $\mathcal{B}$ (and then q is even), or all blocks of a certain parallel class are affine lines. We treat these three cases separately. Afterwards, we also treat the case where there are no affine lines, of course.

2.2.2 Precisely one block of $\mathcal{A}$ is an affine line of $\mathsf{PG}(4, q)$

We continue with the above notation.

The tangent lines at ℓ to the ovals arising from the q-arcs parallel to L by adding ℓ are contained in the common plane $\langle L, x_\infty \rangle$. We claim that these tangent lines coincide. Indeed, consider the projection of $\mathcal{A} \setminus J$ from $\langle J \rangle$ onto some line Z of $\mathsf{PG}(4, q)$, where J is a block parallel to but distinct from L and Z is disjoint from $\langle J \rangle$. Let K be a block not parallel to L. Since the above projection from L is injective, the blocks L, K, J generate $\mathsf{PG}(4, q)$, and since J and K meet in a point, the projection from $\langle J \rangle$ is injective on the set $K \setminus J$. Hence, if two points have the same projection, then they belong to a block parallel to L. Let J' be a second block parallel to L, $J' \neq J$. If $\langle J, J' \rangle$ is 4-dimensional, then the projection from $\langle J \rangle$ is injective on the set $J' \setminus J$. Since $q \geq 4$, we find at least two points z_1, z_2 in K and two points y_1, y_2 in J' such that the projection from $\langle J \rangle$ of z_i coincides with that of y_i, $i \in \{1, 2\}$. For at most one $i \in \{1, 2\}$, we have $y_i = z_i$. Suppose $y_1 \neq z_1$; it follows from the previous that y_1 and z_1 are contained in a block parallel to L, hence $z_1 \in J'$, a contradiction. Hence $\langle J, J' \rangle$ is 3-dimensional. If $\langle J \rangle \cap \langle J' \rangle$ is not tangent to $J \cup \{\ell\}$ at ℓ, then $\langle J' \rangle$ contains some point of J, a contradiction. This shows our claim. We denote the common tangent line at ℓ to all ovals corresponding with blocks parallel to L by T_ℓ.

Now we forget about the previous projections and we project from ℓ onto a solid Σ, where $\ell \notin \Sigma$. Then the previous properties of the blocks parallel to L imply easily that the projection of $\mathcal{A}$ is contained in a cone $\mathcal{C}$ with vertex the projection x_T of T_ℓ and base a plane q-arc. But the projections of all the blocks not parallel to L go through the common point x_L, which is the projection of L in Σ. Let M be the line of Σ through x_T extending the cone to a cone $\mathcal{C}'$ over an oval, if q is odd, and let M and N be the two lines of Σ through x_T extending the cone to a cone $\mathcal{C}'$ over a hyperoval, if q is even.

If q is odd, the line tangent at $x \in L$ to any block K intersecting L in x is projected onto a line contained in the plane tangent to $\mathcal{C}'$ at the line $\langle x_T, x_L \rangle$. Hence all such tangents are contained in a fixed 3-space containing L, and are hence projected from L onto a line of π. That line contains whence all points b_K; it also contains the point x_∞. It follows that either all pencils with vertex on L correspond with points of type b_K, or all parallel classes do.

Next we want to show that, still for q odd, all pencils with vertex on L correspond with points of type b_K. So suppose by way of contradiction that parallel classes correspond with points of type b_K. Consider two arbitrary blocks K_1, K_2 such that $\langle K_1\rangle, \langle K_2\rangle$ meet $\langle L\rangle$ in different points, and suppose that $\langle K_1, K_2\rangle$ is 3-dimensional. Our assumption implies that $\langle L\rangle$ belongs to $\langle K_1, K_2\rangle$, contradicting the injectivity of the projection from $\langle L\rangle$.

Now we use a fourth projection. Let K be an arbitrary block not parallel to L. Let λ be a line of $\mathsf{PG}(4, q)$ skew to $\langle K\rangle$. Then we project the point set $\mathcal{P} \setminus K$ from $\langle K\rangle$ onto λ. It is clear that $\mathcal{P} \cap \langle K, L\rangle = K \cup L$. Hence $L \setminus K$ gets projected onto a unique point p_L of λ, and all other points of $\mathcal{P} \setminus K$ are projected onto points of λ different from p_L. Suppose two points $x, y \in \mathcal{P} \setminus (L \cup K)$ are projected onto the same point a. Then the block K' through x, y contains two points in a 3-space Σ' containing K. If K' is not parallel to K, then also $K \cap K' \in \Sigma'$ and hence all points of K' belong to Σ'. The previous paragraph implies that $K \cap K' \in L$. If K' is parallel to K, then $\langle K, K'\rangle$ is 4-dimensional, and so no further point of K' is projected onto a. Now suppose some further point z of $\mathcal{P} \setminus (L \cup K \cup K')$ is mapped onto a. Then the blocks through x, z and through y, z are not parallel to K and belong to Σ' and so should meet on L, contradicting the choice of z.

So we see that each point of λ is the image of either 0, 1, 2 or $q-1$ points of $\mathcal{A} \setminus K$. So we have to divide $q(q-1)$ points in $q+1$ sets of size 0, 1, 2 and $q-1$, $q \geq 5$. If we have k sets of size $q-1$, then $k(q-1)+2(q+1-k) \geq q^2-q$, implying $k \geq \frac{q^2-3q-2}{q-3} = q - \frac{2}{q-3}$. Hence, since $q \geq 7$, we infer $k = q$ and we can conclude that all blocks meeting L in $K \cap L$ lie in common 3-spaces with K. But, more importantly, the projection is injective on the set of points of any block parallel to K! This injectivity and the reasoning leading to it also hold for q even. This means that, given such a block K', putting $x_K = K \cap L$, $x_{K'} = K' \cap L$, and t being the intersection of the tangent lines to $K \cup \{m_K\}$ and $K' \cup \{m_{K'}\}$ at x_K and $x_{K'}$, respectively (t exists by the assumption that parallel classes correspond with points of type b_K), the projection of K' from t onto any line of $\langle K'\rangle$ not through t is injective, implying that q is even. Now let $q = 5$. Then either the above holds, and we have a contradiction, or $k = q-1 = 4$ and the other two points on λ are images of two pairs of points, necessarily pairs on blocks parallel to K. But this is a contradiction, as these four remaining points of course lie on the remaining block through the intersection point $K \cap L$.

So we have shown that, if q is odd, then the projection from $\langle L\rangle$ onto π of every block except for L of a pencil of blocks containing the block $K \neq L$ and with vertex on L, is contained in a line that contains b_K. If q is even, then we can choose the notation of N and M such that the points b_K correspond with the line N, and the points a_K with M. Remember that in both cases, with each point a_K corresponds the point m_K in $\langle K\rangle$ extending K to an oval. We now claim that, since $q \geq 5$, we have $m_K = m_{K'}$ as soon as K and K' are two parallel blocks not parallel to L. Indeed, suppose by way of contradiction that $m_K \neq m_{K'}$. Then $a_K = a_{K'}$ implies that the line γ of $\mathsf{PG}(4, q)$ joining m_K with $m_{K'}$ intersects $\langle L\rangle$ in some point m. Hence the projection of $K' \cup \{m_{K'}\}$ from $\langle K\rangle$ onto λ is injective except that it agrees on $m_{K'}$ and $x_{K'}$. Hence the projection of $K' \cup \{m_{K'}\}$ from $t := \langle K\rangle \cap \langle K'\rangle$ onto some line in $\langle K'\rangle$ not containing t is injective except for $m_{K'}$ and $x_{K'}$, whose images agree. But the injectivity implies that t is a nucleus of $K' \cup \{m_{K'}\}$, contradicting the fact that there is a unique secant through t. So we can add m_K to every block $K \neq L$ and obtain a projective plane Π when calling the set consisting of ℓ and all such m_K a block B.

The arguments of the previous paragraphs imply that, if K, K' are blocks distinct from L with $x_K = x_{K'}$, then $\langle K, K'\rangle$ is a 3-space. Since L does not belong to that 3-space (as we noted earlier), the tangents to $K \cup \{m_K\}$ and $K' \cup \{m_{K'}\}$ at x_K coincide.

If we project from x_K instead of from ℓ above, then we see that the projection of $(\mathcal{A} \cup B) \setminus \{x_K\}$ lies on a cone with base an oval. Since B lies in a plane (the projection of $B \setminus \{\ell\}$ from ℓ onto Σ yields the line M), it is an oval. Now, if one considers a point $x \in A$, then the tangent lines at x to the ovals through x are contained in a 3-space containing ℓ (since after projection from ℓ, they are contained in a plane), and in a 3-space containing x_K. These 3-spaces are clearly distinct, and so all these tangent lines are contained in a plane (and they fill up the whole plane).

Now we show that q is even. Assume, by way of contradiction, that q is odd. Let $T_1, T_2, \ldots, T_q, T_\infty$ be the $q+1$ tangent lines at the respective points of $L \cup \{\ell\}$ of the q^2+q ovals of Π (where $\ell \in T_\infty$). Note that all these tangents are contained in a 3-space Σ_T. Let x be an arbitrary point of Π not on $\langle L\rangle$. The tangent lines at x of the $q+1$ ovals containing x intersect the respective lines $T_1, \ldots, T_q, T_\infty$ in collinear points $t_1^x, \ldots, t_q^x, t_\infty^x$ (all these points are on the line L_x which is the intersection of Σ_T and the plane containing all the aforementioned tangent lines at x). This way there arise q transversals L_x of the lines $T_1, T_2, \ldots, T_q, T_\infty$ (vary x on one oval of Π and take account of q being odd), and, adding $\langle L\rangle$, we obtain the two systems of generators of a hyperbolic quadric $\mathcal{H}$ in the solid Σ_T. Let O be an oval of Π not containing the point x of Π, with $x \notin L \cup \{\ell\}$. Let $y \in O$, then the line $\langle x, y\rangle$ intersects the tangent T_i, where i is such that T_i is tangent to the oval through x and y. It follows that the projection of O from x into Σ_T is an oval lying completely on $\mathcal{H}$. But if T_j contains $O \cap \langle L\rangle$, then T_j is clearly tangent to the projection of O, a contradiction. Hence q is even.

Let O be an oval of Π and let x, y be two points of O off $\langle L\rangle$. Pick an arbitrary point z of Π not in $O \cup \langle L\rangle$. Let the ovals of Π through x and z, and through y and z have tangent lines T_x and T_y, respectively, at their intersection with $\langle L\rangle$. Let O' be the oval of Π containing z and $O \cap \langle L\rangle$. Then the projection from T_x of $\langle O\rangle$ onto $\langle O'\rangle$, followed by the projection from T_y of $\langle O'\rangle$ onto $\langle O\rangle$ maps O onto itself, fixes every point of $\langle O\rangle \cap \langle O'\rangle$ (and this is the tangent line to O at $O \cap \langle L\rangle$) and maps x to y. Hence O is a translation oval with center $O \cap \langle L\rangle$. By projection from any point of $\langle L\rangle$ it is clear that all ovals in Π are isomorphic.

We now show that $\mathcal{P} \cup B$ is projectively unique given the isomorphism class of translation ovals belonging to Π. We choose a point $p \in L$ and two translation ovals O_1, O_2 of Π containing the translation center p. These ovals generate a 3-space Σ and have a common tangent line at p. Moreover, by the previous paragraph, the tangent line at ℓ to the ovals through ℓ meets Σ in a point t which is the vertex of a cone $\mathcal{C}$ containing O_1, O_2 and such that the generators of that cone define an isomorphism between O_1 and O_2. We see that the configuration O_1, O_2, ℓ, t is projectively unique. Also, let $x \in \mathcal{P} \setminus (L \cup O_1 \cup O_2)$; then $x \notin \Sigma$. We may choose x such that $\langle \ell, x\rangle$ intersects $\mathcal{C}$ at a given point x' (distinct from t and off $O_1 \cup O_2$). The projections from ℓ onto Σ of the ovals of Π through x but not through ℓ are given by the ovals of $\mathcal{C}$ through x' and p; the projection from ℓ onto Σ of the oval of Π containing ℓ and x, with ℓ removed, is the set $\langle x', t\rangle \setminus \{t\}$.

Let O' be such an oval of $\mathcal{C}$ containing x' and p, and let $O' \cap O_1 = \{x_1\}$ and $O' \cap O_2 = \{x_2\}$. Then the intersection of the cone with base O' and vertex ℓ with the plane $\langle x_1, x_2, x\rangle$ is an oval of Π containing x. If $\langle x', t\rangle \cap O_1 = \{y_1\}$ and $\langle x', t\rangle \cap O_2 = \{y_2\}$, then for the given automorphism of $\mathsf{GF}(q)$ the translation oval with translation line $\langle \ell, t\rangle$ and containing ℓ, y_1, y_2, x is an oval of Π containing x. It follows that all ovals through x are determined, and hence $\mathcal{P} \cup B$ is projectively unique.

Since a translation oval is determined by its associated field automorphism, it remains to show that Case (C) really defines a projective plane satisfying the conditions (i), (ii) and (iii) of the Main Result—Projective Planes, and that it contains exactly one line of $\mathsf{PG}(4, q)$.

Let σ be an automorphism of $\mathsf{GF}(q)$ which generates $\mathrm{Aut}\mathsf{GF}(q)$, q even, and read σ as a natural number 2^e, with $e > 0$, so that $x^{\sigma-1}$ is well defined for all $x \in \mathsf{GF}(q)$. Let us consider $\mathsf{PG}(2, q)$ using ordinary homogenous coordinates for the points and lines. We define the following map τ of the points of $\mathsf{PG}(2, q)$ to a point set of $\mathsf{PG}(4, q)$.

$$(x, y, z)^\tau = (x^\sigma, y^\sigma, z^\sigma, xz^{\sigma-1}, yz^{\sigma-1}).$$

The images of the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,0,1)$ and $(0,1,1)$ clearly generate $\mathsf{PG}(4,q)$. Whence (i).

The line $[0,0,1]$ clearly corresponds with the line $X_2 = X_3 = X_4 = 0$ (labeling the coordinates in $\mathsf{PG}(4,q)$ by $X_0, X_1, \ldots, X_4$).

By the symmetry of τ in the first two coordinates we may assume that a generic line M of $\mathsf{PG}(2,q)$ has coordinates $[1,b,c]$, i.e., the points (x,y,z) of the line satisfy $x = by + cz$. Applying τ, we obtain the set

$$\{(b^\sigma y^\sigma + c^\sigma z^\sigma, y^\sigma, z^\sigma, byz^{\sigma-1} + cz^\sigma, yz^{\sigma-1}) : y, z \in \mathsf{GF}(q)\}.$$

We see that this set of points is contained in the plane π with equations

$$\begin{cases} X_0 = b^\sigma X_1 + c^\sigma X_2, \\ X_3 = bX_4 + cX_2. \end{cases}$$

Projecting this set of points from the line spanned by $(1,0,0,0,0)$ and $(0,0,0,1,0)$ into the plane with equations $X_0 = X_3 = 0$, we obtain the set $\{(0,1,0,0,0)\} \cup \{(0, y^\sigma, 1, 0, y) : y \in \mathsf{GF}(q)\}$, which is clearly a translation oval with associated field automorphism σ. This proves (ii).

Suppose now that some point $(u,v,w)^\tau$ is contained in the plane π above. Then it satisfies $X_0 = b^\sigma X_1 + c^\sigma X_2$, hence $u^\sigma = b^\sigma v^\sigma + c^\sigma w^\sigma$, implying $u = bv + cw$, and so (u,v,w) belongs to the line M. This shows (iii).

It remains to show that the only ovals on $\mathsf{PG}(2,q)^\tau$ are the images of the lines distinct from $[0,0,1]$. This will follow as soon as we prove that no five points are contained in a common plane not generated by the image of a line $M \neq [0,0,1]$ of $\mathsf{PG}(2,q)$ (note $q > 4$). Suppose five points $p_1, \ldots, p_5$ of $\mathsf{PG}(2,q)^\tau$ are contained in a common plane α, where α is not a plane spanned by the image of a line of $\mathsf{PG}(2,q)$, distinct from $[0,0,1]$. We may suppose that at most one of the points $p_1, \ldots, p_5$ is contained in the line $[0,0,1]^\tau$. Hence the oval $O_{i,j}$ through p_i, p_j, with $O_{i,j} = N^\tau$ and N a line of $\mathsf{PG}(2,q)$, does not contain any point of α besides p_i, p_j, $1 \le i < j \le 5$. Let the ovals $O_{1,2}$ and $O_{3,4}$ intersect in a point p. Then clearly $O_{1,2}$ and $O_{3,4}$ are contained in the 3-space $\langle \alpha, p \rangle$. It is now easy to see that, using (ii), all points of $\mathsf{PG}(2,q)^\tau$ are contained in that 3-space, contradicting (i).

This completes the case of one affine line, by deleting from the foregoing block B and noting that we may take for B the image of the line $[0,1,0]$ (indeed , one verifies that all choices are projectively equivalent). We obtain Case $(c4)$ of Main Result—Affine Planes. Adding B again, we obtain Case (C) of Main Result—Projective Planes.

It is also easy to check that, if $\sigma = 2$, then in the projective case we obtain the projection of a Veronesean surface $\mathcal{V}_2^4$ from a point n of its nucleus plane, and in the affine case we delete from $\mathcal{V}_2^4$ the conic with nucleus n and then project from n.

2.2.3 Precisely two blocks of $\mathcal{A}$ are affine lines of $\mathsf{PG}(4, q)$

We have already shown in Paragraph 2.2.1 that in this case q is even. Let L and L' be the two affine lines in $\mathcal{B}$, and let ℓ and ℓ' be the points of $\mathsf{PG}(4, q)$ on $\langle L\rangle$ and $\langle L'\rangle$, respectively, but not belonging to $\mathcal{P}$. We have also shown above that L and L' are parallel blocks of $\mathcal{A}$.

Let K be a block intersecting L in x_K and L' in x'_K. We consider the projection of $\mathcal{P} \setminus K$ from $\langle K\rangle$ onto $\lambda := \langle \ell, \ell'\rangle$ (note that λ and $\langle K\rangle$ are indeed skew since otherwise $\mathcal{P}$ is contained in the 3-space $\langle K, \lambda\rangle$). Suppose some point $x \in \mathcal{P} \setminus (K \cup L \cup L')$ is projected onto ℓ. Then every block through x not parallel to L and not parallel to K is contained in $\langle K, L\rangle$, and hence all points of $\mathcal{A}$ are contained in the 3-space $\langle K, L\rangle$, a contradiction. Similarly, no point of $\mathcal{P} \setminus (K \cup L \cup L')$ is projected onto ℓ'. Now suppose three points $x, y, z \in \mathcal{P} \setminus (K \cup L \cup L')$ are projected onto the same point p. The block through x, y is a q-arc $K_{x,y}$. By Paragraph 2.2.1 $K_{x,y}$ is not parallel to L, as otherwise it contains ℓ and ℓ'. Suppose first that $K_{x,y}$ is not parallel to K and hence intersects K in some point u. Then $K_{x,y} \subseteq \langle K, p\rangle$. At least one of L, L' does not contain u, hence is also contained in $\langle K, p\rangle$, and so all points of $\mathcal{A}$ are contained in $\langle K, p\rangle$, a contradiction. Hence x, y, z are contained in a block $K_{x,y}$ parallel to K and again $K_{x,y} \subseteq \langle K, p\rangle$, leading to the same contradiction as above. Hence the number of points of $\mathcal{P} \setminus K$ projected onto ℓ and ℓ' is precisely $q - 1$ (for each), and every other point on λ is the image of at most two points of $\mathcal{P}$. Hence $q^2 - q \le 2(q - 1) + 2(q - 1)$, implying $q^2 \le 5q - 4$, or $q \in \{2, 4\}$, the final contradiction for this case.

2.2.4 At least three blocks of $\mathcal{A}$ are affine lines of $\mathsf{PG}(4, q)$

Here, we know by Paragraph 2.2.1 that all blocks of a parallel class of $\mathcal{A}$ are affine lines, and no other block is an affine line. Let $L_1, L_2, \ldots, L_q$ be the affine lines of $\mathcal{B}$, and put $\ell_i = \langle L_i\rangle \setminus L_i$, $i = 1, 2, \ldots, q$. From Paragraph 2.2.1, we infer that all ℓ_i are contained in a plane together with L_j, for every $j \in \{1, 2, \ldots, q\}$. Taking intersections of two of these planes, we deduce that all ℓ_i are contained in a line L_∞ of $\mathsf{PG}(4, q)$. Consider a point $x \in L_1$ and project $\mathcal{P} \setminus \{x\}$ from x onto some 3-space $\Sigma \not\ni x$. The projections of the blocks through x distinct from L_1, together with the projection of L_∞, and the projections of the L_i, $i = 2, 3, \ldots, q$, are respectively contained in the two systems of generators of a hyperbolic quadric $\mathcal{H}$. Every block K not through x and distinct from L_i, $i = 2, \ldots, q$, is entirely projected into $\mathcal{H}$, which implies that K is contained in a conic $\widetilde{K}$, which is uniquely determined by K and $\mathcal{H}$ (since $q > 4$, we even do not need $\mathcal{H}$ for the uniqueness). Let $\widetilde{k} = \widetilde{K} \setminus K$. Then the projection of the point $\widetilde{k}$ is contained in the unique generator not containing the projection of any point of $\mathcal{A} \setminus \{x\}$. Hence all points $\widetilde{k}$ together with x are contained in a plane. This plane does not contain the line L_1. Considering another choice $x' \in L_1$ for x we see that for all q-arcs K not containing x nor x' the points $\widetilde{k}$ are contained in a line L_{q+1}. Also introducing the points $\widetilde{k}$ for all q-arcs on x, it is now clear that for all q-arcs the points $\widetilde{k}$ are on the line L_{q+1}. Choosing for x a point x'' on L_2 we see that L_{q+1} contains a point of $\langle x, L_\infty \rangle \cap \langle x'', L_\infty \rangle = L_\infty$ which is necessarily the point $L_\infty \setminus \{\ell_1, \ell_2, \ldots, \ell_q\}$.

It is clear that projection from L_∞ of any block K onto the plane $\langle K' \rangle$ of any other block K', with K and K' q-arcs, coincides with K'. Conversely, the projection from $\langle K \rangle$ onto L_∞ of the unique conic containing K' is bijective. It follows that cross-ratios are preserved and hence $\mathcal{P} \cup L_\infty \cup L_{q+1}$ is a rational normal cubic scroll. So we obtain Case $(c1)$ of Main Result—Affine Planes. Remark that a rational normal cubic scroll can be viewed as the projection of a Veronesean surface $\mathcal{V}_2^4$ from a point $u \in \mathcal{V}_2^4$ into a hyperplane $\mathsf{PG}(4,q)$ not containing u.

Since the projective lines corresponding with the blocks which are affine lines do not meet in a point, this model cannot be extended to a projective plane.

2.2.5 All lines of $\mathcal{A}$ are planar q-arcs of $\mathsf{PG}(4,q)$

Let K be any block, and let L be a line of $\mathsf{PG}(4,q)$ skew to $\langle K \rangle$. We project $\mathcal{P} \setminus K$ on L from $\langle K \rangle$. Suppose three points $x, y, z \in \mathcal{P} \setminus K$ are projected onto the same point $p \in L$. If x, y, z are contained in a block D, then all points of $D \setminus K$ are projected onto p (and there are q or $q-1$ such points). If some additional point $u \in \mathcal{P} \setminus (K \cup D)$ is projected onto p, then it is easy to see that $\mathcal{P}$ is contained in the inverse image of p, which is a 3-space, a contradiction. If x, y, z are not contained in a block, then we may suppose that the block D containing x and y is not parallel to K, and so $\langle K, D \rangle$ is 3-dimensional. But z is an additional point of $\mathcal{P}$ contained in $\langle K, D \rangle$, a contradiction. Hence it follows that the pre-image of a point on L with respect to the projection is either empty, or contains 1, 2, $q-1$ or q points. Now note that q and $q-1$ cannot occur both since, if K and an intersecting block D are contained in a 3-space, and also K and a parallel block D', then $\langle D \cap D', K \rangle = \langle D, K \rangle = \langle D', K \rangle$, a contradiction. Also, if the inverse image contains exactly two points, then these two points belong to a block parallel to K.

Suppose first that q does not occur and let m be the number of times $q-1$ occurs. Then $q^2 - q \le m(q-1) + 2(q+1-m)$, which implies $m \ge \frac{q^2-3q-2}{q-3}$. Since $q \ge 5$, we deduce $m \in \{q, q-1\}$. Since the blocks defined by the m pre-images cannot meet off K, we either have m blocks through a point of K, or m blocks of a parallel class distinct from the class of K. If $m = q-1$, then the $q-1 \geq 4$ points remaining lie on a unique block, which is by the foregoing not parallel to K, contradicting a previous observation. So in all cases $m = q$, and we see that either there is a point x_K on K such that the plane of every block through x_K meets $\langle K \rangle$ in at least a line, or there is some parallel class of blocks distinct from the class of K such that the plane of every member of that class intersects $\langle K \rangle$ in a line.

Now suppose that $q-1$ does not occur and let m be the number of times q occurs as size of the pre-image of a point of L. Similarly as above we obtain $m \geq q - 1 - \frac{4}{q-2}$. If $q > 6$, then $m = q-1$. Hence, interchanging the roles of K and any parallel block, we see that in this case every pair of blocks of the parallel class of K is contained in a 3-space.

Now assume that K is contained in a 3-space together with every block through some point $x_K \in K$. Let $K' \in \mathcal{B} \setminus \{K\}$ and let K and K' meet in a point $y \neq x_K$. Then there is always a block $D \neq K$ through x_K contained with K' in a 3-space. Hence D is contained with K in a 3-space, and also with K' in a 3-space. So $\langle D, K \rangle = \langle D, y \rangle = \langle D, K' \rangle$, a contradiction. Consequently this case does not occur. We have shown that for every block K, the blocks contained with K in a 3-space form a unique parallel class of blocks.

This easily implies that there is an involutorial mapping ι on the family of parallel classes of $\mathcal{A}$ such that every member of a parallel class $\mathfrak{P}$ is contained in a 3-space together with every member of the parallel class $\mathfrak{P}^\iota$. We call ι a *pairing*, and we say that a parallel class is *self-paired* if it is fixed under ι.

We now proceed with a case distinction depending on the number of self-pairings.

There are at least three self-pairings, i.e., ι has at least three fixed elements

Note that the planes of the blocks of any self-paired parallel class P contain a common line L_P of $\mathsf{PG}(4,q)$, as otherwise we can find three parallel blocks contained in a 3-space, and then $\mathcal{P}$ belongs to that 3-space, a contradiction.

Let $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3$ be three self-paired parallel classes of blocks, and let $x \in \mathcal{P}$ be arbitrary. Let K_i, $i = 1, 2, \ldots, q+1$, be the blocks containing x, and suppose $K_j \in \mathfrak{P}_j$ for $j \in \{1, 2, 3\}$. There are precisely two solids $\Sigma_j^{(1)}, \Sigma_j^{(2)}$ intersecting $\mathcal{P}$ in precisely K_j, $j = 1, 2, 3$. Clearly, these solids intersect $\langle K_i \rangle$ (if they do not contain that plane) in a tangent to K_i, $i = 1, 2, \ldots, q+1$. Suppose that the solid $\Sigma_1^{(n)}$ intersects $\langle K_i \rangle$ in the tangent $T_i^{(n)}$, $n = 1, 2$, $i = 2, 3, \ldots, q+1$; as $\Sigma_1^{(1)} \neq \Sigma_1^{(2)}$ we have $T_i^{(1)} \neq T_i^{(2)}$. Then all $T_i^{(1)}$, $i = 2, 3, \ldots, q+1$, are contained in the union of the two distinct planes $\Sigma_1^{(1)} \cap \Sigma_2^{(1)}$ and $\Sigma_1^{(1)} \cap \Sigma_2^{(2)}$, which intersect in $T_2^{(1)}$, and also in the two distinct planes $\Sigma_1^{(1)} \cap \Sigma_3^{(1)}$ and $\Sigma_1^{(1)} \cap \Sigma_3^{(2)}$, which intersect in

$T_3^{(1)} \neq T_2^{(1)}$. It follows easily that we may assume that $\Sigma_1^{(1)} \cap \Sigma_2^{(1)} = \Sigma_1^{(1)} \cap \Sigma_3^{(1)} =: \alpha$. The number of distinct tangents $T_i^{(1)}$, $i = 2, 3, \ldots, q+1$, is at least $2 + \frac{q-2}{2} = \frac{q}{2} + 1$, so α contains at least $\frac{q}{2}$ tangents. Assume by way of contradiction that neither $T_1^{(1)}$ nor $T_1^{(2)}$ is contained in α, where $T_1^{(1)}$ and $T_1^{(2)}$ are the tangents of K_1 at x. Then $\alpha \cap \langle K_1 \rangle$ is a point. So we find at least $\frac{q}{2}$ 3-spaces $\langle K_1, L \rangle$, with L a tangent in α, intersecting $\mathcal{P}$ in K_1, clearly a contradiction. Hence we may assume that $T_1^{(1)} \subseteq \alpha$. Clearly $T_1^{(2)}$ is not contained in α. Hence $T_1^{(2)} = \Sigma_1^{(1)} \cap \Sigma_2^{(2)} \cap \Sigma_3^{(2)}$. As $T_i^{(1)} \neq T_1^{(2)}$, $i = 1, 2, \ldots, q+1$, we conclude that $T_1^{(1)}, T_2^{(1)}, \ldots, T_{q+1}^{(1)}$ are contained in α. Likewise, there is a plane β containing $T_i^{(2)}$, for all $i \in \{1, 2, \ldots, q+1\}$, and $\alpha \cap \beta = \{x\}$ (the latter also follows from $\langle K_1, K_2 \rangle = \mathsf{PG}(4, q)$).

We now claim that ι is the identity. Indeed, if not, then let K_4 belong to a parallel class which is paired with the parallel class containing K_5. Let K_4' be a block parallel to K_4 but not containing x. Project $\mathcal{P} \setminus \{x\}$ from x onto a 3-space Σ^* not containing x. The projections of $K_5 \setminus \{x\}$ and K_4' are contained in a plane; the projection of $K_5 \setminus \{x\}$ generates a line T_4 which is tangent to the q-arc C, which denotes the projection of K_4'. Let $i \in \{1, 2, 3\}$. Let A and B denote the projections of $\alpha \setminus \{x\}$ and $\beta \setminus \{x\}$, respectively. Let N_i be the line generated by the projection of $K_i \setminus \{x\}$, $i \in \{1, 2, 3, 6, 7, \ldots, q+1\}$. Then N_i intersects the skew lines A, B and also the q-arc C; hence N_i is completely determined by A, B and $N_i \cap C$. It follows that the plane spanned by $N_i \cap A$ and B intersects C in a tangent T_i, $i = 1, 2, 3$ (indeed, otherwise some N_j, $j \neq i$, meets N_i in a point and so the parallel class of K_i is paired with the parallel class of K_j, a contradiction). Hence there are 4 tangent lines T_1, T_2, T_3, T_4 to C through the point $B \cap \langle C \rangle$. This implies that $B \cap \langle C \rangle$ extends C to a $(q+1)$-arc, see Section 10 of [1]. Similarly, also $A \cap \langle C \rangle$ extends C to a $(q+1)$-arc. It follows that q is even and both points extend C to a hyperoval. But this contradicts the fact that T_4 contains a point of C, and also contains $A \cap \langle C \rangle$ and $B \cap \langle C \rangle$. Hence ι is the identity.

It now also follows that $\alpha \cup \beta$ is the union of the tangent lines at x to the q-arcs of $\mathcal{B}$ through x. Now we again consider the projection from x onto Σ^*, as in the previous paragraph. The arguments there now imply that the projections of $K_4 \setminus \{x\}$ and K_4' are contained in a plane, and , with the same notation as above, the points $A \cap \langle C \rangle$ and $B \cap \langle C \rangle$ extend C to a hyperoval (hence q is even). In $\mathsf{PG}(4, q)$, this means that the intersections a_4 and b_4 of the tangents at x to K_4 with $\langle K_4 \rangle \cap \langle K_4' \rangle$ extend K_4' to a hyperoval. Now interchange the roles of K_4 and K_4', where x is interchanged with some $x' \in K_4'$. Then the tangents at x' of K_4' are precisely the lines $\langle x', a_4 \rangle$ and $\langle x', b_4 \rangle$, and they must intersect $\langle K_4 \rangle \cap \langle K_4' \rangle = \langle a_4, b_4 \rangle$ in the points extending K_4 to a hyperoval. It follows that a_4 and b_4 extend K_4 to a hyperoval, and hence they extend every block parallel to K_4 to a hyperoval. Hence a_4 and b_4 only depend on the parallel class $\mathfrak{P}_4$ containing K_4, and we can do this for the parallel class $\mathfrak{P}_i$, $i = 1, 2, \ldots, q+1$, containing any K_i, obtaining the points a_i, b_i. We may assume that all $a_i \in \alpha$ and all $b_i \in \beta$. By varying the point x, we see that all a_i form a unique line A^* and all b_i form a unique line B^*. If we add either A^* or B^* to $\mathcal{P}$, and define new blocks as the ovals arising from the old blocks by adding the appropriate a_i or b_i, respectively, and also add the appropriate block A^* or B^*, then we obtain a projective plane. By deleting an arbitrary block which is an oval, we have a representation of an affine plane in $\mathsf{PG}(4, q)$ where all lines are q-arcs, except for one, which is an affine line. This brings us back to the situation of Paragraph 2.2.2, and we obtain Case $(c3)$ of the Main Result—Affine Planes. Note that by the last part of Paragraph 2.2.2 Condition (iii) of the Main Result— Affine Planes is satisfied in Case (c3).

There are at most two self-pairings

Since we assume $q \geq 7$, and there are $q+1$ parallel classes, there are at least three nonidentity pairings. Suppose that the distinct parallel classes $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are paired with each other. From the previous considerations we know that every member of $\mathfrak{P}_1$ together with every member of $\mathfrak{P}_2$ generates a 3-space, and that any two members of either $\mathfrak{P}_1$ or $\mathfrak{P}_2$ generate the whole 4-space. If we consider the corresponding sets $\mathfrak{V}_1$ and $\mathfrak{V}_2$ of planes generated by the blocks of $\mathfrak{P}_1$ and $\mathfrak{P}_2$, respectively, and we dualize to sets $\mathfrak{L}_1$ and $\mathfrak{L}_2$ of q lines, respectively, then we see that $\mathfrak{L}_1$ and $\mathfrak{L}_2$ each consists of q members of a system of generators of a common hyperbolic quadric in some 3-space. The dual of this 3-space is the point p contained in all members of $\mathfrak{V}_1 \cup \mathfrak{V}_2$.

Suppose, by way of contradiction, that p, u, v, with $u, v \in \mathcal{P}$, $u \neq v$, are contained in a line. Clearly $p \notin \mathcal{P}$. Hence the line $\langle u, v \rangle$ is contained in the planes of $\mathfrak{V}_1$ and $\mathfrak{V}_2$ that contain either u or v and u and v are contained in the same member of $\mathfrak{V}_1$ and in the same member of $\mathfrak{V}_2$, clearly a contradiction, as a block can not belong to both $\mathfrak{P}_1$ and $\mathfrak{P}_2$.

Hence p extends all members of $\mathfrak{P}_1 \cup \mathfrak{P}_2$ to ovals. If we project $\mathcal{P}$ from p onto some 3-space Σ not containing p, then we obtain q^2 points of a hyperbolic quadric $\mathcal{H}$, lying on $2q$ lines. The unique point of $\mathcal{H}$ not covered by these lines will be denoted by r. Let K be any block not belonging to $\mathfrak{P}_1 \cup \mathfrak{P}_2$. Its projection K' is contained in $\mathcal{H}$ and contains a point of every generator containing the projection of a member of $\mathfrak{P}_1$, and similarly for $\mathfrak{P}_2$. It follows that K' is contained in a conic C' which necessarily contains r. Since there are $q^2 - q$ conics on $\mathcal{H}$ containing r, we see that there is a bijective correspondence between the members of $\mathcal{B} \setminus (\mathfrak{P}_1 \cup \mathfrak{P}_2)$ and the conics on $\mathcal{H}$ through r. Suppose now that K belongs to a parallel class $\mathfrak{P}_3$ which is paired with $\mathfrak{P}_4 \neq \mathfrak{P}_3$. Let p^* be the intersection of the planes of the members of $\mathfrak{P}_3 \cup \mathfrak{P}_4$, and let p' be the projection of p^*. Since there are $2q > q+1$ projections of such planes, not all these projections can contain a common line, and so p' coincides with r. Interchanging the roles of $\mathfrak{P}_1 \cup \mathfrak{P}_2$ and $\mathfrak{P}_3 \cup \mathfrak{P}_4$, we see that $K_i \cup \{p\}$ is a conic for all $K_i \in \mathfrak{P}_i$, $i = 1, 2$.

Now let $(\mathfrak{P}_5, \mathfrak{P}_6)$ be a third pair of distinct parallel classes paired to each other, $\mathfrak{P}_5, \mathfrak{P}_6 \notin \{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{P}_4\}$ and let $K_i \in \mathfrak{P}_i$, $i = 5, 6$. By re-choosing Σ we may suppose that r extends K_i to a conic C_i, $i = 5, 6$. Moreover, since $K_5 \cup K_6$ generates a 3-space (as $\mathfrak{P}_5$ is paired with $\mathfrak{P}_6$), we may even assume that $\Sigma = \langle K_5, K_6 \rangle$. Projecting $\mathcal{P}$ from p^* onto Σ yields a hyperbolic quadric $\mathcal{H}^*$. Clearly, $(C_5 \cup C_6) \subseteq \mathcal{H} \cap \mathcal{H}^*$, so both $\mathcal{H}$ and $\mathcal{H}^*$ belong to the pencil $\mathfrak{B}$ of quadrics in Σ with base $C_5 \cup C_6$.

Since K_5 and K_6 belong to different parallel classes, they meet in some point r'. We now coordinatize the situation using coordinates $(X_0, X_1, X_2, X_3, X_4)$. Let Σ be determined by the equation $X_4 = 0$. Choose $r = (1,0,0,0,0)$ and $r' = (0,1,0,0,0)$. Let $\langle C_5 \rangle$ have equations $X_3 = X_4 = 0$ and let $\langle C_6 \rangle$ have equations $X_2 = X_4 = 0$. We can choose the coordinates of the intersection of the tangent lines of C_5 at r and r' equal to $(0,0,1,0,0)$, and the coordinates of the intersection of the tangent lines of C_6 at r and r' equal to $(0,0,0,1,0)$. Furthermore, we can assume that the point $(1,1,1,0,0)$ belongs to C_5. This pins down the equations of K_5 to $X_3 = X_4 = X_0X_1 - X_2^2 = 0$, whereas the equations of K_6 are $X_2 = X_4 = X_0X_1 - \rho X_3^2$, for some $\rho \in \mathsf{GF}(q) \setminus \{0\}$. By choosing the point $(1,1,1,1,0)$ appropriately, we may assume that ρ is either 1, or a given non-square ρ_0 of $\mathsf{GF}(q)$; if q is even then we may assume that $\rho = 1$, if q is odd, then we may assume that $\rho \in \{1, \rho_0\}$. It follows that the equations of an arbitrary member Q_α, $\alpha \in \mathsf{GF}(q) \cup \{\infty\}$, of the pencil $\mathfrak{B}$ are

$$X_4 = X_2^2 - X_0X_1 + \rho X_3^2 + \alpha X_2X_3 = 0.$$

Note that the point r belongs to every member of the pencil, and that the tangent plane at r to Q_α, for all $\alpha \in \mathsf{GF}(q)$, has equations $X_1 = X_4 = 0$. It follows that Q_α is hyperbolic if and only if the intersection of Q_α with that tangent plane is the union of two distinct lines, i.e., the equation $X_2^2 + \alpha X_2 + \rho = 0$ has exactly two (different) solutions.

Suppose $\mathcal{H} = Q_\theta$ and $\mathcal{H}^* = Q_{\theta^*}$, $\theta, \theta^* \in \mathsf{GF}(q)$. We can assign the coordinates $(0,0,0,0,1)$ to p and $(1,0,0,0,1)$ to p^*. Then $\mathcal{P}$ is contained in the cone $\overline{\mathcal{H}}$ with vertex p and base $\mathcal{H}$, and also in the cone $\overline{\mathcal{H}}^*$ with vertex p^* and base $\mathcal{H}^*$. Hence $\mathcal{P}$ is contained in the intersection $\overline{\mathcal{H}} \cap \overline{\mathcal{H}}^*$, which has equations

$$\begin{cases} 0 &= X_2^2 - X_0X_1 + \rho X_3^2 + \theta X_2X_3, \\ 0 &= X_2^2 - (X_0 - X_4)X_1 + \rho X_3^2 + \theta^* X_2X_3. \end{cases}$$

Now, we also know that no point of $\mathcal{P}$ is projected onto the lines of $\mathcal{H}$ through r, and one verifies that these points are characterized by the fact that their coordinate X_1 is equal to 0. Hence we may assume that every point of $\mathcal{P}$ has X_1-coordinate equal to 1. We can parameterize such points on $\overline{\mathcal{H}}$ using the parameters $a, b, c \in \mathsf{GF}(q)$ as follows:

$$\begin{cases} X_1 &= 1, \\ X_2 &= a, \\ X_3 &= b, \\ X_4 &= c, \\ X_0 &= a^2 + \rho b^2 + \theta ab. \end{cases}$$

Likewise, we can parameterize such points on $\overline{\mathcal{H}}^*$, using the same parameters $a, b, c \in \mathsf{GF}(q)$, anticipating on taking the intersection, as follows:

$$\begin{cases} X_1 &= 1, \\ X_2 &= a, \\ X_3 &= b, \\ X_4 &= c, \\ X_0 &= c + a^2 + \rho b^2 + \theta^* ab. \end{cases}$$

Hence a parameterization of the intersection is given by

$$\begin{cases} X_1 &= 1, \\ X_2 &= a, \\ X_3 &= b, \\ X_4 &= (\theta - \theta^*)ab, \\ X_0 &= a^2 + \rho b^2 + \theta ab. \end{cases}$$

Since the parameters can take q^2 different values, this intersection coincides with $\mathcal{P}$.

Now, if $\theta = \theta^*$, then $\mathcal{P}$ is contained in the 3-space with equation $X_4 = 0$, a contradiction. Hence we can recoordinatize as follows: $X'_i = X_i$, for $i = 1, 2, 3$, $X'_4 = X_4/(\theta - \theta^*)$ and $X'_0 + \theta X'_4 = X_0$. Consequently we see that the set $\mathcal{P}$ is projectively equivalent with the following set of points:

$$\begin{cases} X_1 &= 1, \\ X_2 &= a, \\ X_3 &= b, \\ X_4 &= ab, \\ X_0 &= a^2 + \rho b^2, \end{cases}$$

with $a, b \in \mathsf{GF}(q)$, and which is independent of θ and θ^*. Since $\rho = 1$ if q is even, we have a unique solution in this case, and this must coincide with Case $(c2)$ in the Main

Result—Affine Planes. If q is odd, then $\rho \in \{1, \rho_0\}$, yielding at most two different cases; hence there are precisely two different cases, which are described in Case ($c2$) of the Main Result—Affine Planes.

This completes the case $d = 4$.

2.3 Case $d = 3$

We continue to assume $q \geq 7$.

2.3.1 All blocks are q-arcs

In this case, $\mathcal{P}$ is a q^2-cap of $\mathsf{PG}(3, q)$. Any such cap is , for $q > 2$, contained in an ovoid $\mathcal{O}$ (see Sections 18.3, 18.4 and 18.5 of [2]). Let $\mathcal{O} = \mathcal{P} \cup \{p\}$. By Assumption ($iii$), the planes generated by the blocks must contain p. Hence we obtain Case ($b1$). Clearly, there is no extension to a projective plane.

2.3.2 There is a point $p \in \mathcal{P}$ such that at most two blocks in $\mathcal{B}$ through p are q-arcs

In this case, we claim that all blocks through p are affine lines. Indeed, suppose not and let K be a q-arc in $\mathcal{B}$ containing p. Let $L_1, L_2, L_3 \in \mathcal{B}$ be three affine lines through p. If $K' \neq K$ is some block parallel to K, then K' intersects each of L_1, L_2, L_3. If K' is an affine line, then $\langle L_1, L_2, L_3 \rangle$ is a plane and we see that this plane must contain $\mathcal{P}$, a contradiction. Hence every triple of affine lines of $\mathcal{B}$ through p generates $\mathsf{PG}(3, q)$, and every block not through p is a q-arc.

Suppose first that there is another block through p that is a q-arc, say J, $J \neq K$. Let $L_1, L_2, \ldots, L_{q-1}$ be the blocks through p that are affine lines. By considering a block parallel to K, but distinct from K, we see that there exists a line of $\mathsf{PG}(3, q)$ through p which is not contained in a plane $\langle L_i, L_j \rangle$, $i, j \in \{1, 2, \ldots, q-1\}$, $i \neq j$. Then by Section 10.3 of [1], there are precisely three (when q is even) or two (when q is odd) lines M_1, M_2 and possibly M_3 through p such that no such line lies in a plane together with two lines of the set $\{\langle L_1 \rangle, \langle L_2 \rangle, \ldots, \langle L_{q-1} \rangle\}$. These two or three lines are the only lines through p which are not contained in a plane $\langle L_i, L_j \rangle$, $i, j \in \{1, 2, \ldots, q-1\}$, $i \neq j$. If q is odd, we put $M_3 = M_2$. Let $S \in \mathcal{B}$ be any block parallel to K but distinct from K. Then it is clear that the point $J \cap S$ must be found among the points $\langle S \rangle \cap (M_1 \cup M_2 \cup M_3)$.

Varying S, this implies that J is contained in the union of at most three lines and must hence contain a triple of collinear points, a contradiction. Hence p is contained in q affine lines $L_1, L_2, \ldots, L_q$ of $\mathcal{B}$. Let M_2, M_3 be the lines of $\mathsf{PG}(3, q)$ through p not contained in any plane $\langle L_i, L_j\rangle$, $i, j \in \{1, 2, \ldots, q\}$, $i \neq j$, where $M_2 = M_3$ for q odd and $M_2 \neq M_3$ for q even. Now let C be an arbitrary block not parallel to K, and not through p. Then, as above, the point $C \cap K$ must be found among $\langle C\rangle \cap M_2$ and $\langle C\rangle \cap M_3$. Hence K is contained in the union of at most two lines, a contradiction. This implies that $K = L_{q+1}$ is also an affine line, proving our claim.

Let v_i be the point of $\langle L_i\rangle$ which does not belong to L_i, $i = 1, 2, \ldots, q + 1$. The planes of the $q - 1$ blocks parallel to L_i clearly contain v_i, $i = 1, 2, \ldots, q + 1$; adding v_i to these blocks produces $q - 1$ ovals through v_i. As parallel blocks do not intersect, all these ovals have a common tangent line at v_i, $i = 1, 2, \ldots, q+1$. Let C be one of those ovals (through v_i, for some $i \in \{1, 2, \ldots, q + 1\}$). Let $s \in C \setminus \{v_i\}$. Since different blocks through s do not intersect in more than one point, and since the planes of such blocks do not share a point of type v_j, $j \in \{1, 2, \ldots, q+1\}$ (as otherwise they are both parallel to L_j), all blocks through s that are q-arcs have a common tangent line T at s which is also tangent to the q corresponding ovals.

Let O_1, O_2, O_3 be three of the $q^2 - 1$ ovals obtained from the q-arcs in $\mathcal{B}$ by adding some member of $\{v_1, \ldots, v_{q+1}\}$, and suppose $O_1 \cap O_2 = \{s_3\} \neq \{s_2\} = O_1 \cap O_3 \neq O_2 \cap O_3 = \{s_1\} \neq \{s_3\}$. The three tangents at s_1, s_2, s_3 are pairwise contained in the same plane, but are not all contained in a common plane; hence they have a point n in common. Hence all tangents of the $q^2 - 1$ ovals obtained from the q-arcs in $\mathcal{B}$ contain n. It follows that q is even. Considering the $q + 1$ planes $\langle n, L_i\rangle$, $i = 1, 2, \ldots, q + 1$, and the $q^2 - 1$ planes of the ovals, there arise $q^2 + q$ planes through n none of which contains two points of $\{v_1, v_2, \ldots, v_{q+1}\}$. It follows that the remaining plane through n contains $v_1, v_2, \ldots, v_{q+1}$. Clearly $\{v_1, v_2, \ldots, v_{q+1}\}$ is an oval with nucleus n.

This is Case $(b4)$ of Main Result—Affine Planes. Clearly it uniquely extends to Case (B) of Main Result—Projective planes.

2.3.3 There is some affine line in $\mathcal{B}$ and every point $p \in \mathcal{P}$ is contained in at least three q-arcs belonging to $\mathcal{B}$

Let $x \in \mathcal{P}$. By assumption, there are at least three q-arcs contained in $\mathcal{B}$ through x. Suppose first that there are at least four such q-arcs K_1, K_2, K_3, K_4 through x. We claim that these have a common tangent line at x. Indeed, let $L_i^{(1)}$ and $L_i^{(2)}$, $i = 1, 2, 3, 4$, be the tangent lines at x to K_i. By Assumption (iii), the line $\langle K_i\rangle \cap \langle K_j\rangle$, $i, j \in \{1, 2, 3, 4\}$,

$i \neq j$, is tangent to both K_i and K_j. Hence $|\{L_i^{(1)}, L_i^{(2)}\} \cap \{L_j^{(1)}, L_j^{(2)}\}| = 1$, for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. It is easy to see that this implies our claim. Now suppose that there are exactly three q-arcs of $\mathcal{B}$ through x. Then there are at least three affine lines of $\mathcal{B}$ containing x; by Assumption (iii) any three such lines generate $\mathsf{PG}(3, q)$. Let K be any of the three q-arcs. Then the three planes π_i, $1 = 1, 2, 3$, obtained by joining any two of the three affine lines pairwise intersect $\langle K \rangle$ in three distinct lines L_i, $i = 1, 2, 3$, respectively; hence one of these lines, say L_1, intersects K in a further point y. Joining y with a point x_1 of one of the affine lines in π_1 such that the line $\langle y, x_1 \rangle$ intersects the other affine line of π_1 nontrivially, we see that Assumption (iii) and Assumption (ii) imply that the block containing y and x_1 is an affine line. This then readily implies that $\mathcal{P}$ is contained in π_1, a contradiction. In conclusion, we have shown that the q-arcs of $\mathcal{B}$ through a fixed point $x \in \mathcal{P}$ all admit a common tangent T_x at x.

Note that the arguments in the previous paragraph imply that through any point of $\mathcal{P}$ pass at most two affine lines belonging to $\mathcal{B}$.

Now let $K \in \mathcal{B}$ be a fixed q-arc, and let $x \in \mathcal{P} \setminus K$. By the previous paragraph, there are at least three q-arcs K_1, K_2, K_3 belonging to $\mathcal{B}$ through x intersecting K in points, say x_1, x_2, x_3, respectively. The common tangent T_x and the tangent line at x_i to K_i, say T_{x_i}, are contained in the same plane $\langle K_i \rangle$, $i = 1, 2, 3$, and hence meet in some point y_i. If $y_1 \neq y_2$, then $T_x \subseteq \langle K \rangle$ and so $\langle K \rangle = \langle K_1 \rangle = \langle K_2 \rangle$, a contradiction. Consequently, $T_x, T_{x_1}, T_{x_2}, T_{x_3}$ intersect in a common point n. Since we can also first choose x_1 and x_2, and can find a corresponding $x \in \mathcal{P}$ outside K, we conclude that all common tangents contain the point n. This point n extends all q-arcs of $\mathcal{B}$ to ovals.

Assume, by way of contradiction, that three affine lines $L, L', L'' \in \mathcal{B}$ belong to different parallel classes and have no point in common. Then it is easy to see that $\langle L, L', L'' \rangle$ is a plane containing $\mathcal{P}$, a contradiction. Also the same argument implies that, if L, L', L'' are three affine lines of $\mathcal{B}$ with L parallel to L', but not with L'', then $\langle L \rangle$ and $\langle L' \rangle$ are skew lines in $\mathsf{PG}(3, q)$.

By assumption, there is a block of $\mathcal{A}$ which is an affine line. Let $L_1 \in \mathcal{B}$ be an affine line.

Suppose $\{L_1, L_2, \ldots, L_q\}$ is the parallel class of blocks of $\mathcal{A}$ containing L_1, with $L_1, \ldots, L_k$ affine lines and $L_{k+1}, \ldots, L_q$ q-arcs, $1 \leq k \leq q$. Put $\{\ell_i\} = \langle L_i \rangle \setminus L_i$, $i = 1, 2, \ldots, k$. Then $\ell_i \in \langle L_j \rangle$ and $n \in \langle L_j \rangle$, for all $i = 1, 2, \ldots, k$ and all $j = k+1, \ldots, q$.

First, we assume $1 < k < q$.

Clearly, $n \neq \ell_i$, for any $i = 1, 2, \ldots, k$, since the plane generated by a q-arc belonging to $\mathcal{B}$ and not parallel to L_i would then contain L_i, a contradiction. If the line $\langle n, \ell_i \rangle$, for some $i \in \{1, 2, \ldots, k\}$, would contain a point y of L_j, for some $j \in \{k+1, \ldots, q\}$, then the plane spanned by any block through y, distinct from L_j, which is a q-arc, would contain L_i (and such a block exists by our assumption), a contradiction. It follows that the points $n, \ell_1, \ldots, \ell_k$ are collinear and that the ovals $L_{k+1} \cup \{n\}, \ldots, L_q \cup \{n\}$ have a common tangent $\langle \ell_1, n \rangle$ at n.

Assume that $\ell_1 = \ell_2 \neq \ell_3$. Let z be the intersection of $\langle L_1, L_2 \rangle$ and L_3, and choose $v \in L_1$ and $w \in L_2$ such that v, w, z are collinear. If the block Z through w and v is an affine line, then we see that $\mathcal{P}$ is contained in the plane $\langle L_1, L_2 \rangle$, a contradiction. If Z is a q-arc and if $z \neq \ell_3$, then we obtain a contradiction to Assumption (iii). If Z is a q-arc and $z = \ell_3$, then $\langle Z \rangle$ contains n, ℓ_1, ℓ_3, and so $\langle L_1, L_2 \rangle$ contains Z, also a contradiction. Hence either all points $\ell_1, \ldots, \ell_k$ are distinct and collinear, or they all coincide.

First, let $\ell_1, \ldots, \ell_k$ be distinct collinear points. Put $L_{k+1} = \{x_1, x_2, \ldots, x_q\}$. Let $\overline{N}_i$ be the line of $\mathsf{PG}(3, q)$ which contains x_i and intersects L_1 and L_2, $i = 1, 2, \ldots, q$. By Assumption (iii), $\overline{N}_i$ contains an affine line N_i which belongs to $\mathcal{B}$, $i = 1, 2, \ldots, q$. The blocks $N_1, N_2, \ldots, N_q$ clearly belong to a common parallel class of $\mathcal{A}$. As $q \geq 7$, there is a line $\overline{M}$ of $\mathsf{PG}(3, q)$ distinct from L_1, L_2 which intersects the affine lines N_1, N_2, N_3 nontrivially. Assumption (iii) again implies that $\overline{M}$ contains an affine line $M \in \mathcal{B}$, which is clearly parallel to L_1 and hence belongs to $\{L_3, \ldots, L_k\}$. Hence $k \geq 3$. It follows that $\langle L_1 \rangle$, $\langle L_2 \rangle$,..., $\langle L_k \rangle$, $\overline{N}_1, \ldots, \overline{N}_q, \langle \ell_1, \ell_2 \rangle$ are generators of a hyperbolic quadric $\mathcal{H}$. But $\ell_1, \ldots, \ell_k$ and L_i, $i \in \{k+1, \ldots, q\}$, belong to a common plane and are contained in $\mathcal{H}$, a contradiction.

Hence $\ell_1 = \ell_2 = \cdots = \ell_k =: \ell$. Since $k \geq 2$, we immediately deduce that every block not parallel to L_1 is a q-arc. We claim that $\mathcal{K} = L_{k+1} \cup \cdots \cup L_q$ is a cap. Indeed, if not, then we find three points $x, y, z \in \mathcal{K}$ collinear in $\mathsf{PG}(3, q)$. By Condition (iii), these must be contained in an affine line $W \in \mathcal{B}$, contradicting the fact that it must be a q-arc. Our claim is proved, and $\mathcal{K}$ is a $(q^2 - kq)$-cap of $\mathsf{PG}(3, q)$, which clearly has no point in common with $\langle L_1, L_i \rangle$, for all $i \in \{2, 3, \ldots, k\}$. Let π be a plane of $\mathsf{PG}(3, q)$ containing L_1 but not L_i, for all $i \in \{2, 3, \ldots, k\}$. If $\pi \cap \mathcal{K}$ contains at least two points x, y, then Condition (iii) implies that $\langle x, y \rangle$ intersects $\langle L_1 \rangle$ in ℓ (remember that the block through x and y is a q-arc; in particular no further point of $\mathcal{P}$ lies on $\langle x, y \rangle$). Hence $\pi \cap \mathcal{K}$ contains at most two elements. Further, we claim that the plane $\langle L_1, n \rangle$ does not contain any element of $\mathcal{P}$. Indeed, suppose $x \in \mathcal{P} \setminus L_1$ is contained in $\langle L_1, n \rangle$. Then any block of $\mathcal{A}$ through x not parallel to L_1 is a q-arc K' such that $\langle K' \rangle$ contains n and also ℓ, and hence also every $\langle L_i \rangle$ as it intersects each L_i nontrivially, $i = 1, 2, \ldots, k$, a contradiction. Our claim follows.

It now follows that $q^2 - kq = |\mathcal{K}| \leq 2(q - k + 1)$, which means

$$k \geq q - \frac{2}{q-2}.$$

Since $q \geq 7$, this implies that $k = q$, a contradiction.

Hence it follows that, if $\mathcal{B}$ contains two parallel affine lines L_1, L_2, then all blocks parallel to L_1 are affine lines. Suppose $\{L_1, L_2, \ldots, L_q\}$ is a parallel class of $\mathcal{A}$ and all of $L_1, \ldots, L_q$ are affine lines. Put $\{\ell_i\} = \langle L_i \rangle \setminus L_i$, $i = 1, 2, \ldots, q$. As in the case $1 < k < q$ it is shown that $n \neq \ell_i$, with $i = 1, 2, \ldots, q$. First assume that $\ell_1 = \ell_2 = \cdots = \ell_q =: \ell$. Then $n \notin \langle L_i, L_j \rangle$, $i, j \in \{1, 2, \ldots, q\}$, $i \neq j$, as otherwise $\mathcal{P}$ is contained in the plane $\langle L_i, L_j \rangle$. Clearly we obtain Case $(b2)$ of Main Result—Affine Planes. For q even, this model extends to Case (B) of Main result—Projective Planes.

Next assume that $\ell_1 \neq \ell_2$. Assume by way of contradiction that not all points $\ell_1, \ldots, \ell_q$ are distinct; then without loss of generality we may assume $\ell_1 = \ell_q$. This already implies that all blocks not parallel to L_1 are q-arcs. Condition (iii) implies that the intersection point r of $\langle L_2 \rangle$ and $\langle L_1, L_q \rangle$ coincides with ℓ_2. Choose two points $y_1 \in L_1$ and $y_q \in L_q$ such that y_1, y_q, ℓ_2 are collinear. Let K be the block through y_1 and y_q. Then $\langle K \rangle$ intersects L_2 in a point off $\langle L_1, L_q \rangle$, and contains ℓ_2, hence $\langle K \rangle$ contains L_2, a contradiction. Consequently, $\ell_1, \ell_2, \ldots, \ell_q$ are distinct.

Consider two lines $\overline{M}_1, \overline{M}_2$ of $\mathsf{PG}(3, q)$ intersecting L_1, L_2, L_3 (as affine lines; this is possible since $q \geq 7$). As before, Assumption (iii) implies that $\overline{M}_1, \overline{M}_2$ are the projective completions of affine lines M_1, M_2 belonging to $\mathcal{B}$, and necessarily parallel in $\mathcal{A}$. By the foregoing, all members of the parallel class of M_1 are affine lines $M_1, M_2, \ldots, M_q$. Set $m_i = \langle M_i \rangle \setminus M_i$, $i = 1, 2, \ldots, q$. Clearly, the lines $\langle L_1 \rangle, \langle L_2 \rangle, \ldots, \langle L_q \rangle, \langle M_1 \rangle, \ldots, \langle M_q \rangle$ are generators of a hyperbolic quadric $\mathcal{H}$, the points $\ell_1, \ldots, \ell_q$ are on a line L of $\mathsf{PG}(3, q)$, and the points $m_1, \ldots, m_q$ are on a line M of $\mathsf{PG}(3, q)$. As a q-arc of $\mathcal{B}$ intersects all L_i and M_i, $i = 1, 2, \ldots, q$, it must intersect L and M in $L \cap M$. Hence $L \cap M = \{n\}$. It is now clear that we obtain Case $(b3)$ of Main Result—Affine Planes. Clearly, this model cannot be extended to a projective plane.

Hence we now only have to deal with the case that no parallel class of blocks of $\mathcal{A}$ contains at least two affine lines, but there exists at least one affine line L in $\mathcal{B}$ (the case $k = 1$). Note that there is at most one other affine line M in $\mathcal{B}$, which meets L in a point $x \in \mathcal{P}$. If M does not exist, then, as before in the case $k > 1$ and $\ell_1 = \ell_2 = \cdots = \ell_k$, one shows that the q blocks parallel to L are affine lines, a contradiction.

So we may assume that M exists. As in the case $k > 1$ and $\ell_1 = \ell_2 = \cdots = \ell_k$ one shows that every plane containing L but not M contains at most two points of $\mathcal{P} \setminus L$. Hence

$q^2 \leq (2q-1)+2q$ (the $2q-1$ comes from $L \cup M$, the $2q$ comes from the q planes through L not containing M), which implies $q \leq 3$, a contradiction.

This concludes the case $d = 3$.

Our main results are proved.

3 Remark on the small cases and on Condition (iii)

3.1 The small cases $q = 2, 3, 4, 5$

The case $q = 2$ is trivial.

Now let $\mathcal{P}$ be any cap of size 9 in $\mathsf{PG}(d, 3)$, with $\langle \mathcal{P} \rangle = \mathsf{PG}(d, 3)$ and $d \in \{3, 4, \ldots, 8\}$. Then any bijection of $\mathsf{AG}(2, 3)$ onto $\mathcal{P}$ yields a representation of $\mathsf{AG}(2, 3)$ in $\mathsf{PG}(d, 3)$ satisfying the requirements.

For $q \in \{4, 5\}$, any representation of $\mathsf{AG}(2, q)$ in $\mathsf{PG}(d, q)$ satisfying (i) (but without any restriction on d or q), (ii) and (iii) implies $d \leq 5$. It is certainly possible to handle the remaining cases by hand or by computer, but we have chosen not to do it here to keep the paper at a reasonable length.

3.2 Condition (iii)

If $d \geq 5$ then Condition (iii) is always satisfied for $q > 3$. Indeed, if some point p would be contained in the plane spanned by some block L_1, then we can choose blocks L_2 and L_3 such that $p \in L_2$ and L_1, L_2, L_3 pairwise intersect in distinct points. As every point is contained in a block intersecting $L_1 \cup L_2 \cup L_3$ in three distinct points, we see that $\langle L_1, L_2, L_3 \rangle = \mathsf{PG}(d, q)$, but as $p \in \langle L_1 \rangle \cap \langle L_2 \rangle$, $\langle L_1, L_2, L_3 \rangle$ is at most 4-dimensional, a contradiction.

Let $d = 2$ and consider a set of $q^2 + q + 1$ conics in $\mathsf{PG}(2, q)$ such that the point set of $\mathsf{PG}(2, q)$ together with these conics form a $\mathsf{PG}(2, q)$. Removing one of these conics and all points on it yields a representation of $\mathsf{AG}(2, q)$ in $\mathsf{PG}(2, q)$ satisfying (i) (with $d = 2$ and with no restriction on q), (ii), but not satisfying (iii).

Consider $(c4)$ with $q \neq 2$ in Main Result—Affine Planes and let p be a point of type $(a^\sigma, b^\sigma, c^\sigma, ac^{\sigma-1}, bc^{\sigma-1})$ with $b = 0$ and $c \neq 0$. Now project $\mathcal{P}$ from p onto a 3-space

$\mathsf{PG}(3,q)$ not containing p. Then there arises a representation of $\mathsf{AG}(2,q)$ in $\mathsf{PG}(3,q)$ satisfying (i) (with $d=3$ and $q\neq 2$), (ii), but not satisfying (iii).

Consider (a) with $q\neq 2$ in Main Result—Affine Planes. Let D be a conic on $\mathcal{V}_2^4$, distinct from the removed conic C. Consider a 3-space $\mathsf{PG}(3,q)$ of $\mathsf{PG}(5,q)$ which intersects $\mathcal{V}_2^4$ in D and a point r not on C. Let p be a point of $\mathsf{PG}(3,q)$ not in a conic plane of $\mathcal{V}_2^4$. Now we project $\mathcal{P}$ from p onto a 4-space $\mathsf{PG}(4,q)$ in $\mathsf{PG}(5,q)$ not containing p. Then there arises a representation of $\mathsf{AG}(2,q)$ in $\mathsf{PG}(4,q)$ satisfying (i) (with $d=4$ and $q\neq 2$), (ii), but not satisfying (iii).

These examples show that Condition (iii) is necessary and does not follow from the other conditions.

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