EMBEDDINGS OF HERMITIAN UNITALS INTO PAPPIAN PROJECTIVE PLANES

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Abstract. Every embedding of a hermitian unital with at least four points on a block into any pappian projective plane is standard, i.e. it originates from an inclusion of the pertinent fields. This result about embeddings also allows to determine the full automorphism groups of (generalized) hermitian unitals.

A hermitian unital in a pappian projective plane consists of the absolute points of a unitary polarity of that plane, with blocks induced by secant lines (see Section 2). The finite hermitian unitals of order q are the classical examples of 2-$(q^3+1, q+1, 1)$-designs.

In Section 2 we define and determine the groups of projectivities in hermitian unitals. In fact, we consider generalized hermitian unitals $\mathcal{H}(C|R)$ where $C|R$ is any quadratic extension of fields; separable extensions $C|R$ yield the hermitian unitals, inseparable extensions give certain projections of quadrics. In Section 3 we classify some embeddings of affine quadrangles into affine spaces. This is used in the final section to obtain the following results (Theorem 5.1, Corollary 5.4 with Remark 5.5).

Main Theorem. *For $|R| > 2$ every embedding of $\mathcal{H}(C|R)$ into a projective plane $\mathrm{PG}(2, E)$ over a field E originates from an embedding $C \to E$ of fields.*

Thus the image of such an embedding generates a subplane of $\mathrm{PG}(2, E)$ that is isomorphic to $\mathrm{PG}(2, C)$, and $\mathcal{H}(C|R)$ is embedded naturally into this subplane. The assumption $|R| > 2$ is necessary: $\mathcal{H}(\mathbb{F}_4|\mathbb{F}_2)$ is isomorphic to the affine plane $\mathrm{AG}(2, \mathbb{F}_3)$ over $\mathbb{F}_3$, and this affine plane embeds into its projective closure $\mathrm{PG}(2, \mathbb{F}_3)$ and into many other pappian projective planes, of arbitrary characteristic; see Remark 2.18.

Corollary. *Every finite projective plane $\mathrm{PG}(2, \mathbb{F}_{q^2})$ contains only one copy of the hermitian unital $\mathcal{H}(\mathbb{F}_{q^2}|\mathbb{F}_q)$, up to collineations from $\mathrm{PGL}_3\mathbb{F}_{q^2}$.*

This corollary was proved also by Korchmáros, Siciliano and Szőnyi [20]. They consider a cyclic subgroup of order $q+1$ of the group of projectivities of $\mathcal{H}(\mathbb{F}_{q^2}|\mathbb{F}_q)$ and use the conjugacy of all such subgroups in $\mathrm{PGL}_2\mathbb{F}_{q^2}$. We consider the larger group of all projectivities, which allows to replace the conjugacy statement by Proposition 1.1 below.

Our Main Theorem is also used to determine the full group of automorphisms of a (generalized) hermitian unital (Theorem 5.2).

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1. Embeddings between some permutation groups

If X is a set and G is a subgroup of the symmetric group of X, then (G, X) is called a permutation group. An *embedding* of (G, X) into another permutation group (H, Y) is a pair (α, β) where $\alpha\colon G \to H$ is a monomorphism of groups and $\beta\colon X \to Y$ is an injection such that $\alpha(g) \circ \beta = \beta \circ g$ for every $g \in G$. Then for every $h \in H$ the pair $(i_h \circ \alpha, h \circ \beta)$ is also an embedding of (G, X) into (H, Y), where i_h denotes the inner automorphism of H determined by h.

If $\alpha(G) = H$ and $\beta(X) = Y$, then (α, β) is a permutation isomorphism, and the two permutation groups are permutation isomorphic.

The projective line $\mathrm{PG}(1, F)$ over a field F is the set of all one-dimensional subspaces of the vector space F^2. We identify $\mathrm{PG}(1, F)$ with $\overline{F} := F \cup \{\infty\}$ as usual, where $\infty \notin F$. Then the group PGL_2F consists of all permutations of $\overline{F}$ of the form $x \mapsto (ax + b)/(cx + d)$ with $a, b, c, d \in F$.

For every subgroup M of the multiplicative group $F^\times$ of F, we define $\mathrm{SL}_2^M F := \{A \in \mathrm{GL}_2F \mid \det A \in M\}$. These are the groups between SL_2F and GL_2F, and their images $\mathrm{PSL}_2^M F$ in PGL_2F are the groups between PSL_2F and PGL_2F.

Every embedding $\beta\colon F \to E$ of fields yields an embedding (α, β) of the permutation group $(\mathrm{PSL}_2^M F, \overline{F})$ into $(\mathrm{PGL}_2E, \overline{E})$, where β is extended to $\overline{F}$ by $\beta(\infty) = \infty$. We say that (α, β) and the embeddings $(i_h \circ \alpha, h \circ \beta)$ as above with $h \in \mathrm{PGL}_2E$ *originate* from β.

Proposition 1.1. *Let E and F be fields and let $M \le F^\times$. If $|F| > 3$ or $|M| > 1$, then every embedding of the permutation group $(\mathrm{PSL}_2^M F, \overline{F})$ into $(\mathrm{PGL}_2E, \overline{E})$ originates from an embedding $F \to E$ of fields.*

Proof. We may assume that we have an embedding (α, β) with $\beta(\infty) = \infty$, $\beta(0) = 0$ and $\beta(1) = 1$, because $(\mathrm{PGL}_2E, \overline{E})$ is triply transitive. The stabilizer $(\mathrm{PGL}_2E)_{\infty,0,1}$ is trivial, hence $\alpha(g) \in \mathrm{PGL}_2E$ is determined by its restriction to the set $\beta(\{\infty, 0, 1\}) = \{\infty, 0, 1\}$. Thus the monomorphism α is determined by β, and it suffices to show that the restriction $\beta_{|F}\colon F \to E$ is an embedding of fields.

For $m \in M$, $a \in F^\times$ and $c \in F$, the two permutations $(x \mapsto ma^2x)$ and $(x \mapsto x + c)$ belong to the stabilizer $(\mathrm{PSL}_2^M F)_\infty$ and have the commutator $(x \mapsto x + (1 - ma^2)c)$. By our assumptions on F and M we can achieve that $ma^2 \ne 1$, hence $t_b := (x \mapsto x + b)$ is a commutator in $(\mathrm{PSL}_2^M F)_\infty$ for every $b \in F$. Thus $\alpha(t_b)$ is a commutator in $(\mathrm{PGL}_2E)_\infty = \mathrm{AGL}_1E$, whence $\alpha(t_b) = (y \mapsto y + b')$ for some $b' \in E$. The equation $\alpha(t_b) \circ \beta = \beta \circ t_b$ says that $\beta(x) + b' = \beta(x + b)$ for every $x \in F$. Using $\beta(0) = 0$ we infer that $b' = \beta(b)$. Thus $\beta_{|F}$ is additive (and $\operatorname{char} F = \operatorname{char} E$).

The involution $g = (x \mapsto -1/x) \in \mathrm{PSL}_2F \le \mathrm{PSL}_2^M F$ exchanges 0 and ∞. Hence $\alpha(g) \in \mathrm{PGL}_2E$ has the same property, whence $\alpha(g) = (y \mapsto d/y)$ for some $d \in E$. The equation $\alpha(g) \circ \beta = \beta \circ g$ implies that $d/\beta(x) = \beta(-1/x) = -\beta(1/x)$ for every $x \in F$. Specializing $x = 1$ yields $d = -\beta(1) = -1$, thus $\beta(1/x) = 1/\beta(x)$ for every $x \in F$. Now a theorem of Hua implies that $\beta_{|F}$ is multiplicative (or antimultiplicative, but E is commutative); see Artin [1] Theorem 1.15, p. 37 or Cohn [5] Theorem 9.1.3. Thus $\beta_{|F}\colon F \to E$ is an embedding of fields. □

The following examples show that Proposition 1.1 does not hold if $|F| \leq 3$ and $|M| = 1$. Let E be a field.

For every subset $X \subseteq \overline{E}$ with $|X| = 3$, the stabilizer $(\mathrm{PGL}_2 E)_X$ induces on X the symmetric group of X, and $((\mathrm{PGL}_2 E)_X, X)$ is permutation isomorphic to $(\mathrm{PSL}_2\mathbb{F}_2, \overline{\mathbb{F}_2})$, even if $\mathbb{F}_2$ is not a subfield of E.

If the multiplicative group $E^\times$ contains an element ζ of order three, then $\mathbb{F}_3$ is not a subfield of E. The two permutations $(x \mapsto \zeta x)$ and $(x \mapsto (1-x)/(1+2x))$ act on $X = \{0, 1, \zeta, \zeta^2\} \subseteq \overline{E}$ as a 3-cycle and as a double transposition, respectively. Hence the group generated by these two permutations induces on X the alternating group of X, which is permutation isomorphic to $(\mathrm{PSL}_2\mathbb{F}_3, \overline{\mathbb{F}_3})$.

2. Generalized hermitian unitals and their projectivities

Let $C|R$ be any quadratic (possibly inseparable) extension of fields; the classical example is $\mathbb{C}|\mathbb{R}$. We can write $C = R + \varepsilon R$, with $\varepsilon \in C \smallsetminus R$. There exist $t, d \in R$ such that $\varepsilon^2 - t\varepsilon + d = 0$, since $\varepsilon^2 \in R + \varepsilon R$. The mapping

$$\sigma\colon C \to C\colon x + \varepsilon y \mapsto (x + ty) - \varepsilon y \quad \text{for } x, y \in R$$

is a field automorphism which generates $\mathrm{Aut}_R C$: if $C|R$ is separable, then σ has order 2 and generates the Galois group of $C|R$; if $C|R$ is inseparable, then σ is the identity.

Now we introduce our geometric objects. We consider the pappian projective plane $\mathrm{PG}(2, C)$ arising from the 3-dimensional vector space C^3 over C, and we use homogeneous coordinates $[X, Y, Z] := (X, Y, Z)C$ for the points of $\mathrm{PG}(2, C)$.

Definition 2.1. The *generalized hermitian unital* $\mathcal{H}(C|R)$ is the incidence structure $(U, \mathcal{B})$ with the point set $U := \{[X, Y, Z] \mid \sigma(X)Y + \sigma(Z)Z \in \varepsilon R\}$, and the set $\mathcal{B}$ of *blocks* consists of the intersections of U with secant lines, i.e. lines of $\mathrm{PG}(2, C)$ containing more than one point of U.

Note that U is not empty: it contains $[1, 0, 0]$ and $[0, 1, 0]$. The condition $\sigma(X)Y + \sigma(Z)Z \in \varepsilon R$ is homogeneous, since $\sigma(c)c \in R$ for every $c \in C$.

Using terminology as in [13, 5.1C], one can regard εR as a form parameter, and $(C, \varepsilon R)$ as a form ring relative to id and 1.

In the next proposition, we identify $\mathcal{H}(C|R)$ in classical terms and motivate the name "generalized hermitian unital". The nucleus of a quadric is the projective subspace corresponding to the radical of the associated polar form.

Proposition 2.2. *If $C|R$ is separable, then $\mathcal{H}(C|R)$ is the hermitian unital arising from the skew-hermitian form $h\colon C^3 \times C^3 \to C$ defined by*

$$h\big((X, Y, Z), (X', Y', Z')\big) = \sigma(\varepsilon)\sigma(X)Y' - \varepsilon\sigma(Y)X' + (\sigma(\varepsilon) - \varepsilon)\sigma(Z)Z'\,.$$

In this case, the point set U is the image of

$$\{[X, Y, Z] \mid \sigma(Y)X + \sigma(X)Y + \sigma(Z)Z = 0\}$$

under some element of $\mathrm{PGL}_3 C$.

If $C|R$ is inseparable, then $\mathcal{H}(C|R)$ is the projection of an ordinary quadric Q in some projective space of dimension at least 3 *over C from a subspace of codimension* 1 *in the nucleus of Q.*

Proof. First assume that $C|R$ is separable. The set $C_{(\sigma)} = \{t-\sigma(t) \colon t \in C\}$ is a 1-dimensional subspace of C considered as vector space over R. Hence $\mathcal{H}(C|R)$ is the null set of the pseudo-quadratic form

$$C^3 \to C/C_{(\sigma)} \colon (X,Y,Z) \mapsto (1-\sigma(\varepsilon)\varepsilon^{-1})(\sigma(X)Y + \sigma(Z)Z) \mod C_{(\sigma)}.$$

The first assertion follows in this case from Chapter 10 of [4]. For the convenience of the reader, we include the following direct argument.

A point $[X,Y,Z]$ belongs to $\mathcal{H}(C|R)$ if and only if $\sigma(X)Y+\sigma(Z)Z = \varepsilon k$ for some $k \in R$. We apply σ and obtain $\sigma(Y)X+\sigma(Z)Z = \sigma(\varepsilon)k$. Eliminating k from these two equations, we obtain that $[X,Y,Z]$ belongs to $\mathcal{H}(C|R)$ if and only if

$$\sigma(\varepsilon)\sigma(X)Y - \varepsilon\sigma(Y)X + (\sigma(\varepsilon)-\varepsilon)\sigma(Z)Z = 0\,;$$

that is, if $h\left((X,Y,Z),(X,Y,Z)\right) = 0$. This gives the description via the skew-hermitian form h.

Using $\varepsilon - \sigma(\varepsilon) \neq 0$ we define a collineation of $\mathrm{PG}(2,C)$ by $[X,Y,Z] \mapsto [\frac{\varepsilon-\sigma(\varepsilon)}{\varepsilon}X,Y,Z]$. That collineation maps the set U to the set

$$\{[X,Y,Z] \,|\, \sigma(Y)X + \sigma(X)Y + \sigma(Z)Z = 0\}\,,$$

and we obtain the description by the hermitian form.

Now suppose that $C|R$ is inseparable. Then $\mathcal{H}(C|R)$ is the null set of the generalized pseudo-quadratic form $(X,Y,Z) \mapsto XY + Z^2 \mod \varepsilon R$, in the sense of Pasini [24], and the second assertion follows from [24]. Explicitly, since the field of squares in C is isomorphic to C and contained in R, we can consider R as a vector space over C, where scalars operate by $c \cdot r := c^2 r$. The quadratic form

$$q \colon C \times C \times C \times R \to C \colon (X,Y,Z,w) \mapsto XY + Z^2 + \varepsilon w$$

has the polar form $((X',Y',Z',u'),(X,Y,Z,u)) \mapsto X'Y + Y'X$, which is a degenerate alternating form with radical $\{(0,0)\} \times C \times R$. The condition $XY + Z^2 \in \varepsilon R$ is equivalent to the existence of $w \in R$ such that $q(X,Y,Z,w) = 0$. Thus $\mathcal{H}(C|R)$ is obtained by projection of the quadric Q defined by q along the subspace $\{(0,0,0)\} \times R$, which is a codimension 1 subspace of the radical of the polar form, hence it defines a projective subspace in the nucleus of Q of codimension 1. □

Proposition 2.3. *The isomorphism type of $\mathcal{H}(C|R)$ does not depend on the choice of $\varepsilon \in C \setminus R$. Hence it is determined uniquely by the extension $C|R$.*

(1) *If $\sigma \neq \mathrm{id}$ then the orthogonal space $p^\perp$ with respect to the skew-hermitian form h in 2.2 is the unique tangent through p (i.e. the unique line meeting U just in p), for any $p \in U$.*
(2) *If $\sigma = \mathrm{id}$ then for each point $p \in U$ the line $p + [0,0,1]$ is the unique tangent through p.*

Proof. If $\sigma \neq \mathrm{id}$ then the description by the hermitian form (not the skew-hermitian one) given in Proposition 2.2 shows uniqueness. See [15, Lemma II.2.47, p. 59] for the assertion about the tangents.

Now assume $\sigma = \mathrm{id}$. Consider elements $\gamma, \varepsilon \in C \setminus R$. Then there exist $u, v \in R$ such that $u + v\varepsilon = \varepsilon/\gamma$, and we find that $a := u\gamma/\varepsilon$ coincides with $\gamma v + 1$. If $XY + Z^2 = \varepsilon r$ with $r \in R$ then $aXY + Z^2 = a(XY + Z^2) + (a+1)Z^2 = \gamma u r + \gamma v Z^2$ belongs to γR. Conversely, from $(aX)Y + Z^2 = \gamma s$

with $s \in R$ we infer $XY + Z^2 = \gamma s/a + (1 + 1/a)Z^2 = \varepsilon(s + v)Z^2/u \in \varepsilon R$. So the linear transformation $(X, Y, Z) \mapsto (aX, Y, Z)$ induces a collineation of $\mathrm{PG}(2, C)$ mapping the generalized hermitian unital for ε onto that for γ.

We use the quadratic form $q\colon C^3 \to C\colon (X, Y, Z) \mapsto XY + Z^2$ and its polar form f. Let $p = vC$, and let wC be any point of $\mathrm{PG}(2, C)$ different from vC. The line $vC + wC$ contains another point of U if there exists $c \in C$ such that $(cv + w)C \in U$. This gives the condition $q(cv + w) = c^2q(v) + q(w) + cf(v, w) \in \varepsilon R$; note that $c^2q(v) \in \varepsilon R$ holds by assumption, so the condition actually is $q(w) + cf(v, w) \in \varepsilon R$.

If $vC + wC \leq p^\perp$ then we may assume $w = (0, 0, 1)$. The condition becomes $q(w) \in \varepsilon R$, and is not satisfied for any c. If $vC + wC \not\leq p^\perp$ then we may assume $f(v, w) = 1$, and the condition becomes $q(w) + c \in \varepsilon R$. This is satisfied by any $c \in q(w) + \varepsilon R$. □

So, if $C|R$ is separable, then $\mathcal{H}(C|R)$ is a classical Hermitian curve arising from a non-degenerate Hermitian form in a vector space of dimension 3 over the field C. The aim of the rest of this section is to extend some well-known properties of this curve to the inseparable case, to introduce the group of projectivities for every generalized Hermitian unital and to determine the structure of that group. The latter is new for the separable case, too.

For the convenience of the reader we here summarize, without precise definitions, the well-known properties of classical Hermitian unitals that we shall extend to the inseparable case (proper references to the literature shall be given below in each of the appropriate proofs).

Properties of Hermitian unitals—*Let $\mathcal{H}(C|R)$ be the classical Hermitian unital as defined above, with $C|R$ separable, and let σ be the corresponding Galois involution. Let B be any block of $\mathcal{H}(C|R)$. Then the following assertions hold.*

1^o. *$\mathcal{H}(C|R)$ does not contain O'Nan configurations.*

2^o. *The stabilizer $(\mathrm{PSL}_3C)_\mathcal{H}$ acts doubly transitively on $\mathcal{H}$, and is generated by translations. If $|R| > 2$ then this stabilizer is a simple group.*

3^o. *The block B is a Baer subline in $\mathrm{PG}(2, C)$ and the action of the block stabilizer $((\mathrm{PSL}_3C)_\mathcal{H})_B$ on B is permutation equivalent to $(\mathrm{PSL}_2^N(R), \overline{R})$, where $N = \{\sigma(z)z \mid 0 \neq z \in C\}$.*

We will also show that the group of projectivities of B (defined below) is permutation equivalent to the permutation group mentioned in 3^o above.

We start with analyzing $(\mathrm{PSL}_3C)_\mathcal{H}$.

Theorem 2.4. *The stabilizer $(\mathrm{PGL}_3C)_\mathcal{H}$ of $\mathcal{H} = \mathcal{H}(C|R) = (U, \mathcal{B})$ in PGL_3C is doubly transitive on the point set U, and thus transitive on the set $\mathcal{B}$ of blocks. If $|R| > 2$ then the same holds for the stabilizer $(\mathrm{PSL}_3C)_\mathcal{H}$.*

If $C|R$ is inseparable, we state more explicitly:

(1) *Via multiplication from the left on homogeneous coordinates, the matrices $M_{a,c} := \left(\begin{smallmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & 0 & 1 \end{smallmatrix}\right)$ with $a, c \in C$ and $a + c^2 \in \varepsilon R$ induce a subgroup of $(\mathrm{PSL}_3C)_\mathcal{H}$ which acts transitively on $U \setminus \{[0, 1, 0]\}$.*

(2) *The matrices $W_{a,c} := \left(\begin{smallmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{smallmatrix}\right)$ with $a, c \in C$ and $a + c^2 \in \varepsilon R$ induce a subgroup of $(\mathrm{PSL}_3C)_\mathcal{H}$ which acts transitively on $U \setminus \{[1, 0, 0]\}$.*

(3) *The group generated by* $\{M_{a,c} \mid a+c^2 \in \varepsilon R\} \cup \{W_{a,c} \mid a+c^2 \in \varepsilon R\}$ *induces a subgroup of* $(\mathrm{PSL}_3 C)_{\mathcal{H}}$ *which acts two-transitively on* U.

(4) *The group* $\left\{ \left(\begin{smallmatrix} a^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{smallmatrix}\right) \,\middle|\, a \in C \smallsetminus \{0\} \right\}$ *stabilizes* U*, and induces a subgroup of* $(\mathrm{PGL}_3 C)_{\mathcal{H}}$.

Proof. If $C|R$ is separable, then the stabilizer $(\mathrm{PGL}_3 C)_{\mathcal{H}}$ contains the projective unitary group $\mathrm{PU}_3(C|R)$, which is doubly transitive on U by Witt's theorem, see [31, 7.4] or [10, 10.12]; if $|R| > 2$, then the same holds for $\mathrm{PSU}_3(C|R) \le (\mathrm{PSL}_3 C)_{\mathcal{H}}$, see [31, 10.12] or [10, 11.8, 11.11].

Now assume that $C|R$ is inseparable. Consider $a, c \in C$ with $a + c^2 \in \varepsilon R$. For $(X, Y, Z) \in C^3$ we have $X(aX+Y)+(cX+Z)^2 = XY+Z^2+(a+c^2)X^2 \in XY + Z^2 + \varepsilon R$. So multiplication by $M_{a,c}$ leaves U invariant, and induces an automorphism of $\mathcal{H}(C|R)$. Analogously, multiplication by $W_{a,b}$ induces an automorphism of $\mathcal{H}(C|R)$. The orbits $\{[1, a, c] \mid a, c \in C, a + c^2 \in \varepsilon R\}$ of $[1, 0, 0]$ and $\{[a, 1, c] \mid a, c \in C, a + c^2 \in \varepsilon R\}$ of $[0, 1, 0]$ coincide with $U \smallsetminus \{[0, 1, 0]\}$ and $U \smallsetminus \{[1, 0, 0]\}$, respectively. This proves assertions 1 and 2, and assertion 3 follows.

Now let $a \in C \smallsetminus \{0\}$ be arbitrary. Then $(a^2X)Y+(aZ)^2 = a^2(XY+Z^2) \in R(XY + Z^2)$ yields that U is invariant under the group in assertion 4. □

Remark 2.5. If $C|R$ is inseparable then the existence of c with $a+c^2 \in \varepsilon R$ imposes a restriction on a because $\{c^2 + \varepsilon r \mid c \in C, r \in R\}$ is a proper subset of $R + \varepsilon R$, in general. If $c \in C$ exists such that $c^2 \in a + \varepsilon R$ then c is determined by a because R and $a+\varepsilon R$ have at most one element in common.

The center of the elation induced by $M_{a,c}$ is $[0, a, c]$. That point lies in U precisely if $c^2 \in \varepsilon R$. As each square is in R, we find $c = 0$ and $a \in \varepsilon R$. In the inseparable case, it is therefore no longer true that each elation in the stabilizer of U induces a translation of $\mathcal{H}$ (in the sense of 2.10).

Remark 2.6. Assume that $C|R$ is inseparable. Then the group

$$G := \left\langle M_{a,c}, W_{a,c} \;\middle|\; a, c \in C, a + c^2 \in \varepsilon R \right\rangle$$

generated by the matrices in 2.4.3 is isomorphic to the little projective group of the Moufang set of the polar line $\mathcal{MPL}(C, R, \mathrm{id})$ in the sense of [6, Section 2.4]; we use the isotopic set $(C, \varepsilon R, \mathrm{id})$ which is isotopic to the isotopic set (C, R, id) used in [6, Section 2.4].

In fact, the linear group G acts on the quotient $C^3/[0, 0, 1]$, and that action is faithful (see the proof of 2.15 below). Under this action the generators of G are represented by the members of $\left\{ \left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \mid a \in S + \varepsilon R \right\}$, where $S := \{c^2 \mid c \in C\}$, and the action yields an isomorphism from G onto the little projective group of $\mathcal{MPL}(C, R, \mathrm{id})$.

Proposition 2.7. *The generalized hermitian unital* $\mathcal{H}(C|R)$ *does not contain any O'Nan configurations.*

Proof. For the separable (in particular, the finite) case, this result is well known; see [11, 2.2] (cf. [22, 3.11] for the finite case). So we treat the inseparable case only.

Aiming at a contradiction, we consider an O'Nan configuration in $\mathcal{H}(C|R)$. Using 2.4.3 and 2.4.4 we may assume that $[1, 0, 0]$ and $[0, 1, 0]$ are points of the configuration, and that $[1,0,0]+[0,1,1]$ induces a block of the configuration (passing through $[1,0,0]$).

The two further points on that block are then of the form $[1,x,x]$ and $[1,y,y]$ with $0 \neq x \neq y \neq 0$ and $x+x^2$, $y+y^2$ both in εR. The second block through $[1,0,0]$ is induced by $[1,0,0]+[0,u,v]$ with $uv \neq 0$ and $u \neq v$. We compute the missing two points as $[v,ux,vx]$ and $[v,uy,vy]$, respectively.

These two points lie in U, so $vux+v^2x^2$ and $vuy+v^2y^2$ belong to εR. Subtracting $v^2(x+x^2)$ or $v^2(y+y^2)$ we find $v(u+v)x$ and $v(u+v)y$ both in εR, so $y = rx$ with $r \in R \setminus \{0\}$. Now εR contains $(y+y^2)-r(x+x^2) = (r^2-r)x^2 \in R$, and $r^2-r=0$ follows. This is a contradiction. $\square$

Recall that a *Baer subplane* in a projective plane $\mathcal{P}$ is a proper subplane $\mathcal{S}$ of $\mathcal{P}$ such that every line of $\mathcal{P}$ contains at least one point of $\mathcal{S}$, and every point of $\mathcal{P}$ is on at least one line of $\mathcal{S}$. A *Baer subline* of $\mathcal{P}$ is a set of at least two points that is obtained as the intersection of a line with a Baer subplane of $\mathcal{P}$.

Lemma 2.8. *The line* $[1,0,0]+[0,1,0] = C^2 \times \{0\}$ *of* $\mathrm{PG}(2,C)$ *induces the block* $B = \{[\varepsilon x, y, 0] \mid (0,0) \neq (x,y) \in R^2\}$ *of* $\mathcal{H}(C|R)$*, and each block of* $\mathcal{H}(C|R)$ *is a Baer subline of* $\mathrm{PG}(2,C)$.

Proof. A point $[\varepsilon x, y, 0]$ belongs to $\mathcal{H} = \mathcal{H}(C|R)$ precisely if $\varepsilon xy \in \varepsilon R$. If $y \neq 0$, we can choose it in R and then also $x \in R$. Similarly if $x \neq 0$. This yields the formula for B.

The block B is the line of the Baer subplane Σ given by restricting the coordinates to $\varepsilon R \times R \times R$. This is indeed a Baer subplane as is well known in the separable case (a Baer involution is then $[X,Y,Z] \mapsto [\sigma(X)\varepsilon, \sigma(Y)\sigma(\varepsilon), \sigma(Z)\sigma(\varepsilon)]$). In the inseparable case, consider the equation $aX+bY+cZ=0$ of a line in $\mathrm{PG}(2,C)$. The three elements $a\varepsilon$, b, c are linearly dependent in the two-dimensional vector space C over R. Therefore, there exists $(x,y,z) \in R^3 \setminus \{(0,0,0)\}$ with $(a\varepsilon)x+by+cz=0$, and $[\varepsilon x,y,z]$ is a point on the line and in the Baer subplane. Dually, the point $[a,b,c]$ lies on the line defined by $(\varepsilon x)X + yY + zZ = 0$, with the same choice of $x,y,z \in R$. Every other block is the image of B under some element from $(\mathrm{PGL}_3 C)_{\mathcal{H}}$ by Theorem 2.4. $\square$

Definition 2.9. Let B and B' be blocks of a generalized hermitian unital $\mathcal{H}(C|R)$. If $c \notin B \cup B'$ is a point such that every block joining c to a point $x \in B$ meets B' in a point x', and every block joining c to a point of B' meets B, then the bijection $B \to B' : x \mapsto x'$ is a *perspectivity* in $\mathcal{H}(C|R)$ with center c from B onto B'. A *projectivity* in $\mathcal{H}(C|R)$ is any concatenation of perspectivities. The group of all projectivities of B onto itself will be denoted by Π_B.

In general, there is no perspectivity between two given blocks in a generalized hermitian unital. We construct projectivities as restrictions of automorphisms of $\mathcal{H}(C|R)$. Translations of generalized hermitian unitals play a crucial role here.

Definition 2.10. Let c be a point of the generalized hermitian unital $\mathcal{H}(C|R)$. A *translation* τ of $\mathcal{H}(C|R)$ with center c is an automorphism of $\mathcal{H}(C|R)$ leaving invariant each line through the point c. If τ is not the identity, then c is uniquely determined (by 2.7), and we call it the *center* of τ.

Lemma 2.11. *For $a \in \varepsilon R$, the matrices $M_{a,0}$ and $W_{a,0}$ from 2.4.1 and 2.4.2 induce translations of $\mathcal{H}(C|R)$, with centers $[0,1,0]$ and $[1,0,0]$, respectively.*

These translations generate a subgroup isomorphic to SL_2R in the block stabilizer, and the induced action on the block is permutation equivalent to the standard action of PSL_2R on the projective line $\overline{R} = R \cup \{\infty\}$.

Proof. From 2.4 and 2.5 we know that an elation of $\mathrm{PG}(2,C)$ induces a translation of $\mathcal{H}$ with center $[1,0,0]$ if, and only if, it is induced by $W_{r\varepsilon,0}$ with $r \in R$. Analogously, translations with center $[0,1,0]$ are induced by $M_{r/\varepsilon,0}$ with $r \in R$. The block determined by the line $[1,0,0]+[0,1,0]$ is $B = \{[1,0,0]\} \cup \{[\varepsilon x,1,0] \,|\, x \in R\}$; see 2.8. On this set, the matrices $W_{r\varepsilon,0}$ and $M_{r/\varepsilon,0}$ act as usual (by $x \mapsto x/(1+rx)$ and $x \mapsto x+r$, respectively). This induces a faithful representation as PSL_2R which is permutation equivalent to the natural one.

The group generated by all translations with center $[1,0,0]$ or $[0,1,0]$ contains all translations with centers on that block B, and coincides with the group just discussed. □

Remark 2.12. Lemma 2.11 yields, in particular, that for each point c and each block B through c the group of all translations with center c acts transitively on $B \setminus \{c\}$. For finite unitals, this transitivity property characterizes the hermitian unitals (see [12]). This further justifies the name "generalized hermitian unital" for $\mathcal{H}(C|R)$ also in the inseparable case.

Lemma 2.13. *Let $C|R$ be a quadratic extension of fields, let B be a block of the generalized hermitian unital $\mathcal{H} = \mathcal{H}(C|R)$ and let $c \in B$.*

(1) *Every translation of $\mathcal{H}$ with center c extends to an elation of $\mathrm{PG}(2,C)$ with center c and axis c', the tangent line at c. The group of all translations of $\mathcal{H}$ with center c acts sharply transitively on $B \setminus \{c\}$.*

(2) *Every product of translations of $\mathcal{H}$ can be written as a product $\rho_k \circ \rho_{k-1} \circ \cdots \circ \rho_1$ of translations $\rho_i \neq \mathrm{id}$ of $\mathcal{H}$ such that the center of ρ_i is not on the block $(\rho_{i-1} \circ \cdots \circ \rho_1)(B)$ for $2 \le i \le k$.*

Proof. (1) By the double transitivity of $(\mathrm{PGL}_3)_{\mathcal{H}}$, see Theorem 2.4, it suffices to consider the block B in 2.8 and the point $c = [1,0,0]$; then $c' = [1,0,0]+[0,0,1]$. For every $r \in R$, the matrix $W_{r\varepsilon,0}$ induces a (c,c')-elation η_r that leaves U invariant, and induces a translation of $\mathcal{H}$ with center c; see Example 2.11. Clearly, the group $\{\eta_r \,|\, r \in R\}$ acts sharply transitively on $B \setminus \{c\} = \{[\varepsilon r,1,0] \,|\, r \in R\}$.

If $\mathcal{H}$ would admit any other translation with center c, we could find a non-trivial translation τ with center c fixing a point $p \in B \setminus \{c\}$. If τ fixes each block through p then every point $x \in U \setminus B$ is fixed by τ, and then also every point on B is fixed. So there is a point $x \in U \setminus B$ such that $\tau(x)$ is not on the block joining x to p. For any third point y on that block, the set $\{c,p,x,y,\tau(x),\tau(y)\}$ forms an O'Nan configuration in $\mathcal{H}$, contradicting Proposition 2.7.

(2) Let ρ be a translation with center c on the block B. For the following construction see Figure 1. Choose an arbitrary point $d \notin B$, and let τ be the

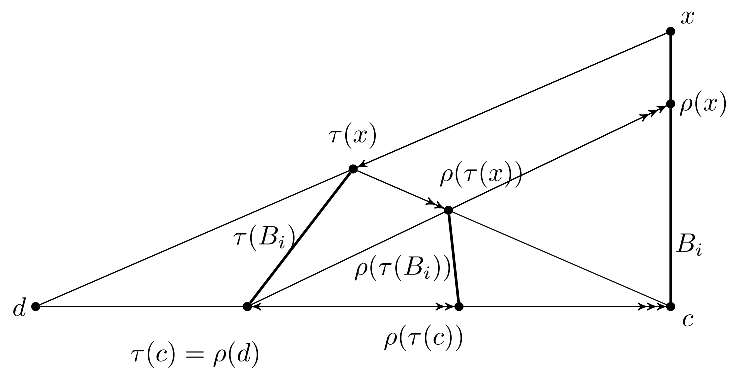

FIGURE 1. Constructing translations

(non-trivial) translation with center d and such that $\tau(c) = \rho(d)$; see (1). Now $d \notin B$ by our choice, and $c \notin \tau(B)$ because $c \in B$ and τ is a non-trivial translation with center d outside B. The conjugate $\rho \circ \tau^{-1} \circ \rho^{-1}$ is a translation with center $\rho(d)$ not on $\rho(\tau(B))$ because $d \notin \tau(B)$. We can thus replace $\rho = (\rho \circ \tau^{-1} \circ \rho^{-1}) \circ \rho \circ \tau$ by the product of three translations that induce perspectivities from B to $\tau(B)$, from $\tau(B)$ to $\rho(\tau(B))$ and from there to B. Repeated replacements of this type yield a product representation as required. $\square$

Proposition 2.14. *Let $C|R$ be a quadratic extension of fields. The action of the stabilizer of a block of $\mathcal{H} = \mathcal{H}(C|R)$ in $(\mathrm{PSL}_3C)_{\mathcal{H}}$ is permutation equivalent to $(\mathrm{PSL}_2^N R, \overline{R})$, where $N := \{\sigma(z)z \mid 0 \neq z \in C\}$ is the norm group of $C|R$.*

Proof. By the double transitivity of $\mathrm{Aut}\,\mathcal{H}$, see Theorem 2.4, it suffices to consider the block B induced by $[1, 0, 0] + [0, 1, 0]$. We have already seen in 2.11 that the groups of translations of $\mathcal{H}$ with center $p = [1, 0, 0]$ or $q = [0, 1, 0]$ are induced by the matrices $W_{r\varepsilon,0}$ or $M_{r/\varepsilon,0}$, respectively, with $r \in R$, and that the action of the group generated by these two groups of translations is permutation equivalent to the action of PSL_2R on $\overline{R}$.

In order to determine the full stabilizer of B in $(\mathrm{PGL}_3C)_{\mathcal{H}}$, it remains to study the stabilizer of the points p and q, and its action on B. Consider $M \in (\mathrm{GL}_3C)_{\mathcal{H}}$ and assume that M fixes both p and q. Then M also fixes the tangents $p+[0, 0, 1]$ and $q+[0, 0, 1]$ (see 2.3), and then the point $[0, 0, 1]$. So M is a diagonal matrix, and we may assume that $M = \mathrm{diag}(a, 1, c)$ with $a, c \in C \setminus \{0\}$. So $[X, Y, Z]$ is mapped to $[aX, Y, cZ]$.

Evaluating the condition $M(U) = U$ at points $[X, Y, 0] \in U$, we see that $\sigma(a) \in R$. For $[X, Y, 1] \in U$ we then obtain the condition $a\sigma(X)Y + \sigma(c)c \in \varepsilon R \iff \sigma(X)Y+1 \in \varepsilon R$ which yields $-a+\sigma(c)c \in R\cap\varepsilon R$. The intersection is trivial, and $M = \mathrm{diag}(\sigma(c)c, 1, c)$. Conversely, every such M stabilizes U.

The matrix $\mathrm{diag}(\sigma(c)c, 1, c)$ induces on the block B the map $[X, Y, 0] \mapsto [\sigma(c)cX, Y, 0]$. Now $\mathrm{diag}(\varepsilon, 1, 1)$ induces a collineation of PG_2C that maps B to $\{[x, y, 0] \mid (x, y) \in R^2 \setminus \{(0, 0)\}\}$, while conjugation with that matrix maps $M_{r/\varepsilon,0}$ and $W_{r\varepsilon,0}$ to $M_{r,0}$ and $W_{r,0}$, respectively, and leaves $\mathrm{diag}(\sigma(c)c, 1, 1)$

fixed. Restriction to B now gives the equivalence of permutation groups, as claimed. $\square$

Proposition 2.15. *Let $C|R$ be a quadratic extension of fields with $|R| > 2$. Then the translations of $\mathcal{H} = \mathcal{H}(C|R)$ generate the group $(\mathrm{PSL}_3C)_{\mathcal{H}}$, and that group is simple.*

Proof. If $C|R$ is separable, this follows from [10, Theorem 11.15] as $|R| > 2$. So we consider the inseparable case.

We first show that $(\mathrm{PSL}_3C)_{\mathcal{H}}$ acts faithfully on the quotient $C^3/[0,0,1]$. Indeed, the point $[0,0,1]$ is fixed because it lies on every tangent (cf. 2.3), and an element of $(\mathrm{PSL}_3C)_{\mathcal{H}}$ acts trivially on the quotient if, and only if, it is induced by a matrix of the form $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & v & 1 \end{smallmatrix}\right)$ with $u, v \in C$ such that $XY + Z^2 \in \varepsilon R \iff XY + (uX + vY + Z)^2 \in \varepsilon R$. Evaluating the latter condition for points with $Z = 0$, we obtain $(uX + vY)^2 \in R \cap \varepsilon R$ whenever $XY \in \varepsilon R$. This implies $u = 0 = v$.

We now claim that the stabilizer $(\mathrm{PSL}_3C)_{\mathcal{H}}$ coincides with the subgroup G generated by the set $\{M_{a,c} \mid a + c^2 \in \varepsilon R\} \cup \{W_{a,c} \mid a + c^2 \in \varepsilon R\}$ from 2.4. Since both $(\mathrm{PSL}_3C)_{\mathcal{H}}$ and its subgroup G are doubly transitive on the points of $\mathcal{H}$, it suffices to prove that the stabilizers $G_{p,q}$ and $(\mathrm{PSL}_3C)_{\mathcal{H},p,q}$ of the two points $p = [1,0,0]$ and $q = [0,1,0]$ coincide. Consider an element $g \in (\mathrm{PSL}_3C)_{\mathcal{H},p,q}$; this is induced by a diagonal matrix of the form $D_b := \mathrm{diag}(b, 1/b, 0)$ with no restriction on $b \in C \setminus \{0\}$.

On the quotient $C^3/[0,0,1]$, the matrices $M_{a,c}$, $W_{a,c}$ and D_b act as $\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1} \end{smallmatrix}\right)$, respectively. Invariance of $\mathcal{H}$ means the restriction $a \in \{x^2 + \varepsilon r \mid x \in C, r \in R\}$. The latter set is closed under taking additive and multiplicative inverses, but not under multiplication (unless every element of R is a square in C). So the group induced by G on the quotient contains $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & a^{-1} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ -a & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & a^{-1} \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ for each $a \in \{x^2 + \varepsilon r \mid x \in C, r \in R\}$. It remains to show that the set $\{x^2 + \varepsilon r \mid x \in C, r \in R\}$ multiplicatively generates $C \setminus \{0\}$. Let $r + \varepsilon s$ with $r, s \in R$ be an arbitrary nonzero element of C. If $s = 0$, then we can write $r = (\varepsilon r)(\varepsilon\varepsilon^{-2})$, noting that $\varepsilon^2 \in R$. If $s \neq 0$, then $r + \varepsilon s = (\varepsilon s)(1 + \varepsilon(rs^{-1}\varepsilon^{-2}))$.

This shows that $G = (\mathrm{PSL}_3C)_{\mathcal{H}}$. We now show that G is perfect. Indeed, one verifies (in the faithful representation on the quotient) that $\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} b^{-1} & 0 \\ 0 & b \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} b^{-1} & 0 \\ 0 & b \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ (b^2+1)a & 1 \end{smallmatrix}\right)$ which implies that the derived group of G contains the generators. So G is perfect, acts primitively (in fact, two-transitively) on U, and is generated by the conjugates of a normal abelian subgroup of the stabilizer of p (namely, the group induced by the matrices $W_{r\varepsilon,0}$). Iwasawa's criterion therefore yields that $G = (\mathrm{PSL}_3C)_{\mathcal{H}}$ is simple.

The translations generate a normal subgroup of that simple group, hence they generate $(\mathrm{PSL}_3C)_{\mathcal{H}}$. $\square$

We remark that, in the inseparable case, the group $(\mathrm{PSL}_3C)_{\mathcal{H}}$ is isomorphic to a proper subgroup of PSL_2C but not to PSL_2C itself, in general, cf. 2.5.

Theorem 2.16. *Let $C|R$ be a quadratic extension of fields and let B be a block of the generalized hermitian unital $\mathcal{H}(C|R)$.*

1. *The group Π_B of all projectivities in $\mathcal{H}(C|R)$ of B onto itself is permutation isomorphic to the permutation group $(\mathrm{PSL}_2^N R, \overline{R})$, where $N := \{\sigma(z)z \,|\, 0 \neq z \in C\}$ is the norm group of $C|R$.*
2. *If $|R| > 2$ then $\mathcal{H}(C|R)$ admits projectivities of B onto every other block.*

Proof. First we prove (1) if $R = \mathbb{F}_2$. Then $C = \mathbb{F}_4$, $N = \{1\}$ and $\mathcal{H}(\mathbb{F}_4|\mathbb{F}_2)$ is isomorphic to the affine plane $\mathrm{AG}(2, \mathbb{F}_3)$ over $\mathbb{F}_3$. This hermitian unital admits perspectivities only between blocks that form parallel lines in the affine plane. There are projectivities (in fact, products of three perspectivities) that act as transpositions on B, and Π_B is isomorphic to the symmetric group on three symbols, i.e. to $\mathrm{PGL}_2\mathbb{F}_2 = \mathrm{PSL}_2^N\mathbb{F}_2$.

Now let $|R| > 2$ and let G be the group of collineations of $\mathcal{H} = \mathcal{H}(C|R)$ generated by all translations. We have $G = (\mathrm{PSL}_3 C)_{\mathcal{H}}$ by Proposition 2.15. Hence G is transitive on the set of blocks of $\mathcal{H}$ by Theorem 2.4. By Lemma 2.13 (2) the restriction of every element $\gamma \in G$ to B is a projectivity of B onto $\gamma(B)$, whence assertion (2) holds.

Let G_B be the stabilizer of B in G. The restriction of every element of G_B to B belongs to Π_B. We show that Π_B, as a permutation group on B, coincides with the action of G_B on B (modulo the kernel of that action).

Every $\pi \in \Pi_B$ is the product of perspectivities $\rho\colon B_1 \to B_2$ between blocks B_1 and B_2. Let c be the center of ρ, let c' be the tangent to $\mathcal{H}$ at c and denote by η the unique elation η of $\mathrm{PG}(2, C)$ with center c and axis c' mapping some point $b \in B_1$ to $\rho(b) \in B_2$. Then the absence of O'Nan configurations in $\mathcal{H}$ (see 2.7) forces that $\eta(B_1) = B_2$, hence ρ is the restriction of η to B_1. Moreover, η induces a translation of $\mathcal{H}$ by Lemma 2.13 (1), hence the restriction of η to $\mathcal{H}$ belongs to G. Thus π is the restriction to B of some element of G_B.

Now assertion (1) follows from Proposition 2.14. □

Corollary 2.17. *For every block B of the finite hermitian unital $\mathcal{H}(\mathbb{F}_{q^2}|\mathbb{F}_q)$, the group Π_B of all projectivities of B onto itself is permutation isomorphic to $(\mathrm{PGL}_2\mathbb{F}_q, \overline{\mathbb{F}_q})$.*

Proof. The norm $\mathbb{F}_{q^2} \to \mathbb{F}_q : c \mapsto \sigma(c)c = c^{1+q}$ is surjective, hence $N = \mathbb{F}_q^\times$ and $\mathrm{PSL}_2^N\mathbb{F}_q = \mathrm{PGL}_2\mathbb{F}_q$. □

Remark 2.18. The assumption $|R| > 2$ in 2.16 (2) excludes the smallest hermitian unital $\mathcal{H}(\mathbb{F}_4|\mathbb{F}_2)$, which is isomorphic to the affine plane $\mathrm{AG}(2, \mathbb{F}_3)$ over $\mathbb{F}_3$. The translations of $\mathcal{H}(\mathbb{F}_4|\mathbb{F}_2) \cong \mathrm{AG}(2, \mathbb{F}_3)$ are point reflections in the affine plane. They generate a group of order 18 (isomorphic to $\mathbb{F}_3^2 \rtimes \mathrm{C}_2$) which has four orbits on the set of blocks, namely the parallel classes of the affine plane.

Embeddings of $\mathrm{AG}(2, \mathbb{F}_3)$ into $\mathrm{PG}(2, \mathbb{C})$ are well known: the nine points of $\mathrm{AG}(2, \mathbb{F}_3)$ are the inflection points of a nonsingular cubic; see [7, Thm. 2, Thm. 3]. In fact, $\mathrm{AG}(2, \mathbb{F}_3)$ embeds into pappian projective planes of arbitrary characteristic, and all embeddings of $\mathrm{AG}(2, \mathbb{F}_3)$ into desarguesian projective planes are known; see [23, Thm. 2], [27], [3], [25], [16].

3. Embeddings of certain affine quadrangles

Our incidence structures have no repeated lines (or blocks), hence we can consider the lines as subsets of the point set. An *embedding* of such an incidence structure $(\mathcal{P}, \mathcal{L})$ into a second one, say $(\mathcal{P}', \mathcal{L}')$, is an injective mapping $\theta \colon \mathcal{P} \to \mathcal{P}'$ such that for every line $L \in \mathcal{L}$ the image $\theta(L)$ is contained in some line from $\mathcal{L}'$. The embedding θ is called *full* if $\theta(L) \in \mathcal{L}'$ for every $L \in \mathcal{L}$.

Let F be a field and let $\mathrm{Q}(4, F)$ be the orthogonal generalized quadrangle defined by a non-degenerate quadratic form of (maximal) Witt index 2 on the vector space F^5. Then $\mathrm{Q}(4, F)$ has, by definition, a full embedding into the 4-dimensional projective space $\mathrm{PG}(4, F)$. Let H be a hyperplane of $\mathrm{PG}(4, F)$ intersecting $\mathrm{Q}(4, F)$ in a non-degenerate ruled quadric, and denote by

$$\mathrm{AQ}(4, F) := \mathrm{Q}(4, F) \setminus H$$

the incidence structure obtained from $\mathrm{Q}(4, F)$ by removing all points and lines that are contained in H. Then $\mathrm{AQ}(4, F)$ is an *affine quadrangle* fully embedded into the affine space $\mathrm{AG}(4, F)$. We call this embedding the *standard embedding* of $\mathrm{AQ}(4, F)$.

The following properties of $\mathrm{Q}(4, F)$ are best proved by considering its standard embedding in $\mathrm{PG}(4, F)$.

Fact 3.1. (1) Let L, M be two non-intersecting lines of $\mathrm{Q}(4, F)$. *Then the subspace of* $\mathrm{PG}(4, F)$ *generated by* L *and* M *is a* 3*-space and intersects* $\mathrm{Q}(4, F)$ *in a full non-thick subquadrangle* Q *of* $Q(4, F)$*,* i.e. all points of $Q(4, F)$ on a line of Q belong to Q *(full)* and each point of Q is on exactly two lines of Q *(non-thick).*

(2) *Such a subquadrangle contains two classes of lines: two lines belong to different classes if and only if they intersect nontrivially.* We call each class a *regulus.* Such a regulus is in fact one family of generators of a hyperbolic quadric in 3-space, hence determined by any three of its elements. However, *if we insist that the regulus is contained in* $\mathrm{Q}(4, F)$*, then it is determined by any two of its elements, since the intersection of the* 3*-space generated by these two elements with the generalized quadrangle* $\mathrm{Q}(4, F)$ *is exactly that ruled quadric.*

(3) *Two distinct full non-thick subquadrangles that have at least one line in common have exactly two lines in common, namely one of each regulus of either subquadrangle.* This follows readily from the fact that two distinct 3-spaces in $\mathrm{PG}(4, F)$ intersect in a plane, and each plane of a projective 3-space containing at least one line of a ruled nondegenerate quadric Q^* is a tangent plane and hence contains exactly two lines of Q^*. □

Definition 3.2. If N and N' are two lines of a regulus of $\mathrm{Q}(4, F)$, then that regulus is determined uniquely, and we denote it by $\mathcal{R}(N, N')$.

Theorem 3.3. *Let* F *be field with* $|F| > 2$*. Then up to collineations from* $\mathrm{A\Gamma L}_4 F$*, the standard embedding of* $\mathrm{AQ}(4, F)$ *is the only full embedding of* $\mathrm{AQ}(4, F)$ *into the affine space* $\mathrm{AG}(4, F)$ *with the property that any two lines of* $\mathrm{AQ}(4, F)$ *which do not intersect in* $\mathrm{Q}(4, F)$ *are not parallel in* $\mathrm{AG}(4, F)$*.*

Proof. It is convenient to write $\mathrm{AQ}(4,F) = (\mathcal{P},\mathcal{L})$; here $\mathcal{L}$ is a subset of the line set of $\mathrm{Q}(4,F)$. Let θ be a full embedding of $\mathrm{AQ}(4,F)$ into $\mathrm{AG}(4,F)$, as defined above. Then θ induces an injective mapping of $\mathcal{L}$ into the line set of $\mathrm{AG}(4,F)$.

For $L \in \mathcal{L}$, we denote by ∞_L the point of $\mathrm{Q}(4,F)$ incident with L but not belonging to $\mathrm{AQ}(4,F)$, and we set $\overline{L} = L \cup \{\infty_L\}$. We also denote by $\infty_{\theta(L)}$ the unique point on $\theta(L)$ belonging to the projective completion $\mathrm{PG}(4,F)$ of $\mathrm{AG}(4,F)$, but not to $\mathrm{AG}(4,F)$ itself. We assume for $L,M \in \mathcal{L}$ with $\overline{L} \cup \overline{M} = \emptyset$ that $\theta(L)$ is not parallel to $\theta(M)$ in $\mathrm{AG}(4,F)$, and we refer to this assumption as Assumption $(*)$.

Our basic aim is to show that, if $L,M \in \mathcal{L}$ with $\infty_L = \infty_M$, then $\theta(L)$ is parallel with $\theta(M)$. This enables us to define $\theta(\infty_L)$ as the unique point of the projective completion $\mathrm{PG}(4,F)$ of $\mathrm{AG}(4,F)$ corresponding with the direction of $\theta(L)$. Then we also show that, if $L,M,N \in \mathcal{L}$ with ∞_L, ∞_M and ∞_N collinear in $Q(4,F)$, then $\theta(\infty_L)$, $\theta(\infty_M)$ and $\theta(\infty_N)$ are also collinear.

The set $\{\infty_L \mid L \in \mathcal{L}\}$ is the point set of a full non-thick subquadrangle Q of $\mathrm{Q}(4,F)$.

Let $L,M \in \mathcal{L}$ with $\infty_L = \infty_M \in Q$, and suppose that $\theta(L)$ is not parallel with $\theta(M)$, i.e. $\infty_{\theta(L)} \neq \infty_{\theta(M)}$. Let X be any of the two lines of Q incident with ∞_L. Let K be any line of $\mathrm{AQ}(4,F)$ intersecting X within $\mathrm{Q}(4,F)$, and with $\infty_K \neq \infty_L$. Then L and K are contained in a unique non-thick full subquadrangle $Q_{L,K}$ of $\mathrm{Q}(4,F)$. By 3.1, the subquadrangle $Q_{L,K}$ has precisely two lines in common with Q, namely X and some other line Y. By Assumption $(*)$, the image under θ of the lines of $Q_{L,K}$ belonging to $\mathrm{AQ}(4,F)$ form a grid, so they constitute the affine part of a hyperbolic quadric intersecting the 3-space $\mathrm{PG}(3,F)_\infty$ at infinity in two lines. One of the latter is the line A_L defined by $\infty_{\theta(L)}$ and $\infty_{\theta(K)}$. That line also contains $\infty_{\theta(Z)}$ for each line Z belonging to the regulus $\mathcal{R}(L,K)$. There is a unique point ∞_{A_L} of A_L not of the form $\infty_{\theta(Z)}$, with $Z \in \mathcal{R}(L,K)$. Interchanging the roles of L and M, we can similarly define A_M. We first claim that $A_L \neq A_M$. Indeed, if not, by Assumption $(*)$, $A_L = A_M$ is equivalent with $\infty_{\theta(M)} = \infty_{A_L}$. Now, in $Q_{M,K}$ (similarly defined as $Q(L,K)$), there is a unique line Z' concurrent with X and Y. Assumption $(*)$ yields that $\infty_{\theta(Z')} \neq \infty_{\theta(M)}$, hence $\infty_{\theta(Z')} = \infty_{\theta(Z)}$ for some $Z \in \mathcal{R}(L,K)$. As Z' does not intersect any such line, this contradicts Assumption $(*)$. Our claim follows.

Hence $A_L \neq A_M$. But since these lines intersect in $\infty_{\theta(K)}$, they span a plane π_X entirely contained in $\mathrm{PG}(3,F)_\infty$. Now we consider the regulus $\mathcal{R}(Z',Y')$ where Y' is the unique line of Q distinct from X and incident with ∞_L. This regulus intersects the regulus $\mathcal{R}(L,K)$ in some line Z. Hence the line $A_{Y'}$ spanned by $\infty_{\theta(Z')}$ and $\infty_{\theta(Z)}$ contains a unique point $\infty_{A'_Y}$ that is not of the form $\infty_{\theta(U)}$, for some $U \in \mathcal{R}(Z,Z') \setminus \{Y'\}$. Now we see that each line A_U spanned by $\infty_{\theta(L)}$ and $\infty_{\theta(U)}$, for U ranging over $\mathcal{R}(Z,Z') \setminus \{Y'\}$, contains a unique point ∞_{A_U} that is not of the form $\infty_{\theta(W)}$, for some line W of $AQ(4,F)$ intersecting X, but not incident with ∞_L. It now easily follows that all the points ∞_{A_U} are contained in a line L_X of π_X. Moreover, since we can interchange the roles of L and K, we see that all points $\infty_{\theta(N)}$, for N ranging over the set of lines of $\mathrm{AQ}(4,F)$ incident with ∞_K, are contained in a unique line L_K, of which exactly two points are not of the form $\infty_{\theta(N)}$. One point is the intersection with L_X, the other point, P_π, is the intersection with the line L_M spanned by $\infty_{\theta(L)}$ and $\infty_{\theta(M)}$. It follows that all points of L_M but P_π and $\infty_{A_{Y'}}$ are of the form $\infty_{\theta(V)}$, for V ranging over all lines of $\mathrm{AQ}(4,F)$ incident with ∞_L.

We have shown that the mapping $\rho_X\colon N \mapsto \infty_{\theta(N)}$ is a bijection from the set of lines of $\mathrm{AQ}(4,F)$ concurrent with X in $Q(4,F)$ to the set of points of π_X distinct from P_π and not on L_X. Moreover, reguli correspond under ρ_X to lines not through P_π, and line pencils correspond to lines through P_π.

We now can similarly define π_Y and $\pi_{Y'}$, which intersect π_X in respective lines through P_π. The planes π_Y and $\pi_{Y'}$ necessarily intersect in their common line $L_Y = L_{Y'}$. Varying X over Q, we see that the mapping $\rho\colon \mathcal{L} \to \mathrm{PG}(3,F)_\infty\colon N \mapsto \infty_{\theta(N)}$ is, as the union of all maps ρ_X with X ranging over all lines of Q, a bijection from $\mathcal{L}$ onto the set of points of $\mathrm{PG}(3,F)_\infty$ off the lines L_X and L_Y, for two arbitrary intersecting lines X and Y of Q.

Now let L, X and Y' be as before. Let L' be a line of $\mathrm{AQ}(4,F)$ intersecting Y', but not X, in $\mathrm{Q}(4,F)$. Then $\theta(L')$ does not intersect π_X, and hence $\theta(L')$ has a unique intersection point b with the 3-space S of $\mathrm{AG}(4,F)$ determined by L and π_X. Let B be the unique line of $\mathrm{AQ}(4,F)$ incident with b and intersecting L. Then, by the foregoing, $\infty_{\theta(B)}$ is not contained in π_X whereas $\theta(B)$ contains two distinct points of S, namely $\theta(b)$ and some point on $\theta(L)$. This contradiction finally shows that $\infty_{\theta(L)} = \infty_{\theta(M)}$.

Now considering again reguli through K and L (M, respectively), we easily see that the set of points $\infty_{\theta(N)}$, for N ranging over all lines of $\mathrm{AQ}(4,F)$ concurrent with X in $\mathrm{Q}(4,F)$, coincides with the set of points of a line of $\mathrm{PG}(3,F)_\infty$. Hence we obtain a full embedding of $\mathrm{Q}(4,F)$ into $\mathrm{PG}(4,F)$ and the result follows from Dienst's main theorem in [8] and the fact that the projective group of automorphisms of $\mathrm{Q}(4,F)$ acts transitively on full non-thick subquadrangles. $\square$

We need a slightly more general result (cf. [26, 2.10]), which will follow from the previous one and the following two lemmas.

Lemma 3.4. *The geometry* $\mathrm{AQ}(4,F)$ *is connected, for all fields* F.

Proof. If $|F| = 2$, then each line has exactly two points, so we can consider $\mathrm{AQ}(4,F)$ as a graph; it is easy to see that $\mathrm{AQ}(4,F)$ is the complete bipartite graph of valency 3.

Suppose now that $|F| > 2$. Let x, y be any two points of $\mathrm{AQ}(4,F)$. We will show that there is a path in $\mathrm{AQ}(4,F)$ connecting them. We may assume that they are not collinear in $\mathrm{AQ}(4,F)$, and hence neither in $\mathrm{Q}(4,F)$. Let L be a line through x. If, in $\mathrm{Q}(4,F)$, the unique point z on L that is collinear with y belongs to $\mathrm{AQ}(4,F)$, then we are done. So suppose that z belongs to $\mathrm{Q}(4,F)$ but not to $\mathrm{AQ}(4,F)$. Let X be a line through z in $\mathrm{Q}(4,F)$ that does not belong to $\mathrm{AQ}(4,F)$, let $z' \neq z$ be a point on X, and let M be a line through z' belonging to $\mathrm{AQ}(4,F)$. Then the respective unique points x', y' on M collinear with x and y belong to $\mathrm{AQ}(4,F)$ (as z' is collinear to neither of x, y), and we have the path x, x', y', y. $\square$

Lemma 3.5. *Let F be a field with $|F| > 2$, and let X and Y be two intersecting lines of* $\mathrm{Q}(4,F)$ *that do not belong to* $\mathrm{AQ}(4,F)$. *Then the geometry* $\mathrm{AQ}_{X,Y}(4,F)$ *obtained from* $\mathrm{AQ}(4,F)$ *by deleting all lines that meet the union* $X \cup Y$ *in* $\mathrm{Q}(4,F)$ *is a connected geometry.*

Proof. By the previous lemma, we only need to show that any two points x, y of $\mathrm{AQ}(4,F)$ are connected in $\mathrm{AQ}_{X,Y}(4,F)$ if they are incident with a line of $\mathrm{Q}(4,F)$ that intersects X or Y. Let L be the line joining two such points x and y. We may assume that, in $\mathrm{Q}(4,F)$, L and X intersect in a point z.

First we suppose that $z \notin Y$. Let Y' be the (unique) line through z, different from X and belonging to $\mathrm{Q}(4,F)$, but not to $\mathrm{AQ}(4,F)$. Let $z' \neq z$ be a point on Y', and let M be a line through z' belonging to $\mathrm{AQ}(4,F)$, but not to the regulus $\mathcal{R}(L,Y)$; such a line M exists since $|F| > 2$. Then the respective unique points x', y' on M collinear with x and y belong to $\mathrm{AQ}(4,F)$ (as z' is collinear to neither of x, y), and we have the path x, x', y', y. The lines through x, x' and through y, y', respectively, do not meet X (as there are no triangles in $\mathrm{Q}(4,F)$) nor Y (as they intersect every member of $\mathcal{R}(L,M) \not\ni Y$), and also M does not meet $X \cup Y$. Hence x is joined to y in $\mathrm{AQ}_{X,Y}(4,F)$.

If $z \in Y$, then we choose a regulus $\mathcal{R}$ through L such that X intersects every member of $\mathcal{R}$ (then X is called a *transversal of* $\mathcal{R}$) and we choose $M \in \mathcal{R} \smallsetminus \{L\}$. Then the points x', y' of M on a transversal of $\mathcal{R}$ together with x, y, respectively, belong to $\mathrm{AQ}(4,F)$. By the previous paragraph, we find a path in $\mathrm{AQ}_{X,Y}(4,F)$ connecting x' and y'. Since the above mentioned transversals do not meet $X \cup Y$, we can extend that path to join x and y. □

Corollary 3.6. *Let V be a vector space over a field F with $|F| > 2$ and $\dim_F V \geq 4$. Then up to collineations from* $\mathrm{A\Gamma L}(V,F)$, *the standard embedding of* $\mathrm{AQ}(4,F)$ *into* $\mathrm{AG}(4,F) \leq \mathrm{AG}(V,F)$ *is the only full embedding of* $\mathrm{AQ}(4,F)$ *into the affine space* $\mathrm{AG}(V,F)$ *with the property that any two lines of* $\mathrm{AQ}(4,F)$ *which do not intersect in* $\mathrm{Q}(4,F)$ *are not parallel in* $\mathrm{AG}(V,F)$.

Proof. We embed $\mathrm{AQ}(4,F)$ into $\mathrm{AG}(V,F)$ and identify points and lines of $\mathrm{AQ}(4,F)$ with their images in $\mathrm{AG}(V,F)$.

Let Q be a non-thick full subquadrangle of $\mathrm{AQ}(4,F)$ (this arises from a non-thick full subquadrangle of $\mathrm{Q}(4,F)$ that contains two lines X, Y of $\mathrm{Q}(4,F)$ which do not belong to $\mathrm{AQ}(4,F)$). Then, since Q is determined by two non-intersecting lines, it is entirely contained in a 3-space $\mathrm{AG}(3,F)$ of $\mathrm{AG}(V,F)$. Let x be a point of $\mathrm{AQ}(4,F)$ not in Q. If we show that every point of $\mathrm{AQ}(4,F)$ is contained in the affine space A generated by $\mathrm{AG}(3,F)$ and x, then $\mathrm{AQ}(4,F)$ is certainly contained in an affine 4-space; hence we can apply Theorem 3.3 and the result follows.

Clearly, all points of $\mathrm{AQ}(4,F)$ collinear with x, but not collinear with any point of $X \cup Y$ in $\mathrm{Q}(4,F)$ belong to A. By connectivity of the geometry $\mathrm{AQ}_{X,Y}(4,F)$, see Lemma 3.5, all points of $\mathrm{AQ}(4,F)$ belong to A, and the assertion follows. □

4. Unitals from the affine quadrangles $\mathrm{AQ}(4, R)$

Let $C|R$ be a quadratic extension of fields. The projective plane $\mathrm{PG}(2, C)$ can be considered as the projective closure of the affine plane $\mathrm{AG}(2, C)$. The latter is constructed from the 2-dimensional vector space $V_C = C^2$ over C: the points are the pairs $(X, Y) \in C \times C$, and the lines are the sets of points satisfying an equation of the form $AX + BY + D = 0$ with $(A, B) \neq (0, 0)$. The set $V_R = R^4$ is a 4-dimensional vector space over R, and $\beta\colon V_C \to V_R\colon (X, Y) = (x_0 - \varepsilon^{-1}x_1, y_0 + y_1) \mapsto (x_0, x_1, y_0, y_1)$ is an R-linear identification between V_C and V_R. The affine line in V_C with equation $AX + BY + D = 0$, where $A = a_0 + \varepsilon a_1$, $B = b_0 + \varepsilon b_1$, $D = d_0 + \varepsilon d_1$ is mapped by β onto the affine plane in V_R with equations

$$\begin{cases} a_0x_0 - (td^{-1}a_0 + a_1)x_1 + b_0y_0 - tdb_1y_1 + d_0 = 0, \\ a_1x_0 + d^{-1}a_0x_1 + b_1y_0 + b_0y_1 + tb_1y_1 + d_1 = 0. \end{cases}$$

The points at infinity of such a plane form a line in the projective space $\mathrm{PG}(V_R)$, and the set of all such lines forms a line spread of $\mathrm{PG}(V_R)$, which we call the ABB spread of β. (ABB stands for André, Bose and Bruck.)

Recall that in $\mathrm{PG}(2, C)$ we use homogeneous coordinates $[X, Y, Z]$ to denote the point $(X/Z, Y/Z)$ of $\mathrm{AG}(2, C)$ if $Z \neq 0$, and the point at infinity corresponding with the slope Y/X if $Z = 0$ and $X \neq 0$, and the point at infinity of the Y-axis if $X = Z = 0$.

We now prove that generalized hermitian unitals $\mathcal{H}(C|R)$ are equivalent to affine quadrangles isomorphic to $\mathrm{AQ}(4, R)$ whose grid at infinity shares a regulus with an ABB spread related to $C|R$. Note that the line at infinity induces the block $B = \{[\varepsilon x, y, 0] \mid (0, 0) \neq (x, y) \in R^2\}$.

Proposition 4.1. *The image of $\mathcal{H}(C|R) \setminus B$ under β consists of the points of an affine quadrangle $\mathrm{AQ}(4, R)$ (images of the affine points) together with one regulus $\mathcal{R}$ of a grid in $\mathrm{PG}(V_R)$ completing it to $\mathrm{Q}(4, R)$ (images of the points at infinity). Conversely, if $\mathrm{AQ}(4, R)$ is an affine quadrangle in $\mathrm{PG}(4, R)$ such that one of the reguli of its completion to $\mathrm{Q}(4, R)$ is a subset of the ABB spread, then the point set of $\mathrm{AQ}(4, R)$ is projectively equivalent to the set of affine points of $\mathcal{H}(C|R)$.*

Two blocks B_1, B_2 in $\mathcal{H}(C|R) \setminus B$ meet in a point of B if, and only if, there is a line L in the regulus $\mathcal{R}$ meeting both images $\beta(B_1 \setminus B)$ and $\beta(B_2 \setminus B)$.

Proof. A point $[X, Y, 1] = [x_0 - \varepsilon^{-1}x_1, y_0 + \varepsilon y_1, 1]$ belongs to $\mathcal{H} := \mathcal{H}(C|R)$ if and only if

$$\sigma(x_0 - \varepsilon^{-1}x_1)(y_0 + \varepsilon y_1) + 1 \in \varepsilon R.$$

Noting that $\sigma(\varepsilon)^{-1} = d^{-1}\varepsilon$ and $\varepsilon^2 = t\varepsilon - d$, we compute the left hand side as $\sigma(x_0 - \varepsilon^{-1}x_1)(y_0 + \varepsilon y_1) + 1 = x_0y_0 + x_1y_1 + 1 + \varepsilon(x_0y_1 - d^{-1}x_1y_0 - td^{-1}x_1y_1)$. The latter belongs to εR if and only if $x_0y_0 + x_1y_1 + 1 = 0$, which is, reading $x_0, \ldots, y_1$ as variables, precisely the equation of an affine quadric isomorphic to $\mathrm{AQ}(4, R)$. The equation of the points at infinity of that quadric reads $x_0y_0 + x_1y_1 = 0$. Now, the points at infinity of $\mathcal{H}$, which have coordinates $[0, 1, 0]$ and $[1, k\varepsilon, 0]$ with $k \in R$, are given by the directions of the lines with equation $X = 0$ and $Y - \varepsilon kX = 0$. These give rise to the lines of $\mathrm{PG}(V_R)$ with equations $x_0 = 0 = x_1$ and $y_0 + kx_1 = 0 = y_1 - kx_0$, and these lines constitute indeed one regulus $\mathcal{R}$ of the ruled quadric with equation $x_0y_0 + x_1y_1 = 0$.

In the previous paragraph, we have found the interpretation in $\mathrm{PG}(2, C)$ of the (affine) quadric in V_R with equation $x_0y_0 + x_1y_1 + 1 = 0$. Now let Q be any affine quadric in V_R whose structure at infinity is a ruled quadric where one regulus is a subset of the ABB spread of β. By the 3-transitivity of the automorphism group of $\mathrm{AG}(2, C)$ on its set of points at infinity, and hence the 3-transitivity of the automorphism group of the ABB spread on its set of lines, we may assume that the regulus at infinity of Q which belongs to the ABB spread is $\mathcal{R}$. Then the equation of Q is of the form $x_0y_0 + x_1y_1 + k_1 + k_2x_0 + k_3x_1 + k_4y_0 + k_5y_1 = 0$. The translation $(x_0, x_1, y_0, y_1) \mapsto (x_0 - k_4, x_1 - k_5, y_0 - k_2, y_1 - k_3)$ is an automorphism of $\mathrm{AG}(2, C)$, and transforms the equation into $x_0y_0 + x_1y_1 + \ell = 0$, with $\ell = k_1 + k_2k_4 + k_3k_5$. If Q is an affine quadrangle then $\ell \neq 0$, and the automorphism $(X, Y) \mapsto (X, \ell^{-1}Y)$ of $\mathrm{AG}(2, C)$ maps Q to the set of affine points of $\mathcal{H}(C|R)$. $\square$

Proposition 4.2. *Let $C|R$ and $C'|R'$ be quadratic extensions of fields. Then the generalized hermitian unitals $\mathcal{H}(C|R)$ and $\mathcal{H}(C'|R')$ are isomorphic (as incidence structures) if, and only if, there exists an isomorphism between C and C' mapping R onto R'.*

Proof. The "if" part follows from 2.3, so we show the converse implication.

We begin by reconstructing the affine plane $\mathrm{AG}(2, R)$ from the unital $\mathcal{H} := \mathcal{H}(C|R)$. Let q be a point of $\mathcal{H}$. Let π_q be the set of blocks of $\mathcal{H}$ containing q. Consider two blocks $B_0, B_1 \in \pi_q$. We define the subset $L(B_0, B_1)$ as the set containing B_0, B_1 and each block $B \in \pi_q \smallsetminus \{B_0, B_1\}$ with the property that no block outside π_q meets B_0, B_1, and B. We claim that $L(B_0, B_1) = L(B'_0, B'_1)$, for every choice of two blocks $B'_0, B'_1 \in L(B_0, B_1)$.

Indeed, we can do this by an explicit calculation. As $\mathrm{Aut}\,\mathcal{H}$ is two-transitive, we may take $q = [0, 1, 0]$ and B_0 as the block induced by $q + p$ with $p = [1, 0, 0]$. Each block in π_q is then induced by a line of the form $q + [1, 0, Z]$ with $Z \in C$ because $q + [0, 0, 1]$ is the unique tangent through q.

The maps $\rho_A \colon [X, Y, Z] \mapsto [X, \sigma(A)AY, AZ]$ with $A \in C \smallsetminus \{0\}$ form a group H_q of automorphisms of $\mathcal{H}$ fixing q, stabilizing the block B_0, and acting transitively on $\pi_q \smallsetminus \{B_0\}$. So we may assume $B_1 = q + [1, 0, 1]$. For $Z \in C$, let B_Z be the block induced by the line $q + [1, 0, Z]$; then $\pi_q = \{B_Z \mid Z \in C\}$.

A general point of B_1 is $p_r := [1, \varepsilon r - 1, 1]$, with $r \in R$. By the existence of all translations with center q, the members of $L(B_0, B_1)$ are exactly the blocks through q that do not meet any block through p and p_r, with $r \in R$. For $W = \sigma(u + \varepsilon v)$ with $u, v \in R$, the intersection point $[W, \varepsilon r - 1, 1]$ of the lines $p + p_r$ and $q + [W, 0, 1]$ belongs to $\mathcal{H}$ precisely if $u = 1 - dvr$. If $W \notin R$ then $v \neq 0$, and we find $r = \frac{1-u}{dv} \in R$ such that the blocks induced by $p + p_r$ and $q + [W, 0, 1]$ do meet. If $W \in R$ then $v = 0$, and the blocks only meet if $W = 1$, and $q + [W, 0, 1]$ induces the block B_1. So $L(B_0, B_1) = \{B_r \mid r \in R\}$.

Now $\{\rho_a \mid a \in R \smallsetminus \{0\}\}$ is a subgroup of H_q fixing B_0, stabilizing $L(B_0, B_1)$ and acting transitively on $L(B_0, B_1) \smallsetminus \{B_0\}$. Interchanging the roles of B_0 and B_1, we see that some subgroup of the stabilizer of q in $(\mathrm{PSL}_3 C)_{\mathcal{H}}$ stabilizes $L(B_0, B_1)$ and acts two-transitively on $L(B_0, B_1)$. Our claim is proved.

For $S, T \in C$ with $B \neq 0$ we put $L_{A,T} := \{B_{S+Tu} \mid u \in R\}$; note that $L(B_0, B_1) = L_{0,1}$. Then $\rho_T(L(B_0, B_1)) = L_{0,T}$. It is routine to check that the map

$$\tau_S \colon [X, Y, Z] \mapsto [X, -\sigma(S)SX + Y + \sigma(S)(\sigma(\varepsilon)^{-1}\varepsilon - 1)Z, SX + Z]$$

is an automorphism of $\mathcal{H}$ fixing q, with $B_{S+W} = \tau_S(B_W)$ and thus $L_{S,T} = \tau_S(L_{0,T}) = \tau_S(\rho_T(L_{0,1})) = \tau_S(\rho_T(L(B_0, B)))$. This yields that $L_{S,T} = L(B', B'')$ for two blocks $B', B'' \in \pi_q$. Conversely, any two blocks B_V, B_W through q are contained in the set $L_{V,W-V} = \{B_{V+(W-V)u} \mid u \in R\}$. Clearly, π_q endowed with this family of subsets is an affine plane Π_q isomorphic to $\mathrm{AG}(2, R)$.

We identify the block $B_W \in \pi_q$ with $W \in C$, use the two-dimensional vector space C over R as model for the affine plane $\mathrm{AG}(2, R)$, and study the action of τ_S and ρ_A; for $S \in C$ and $A \in C \setminus \{0\}$. We have seen above that τ_S induces the translation $W \mapsto S + W$. Writing $A = a + \varepsilon b$ with $a, b \in R$, we obtain the matrix describing ρ_A with respect to the basis $1, \varepsilon$ as $\left(\begin{smallmatrix} a & -db \\ b & a+tb \end{smallmatrix}\right)$. The characteristic roots of that matrix are A and $\sigma(A)$, with eigenspaces $(\sigma(\varepsilon), -1)R$ and $(\varepsilon, -1)R$, respectively (these coincide if σ is the identity, i.e. in the inseparable case). We interpret these points as points at infinity over the algebraic closure of R, and refer to them as the *cyclic points*.

Now let B be an arbitrary block of $\mathcal{U}$ not through q. The set of blocks through q meeting B will be referred to as a *circle* in Π_q. We claim that every circle contains the cyclic points. More exactly, the intersection of every circle with the line at infinity is the set of cyclic points.

Indeed, for $A, S \in C$, the collineations ρ_A and τ_S preserve the cyclic points. Therefore, it is enough to show the claim for B meeting B_0 and B_1 (with the notation of the first part of the proof). Since we have all translations, we may assume that B contains $[1, 0, 0]$.

So let B be the block through the points $[1, 0, 0]$ and the arbitrary point $[1, \varepsilon r - 1, 1]$ of B_1, for some fixed $r \in R$. The points W of Π_P on the circle determined by B correspond to the points $[1, W(\varepsilon r - 1), W]$ of B. Expressing that such a point belongs to $\mathcal{H}$, we obtain the necessary and sufficient condition $\sigma(W)(\varepsilon r - 1) + \sigma(W)W \in \varepsilon R$. Writing $W = u + \varepsilon v$ with $u, v \in R$, we translate that condition into

$$u - tv + dvr + u^2 + tuv + dv^2 = 0\,,$$

which represents a conic in Π_q whose points at infinity are given as $(u, v)R$ satisfying $u^2 + tuv + dv^2 = 0$; these are just the cyclic points.

Hence the circles determine the field C: just add the slopes corresponding to the points at infinity of any circle to R and this generates the field C. If $\mathcal{H}$ and $\mathcal{H}'$ are isomorphic unitals, then the planes Π_q and, with similar notation, $\Pi_{q'}$ are isomorphic, so the fields R and R' are isomorphic. Moreover, there is an isomorphism between Π_q and $\Pi_{q'}$ which also maps circles to circles. Since C and C' are determined by the points a infinity of the circles, the assertion follows. □

5. Embeddings of generalized hermitian unitals

The generalized hermitian unital $\mathcal{H}(C|R)$ defined in Section 2 has, by definition, an embedding into the projective plane $\mathrm{PG}(2, C)$; this is the *standard* *embedding*. Every embedding $C \to E$ of fields yields an embedding θ of $\mathcal{H} = \mathcal{H}(C|R)$ into $\mathrm{PG}(2, E)$, and we can compose θ with any collineation from $(\mathrm{P\Gamma L}_3 C)_{\mathcal{H}}$ on the right and with any collineation from $\mathrm{P\Gamma L}_3 E$ from the left. We say that these embeddings *originate* from the embedding $C \to E$ of fields.

Theorem 5.1. *Let E be a field and let $C|R$ be a quadratic extension of fields with $|R| > 2$. Then every embedding of the generalized hermitian unital $\mathcal{H}(C|R)$ into the projective plane $\mathrm{PG}(2, E)$ originates from an embedding $C \to E$ of fields.*

Proof. We identify $\mathcal{H} := \mathcal{H}(C|R)$ with its image in $\mathrm{PG}(2, E)$. Every perspectivity $\rho\colon B \to B'$ in $\mathcal{H}$ is the restriction of a unique perspectivity $\rho_E\colon L \to L'$, where L and L' are the lines of $\mathrm{PG}(2, E)$ containing B and B', respectively. Thus the group Π_B of all projectivities in $\mathcal{H}$ of B onto itself has a natural embedding (of permutation groups) into the group of projectivities of L onto itself in $\mathrm{PG}(2, E)$, i.e. into the permutation group $(\mathrm{PGL}_2(E), \overline{E})$. By Theorem 2.16, the permutation group (Π_B, B) is permutation isomorphic to $(\mathrm{PSL}_2^N R, \overline{R})$. Now we apply Proposition 1.1 (if $|R| = 3$, then $|N| = 2$ and $\mathrm{PSL}_2^N R = \mathrm{PGL}_2 \mathbb{F}_3$, see 2.17): the field R embeds into E such that B is a projective subline over R, i.e. the intersection of a line with the projective subplane coordinatized by R with respect to a suitable quadrangle. By Theorem 2.16.2 the same embedding of R is used for each block of $\mathcal{H}$, since every projectivity is induced by an element of $\mathrm{PGL}_3 E$, and thus by an R-linear map.

Now let B and B' be two intersecting blocks of $\mathcal{H}$, say $B \cap B' = \{x\}$, and let L and L' be the lines of $\mathrm{PG}(2, E)$ containing B and B', respectively. By Theorem 2.16 there is a projectivity ρ of $\mathcal{H}$ from B to B', and Π_B is transitive on B. Hence we may assume that ρ fixes x, and so does the projectivity $\rho_E\colon L \to L'$ extending ρ. Since ρ_E fixes x, it is a perspectivity, with some center c. Select two points $y, z \in B \setminus \{x\}$ and let y' and z' be the intersection points of B' with the lines of $\mathrm{PG}(2, E)$ defined by c, y and c, z, respectively. Then the projective subplane π generated by $B \cup \{y', z'\}$ contains all points of B'; moreover B and B' are full lines in π.

We have seen above that each block of $\mathcal{H}$ is a projective subline over the subfield R of E. Fix a block B of $\mathcal{H}$ and consider the affine plane $\mathrm{AG}(2, E) \subseteq \mathrm{PG}(2, E)$ obtained by removing the line containing B. Then $\mathrm{AG}(2, E)$ can be seen as a vector space of dimension 2 over E, and hence also as a vector space V of (possibly infinite) dimension $2 \cdot \dim_R E$ over the field R. Each block $B' \neq B$ which meets B becomes a full affine 1-space of V. We now forget about the vector space structure of V and only consider its affine space structure $\mathrm{AG}(V)$ over R. Then the points of $\mathcal{H} \setminus B$ (the unital with the points of the block B removed) are points of $\mathrm{AG}(V)$, and all blocks intersecting B are full lines of $\mathrm{AG}(V)$. These blocks put the structure of $\mathrm{AQ}(4, R)$ on $\mathcal{H} \setminus B$, see [29, 5.1, 5.2]. We have thus derived that $\mathrm{AQ}(4, R)$ is fully embedded in $\mathrm{AG}(V)$.

Now note that parallel lines of $\mathrm{AG}(V)$ are contained in parallel lines of $\mathrm{AG}(2, E)$. Hence, if two blocks of $\mathcal{H} \setminus B$ intersecting B are on two non-parallel lines of $\mathrm{AG}(2, E)$, then the corresponding lines in $\mathrm{AG}(V)$ are non-parallel. Now suppose two blocks B_1, B_2 of $\mathcal{H} \setminus B$ intersecting B are on parallel lines of $\mathrm{AG}(2,E)$, but define non-intersecting lines in $\mathrm{Q}(4,R)$ (the natural extension of $\mathrm{AQ}(4,R)$). From Proposition 4.1 we know that the corresponding lines L_1 and L_2 of $\mathrm{AQ}(4,R)$ intersect the same member X of the regulus $\mathcal{R}$, and they define (cf. 3.2) a regulus $\mathcal{R}(L_1,L_2)$ with X as a transversal. Let M_1,M_2 be two other transversals. Then M_1,M_2 are lines of $\mathrm{AQ}(4,R)$. Since $\mathcal{H}$ does not contain an O'Nan configuration (see Proposition 2.7), the blocks of $\mathcal{H}$ corresponding to M_1,M_2 intersect B in distinct points. Hence the corresponding lines in $AG(V)$ are skew (and we denote these lines also by M_1 and M_2). It follows that also the lines of $\mathrm{AG}(V)$ corresponding to B_1 and B_2 are skew, as they both intersect both M_1 and M_2 in different points, and these four points of intersection are not coplanar.

Hence we can apply Corollary 3.6 and find an affine 4-space $\mathrm{AG}(4,R)$ in $\mathrm{AG}(V)$ containing $\mathcal{H}\setminus B$. Moreover, inside $\mathrm{AG}(4,R)$, the points of $\mathcal{H}\setminus B$ and the blocks intersecting B form a standard embedding of $\mathrm{AQ}(4,R)$. Hence there exists an embedding of $\mathrm{Q}(4,R)$ into the projective completion $\mathrm{PG}(4,R)$ of $\mathrm{AG}(4,R)$ such that $\mathrm{AQ}(4,R)$ consists of the points of $\mathrm{Q}(4,R)$ not contained in a full non-thick subquadrangle Q of $\mathrm{Q}(4,R)$. Denote by $\mathrm{PG}_\infty(4,R)$ the projective 3-space $\mathrm{PG}(4,R)\setminus\mathrm{AG}(4,R)$.

Let $\mathrm{PG}(V)$ be the projective completion of $\mathrm{AG}(V)$; we call the elements of $\mathrm{PG}_\infty(V) := \mathrm{PG}(V)\setminus\mathrm{AG}(V)$ the elements at infinity of $\mathrm{AG}(V)$. It is well known (and easy to see) that the lines of $\mathrm{AG}(2,E)$ correspond to affine subspaces of $\mathrm{AG}(V)$, and that the subspaces at infinity of these subspaces form a partition (or *spread*) Σ of the projective space $\mathrm{PG}_\infty(V)$ (these projective subspaces are mutually complementary; they have projective dimension $\dim_R E-1$). Note that $\mathrm{PG}_\infty(4,R)$ is a subspace of $\mathrm{PG}_\infty(V)$.

We now show that Σ induces a spread Ξ of 1-spaces in $\mathrm{PG}_\infty(4,R)$ containing the regulus $\mathcal{R}$ as a subset. First we remark that $\mathrm{PG}_\infty(4,R)$ is not contained in any member of Σ, as otherwise $\mathcal{H}\setminus B$ would be contained in a line of $\mathrm{PG}(2,E)$, which is a contradiction. It now suffices to show that no member of Σ intersects $\mathrm{PG}_\infty(4,R)$ in just a point, and that the lines of $\mathcal{R}$ are contained in members of Σ.

We start with the latter. We already know (see 4.1) that the blocks of $\mathcal{H}$ intersecting B in some point (i.e. corresponding to lines of $\mathrm{AQ}(4,R)$ intersecting the same line X of $\mathcal{R}$) are contained in lines of $\mathrm{PG}(2,E)$ intersecting B in the *same* point $b\in B$. Hence the element of Σ corresponding with the point b contains X.

Now assume that $S\in\Sigma$ intersects $\mathrm{PG}_\infty(4,R)$ in a single point x. Then x does not lie on the quadric $\mathrm{Q}(4,R)$ because the quadric's points at infinity are covered by spread elements that meet $\mathrm{PG}_\infty(4,R)$ in elements of the regulus. So x is contained in a 3-space of $\mathrm{PG}(4,R)$ intersecting $\mathrm{Q}(4,R)$ in a non-degenerate quadric, and is, therefore, not a nucleus for $\mathrm{Q}(4,R)$. Thus there exists a line Z of $\mathrm{PG}(4,R)$ through x intersecting $\mathrm{AQ}(4,R)$ in exactly two points. Hence the line of $\mathrm{PG}(2,E)$ corresponding with the subspace of $\mathrm{PG}(V)$ generated by S and Z intersects $\mathcal{H}$ in exactly two points; this is a contradiction to the fact that $\mathcal{H}$ is embedded in $\mathrm{PG}(2,E)$.

We fix one of the points of $\mathrm{AG}(4,R)$ as the origin of the affine space $\mathrm{AG}(V)$. Then the points of $\mathrm{AG}(4,R)$ form a vector subspace T of V, the elements of Σ are the lines of $\mathrm{AG}(2,E)$ through the origin, and the members of Ξ form a spread (of 2-dimensional vector subspaces) in T; these are the lines through the origin of an affine plane $(T, \{X+t \mid X \in \Xi, t \in T\})$. Each element of Ξ is induced by an element of Σ, and the elements of Σ form the points at infinity in the projective completion of $\mathrm{PG}(2,E)$. Those elements of Σ that induce elements of Ξ now form a subset of the line at infinity that completes $(T, \{X+t \mid X \in \Xi, t \in T\})$ to a projective plane π. The plane π is a subplane of the pappian plane $\mathrm{PG}(2,E)$, and thus isomorphic to $\mathrm{PG}(2,C')$, where C' is a field extension of R. As every element of Ξ is a two-dimensional vector subspace of T, the extension $C'|R$ has degree two.

By Proposition 4.1, the unital $\mathcal{H}$ is naturally embedded into $\mathrm{PG}(2,C')$. Now, by Proposition 4.2, the extensions $C|R$ and $C'|R$ are isomorphic. The embedding of $\pi \cong \mathrm{PG}(2,C)$ into $\mathrm{PG}(2,E)$ now yields an embedding of C into E, and the embedding of $\mathcal{H}$ into $\pi \cong \mathrm{PG}(2,C)$ is standard. □

Theorem 5.2. *Let $C|R$ be a quadratic extension of fields, pick $\varepsilon \in C \setminus R$, and consider the generalized hermitian unital $\mathcal{H} = \mathcal{H}(C|R)$. Then $\operatorname{Aut}\mathcal{H} = (\mathrm{P\Gamma L}_3 C)_{\mathcal{H}}$; more explicitly, we have:*

(1) *If $C|R$ is separable then $\operatorname{Aut}\mathcal{H} = \mathrm{P\Gamma U}_3(C|R)$ is induced by the group $\Gamma\mathrm{U}_3(C|R)$ of semi-similitudes of the skew hermitian form h in 2.2.*

(2) *In any case, the group $\operatorname{Aut}\mathcal{H}$ is the product of the simple group $(\mathrm{PSL}_3 C)_{\mathcal{H}}$ and the stabilizer of $[1,0,0]$ and $[0,1,0]$ in $(\mathrm{P\Gamma L}_3 C)_{\mathcal{H}}$. That stabilizer is induced by the group consisting of all semilinear maps $(X,Y,Z) \mapsto (a\gamma(X), b\gamma(Y), \gamma(Z))$ with $\gamma \in \operatorname{Aut} C$ and $a,b \in C \setminus \{0\}$ such that $\sigma(a)b$ equals the unique element in $(1+\varepsilon R) \cap \frac{\varepsilon}{\gamma(\varepsilon)}R$.*

   In particular, the linear elements of that group satisfy $\sigma(a)b = 1$. In the inseparable case, we thus obtain $(\mathrm{PGL}_3 C)_{\mathcal{H}} = (\mathrm{PSL}_3 C)_{\mathcal{H}}$. In the separable case, we have $(\mathrm{PGL}_3 C)_{\mathcal{H}}/(\mathrm{PSL}_3 C)_{\mathcal{H}} \cong R^\times$.

(3) *An automorphism of C occurs as the companion of a semilinear map in $(\Gamma\mathrm{L}_3 C)_{\mathcal{H}}$ if, and only if, it centralizes σ. In particular, in the inseparable case there is no restriction on such a companion, and $\operatorname{Aut}\mathcal{H} = (\mathrm{P\Gamma L}_3 C)_{\mathcal{H}} = (\mathrm{PSL}_3 C)_{\mathcal{H}} \rtimes \operatorname{Aut} C$.*

Proof. Let $\eta\colon [X,Y,Z] \mapsto [X,Y,Z]$ denote the standard embedding of $\mathcal{H}$ into $\mathrm{PG}(2,C)$. Every automorphism $\alpha \in \operatorname{Aut}\mathcal{H}$ yields an embedding $\eta \circ \alpha$, and Theorem 5.1 says that there exists $\alpha' \in \mathrm{P\Gamma L}_3 C$ such that $\eta \circ \alpha = \alpha' \circ \eta$. So α is the restriction of α', and α' is an element of the stabilizer $(\mathrm{P\Gamma L}_3 C)_{\mathcal{H}}$. Assertion 1 is a known result; see [29, 8.1], cp. [30, 6.1, 5.5] for an alternative approach if $\operatorname{char} R \neq 2$.

As $(\mathrm{PSL}_3)_{\mathcal{H}}$ acts two-transitively on U (see 2.4), the full group $(\mathrm{P\Gamma L}_3 C)_{\mathcal{H}}$ is the product of $(\mathrm{PSL}_3 C)_{\mathcal{H}}$ and the stabilizer H of $p = [1,0,0]$ and $q = [0,1,0]$ in $(\mathrm{P\Gamma L}_3 C)_{\mathcal{H}}$. Obviously, the semilinear maps of the form given in assertion 2 belong to H.

From 2.3 we know that the tangents to U in p and q, respectively, intersect in $[0,0,1]$. Thus that point is fixed by H, as well. We choose semilinear representatives that actually fix the vector $(0,0,1)$. It thus remains to determine the semilinear maps of the form $(X,Y,Z) \mapsto (a\gamma(X), b\gamma(Y), \gamma(Z))$ with $a,b \in C \setminus \{0\}$ and $\gamma \in \operatorname{Aut} C$ such that U is invariant.

The block B joining p and q consists of q and the points of the form $[1, \varepsilon r, 0]$ with $r \in R$. The condition that these stay in U amounts to $\sigma(a)b\,\gamma(\varepsilon R) = \varepsilon R$. We infer $\gamma(R) = R$ and $\sigma(a)b\,\gamma(\varepsilon) \in \varepsilon R$.

Points of $U \smallsetminus B$ are of the form $[X, Y, 1]$ with $\sigma(X)Y \in -1+\varepsilon R$. Invariance of U now leads to $-1 + \varepsilon R = \sigma(a)b\,\gamma(-1 + \varepsilon R) = -\sigma(a)b + \varepsilon R$, so $\sigma(a)b \in (1 + \varepsilon R) \cap \frac{\varepsilon}{\gamma(\varepsilon)}R$. As $\gamma(\varepsilon) \notin R$, that intersection consists of precisely one element, and the product is determined uniquely by γ. If $\gamma = \mathrm{id}$ then $\sigma(a)b = 1$. This completes the proof of assertion 2.

Assertion 3 follows from the known fact that the centralizer of σ in $\operatorname{Aut} C$ consists of those automorphisms that leave R invariant, see [17, Proof of 1.3] or [18, Proof of 1.3]. □

Remark 5.3. Let $C|R$ be a quadratic extension of fields, pick $\varepsilon \in C \smallsetminus R$, and consider the generalized hermitian unital $\mathcal{H}(C|R)$. On the set of all embeddings of $\mathcal{H}(C|R)$ into $\mathrm{PG}(2, E)$, the group $\operatorname{Aut}\mathrm{PG}(2, E) = \mathrm{P\Gamma L}_3E$ acts from the left and $\operatorname{Aut}\mathcal{H}(C|R)$ acts from the right. Since $\operatorname{Aut}\mathcal{H}(C|R)$ is induced by collineations of $\mathrm{PG}(2, E)$, it suffices to compose the standard embedding on the left with elements from $\mathrm{P\Gamma L}_3E$.

If each automorphism of E leaves the images of both R and C in E invariant, then every element of $\operatorname{Aut} E$ occurs as the companion of an automorphism of $\mathcal{H}(C|R)$, see 5.2.3. In this case, the images of $\mathcal{H}(C|R)$ under embeddings into $\mathrm{PG}(2, E)$ form a single orbit under PGL_3E.

Corollary 5.4. *Let $q > 2$ be a power of a prime p. Every embedding of the finite hermitian unital $\mathcal{H}(\mathbb{F}_{q^2}|\mathbb{F}_q)$ into the pappian projective plane $\mathrm{PG}(2, E)$ over any field E is the composition of the standard embedding with some collineation from $\mathrm{P\Gamma L}_3E$, and the images of these embeddings form a single orbit under the group PGL_3E. In particular, this holds if $E = \mathbb{F}_{q^2}$.*

Proof. We apply Theorem 5.1 with $C = \mathbb{F}_{q^2}$ and obtain all embeddings. Every finite subfield of E is invariant under each automorphism of E. According to 5.3, there is just one orbit under PGL_3E. □

Remark 5.5. The assertion of Corollary 5.4 remains true if $q = 2$ and $E = \mathbb{F}_4$; this is proved in [14, Cor. 11.2].

In the case $E = \mathbb{F}_{q^2}$, an alternative argument for the assertion of Corollary 5.4 can be based on the observation that every block is a Baer subline (see 2.8): the assertion follows from [21] or [9], see also [2, 7.1, 7.2].

Let X be a set of points of a projective space. The *tangents* of X at $p \in X$ are the lines L of the projective space with $L \cap X = \{p\}$.

Corollary 5.6. *Let E be an infinite field and let $C|R$ be a quadratic extension of fields with $|R| > 2$. Then every embedding of the generalized hermitian unital $\mathcal{H}(C|R)$ into the projective plane $\mathrm{PG}(2, E)$ with less than $|E|$ tangents at some point p of the image of $\mathcal{H}(C|R)$ originates from an isomorphism $C \to E$ of fields. In particular, there is just one tangent at p.*

Proof. By Theorem 5.1 the embedding originates from an embedding $C \to E$ of fields. Let C' be the image of C in E. All lines through p not belonging to the subplane $\mathrm{PG}(2, C')$ are tangents. Hence there are at least $|\overline{E} \smallsetminus \overline{C'}| = |E \smallsetminus C'|$ tangents at p, and we infer that $|E \smallsetminus C'| < |E|$.

If $E \smallsetminus C'$ is not empty, then it contains an additive coset of C', hence $|C'| \leq |E \smallsetminus C'| < |E| = |C'| + |E \smallsetminus C'| \leq 2|E \smallsetminus C'| < |E|$, which is absurd. Therefore $E = C'$, and the embedding $C \to E$ is an isomorphism. Uniqueness of the tangent now follows from 2.3 (it is a classical result in the separable case, see [15, Lemma II.2.47]). □

Remark 5.7. Some infinite fields admit proper field endomorphisms, i.e. they are isomorphic to proper subfields. For example, for every field F the rational function field $F(t)$ is isomorphic to its subfield $F(t^n)$, for every integer $n \neq 0$, and the power series field $F((t))$ is isomorphic to $F((t^n))$. If F is a non-perfect field of characteristic p, then F is isomorphic to its proper subfield F^p via the Frobenius endomorphism $x \mapsto x^p$. Moreover, the field $\mathbb{C}$ of complex numbers admits many proper field endomorphisms, and the same is true for every algebraically closed field of infinite transcendency degree over its prime field; see [28, 14.9].

If $\varphi\colon E \to E$ is a proper endomorphism of the field E, then $\mathrm{PG}(2, \varphi(E))$ is a proper subplane of $\mathrm{PG}(2, E)$ with $|E|$ tangents at each point. Every incidence structure embeddable into $\mathrm{PG}(2, E)$ embeds also into $\mathrm{PG}(2, \varphi(E))$, with $|E|$ tangents in $\mathrm{PG}(2, E)$ at each point. This shows that some cardinality bound (like the one for the number of tangents) is needed in Corollary 5.6. even if we embed $\mathcal{H}(C|R)$ in $\mathrm{PG}(2, C)$.

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