FINITE SUBUNITALS OF THE HERMITIAN UNITALS

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Abstract. Every subunital of any hermitian unital is itself a hhermermitian unital, embedded by field restriction (whatever this means here, to be clarefied).

A hermitian unital in a pappian projective plane consists of the absolute points of a unitary polarity of that plane, with blocks induced by secant lines (see Section 1). The finite hermitian unitals of order q are the classical examples of 2-$(q^3+1, q+1, 1)$-designs. In fact, we consider generalized hermitian unitals $\mathscr{H}(C|R)$ where $C|R$ is any quadratic extension of fields; separable extensions $C|R$ yield the hermitian unitals, inseparable extensions give certain projections of quadrics.

1. Generalized hermitian unitals and Baer subplanes

Let $C|R$ be any quadratic (possibly inseparable) extension of fields; the classical example is $\mathbb{C}|\mathbb{R}$. We can write $C = R + \varepsilon R$, with $\varepsilon \in C \smallsetminus R$. There exist $t, d \in R$ such that $\varepsilon^2 - t\varepsilon + d = 0$, since $\varepsilon^2 \in R + \varepsilon R$. For convenience, we can assume that $t = 0$ if $\mathrm{char}(K) \neq 2$ (by replacing ε with $\varepsilon - \frac{1}{2}t$). The mapping

$$\sigma \colon C \to C \colon x + \varepsilon y \mapsto (x + ty) - \varepsilon y \quad \text{for } x, y \in R$$

is a field automorphism which generates $\mathrm{Aut}_R C$: if $C|R$ is separable, then σ has order 2 and generates the Galois group of $C|R$; if $C|R$ is inseparable, then σ is the identity.

Now we introduce our geometric objects. We consider the pappian projective plane $\mathrm{PG}(2, C)$ arising from the 3-dimensional vector space C^3 over C, and we use homogeneous coordinates $[X, Y, Z] := (X, Y, Z)C$ for the points of $\mathrm{PG}(2, C)$.

Definition 1.1. The *generalized hermitian unital* $\mathscr{H}(C|R)$ is the incidence structure $(U, \mathscr{B})$ with the point set $U := \{[X, Y, Z] \mid X^\sigma Y + Z^\sigma Z \in \varepsilon R\}$, and the set $\mathscr{B}$ of *blocks* consists of the intersections of U with secant lines, i.e. lines of $\mathrm{PG}(2, C)$ containing more than one point of U.

Note that U is not empty: it contains $[1, 0, 0]$ and $[0, 1, 0]$. The condition $X^\sigma Y + Z^\sigma Z \in \varepsilon R$ is homogeneous, since $c^\sigma c \in R$ for every $c \in C$.

In the next proposition, we identify $\mathscr{H}(C|R)$ in classical terms and motivate the name "generalized hermitian unital". The nucleus of a quadric is the projective subspace corresponding to the radical of the associated polar form.

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Proposition 1.2 (see [2]). *If $C|R$ is separable, then $\mathscr{H}(C|R)$ is the hermitian unital arising from the skew-hermitian form $h\colon C^3 \times C^3 \to C$ defined by*

$$h\left((X,Y,Z),(X',Y',Z')\right) = \varepsilon^\sigma X^\sigma Y' - \varepsilon Y^\sigma X' + (\varepsilon^\sigma - \varepsilon) Z^\sigma Z' .$$

If $C|R$ is inseparable, then $\mathscr{H}(C|R)$ is the projection of an ordinary quadric Q in some projective space of dimension at least 3 from a subspace of codimension 1 in the nucleus of Q.

2. Main Result

Theorem 2.1. *Let $(U,\mathscr{B})$ be a finite subunital of order t of the generalized hermitian unital $\mathscr{H}(C|R)$. Then $(U,\mathscr{B})$ is a standard embedded hermitian unital, i.e., $C|R$ is separable and coordinates can be chosen such that $\mathscr{H}(C|R)$ has equation $XY^\theta + YX^\theta = ZZ^\theta$, with θ the involution in the Galois group related to $C|R$, the finite field $\mathbb{F}_{t^2}$ is isomorphic to a subfield F of C and θ induces $x \mapsto x^t$ in F; in other words $F \cap R$ is a field of order t. In particular, it follows that a finite unital of order t embedded in a Hermitian unital of order q satisfies $t^3 \leq q$.*

3. Proof of Theorem 2.1

We will use the following properties of hermitian unitals:

(*) If three blocks though a given point p intersect two disjoint blocks B and B' not containing p, then each block through p intersecting either of B, B' intersect both B and B'.

(**) If three blocks though a given point p intersect a block B not through p, then for each point z on either of the three blocks, $z \neq p$, there exists a (unique) block containing z and intersecting the three blocks in three distinct points.

(***) If three blocks though a given point p intersect two disjoint blocks B and B' not containing p, then the intersection of the lines containing B and B' in the standard embedding is contained in the tangent line at p.

We suppose $t > 2$.

We first claim (Theo's observation) that two blocks of $(U,\mathscr{B})$ which have no point of U in common, correspond to disjoint blocks of $\mathscr{H}(C|R)$. Indeed, suppose for a contradiction that two blocks $B_1, B_2 \in \mathscr{B}$ are disjoint in U, but that their extensions to $\mathscr{H}(C|R)$ contain a common point x. The lack of O'Nan configurations in $\mathscr{H}(C|R)$ implies that two arbitrary blocks of $(U,\mathscr{B})$ both intersecting $B_1 \cup B_2$ in exactly two points have no points off $B_1 \cup B_2$ in common. Hence the number of points in U lying on a block intersecting $B_1 \cup B_2$ in exactly two points is equal to $(t+1)^2(t-1) > t^3+1$, a contradiction. The claim is proved.

Now let $p \in U$ be arbitrary, and let $B \in \mathscr{B}$ be such that $p \notin B$. Let $B_0, B_1, \ldots, B_t$ be the blocks of $(U,\mathscr{B})$ containing p and intersecting B nontrivially, say in $x_0, x_1, \ldots, x_t$, respectively. Let x be an arbitrary point on $B_0 \setminus \{p, x_0\}$. We claim that at least one block of $(U,\mathscr{B})$ contains x and intersects $B_1 \cup B_2 \cup \cdots \cup B_t$ in at least two points. Indeed, if not, then there are t^2 blocks through x different from B_0, a contradiction. So let B_x be a block of $(U, \mathscr{B})$ containing at least three points of $B_0 \cup B_1 \cup \cdots \cup B_t$, among which x. We note that B_x and B are disjoint by the lack of O'Nan configurations. For the same reason they are also disjoint in $\mathscr{H}(C|R)$. It then follows from (*) and our first claim that B_x intersects every B_i, $i \in \{0, 1, \ldots, t\}$, and the intersection point belongs to U. Hence we have shown (**), which is equivalent to Wilbrink's second condition (the block is indeed unique by the absence of O'Nan configurations).

Now let θ be the translation of $\mathscr{H}(C|R)$ with centre p mapping x_0 to x. Let y be any point of U not on B_0. Since B was arbitrary, we may assume that $y \in B$, so without loss of generality $y = x_1$. By the uniqueness in (**), θ maps x_1 to the intersection $B_x \cap B_1$. Since this intersection point belongs to U, it follows that θ preserves U. Hence $(U, \mathscr{B})$ admits all translations and hence is Hermitian by the main result of [1].

Now consider the (standard) embedding of $\mathscr{H}(C|R)$ in the projective plane $\mathrm{PG}(2, C)$. Then also $(U, \mathscr{B})$ is embedded in $\mathrm{PG}(2, C)$ and so by [2] there is a subfield $F \leq C$ of order t^2 and a subplane $\pi \cong \mathrm{PG}(2, F)$ containing U. Hence there is a polarity ρ_π of π with absolute point set U. We now show that ρ_π extends to a polarity ρ of $\mathrm{PG}(2, C)$ with absolute point set $\mathscr{H}(C|R)$. (In particular, $C|R$ is separable.)

Given the discussion above, it immediately follows from (***) that the tangent line to U at a point $p \in U$ coincides with the tangent line at p to $\mathscr{H}(C|R)$. This already implies that not all tangent lines to $\mathscr{H}(C|R)$ contain the same point and so $C|R$ is separable. Hence there is a polarity ρ of $\mathrm{PG}(2, C)$ associated to $\mathscr{H}(C|R)$. Since U contains a quadrangle, and points of U are mapped onto lines of π under the action of ρ, we see that ρ preserves π. Since tangent lines to $(U, \mathscr{B})$ and $\mathscr{H}(C|R)$ coincide in π, we see that $\rho_{|\pi} \equiv \rho_\pi$. Hence the involution θ of the Galois group related to $C|R$ preserves F and induces $x \mapsto x^t$ in F.

In particular, if C is finite of order q^2, then F is unique with given order t^2 and $\theta : x \mapsto x^q$ is not trivial on F, which means that F is not contained in the unique subfield R of order q; hence C is an extension of F of odd degree d.

This proves our main result completely for $t \neq 2$. For $t = 2$ we use Markus' arguments.

References

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