Innovations in Incidence Geometry
Algebraic, Topological and Combinatorial

vol. 20, no. 2-3, 2023
https://doi.org/10.2140/iig.2023.20.247

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On inclusions of exceptional long root geometries of type E

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We prove the uniqueness of the inclusion of the long root geometries of type E_6 and E_7 as full embeddings in the one of type E_8; the latter always arises as an equator geometry, the former as an intersection of two appropriate such equator geometries. Along the way, several other embedding results are obtained, notably featuring the subsequent point residuals of the above geometries.

1. Introduction

Equator geometries are subgeometries of the geometries related to spherical buildings, roughly speaking by taking the union of the equators, if any, of two opposite flags — the poles — in all apartments through these flags. That notion was first used in [Kasikova and Van Maldeghem 2013], and systematically and in full generality introduced and studied in [Van Maldeghem and Victoor 2019]. An equator geometry is always a geometry related to the residue of either of its poles in the corresponding building. Thinking in terms of roots, an equator geometry restricts the root system to the set of roots perpendicular to a given direction, which need not be the direction of a root. As a consequence, the long root geometries of the (split) spherical buildings are a natural home for equator geometries as the orthogonality relation is very present in these geometries. Indeed, every pair of opposite points of a long root geometry admits an equator geometry, which is then isomorphic to the long root geometry related to a point residue (see also Section 2F). In most cases an equator geometry is not only a subgeometry, but also a subspace; in incidence geometrical terms we obtain a full embedding of one geometry in the other. It appears from [De Schepper et al. 2022] that the language of equator geometries is very convenient to describe *all* the full embeddings of certain type. And despite the connection of equator geometries with residues of the underlying building, embeddings of geometries not related to any residue at all can also sometimes be adequately described using equator geometries. For instance, it is shown in

MSC2020: primary 51E24; secondary 51B25.
Keywords: long root geometries, exceptional Tits buildings.

[De Schepper et al. 2022] that any (fully) embedded long root geometry of type F_4 in a long root geometry of type E_7 arises as the intersection of two equator geometries.

In the present paper, the aim is to describe all embeddings of long root geometries of exceptional type E_6 and E_7 inside the long root geometry of type E_8. We describe this situation now in some more detail, using the (standard) notation that we will introduce in Section 2.

Let p, q be opposite points in a geometry $\Delta \cong \mathsf{E}_{8,8}(\mathbb{K})$, with $\mathbb{K}$ any (commutative) field, and consider the set of points $E(p, q)$ symplectic to both p and q. Equipped with the singular lines of Δ contained in $E(p, q)$, the set $E(p, q)$ is an *equator geometry* (*with poles* p, q), and is isomorphic to $\mathsf{E}_{7,1}(\mathbb{K})$ (see also Definition 3.8 and Lemma 3.9). This can be explained briefly by the fact that there is a bijection between the points of $E(p, q)$ and the symplecta of $\mathsf{E}_{8,8}(\mathbb{K})$ containing p, that is, the elements of type 1 in the point residue $\mathrm{Res}_\Delta(p) \cong \mathsf{E}_{7,7}(\mathbb{K})$. Informally speaking, a special case of our main result reads as follows (see Main Result 4.1 for a precise statement).

Main Result 1.1. *Let* Γ *be the long root geometry* $\mathsf{E}_{7,1}(\mathbb{K})$ *fully embedded in a long root geometry* $\Delta \cong \mathsf{E}_{8,8}(\mathbb{K})$. *Then there are opposite points* p, q *in* Δ *such that* $\Gamma = E(p, q)$. *Consequently, the embedding of* Γ *in* Δ *is projectively unique.*

The long root geometry $\Upsilon \cong \mathsf{E}_{6,2}(\mathbb{K})$ also embeds in a projectively unique way in the long root geometry $\Gamma \cong \mathsf{E}_{7,1}(\mathbb{K})$, also as an equator geometry; see Proposition 6.14 of [De Schepper et al. 2022]. However, the poles are not points, but subgeometries corresponding to vertices of type 7. A natural question then is whether, if we embed Υ in Δ, it is always contained in a subgeometry isomorphic to $\mathsf{E}_{7,1}(\mathbb{K})$? The answer is yes. An informal statement is given below, and for a precise statement we refer to Main Result 5.1:

Main Result 1.2. *Let* Υ *be the long root geometry* $\mathsf{E}_{6,2}(\mathbb{K})$ *fully embedded in a long root geometry* $\Delta \cong \mathsf{E}_{8,8}(\mathbb{K})$. *Then there exist pairs of opposite points* p, q *and* r, s *in* Δ *such that* $\Upsilon = E(p, q) \cap E(r, s)$, *i.e.*, Υ *is the intersection of two equator geometries isomorphic to* $\mathsf{E}_{7,1}(\mathbb{K})$. *Consequently, the embedding of* Υ *in* Δ *is projectively unique.*

The points p, r and q, s can be chosen collinear, so that $\mathsf{E}_{6,2}(\mathbb{K})$ could be viewed as the (appropriate connected component of the) equator geometry $E(L, M)$ of two opposite lines L, M of $\mathsf{E}_{8,8}(\mathbb{K})$.

A large part of the proof of these results consists in showing that the given embedding is *isometric*, that is, the relative position of two points in the subgeometry is the same as that in the ambient geometry (relative position meaning "being collinear", "being symplectic", "being special" and "being opposite"). To achieve that, we take an inductive approach, considering point residues. For Main Result

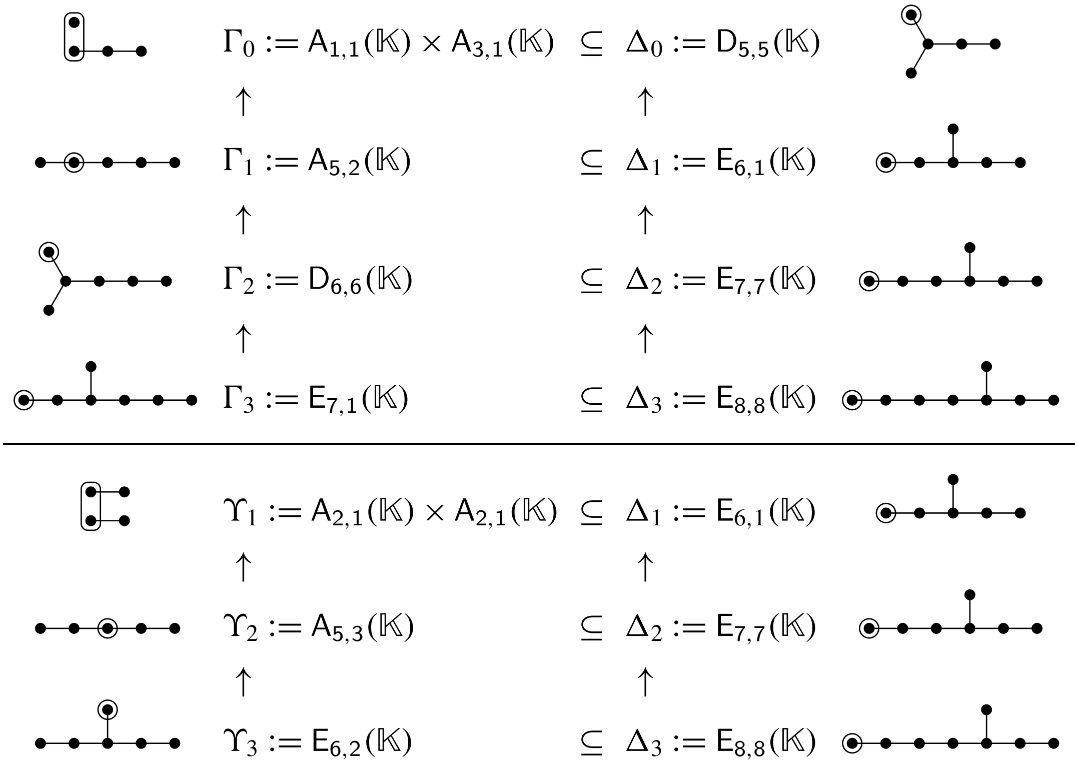

Figure 1. Sequence of full embeddings of point-line geometries described in Main Result 1.1 (top) and Main Result 1.2 (bottom).

1.1, this gives rise to the sequence of full embeddings of point-line geometries shown at the top of Figure 1, while for Main Result 1.2 the sequence at the bottom appears. (An arrow points from a parapolar space to its point residual.)

We will show that, for each $i \in \{1, 2, 3\}$ and each $j \in \{2, 3\}$, every embedding of Γ_i and Υ_j in Δ_i and Δ_j, respectively, is isometric, projectively unique and corresponds to an equator geometry. This is not the case for $i = 0$ and $j = 1$: there exist full embeddings of $\Gamma_0 \cong \mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{3,1}(\mathbb{K})$ in $\Delta_0 \cong \mathsf{D}_{5,5}(\mathbb{K})$ which are not isometric. However, we can prove directly (not using the point residues) that the embedding of $\Gamma_1 \cong \mathsf{A}_{5,2}(\mathbb{K})$ in $\Delta_1 \cong \mathsf{E}_{6,1}(\mathbb{K})$ is isometric, see Lemma 4.3; so we will limit us to studying the isometric embeddings of Γ_0 in Δ_0. There also exist full embeddings of $\Upsilon_1 \cong \mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$ in $\Delta_1 \cong \mathsf{E}_{6,1}(\mathbb{K})$ which are not isometric. The latter are classified in Proposition 5.7 up to the point that we need it to show that $\Upsilon_2 \cong \mathsf{A}_{5,3}(\mathbb{K})$ embeds isometrically in $\Delta_2 \cong \mathsf{E}_{7,7}(\mathbb{K})$.

Structure of the paper. In the preliminaries (Section 2) we gather the basics on the general notions (such as Lie incidence geometries, (para)polar spaces, full embeddings of such geometries and long root geometries) needed in this paper. Specific properties on the Lie incidence geometries we will encounter can be found in the Appendix, they could also be found partially in, for instance, [De Schepper et al. 2022], but as we will use these properties frequently we included them for the convenience of the reader.

In Section 3 we give a description of the equator geometries we study, including proofs that the defined geometry is of the type that we aim for. After that we then study the full embeddings of Γ_i in Δ_i (Section 4) and of Υ_i in Δ_i (Section 5), which arise as (intersections of) equator geometries if isometric (which is automatically the case, except for Γ_0 in Δ_0 and for Υ_1 in Δ_1, as explained above).

2. Preliminaries

We fix notation and introduce all relevant terminology. We assume that the reader is familiar with the basic theory of abstract buildings, Coxeter groups and Dynkin diagrams [Bourbaki 1968] and refer to the literature (for instance [Tits 1974]) for precise definitions and details; or to the introduction of [De Schepper et al. 2022]. We say that a spherical building is *split* if it arises from a split algebraic group. We will only be concerned with buildings whose Coxeter diagram is simply laced, and all these buildings are automatically split (whenever they are irreducible, have rank at least 3, and are defined over a field).

2A. ***Abstract point-line geometries.*** Let $\Gamma = (X, \mathscr{L})$ be a point-line geometry (X is the set of points, the set of lines $\mathscr{L}$ is a subset of the power set of X, and incidence is given by symmetrised inclusion). To exclude trivial cases, we assume $|\mathscr{L}| \geq 2$. We also assume that each line has at least three points.

Points $x, y \in X$ contained in a common line are called *collinear*, denoted $x \perp y$; the set of all points collinear to x is denoted by $x^\perp$. We will always deal with situations where every point is contained in at least one line, so $x \in x^\perp$. The *collinearity graph* of Γ is the graph on X with collinearity as adjacency relation. The *distance* δ between two points $p, q \in X$ (denoted $\delta_\Gamma(p, q)$, or $\delta(p, q)$ if no confusion is possible) is the distance between p and q in the collinearity graph, where $\delta(p, q) = \infty$ if there is no such path. If $\delta := \delta(p, q)$ is finite, then a *geodesic path* or a *shortest path* between p and q is a path of length δ between them in the collinearity graph. The *diameter* of Γ (denoted $\operatorname{Diam}\Gamma$) is the diameter of the collinearity graph. We say that Γ is *connected* if every pair of vertices is at finite distance from one another. The point-line geometry Γ is called a *partial linear space* if each pair of distinct points is contained in at most one line.

A *subspace* of Γ is a subset S of X such that, if $x, y \in S$ are collinear and distinct, then all lines containing both x and y are contained in S. A subspace S is called *convex* if, for any pair of points $\{p, q\} \subseteq S$, every point occurring in a shortest path between p and q in the collinearity graph is contained in S; it is *singular* if $\delta(p, q) \leq 1$ for all $p, q \in S$. The intersection of all convex subspaces of Γ containing a given subset $S \subseteq X$ is called the *convex closure* of S (this is well defined since X is a convex subspace). For $S \subseteq X$, we denote by $\langle S \rangle$ the subspace generated by S, it is the intersection of all subspaces containing S (again, this is well defined since X is a subspace). If S consists of two distinct collinear points p and q contained in a unique line L, then $\langle S\rangle = L$ is sometimes briefly denoted by pq. Two singular subspaces S_1 and S_2 are called *collinear* if $S_1 \cup S_2$ is a set of pairwise collinear points, and if so, we write $\langle S_1, S_2\rangle$ instead of $\langle S_1 \cup S_2\rangle$. In the geometries that we will consider, that is, parapolar spaces, the subspace generated by a set of mutually collinear points is always a singular subspace.

2B. *Polar spaces.* Abstractly, a (nondegenerate, thick) *polar space* $\Gamma = (X, \mathscr{L})$ is a point-line geometry satisfying the following four axioms, due to Buekenhout and Shult [1974], which simplify the axiom system in [Tits 1974].

(PS1) Every line contains at least three points, i.e., every line is *thick*.

(PS2) No point is collinear to every other point.

(PS3) Every nested sequence of singular subspaces is finite.

(PS4) The set of points incident with a given arbitrary line L and collinear to a given arbitrary point p is either a singleton or coincides with L.

We will assume that the reader is familiar with the basic theory of polar spaces, see for instance [Buekenhout and Cohen 2013]. Let us recall that every polar space, as defined above, is a partial linear space and has a unique *rank*, given by the length of the longest nested sequence of singular subspaces (including the empty set); the rank is always assumed to be finite by (PS3) and at least 2 since we always have a sequence $\varnothing \subseteq \{p\} \subseteq L$, for a line $L \in \mathscr{L}$ and a point $p \in L$.

Now let $\Gamma = (X, \mathscr{L})$ be a polar space of rank $r \geq 2$. It is well known that the maximal singular subspaces are projective spaces of dimension $r-1$ (and so two arbitrary points of Γ are contained in at most one line). Moreover, there is a (not necessarily finite) constant t such that every singular subspace of dimension $r-2$ is contained in exactly $t+1$ maximal singular subspaces. If $t = 1$, then we say that Γ is of *hyperbolic type*, or is a *hyperbolic* polar space. In this paper, all polar spaces we encounter will be hyperbolic. A hyperbolic polar space is isomorphic to one of the following.

$r = 2$: $\mathscr{L}$ consists of two disjoint systems of lines, each covering the point set, such that two lines intersect nontrivially (hence in exactly one point) if, and only if, they belong to different systems. A typical example is a ruled nondegenerate quadric in a projective 3-space.

$r = 3$: X is the set of lines of a 3-dimensional projective space $\mathsf{PG}(3, \mathbb{L})$ over a noncommutative skew field $\mathbb{L}$. The members of $\mathscr{L}$ are the (full) planar line pencils in $\mathsf{PG}(3, \mathbb{L})$.

$r \geq 3$: X is the point set of a nondegenerate hyperbolic quadric Q in $\mathsf{PG}(2r-1, \mathbb{K})$, $\mathbb{K}$ a (commutative) field. The lines are the lines of $\mathsf{PG}(2r-1, \mathbb{K})$ entirely contained in Q. Note that a standard equation for Q is given by

$$X_{-1}X_1 + X_{-2}X_2 + \cdots + X_{-r}X_r = 0.$$

A maximal singular subspace of a hyperbolic polar space is also called a *generator*. The family of generators of each hyperbolic polar space of rank r is the disjoint union of two systems of generators, called the *natural systems*, such that two generators intersect in a singular subspace of odd codimension in each of them if, and only if, they belong to different systems (the *codimension* of a subspace U in a projective space W is just $\dim W - \dim U$).

We will use some notions of the theory of buildings in polar spaces. For instance, two subspaces are called *opposite* if no point of their union is collinear to every point of this union; in particular two points are opposite if, and only if, they are not collinear and two maximal singular subspaces are opposite if, and only if, they are disjoint.

2C. ***Parapolar spaces.*** Parapolar spaces are point-line geometries that are designed to model the Grassmannians of spherical buildings, see also Section 2E. They were introduced by Cooperstein [1977]. A standard reference is [Shult 2011]. A point-line geometry $\Gamma = (X, \mathscr{L})$ is a *parapolar space* if it satisfies the following axioms.

(PPS1) There is a line L and a point p such that no point of L is collinear to p.

(PPS2) The geometry is connected.

(PPS3) Let x, y be two points at distance 2. Then either there is a unique point collinear with both, or the convex closure of $\{x, y\}$ is a polar space. Such polar spaces are called *symplecta*, or *symps* for short.

(PPS4) Each line is contained in a symplecton.

A pair $\{x, y\}$ of points with $x^\perp \cap y^\perp = \{z\}$ is called *special* and we denote this $z = x \bowtie y$; we also say that x *is special to* y. The set of points special to x is denoted by $x^{\bowtie}$. A pair of points $\{x, y\}$ at distance 2 from one another and contained in a (necessarily unique) symp is called *symplectic* and we write $x \perp\!\!\!\perp y$; we also say that x *is symplectic to* y. The set of points contained in a symp together with x is denoted by $x^{\perp\!\!\!\perp}$; note that this hence also includes $x^\perp$ by (PPS4). A parapolar space without special pairs of points is called *strong*. Due to (PPS4) and the fact that symps are convex subspaces isomorphic to polar spaces, each parapolar space is automatically a partial linear space and, by (PPS1), it is not a polar space. Note that the symps are not required to all have the same rank. A *para* is a proper convex subspace of Γ, whose points and lines form a parapolar space themselves. The set of symps of a para is a subset of the set of symps of Γ.

As alluded to in the introduction, we will often make use of point residuals. If $\Gamma = (X, \mathscr{L})$ is a parapolar space whose symps have rank at least 3, this is defined as follows. For a point $p \in X$, we define the point residual of Γ at p, denoted by $\mathrm{Res}_\Gamma(p)$, as the point-line geometry $(X_p, \mathscr{L}_p)$, where X_p is the set of lines of $\mathscr{L}$ containing p, and an element of $\mathscr{L}_p$ is the set of lines through p in a singular plane through p.

2D. ***Embeddings of point-line geometries in each other.*** Consider two point-line geometries $\Gamma = (X', \mathscr{L}')$ and $\Delta = (X, \mathscr{L})$. We say that Γ is *embedded* in Δ if $X' \subseteq X$ and for each $L' \in \mathscr{L}'$, there is a line $L \in \mathscr{L}$ with L' (viewed as subset of X') contained in L (viewed as a subset of X). The embedding is called *full* if $\mathscr{L}' \subseteq \mathscr{L}$, i.e., $L' \subseteq X'$ coincides with $L \subseteq X$ in the foregoing. Collinearity in Γ and Δ will respectively be denoted by $\perp_\Gamma$ and $\perp_\Delta$. Note that, if Γ is (not necessarily fully) embedded in Δ, then $\delta_\Delta(p,q) \leq \delta_\Gamma(p,q)$ for all points $p, q \in X'$. An embedding is (*point-*)*isometric* if $\delta_\Gamma(p,q) = \delta_\Delta(p,q)$ for all points $p, q \in X'$; in particular, $\perp_\Gamma$ and $\perp_\Delta$ coincide on $\Gamma \times \Gamma$.

Next, suppose additionally that $\Gamma = (X', \mathscr{L}')$ and $\Delta = (X, \mathscr{L})$ are parapolar spaces and let $\xi = (X'', \mathscr{L}'')$ be a symplecton of Γ, a convex subspace of Γ which is isomorphic to a polar space. Since Γ embeds fully in Δ, also ξ embeds fully in Δ. The following fact says that there are two ways in which ξ can embed in Δ.

Fact 2.1 [De Schepper et al. 2022, Lemmas 3.19 and 3.20]. *Either ξ embeds isometrically in Δ ($x \perp_\Gamma y$ if and only if $x \perp_\Delta y$ for each $x, y \in \xi$), or ξ embeds in a singular subspace of Δ ($x \perp_\Delta y$ for each $x, y \in \xi$). In the former case, ξ embeds isometrically in a symplecton Σ of Δ, uniquely determined by any two noncollinear points of ξ; and if Σ is viewed as a quadric embedded in a projective space $\mathbb{P}$, then ξ arises as the intersection of Σ with a subspace of $\mathbb{P}$.*

Notation 2.2. *If ξ embeds isometrically in Δ, we call ξ an* ***isometric symp****; if ξ embeds in a singular subspace of Δ, we refer to ξ as a* ***singular symp****. If x, y are two points of Γ which are symplectic in both Γ and Δ, then we denote by $\xi(x, y)$ the symp of Γ determined by x and y and by $\Sigma(x, y)$ the symp of Δ determined by x and y. We will also use $\xi(L, M)$ for $\xi(x, y)$, if the lines L, M intersect and contain x, y, respectively (and x and y are symplectic). In general we will use ξ for symps in Γ and Σ for symps in Δ. This should add to the clarity of the arguments.*

Note that, if Γ embeds isometrically in Δ, then each symp is isometric. The converse is not automatically true but in all cases we will encounter, preserving both "collinearity" and "being symplectic" allows one to prove that also the other distances (if any) are preserved, as well as "being special".

We also mention the following straightforward observation.

Lemma 2.3. *Let $\Psi = (X, \mathscr{L})$ and $\Psi' = (X', \mathscr{L}')$ be connected point-line geometries with Ψ fully embedded in Ψ' such that for each point $p \in X$, each member of $\mathscr{L}'$ containing p also belongs to $\mathscr{L}$. Then $\Psi = \Psi'$.*

2E. ***Lie incidence geometries.*** Let Δ be a (thick) spherical building, not necessarily irreducible. Let n be its rank, let S be its type set and let $J \subseteq S$. Then we define a point-line geometry $\Gamma = (X, \mathscr{L})$ as follows. The point set X is just the set of flags of Δ of type J. Each member of $\mathscr{L}$ is given by the elements F of X that complete a given flag F' of type $S \setminus \{s\}$, with $s \in J$, to a chamber; that is, $F \cup F'$ is a chamber (note that several distinct flags F' can give rise to the same line of Δ). The geometry Γ is called a *Lie incidence geometry*. For instance, if Δ has type A_n, and $J = \{1\}$ (remember we use Bourbaki labelling), then Γ is the point-line geometry of a projective space. If X_n is the Coxeter type of Δ and Γ is defined using $J \subseteq S$ as above, then we say that Γ has *type* $\mathsf{X}_{n,J}$ and we write $\mathsf{X}_{n,j}$ if $J = \{j\}$.

Most Lie incidence geometries are parapolar spaces. In particular, with the notation of Section 2E, if $|J| = 1$, then we either have a projective space if $\mathsf{X} = \mathsf{A}$ and J is either $\{1\}$ or $\{n\}$, a polar space if $\mathsf{X} \in \{\mathsf{B}, \mathsf{C}, \mathsf{D}\}$ and $J = \{1\}$, or a parapolar space in all other cases, taking into account though that $\mathsf{A}_{3,2} = \mathsf{D}_{3,1}$. The hyperbolic polar spaces correspond precisely to the Lie incidence geometries $\mathsf{D}_{n,1}$. For basic properties of parapolar spaces such as the facts that the intersections of symps are singular subspaces, and also that the set of points collinear to a given point x and belonging to a symp $\xi \not\ni x$ is a singular subspace, we refer to [Shult 2011].

If the building Δ is irreducible and its diagram X_n is simply laced, with $n \geq 3$, then the classification in [Tits 1974] implies that Δ is unambiguously defined by a (skew) field $\mathbb{K}$, which is necessarily a field if X_n contains D_4 as a subdiagram. We denote Δ by $\mathsf{X}_n(\mathbb{K})$. The corresponding Lie incidence geometry of type $\mathsf{X}_{n,J}$, where $J \subseteq S$, is denoted by $\mathsf{X}_{n,J}(\mathbb{K})$. By a *flag* or a *chamber* of Γ we mean a set of objects of Γ corresponding to a flag or chamber of the underling building Δ. By an *apartment* of Γ we also mean the set of objects of Γ contained in an apartment of Δ.

2F. ***Long root geometries.*** Long root geometries are special Lie incidence geometries related to split irreducible spherical buildings. The original, algebraic definition takes as point set the set of root groups corresponding to the long roots of the underlying root system and as set of lines the family of sets consisting of such root groups, each maximal relative to the property that their union forms a group [Timmesfeld 2001]. It turns out that long root geometries thus defined are just the Lie incidence geometries of type $\mathsf{X}_{n,J}$, where $J \subseteq S$ is the set of types corresponding to the roots of a fundamental system not perpendicular to the highest root. Explicitly they are the Lie incidence geometries of types

$$\mathsf{A}_{n,\{1,n\}},\ \mathsf{B}_{n,2},\ \mathsf{C}_{n,1},\ \mathsf{D}_{n,2},\ \mathsf{E}_{6,2},\ \mathsf{E}_{7,1},\ \mathsf{E}_{8,8},\ \mathsf{F}_{4,1}\ \text{and}\ \mathsf{G}_{2,2}$$

related to split spherical buildings. These geometries all share some intriguing properties, and they are so to speak the prototypes of nonstrong parapolar spaces when their rank is at least 3. A lot of information about long root geometries can be found in Shult's book [2011], Chapter 17.

Long root geometries satisfy certain regularity properties. One of those, when the geometry is a parapolar space, is the following. Consider the set of points $E(p,q)$ symplectic to two given opposite points p,q. Then the set $I(p,q)$ of points symplectic to each point of $E(p,q)$ carries the structure of a projective line over the base field: it is defined by an orbit of a full root group with centre in $I(p,q)$. The set $I(p,q)$ is called an *imaginary line*. A further property is that, when a point x is not opposite at least three members of $I(p,q)$, then it is not opposite each member of $I(p,q)$.

2G. ***Projectivities.*** The main results of the present paper classify embeddings of subgeometries "up to projective equivalence". In order to define this, we first need to define a projectivity. The *group of projectivities* of a Lie incidence geometry Δ containing projective planes as (not necessarily maximal) singular subspaces, is the group of collineations of Δ generated by all collineations each of which pointwise fixes some line or elementwise fixes a full line pencil of a singular plane. This amounts to the universal Chevalley group of respective type. A *projectivity* of Δ is a member of the group of projectivities of Δ. Then two embeddings Γ' and Γ'' in Δ are *projectively equivalent* if there exists a projectivity of Δ mapping Γ' bijectively to Γ''. A projectivity will also be referred to as a *linear* automorphism.

3. Equator geometries

Generally speaking, an *equator* of a Lie incidence geometry Δ is the set of points lying at equal distance from two given opposite flags F, F' along a shortest path connecting these flags, which are called the *poles* of the equator. The distance is measured in the incidence graph of the building, or in a truncation of it to certain types. We will always be able to define a given equator using incidence geometric properties. As an example, consider Definition 3.8; the graph is the incidence graph restricted to points and symps, but a more geometric definition is just to say that the equator consists of the points contained in respective symps together with x and x'. It is in that spirit that we will always define the individual equator geometries. (Hence, in principle, pairs of opposite flags can define different equator geometries, depending on the truncation of the incidence graph, but this will have no importance to us.)

There are various ways to furnish this set with lines so that it becomes a point-line geometry, called the *equator geometry* (*with poles F and F'*), denoted $E(F,F')$. The standard way is to just consider the lines of Δ completely contained in it (and we shall always do it in this way). The point-line geometries thus obtained are again Lie incidence geometries, they are related to the building $\mathrm{Res}_\Delta(F)$.

We would also like to warn the reader that the poles of an equator are not necessarily unique. We will see explicit examples of this phenomenon, see for instance Proposition 4.2.

We now define and discuss the equator geometries relevant for this paper. In each of these cases, F and F' will consist of one element. For more examples we refer to [Van Maldeghem and Victoor 2019].

3A. *Equator geometries isomorphic to* $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{3,1}(\mathbb{K})$ *in* $\mathsf{D}_{5,5}(\mathbb{K})$.

Definition 3.1. Let U, U' be opposite maximal 3-spaces of $\Delta_0 \cong \mathsf{D}_{5,5}(\mathbb{K})$. The point set of the *equator geometry* $E(U, U')$ *with poles* U, U' is given by the set of points of Δ_0 collinear to simultaneously a plane of U and a plane of U', equipped with the lines of Δ_0 entirely contained in it.

To see that $E(U, U')$ is isomorphic to $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{3,1}(\mathbb{K})$, we will work with Δ_0^*, the polar space isomorphic to $\mathsf{D}_{5,1}(\mathbb{K})$ corresponding to Δ_0. The poles U and U' correspond to opposite lines L, L' of Δ_0^*; the point set of $E(L, L')$ is the set of 4-spaces of Δ_0^* of one natural system of generators intersecting both L and L' in (necessarily collinear) points of Δ_0^*. Observe that two such 4-spaces of Δ_0^* correspond to collinear points of Δ_0 if they meet each other in a plane.

Lemma 3.2. *Let U and U' be opposite maximal* 3*-spaces of* $\Delta_0 \cong \mathsf{D}_{5,5}(\mathbb{K})$. *Then, as a point-line geometry*, $E(U, U')$ *is a subspace isomorphic to* $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{3,1}(\mathbb{K})$.

Proof. As mentioned above, we work in Δ_0^*. We denote the set of 4-spaces corresponding to the points of Δ_0 by Υ. With the above notation, $L^\perp \cap L'^\perp$ is a polar space isomorphic to $\mathsf{D}_{3,1}(\mathbb{K})$, which is contained in $T^\perp$, for each line T intersecting both L and L' nontrivially. Let $\mathscr{T}$ be the set of such lines and let Υ' be the set of planes of the polar space $L^\perp \cap L'^\perp$ of one family of generators, namely the family consisting of the planes that, together with a line $T \in \mathscr{T}$, generate a member of Υ (note that this does not depend on $T \in \mathscr{T}$). We already see from this that $E(U, U')$ is a subspace. Hence we can write an arbitrary member of $E(U, U')$ as the span of a member π of Υ' and a member T of $\mathscr{T}$, and we identify it with the couple (π, T). Hence $E(U, U')$ is already set-theoretically the direct product of $\mathsf{A}_{3,1}(\mathbb{K})$ and $\mathsf{A}_{1,1}(\mathbb{K})$, as $\mathscr{T}$ clearly has the structure of $\mathsf{A}_{1,1}(\mathbb{K})$. It is also clear that (π, T) and (π', T) are always collinear (since π and π' intersect in a point or coincide, hence $\langle \pi, T\rangle$ and $\langle \pi', T\rangle$ intersect in a plane or coincide), and so are (π, T) and (π, T'). It remains to show that (π, T) and (π', T') are not collinear if $\pi \neq \pi'$ and $T \neq T'$. And indeed, $\langle \pi, T\rangle \cap \langle \pi', T'\rangle \subseteq T^\perp \cap T'^\perp = L^\perp \cap L'^\perp$, implying $\langle \pi, T\rangle \cap \langle \pi', T'\rangle \subseteq \pi \cap \pi'$. This proves the lemma. □

Remark 3.3. The fact that $E(U, U')$ is a subspace, together with the fact that all noncollinear point pairs of both Γ_0 and Δ_0 are symplectic in their respective geometry, implies that the embedding of $\mathsf{A}_{1,1}(\mathbb{K})\times\mathsf{A}_{3,1}(\mathbb{K})$ in $\mathsf{D}_{5,5}(\mathbb{K})$ as an equator geometry is isometric.

3B. ***Equator geometries isomorphic to*** $\mathsf{A}_{5,2}(\mathbb{K})$ ***in*** $\mathsf{E}_{6,1}(\mathbb{K})$**.** For notation and terminology regarding the parapolar space $\mathsf{E}_{6,1}(\mathbb{K})$, we refer to Section A2. In particular, we use the notation $4'$*-space for a nonmaximal singular* 4*-space*.

Definition 3.4. Let W, W' be opposite 5-spaces of $\Delta_1\cong\mathsf{E}_{6,1}(\mathbb{K})$. The point set of the *equator geometry* $E(W, W')$ *with poles* W, W' is given by the set of points of Δ_1 simultaneously collinear to a 3-space of W and a 3-space of W', equipped with the lines of Δ_1 entirely contained in it.

Lemma 3.5. *Let* W, W' *be opposite* 5*-spaces of* $\Delta_1\cong\mathsf{E}_{6,1}(\mathbb{K})$*. Then, as a point-line geometry, the equator geometry* $E(W, W')$ *is a subspace isomorphic to* $\mathsf{A}_{5,2}(\mathbb{K})\cong\mathsf{A}_{5,4}(\mathbb{K})$.

Proof. By definition, each point p of $E(W, W')$ is Δ_1-collinear to a unique 3-space W_p of W. We first claim that the map $p\mapsto W_p$ is bijective onto the set of 3-spaces of W. Let S be a 3-space in W. Then there is a unique 4-space U containing S (see Fact A.14). Consider an arbitrary line $L\subseteq S$; let $L'\subseteq W'$ be the unique line of W' each point of which is collinear to a point of L (see Fact A.11). Let Σ be the unique symp through L and L'. Since $\Sigma\cap U$ contains L, it contains some plane π through L in U (see Fact A.8). In Σ, we see that there is a point $p\in\pi$ collinear to L'. Since $p^\perp\cap W'$ contains a line, it is a 3-space by the same fact. Hence $p\in E(W, W')$ and $W_p=S$. This shows surjectivity. As for injectivity, suppose $p'\in U$ also belongs to $E(W, W')$ and set $M=p^\perp\cap p'^\perp\cap W'$. Then M contains a line collinear to the line pp', which intersects S in a point, contradicting Fact A.11 once again. The claim is proved.

Now take a line $L=pq$ of $E(W, W')$. The previous paragraph yields $W_p\neq W_q$. Hence we can take a point $p'\in W_p\setminus W_q$ and consider the symp Σ_L determined by q and p'. Then Σ_L contains p and intersects W in the $4'$-space $V_L:=\langle p', W_q\rangle$. In Σ_L, p is collinear to a 3-space of V_L and hence $W_p\subseteq V_L$. Therefore, W_p and W_q share a plane π_L. Inside Σ_L it is easily seen that for each point r of L, we have $\pi_L\subseteq W_r\subseteq V_L$, and conversely, each 3-space incident with both π_L and V_L is collinear to a point on L. So the lines of $E(W, W')$ correspond to lines of the 3-space Grassmannian of W, and each line of $E(W, W')$ corresponds bijectively to a line of that Grassmannian.

Finally, consider two points p and q in $E(W, W')$ such that $W_p\cap W_q$ is a plane. We show that p and q are collinear in Δ_1, and hence in $E(W, W')$. Indeed, if not, they are symplectic and the symp they determine contains $W_p\cap W_q$ (which is a plane) and $W'_p\cap W'_q$ (which is at least a line). Therefore, $W_p\cap W_q$ contains a point which is collinear to a line of W', contradicting Fact A.11. □

3C. *Equator geometries isomorphic to* $\mathsf{D}_{6,6}(\mathbb{K})$ *in* $\mathsf{E}_{7,7}(\mathbb{K})$.

Definition 3.6. Let Σ, Σ' be opposite symps of $\Delta_2 \cong \mathsf{E}_{7,7}(\mathbb{K})$. The point set of the *equator geometry* $E(\Sigma, \Sigma')$ with poles Σ, Σ' is given by the set of points of Δ_2 simultaneously collinear to a $5'$-space of Σ and a $5'$-space of Σ', equipped with the lines of Δ_2 entirely contained in it.

Lemma 3.7. *Let* Σ, Σ' *be opposite symps of* $\Delta_2 \cong \mathsf{E}_{7,7}(\mathbb{K})$. *Then, as a point-line geometry, the equator geometry* $E(\Sigma, \Sigma')$ *is a subspace isomorphic to* $\mathsf{D}_{6,6}(\mathbb{K})$.

Proof. By definition, each point p of $E(\Sigma, \Sigma')$ corresponds to a unique $5'$-space U_p of Σ. We first claim that the mapping $p \mapsto U_p$ is bijective onto the $5'$-spaces of Σ. Surjectivity is proved in almost exactly the same fashion as in the proof of Lemma 3.5. We now show injectivity. Let p, q be distinct points with $U_p = U_q$. Then $\langle p, U_p\rangle$ and $\langle q, U_q\rangle$ are 6-spaces sharing a $5'$-space, hence they coincide. Set $M = p^\perp \cap q^\perp \cap \Sigma'$. If $M \neq \varnothing$ then M contains a line collinear to the line pq, which intersects S in a point, contradicting Fact A.18. Now assume $M = \varnothing$. Then the symp containing a point x of $p^\perp \cap \Sigma'$ and q contains p and at least a 4-space in $q^\perp \cap \Sigma'$, implying M has dimension at least 3. The claim is proved.

Now let $L = pq$ be a line of $E(\Sigma, \Sigma')$. The previous paragraph implies $U_p \neq U_q$, and then the argument in the last sentence, modified by choosing x in $U_p \setminus U_q$, implies that $U_p \cap U_q$ is a 3-space. Now for any $5'$-space of Σ through $U_p \cap U_q$, the unique 6-space through it meets the 5-space $\langle L, U_p \cap U_q\rangle$ in a 4-space and hence it meets L in a unique point; conversely, for each point z on L, Fact A.15 implies that $z^\perp \cap \Sigma$ is a $5'$-space containing $U_p \cap U_q$.

Now let p and q be two points of $E(\Sigma, \Sigma')$ such that $U_p \cap U_q$ is a 3-space. Suppose for a contradiction that p and q are not collinear in $E(\Sigma, \Sigma')$, i.e., they are not Δ_2-collinear. Then p and q are symplectic, since $p^\perp \cap q^\perp$ contains the 3-space $U_p \cap U_q$. Set $\Sigma_{pq} := \Sigma(p, q)$. Since p and q are noncollinear points of Σ_{pq} collinear to a $5'$-space of Σ', it follows from the symp–symp relations of Δ_2 (see Fact A.16, in particular (iv) and (v)) that $\Sigma_{pq} \cap \Sigma'$ is nonempty. But then a point in $\Sigma_{pq} \cap \Sigma'$ is collinear to more than a unique point of Σ (note that $\Sigma \cap \Sigma_{pq}$ is a 5-space), a contradiction to the fact that Σ and Σ' are opposite (see Fact A.16(v)). So p and q are collinear indeed. We obtain that $E(\Sigma, \Sigma')$ is isomorphic to $\mathsf{D}_{6,6}(\mathbb{K})$. $\square$

3D. *Equator geometries isomorphic to* $\mathsf{E}_{7,1}(\mathbb{K})$ *in* $\mathsf{E}_{8,8}(\mathbb{K})$.

Definition 3.8. Let x, x' be opposite points of $\Delta_3 \cong \mathsf{E}_{8,8}(\mathbb{K})$. The point set of the *equator geometry* $E(x, x')$ with poles x, x' is given by the set of points of Δ_3 symplectic to both x and x', equipped with the lines of Δ_3 entirely contained in it.

Lemma 3.9. *Let* x, x' *be opposite points of* $\Delta_3 \cong \mathsf{E}_{8,8}(\mathbb{K})$. *Then, as a point-line geometry, the equator geometry* $E(x, x')$ *is a subspace isomorphic to* $\mathsf{E}_{7,1}(\mathbb{K})$.

Proof. By construction, each point p of $E(x, x')$ corresponds to a unique symp Σ_p through x and hence a unique symp of $\mathrm{Res}_{\Delta_3}(x) \cong \mathsf{E}_{7,7}(\mathbb{K})$, i.e., a point of $\mathsf{E}_{7,1}(\mathbb{K})$. By Fact A.22(iv) the mapping $p \mapsto \Sigma_p$ is bijective.

Suppose two points p, q of $E(x, x')$ are collinear. By the point–symp relations (Fact A.22) and the fact that $p \perp q$, p is collinear to either a unique $6'$-space $U \ni q$ of Σ_q or a unique line $L \ni q$ of Σ_q. In the second case, $p \perp\!\!\!\perp x$ implies $x \perp L$ by Fact A.22(ii), contradicting $x \perp\!\!\!\perp q$. Hence p is collinear to a $6'$-space U of Σ_q. Looking in Σ_q, we see that x is collinear to a 5-space U_x of U. It follows that $\Sigma_p \cap \Sigma_q$ is the 6-space $\langle x, U_x \rangle$. Note that $\langle p, q, U_x \rangle$ is a 7-space of Δ_3 and that hence, for each point $r \in pq$, the symp Σ_r contains $\langle x, U_x \rangle$ too; moreover, each symp containing $\langle x, U_x \rangle$ shares a point with pq, as can be deduced from the symp–max relations (Fact A.12) of $\mathsf{E}_{6,1}(\mathbb{K})$, which we obtain by considering the residue of a line in U_x.

Conversely, suppose now that Σ_p and Σ_q share a 6-space. Suppose for a contradiction that p and q are not collinear. Put $V_{pq} = p^\perp \cap q^\perp \cap \Sigma_p \cap \Sigma_q$ and note that $\dim V_{pq} \geq 4$. Therefore, $p \perp\!\!\!\perp q$ and the unique symp Σ_{pq} of Δ_3 containing p, q also contains V_{pq}. Consider the position of the point x' with respect to Σ_{pq}. Since x' is symplectic to p and q by definition and p and q are not collinear by assumption, options (iii) and (iv) of Fact A.22 are ruled out. Since x' is special to the points of V_{pq}, option (i) of Fact A.22 is also not possible. The only remaining possibility is (ii) of Fact A.22, where x' is collinear to a unique line L of Σ_{pq}. But then x' would be symplectic to the points of $L^\perp \cap V_{pq}$, a contradiction. We conclude that p and q are collinear. □

4. Uniqueness of equator geometry isomorphic to $\mathsf{E}_{7,1}(\mathbb{K})$ in $\mathsf{E}_{8,8}(\mathbb{K})$

The goal of this section is to show that a geometry isomorphic to $\mathsf{E}_{7,1}(\mathbb{K})$ fully embedded in $\mathsf{E}_{8,8}(\mathbb{K})$ always arises as an equator geometry $E(x, x')$ for two opposite points x, x' of $\mathsf{E}_{8,8}(\mathbb{K})$ (see Definition 3.8). As a consequence, the embedding is unique up to projectivity. We accomplish this inductively, proving the analogues for consecutive point residuals. This gives us the sequence of full embeddings as depicted at the top of Figure 1 in the introduction, and leads to the following main theorem (for the definitions of equator geometries, see Definitions 3.1, 3.4, 3.6 and 3.8; for properties of the Lie incidence geometries, we refer to the Appendix).

Main Result 4.1. *Let $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$ be point-line geometries that are isomorphic to $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{3,1}(\mathbb{K})$, $\mathsf{A}_{5,2}(\mathbb{K})$, $\mathsf{D}_{6,6}(\mathbb{K})$ and $\mathsf{E}_{7,1}(\mathbb{K})$, respectively; and let $\Delta_0, \Delta_1, \Delta_2, \Delta_3$ be point-line geometries isomorphic to $\mathsf{D}_{5,5}(\mathbb{K})$, $\mathsf{E}_{6,1}(\mathbb{K})$, $\mathsf{E}_{7,7}(\mathbb{K})$ and $\mathsf{E}_{8,8}(\mathbb{K})$, respectively. Let $i \in \{0, 1, 2, 3\}$ and suppose Γ_i is fully embedded in Δ_i. If $i = 0$, suppose additionally that this embedding is isometric. Then this embedding is projectively unique and arises as an equator geometry, where the* *poles are elements of the type of the points of the long root geometry* (*an element of type* 2, *type* 2, *type* 1, *and type* 8, *respectively*).

4A. ***Full isometric embeddings of*** $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{3,1}(\mathbb{K})$ ***in*** $\mathsf{D}_{5,5}(\mathbb{K})$**.** We first study the full isometric embeddings of a geometry Γ_0 isomorphic to $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{3,1}(\mathbb{K})$, in the half spin geometry Δ_0 isomorphic to $\mathsf{D}_{5,5}(\mathbb{K})$. Our aim is to show that such an embedding arises as an equator geometry (see Definition 3.1).

Let Π denote the set of singular 3-spaces of Γ_0 and let Λ denote the set of maximal singular 1-spaces of Γ_0. By definition, each point of Γ_0 is contained in a unique element of Π and a unique member of Λ.

Proposition 4.2. *Suppose* $\Gamma_0 \cong \mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{3,1}(\mathbb{K})$ *embeds fully and isometrically in* $\Delta_0 \cong \mathsf{D}_{5,5}(\mathbb{K})$. *Then there exist opposite maximal singular* 3*-spaces* U, U' *in* Δ_0 *such that* Γ_0 *coincides with the equator geometry* $E(U, U')$. *Moreover*, *if* V, V' *are opposite* 3*-spaces of* Δ_0 *such that* $\Gamma_0 = E(V, V')$, *then the lines corresponding to* U, U', V, V' *are on a regulus of* Δ_0^* (*the corresponding points in the long root geometry* $\mathsf{D}_{5,2}(\mathbb{K})$ *are points on an imaginary line*).

Proof. Assume first for a contradiction that some member S of Π is contained in a singular 4-space W and consider a point x of Γ_0 not in S. In Δ_0^*, the point x corresponds to a 4-space V_x, and the space W corresponds to a $4'$-space W^*. If $W^* \cap V_x = \varnothing$, then $x^\perp \cap W = \varnothing$, a contradiction. If $W^* \cap V \neq \varnothing$, then $x^\perp \cap W$ contains a plane, implying $x^\perp \cap S$ contains a line, also a contradiction since this would mean that the embedding is not isometric. Hence each member of Π is a maximal singular subspace. It is easy to see that distinct members of Π are opposite. We let Π^* be the corresponding set of lines of Δ_0^*. We have to show that Π^* consists of an (entire) regulus. Let X^* be the set of 4-spaces of Δ_0^* corresponding to the point set of Γ_0, and we speak of collinear members if the corresponding points of Γ_0 are collinear.

Take $L_1, L_2 \in \Pi^*$ arbitrary, $L_1 \neq L_2$. For each member $V_1 \in X^*$ through L_1, there exists a unique $V_2 \in X^*$ through L_2 collinear to V_1. Denote the regulus containing L_1 and L_2 by $\mathscr{R}$. Then containment is a bijective correspondence between the members of $\mathscr{R}$ and the members of X^* containing $V_1 \cap V_2$, and these members correspond to the points on the line of Γ_0 defined by V_1 and V_2. Replacing L_2 with any other member of $\mathscr{R} \setminus \{L_1\}$, this argument also proves that X^* consists of all 4-spaces containing some member of $\mathscr{R}$. This shows that $\Pi^* = \mathscr{R}$.

In view of the discussion following Definition 3.1, it is clear that Γ_0 coincides with $E(U, U')$, for each pair of maximal 3-spaces of Δ_0 with U and U' corresponding in Δ_0^* to lines of the opposite regulus to $\mathscr{R}$. □

4B. ***Full embeddings of*** $\mathsf{A}_{5,2}(\mathbb{K})$ ***in*** $\mathsf{E}_{6,1}(\mathbb{K})$**.** Next, we study the full embeddings of a geometry Γ_1 isomorphic to $\mathsf{A}_{5,2}(\mathbb{K})$ in a geometry Δ_1 isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$.

The former is a strong parapolar space of diameter 2 with symps isomorphic to $\mathsf{D}_{3,1}(\mathbb{K})$, its properties can be derived from the properties of projective spaces (the points of Γ_1 are the lines of $\mathsf{PG}(5,\mathbb{K}) \cong \mathsf{A}_{5,1}(\mathbb{K})$). For details about Δ_1 we refer to Section A3 in the Appendix. Our aim is to show that Γ_1 arises as an equator geometry of Δ_1 (see Definition 3.4).

We head off by showing that each full embedding of Γ_1 in Δ_1 is isometric.

Lemma 4.3. *Suppose* $\Gamma_1 \cong \mathsf{A}_{5,2}(\mathbb{K})$ *is fully embedded in* $\Delta_1 \cong \mathsf{E}_{6,1}(\mathbb{K})$. *Then the embedding of* Γ_1 *in* Δ_1 *is isometric.*

Proof. If every pair of points of Γ_1 is collinear in Δ_1, then Γ_1 is contained in a singular subspace, which has dimension at most 5, contradicting the fact that Γ_1 contains singular subspaces of dimension 4 which intersect in only a point. Hence some point pair of Γ_1, say $\{p, q\}$, is symplectic in Δ_1, and also in Γ_1 of course. We denote the line in $\mathsf{PG}(5,\mathbb{K})$ corresponding to a point x of $\Gamma_1 \cong \mathsf{A}_{5,2}(\mathbb{K})$ by L_x. Then L_p and L_q span a 3-space. The symp $\xi(p, q)$ of Γ_1 is isometrically embedded in the symp $\Sigma(p, q)$ of Δ_1 by Fact 2.1 (see also the notation below). Let L be a line of Γ_1 through q not contained in $\xi(p, q)$. We claim that L is not contained in $\Sigma(p, q)$.

Indeed, suppose for a contradiction it is. Note that p is Δ_1-collinear to a unique point w of L (clearly, $q \neq w$), so the symp $\xi(p, w)$ embeds in a singular 5-space S of Δ_1. The corresponding 3-space $\langle L_p, L_w\rangle$ in $\mathsf{PG}(5,\mathbb{K})$ then meets $\langle L_p, L_q\rangle$ in a plane π, generated by L_p and the point $L_q \cap L_w$. Let v be any point of $\xi(p, w)$, not Γ_1-collinear to p. Then $L_v \subseteq \langle L_p, L_w\rangle$ meets π in a point not on L_p. Now take a point q' in $\xi(p, q)$ such that $L_{q'}$ meets π precisely in the point $L_v \cap \pi$. Then q' is Γ_1-collinear to v and not Γ_1-collinear to p. Since p and q' are not Δ_1-collinear either, we obtain that $v \in p^{\perp_{\Delta_1}} \cap q'^{\perp_{\Delta_1}} \subseteq \Sigma(p, q') = \Sigma(p, q)$. Since v was any point in $\xi(p, w) \setminus \{p^{\perp_{\Gamma_1}}\}$ and the latter generates S, we obtain that S is a singular 5-space in $\Sigma(p, q) \cong \mathsf{D}_{5,1}(\mathbb{K})$, a contradiction. The claim follows.

Let x be a point of Γ_1 which is Γ_1-symplectic to p. We show that x is also Δ_1-symplectic to p. Now either x is Γ_1-collinear to a plane π of $\xi(p, q)$, or it is Γ_1-symplectic to all points of $\xi(p, q)$. In the first case, π contains a point q' not Γ_1-collinear to p and the above argument with q' instead of q applies. In the second case, each Γ_1-line through x contains a unique point x' which is collinear to a plane π' of $\xi(p, q)$, and we can choose this such that $p \notin \pi$. We just proved that $\xi(p, x')$ is isometric, and replacing q with such x' in the above argument, we see that also $\xi(p, x)$ is isometric. Connectivity of the graph on the points of Γ_1, adjacent if symplectic, now completes the proof of the lemma. □

Knowing that the embedding of Γ_1 in Δ_1 is isometric, we can now show the analogue of Proposition 4.2 for Γ_1 in Δ_1.

Proposition 4.4. *Suppose* $\Gamma_1 \cong \mathsf{A}_{5,2}(\mathbb{K})$ *is fully embedded in* $\Delta_1 \cong \mathsf{E}_{6,1}(\mathbb{K})$. *Then there are opposite* 5*-spaces* W *and* W' *in* Δ_1 *such that* Γ_1 *coincides with the equator* *geometry $E(W, W')$. Moreover, if V, V' are opposite 5-spaces of Δ_1 such that $\Gamma_1 = E(V, V')$, then V, V' are maximal singular 5-spaces of the unique Segre variety $\mathscr{S}_{1,5}(\mathbb{K})$ in Δ_1 determined by W and W' (the points corresponding to W, W', V, V' in the long root geometry $\mathsf{E}_{6,2}(\mathbb{K})$ are on an imaginary line).*

Proof. By Lemma 4.3, $\perp_{\Delta_1} = \perp_{\Gamma_1}$, so we will denote the collinearity relation in both geometries just by $\perp$. Let p be a point of Γ_1 and ξ a symp of Γ_1 such that $p^\perp \cap \xi$ is empty. Then, if Σ is the unique symp of Δ_1 containing ξ, also $p^\perp \cap \Sigma = \varnothing$, for otherwise the $4'$-space $p^\perp \cap \Sigma$ would intersect ξ in at least a point; recall from Fact 2.1 that we can think of Σ as a hyperbolic quadric in $\mathsf{PG}(9, \mathbb{K})$ and of ξ as the intersection of that quadric with a 5-dimensional subspace of $\mathsf{PG}(9, \mathbb{K})$.

Step 1. Determining poles W, W' for the equator geometry. Define Δ_0^p as the point-line geometry induced by the points of $p^\perp$ which are close to Σ and let Γ_0^p be the subgeometry $\Delta_0^p \cap \Gamma_1$. Note that by the argument of the previous paragraph, the points of Γ_0^p are also close to ξ. Clearly, $\Gamma_0^p \cong \mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{3,1}(\mathbb{K})$ is fully embedded in $\Delta_0^p \cong \mathsf{D}_{5,5}(\mathbb{K})$, and this embedding is isometric by Lemma 4.3. As such, Proposition 4.2 yields opposite 3-spaces U and U' in Δ_0^p such that $\Gamma_0^p = E(U, U')$, i.e., each point of Γ_0^p is collinear to a plane of U and U'. Now each point y of Σ (resp. ξ) is collinear to a symp of Δ_0^p (resp. Γ_0^p), namely $p^\perp \cap y^\perp$ (resp. $p^\perp \cap y^\perp \cap \Gamma_0'$), and the induced map is a bijection since each symp of Δ_1 (resp. of Γ_1) through p meets Σ (resp. ξ) in a point. Moreover, by Fact A.13 this map, which is given by collinearity, induces an isomorphism between Δ_0^p and Σ. Therefore, there are unique singular lines M and M' in Σ which are collinear to U and U', and M and M' are opposite in Σ.

Consider the singular 5-spaces $W := \langle U, M\rangle$ and $W' := \langle U', M'\rangle$. We claim that W and W' are opposite 5-spaces in Δ_1. Firstly, they are disjoint, since by convexity their intersection would belong to Σ (and since $\Sigma \cap W = M$ and $\Sigma \cap W' = M'$ are disjoint, this is not possible). Secondly, if not opposite, then the 5–5 relations (Fact A.10) imply that there is a unique plane π in W collinear to a plane π' in W'. This plane π shares at least a point z with U, and $\pi' \cap U'$ is then necessarily the unique point z' of U' collinear to z. Now take a point $y \in U \setminus \pi$. Then the symp determined by y and z' contains π and hence it intersects W in a $4'$-space by the point–5 relations. The latter $4'$-space shares at least a plane with U and therefore z is not the unique point of U collinear to z' after all, a contradiction. The claim follows.

Step 2. Showing that $\Gamma_1 = E(W, W')$. To this end, take any point y in ξ and consider the symp $\xi' = \xi(p, y)$ of Γ_1 determined by p, y and let Σ' be the symp of Δ_1 in which ξ' embeds isometrically, which is also determined by p and y. As explained in Step 1, ξ' meets Γ_0^p in the hyperbolic polar space G of rank 2 and Σ' meets Δ_0^p in a symp $\overline{\Sigma}$ isomorphic to $\mathsf{D}_{4,1}(\mathbb{K})$. Since $G \subseteq E(U, U')$, there are unique lines L and L' in U and U', respectively, with $L \perp G \perp L'$. Therefore, $\overline{\Sigma}$ contains L and L', and hence $L \perp y \perp L'$. We claim that y is also collinear to M and M'. If not, consider a point $y' \in M$ not collinear to y. Then Σ, the unique symp containing y and y', would also contain L, a contradiction. The claim follows. Now each point of ξ' is collinear to L and hence, by the point–5 relations (Fact A.11), is collinear to a 3-space of W. Since each point of Γ_1 lies on a symp through p which meets ξ in a point y', and since W and W' play the same role, we obtain that $\Gamma_1 \subseteq E(W, W')$. Let $z \in \Gamma_1$ be arbitrary. Since each point of $E(W, W')$ collinear to z is collinear to a plane of W_z and a plane of W'_z, it follows that $\mathrm{Res}_{E(W,W')}(z)$, and in particular $\mathrm{Res}_{\Gamma_1}(z)$, is contained in the equator geometry of $\mathrm{Res}_{\Delta_1}(z)$ having as poles the 3-spaces corresponding to $\langle z, W_z\rangle$ and $\langle z, W'_z\rangle$. Lemmas 4.2 and 4.3 imply that $\mathrm{Res}_{E(W,W')}(z) = \mathrm{Res}_{\Gamma_1}(z)$. By Lemma 2.3, we conclude that $\Gamma_1 = E(W, W')$.

The last statement follows from the construction and from the last statement of Proposition 4.2. □

We record a consequence of this that will be useful in the next subsection, when studying the embedding of $\Gamma_2 \cong \mathsf{D}_{6,6}(\mathbb{K})$ in $\Delta_2 \cong \mathsf{E}_{7,7}(\mathbb{K})$.

Corollary 4.5. *Suppose $\Gamma_1 \cong \mathsf{A}_{5,2}(\mathbb{K})$ is fully embedded in $\Delta_1 \cong \mathsf{E}_{6,1}(\mathbb{K})$. Then, for two distinct symps ξ_1 and ξ_2 of Γ_1, embedded in respective symps Σ_1 and Σ_2 of Δ_1, we have that $\Sigma_1 \neq \Sigma_2$. Moreover, $\Sigma_1 \cap \Sigma_2$ is a point if and only if $\xi_1 \cap \xi_2$ is a point. If p is a point of Γ_1 such that $p^\perp \cap \xi_1 = \varnothing$, then $p^\perp \cap \Sigma_1 = \varnothing$.*

Proof. By Proposition 4.4, there are opposite 5-spaces W and W' in Δ_1 such that $\Gamma_1 = E(W, W')$. This yields unique lines L_i in W and L'_i in W' with $L_i \perp \xi_i \perp L'_i$, for $i = 1, 2$. Clearly, $L_i \cup L'_i \subseteq \Sigma_i$; moreover, L_i, L'_i and ξ_i generate Σ_i and $\Sigma_i \cap W = L_i$. The correspondence between $\Gamma_1 = E(W, W')$ and W is such that, if $\xi_1 \cap \xi_2$ is a unique point p, then L_1 and L_2 are disjoint, and if $\xi_1 \cap \xi_2$ is a plane, then L_1 and L_2 intersect in a point. In particular, since $\Sigma_i \cap W = L_i$, we have that $\xi_1 \neq \xi_2$ implies $\Sigma_1 \neq \Sigma_2$. So, if $\xi_1 \cap \xi_2$ is a plane, the symp–symp relations of Δ_1 immediately imply that $\Sigma_1 \cap \Sigma_2$ is a 4-space. If $\xi_1 \cap \xi_2 = \{p\}$, suppose for a contradiction that Σ_1 and Σ_2 share a 4-space $V \ni p$. Then a point $q_1 \perp p$ of ξ_1 is Γ_1-collinear to a plane of ξ_2 and Δ_1-collinear to a 3-space of $\Sigma_2 \cap \Sigma_1$, and one can easily choose q_1 in such a way that those two singular subspaces of Σ_2 share exactly p. Hence q_1 is Δ_1-collinear to a 5-space of Σ_2, obviously a contradiction.

For the final statement, see the first paragraph of the proof of Proposition 4.4. □

4C. *Full embeddings of* $\mathsf{D}_{6,6}(\mathbb{K})$ *in* $\mathsf{E}_{7,7}(\mathbb{K})$. We study the full embeddings of a geometry Γ_2 isomorphic to $\mathsf{D}_{6,6}(\mathbb{K})$ in a geometry Δ_2 isomorphic to $\mathsf{E}_{7,7}(\mathbb{K})$. The former is a strong parapolar space of diameter 3 with symps isomorphic to polar spaces isomorphic to $\mathsf{D}_{4,1}(\mathbb{K})$; for more details we refer to Section 3.2 of [De Schepper et al. 2022]. (The properties of $\mathsf{D}_{6,6}(\mathbb{K})$ can also be verified using the corresponding polar space $\mathsf{D}_{6,1}(\mathbb{K})$.) For details about the latter geometry, $\mathsf{E}_{7,7}(\mathbb{K})$, we refer to Section A3 in the Appendix. Again, our aim is to show that Γ_2 arises as an equator geometry of Δ_2 (see Definition 3.6). We first show that each full embedding of Γ_2 in Δ_2 is isometric.

Lemma 4.6. *Suppose $\Gamma_2 \cong \mathsf{D}_{6,6}(\mathbb{K})$ is fully embedded in $\Delta_2 \cong \mathsf{E}_{7,7}(\mathbb{K})$. Then the embedding of Γ_2 in Δ_2 is isometric.*

Proof. Let p, q be points of Γ_2 and suppose for a contradiction that $d_{\Delta_2}(p, q) < d_{\Gamma_2}(p, q)$. By definition, Γ_2-collinear points are Δ_2-collinear. So suppose first that p and q are symplectic in Γ_2. Then they are contained in a symp ξ of Γ_2. Since no singular subspace of Δ_2 is large enough to contain a polar space isomorphic to $\mathsf{D}_{4,1}(\mathbb{K})$, ξ embeds isometrically in a symp of Δ_2. In particular, p and q are also symplectic in Δ_2.

Now suppose p and q are Γ_2-opposite points and consider an arbitrary line L of Γ_2 through p. The line L contains a unique point r which is Γ_2-symplectic to q and hence also Δ_2-symplectic. Let ξ be the symp of Γ_2 determined by q and r and let Σ be the corresponding symp in Δ_2. Observe that $p^{\perp_{\Gamma_2}} \cap \xi = \{r\}$ because p and q are Γ_2-opposite. Hence, in $\mathrm{Res}_{\Delta_2}(r)$, the point corresponding to pr is far from the symp corresponding to Σ, by Corollary 4.5. Hence $p^{\perp_{\Delta_2}} \cap \Sigma = \{r\}$. Fact A.15(i), together with r and q being Δ_2-symplectic by the above, implies that p and q are Δ_2-opposite. □

Knowing this, we can show that a fully embedded Γ_2 in Δ_2 arises as an equator geometry. The global strategy of the proof is the same as that of Proposition 4.4, yet locally the proofs have differences.

Proposition 4.7. *Suppose $\Gamma_2 \cong \mathsf{D}_{6,6}(\mathbb{K})$ is fully embedded in $\Delta_2 \cong \mathsf{E}_{7,7}(\mathbb{K})$. Then there are opposite symps Σ and Σ' in Δ_2 such that Γ_2 coincides with the equator geometry $E(\Sigma, \Sigma')$. Moreover, if Σ'', Σ''' are opposite symps of Δ_2 such that $\Gamma_2 = E(\Sigma'', \Sigma''')$, then the points corresponding to $\Sigma, \Sigma', \Sigma'', \Sigma'''$ in the long root geometry $\mathsf{E}_{7,1}(\mathbb{K})$ are on an imaginary line.*

Proof. Let p, q be opposite points of Γ_2, which are hence also opposite in Δ_2 by Lemma 4.6. The same lemma allows us to speak about distances without referring to Γ_2 or Δ_2. In particular, we use $\perp$ to denote the collinearity relation.

Step 1. Determining poles Σ, Σ' for the equator geometry. Define Δ_1^p as the point-line geometry induced by the points of $p^\perp$ which are at distance 2 of q and Γ_1^p as the subgeometry of Δ_1^p obtained by intersecting Δ_1^p with Γ_2. Then $\Gamma_1^p \cong \mathsf{A}_{5,2}(\mathbb{K})$ is fully embedded in $\Delta_1^p \cong \mathsf{E}_{6,1}(\mathbb{K})$. Likewise, we define Δ_1^q and Γ_1^q with respect to q. Proposition 4.4 yields opposite 5-dimensional subspaces W and W' in Δ_1^p such that $\Gamma_1^p = E(W, W')$, i.e., each point of Γ_1^p is collinear to a 3-space of W and W'. Note that W and W' are $5'$-spaces in Δ_2 as they are nonmaximal (they are collinear to p). By Fact A.17 and its analogue for Γ_2, collinearity induces an isomorphism ρ between Δ_1^p and Δ_1^q; which, restricted to Γ_1^p, gives an isomorphism between Γ_1^p and Γ_1^q. Define $V = \rho(W)$ and $V' = \rho(W')$. Then the $5'$-spaces V and V' are opposite in Δ_1^q. Taking a pair of noncollinear points $r \in W$ and $s \in V$, we obtain that the unique symp Σ determined by r and s contains $U \cup W$, since r is collinear to a hyperplane of V, and s to a hyperplane of W. Likewise, there is a unique symp Σ' containing $U' \cup V'$.

We claim that Σ and Σ' are opposite symps of Δ_2. Firstly, suppose there would be a point $z \in \Sigma \cap \Sigma'$. Now z, contained in $\Sigma \cap \Sigma'$, is collinear to a hyperplane W_z of W and a hyperplane W'_z of W', implying that p and z are symplectic: firstly, $p \neq z$ because $p \perp W$; secondly, $z \notin p^\perp$, as this would yield a point z' in Δ_1^p collinear to W_z and W'_z, a contradiction to the point–5 relations in Δ_1^p (Fact A.11). The unique symp determined by p and z contains W_z and W'_z, contradicting the fact that W and W' are opposite 5-spaces in Δ_1^p. So $\Sigma \cap \Sigma' = \varnothing$ indeed. Secondly, if Σ and Σ' are not opposite, then the symp–symp relations (Fact A.16) imply that there is a symp Σ'' meeting both Σ and Σ' in 5-spaces Z and Z', respectively. Then Z and W share at least a point z since they are 5-dimensional subspaces of different types in Σ. Since $z^\perp \cap Z'$ is a 4-space, the point–symp relations of Δ_2 (Fact A.15) imply that $z^\perp \cap \Sigma'$ is a $5'$-space. However, z is collinear to a unique point of W', which yields the absurdity that the two $5'$-spaces W' and $z^\perp \cap \Sigma'$ of Σ' would share exactly one point. The claim follows.

Step 2. Showing that $\Gamma_2 = E(\Sigma, \Sigma')$. Since clearly $p^{\perp_{\Gamma_2}} \cup q^{\perp_{\Gamma_2}}$ generates Γ_2 as a subspace of itself, and hence as a subspace of Δ_2, by [Blok and Brouwer 1998; Cooperstein and Shult 1997], and since equator geometries are subspaces, it already follows that $\Gamma_2 \subseteq E(\Sigma, \Sigma')$. We now show equality.

Let $z \in \Gamma_2$ be arbitrary and let $z' \in \Gamma_2$ be Γ_2-collinear to z. Let W_z and W'_z be the respective $5'$-spaces $z^\perp \cap \Sigma$ and $z^\perp \cap \Sigma'$. Then, since $z, z' \in E(\Sigma, \Sigma')$, it follows from Lemma 3.7 that $z'^\perp \cap W_z$ and $z'^\perp \cap W'_z$ are 3-spaces of W_z and W'_z, respectively. Hence $\mathrm{Res}_{\Gamma_2}(z)$ fully embeds in the equator geometry of $\mathrm{Res}_{\Delta_2}(z)$, with poles the 5-spaces corresponding to $\langle z, W_z\rangle$ and $\langle z, W'_z\rangle$. By Proposition 4.4, we obtain $\mathrm{Res}_{\Gamma_2}(z) = E(W_z, W'_z)$. Now Lemma 2.3 shows that $\Gamma_2 = E(\Sigma, \Sigma')$.

The last statement follows from the construction and from the last statement of Proposition 4.4. □

4D. *Full embeddings of* $\mathsf{E}_{7,1}(\mathbb{K})$ *in* $\mathsf{E}_{8,8}(\mathbb{K})$. Finally, we study the full embeddings of a geometry Γ_3 isomorphic to $\mathsf{E}_{7,1}(\mathbb{K})$ in a geometry Δ_3 isomorphic to $\mathsf{E}_{8,8}(\mathbb{K})$. Both are nonstrong parapolar spaces of diameter 3. For more details about these geometries, we refer to Section 3.4 of [De Schepper et al. 2022] and Section A4. Our aim is to show that Γ_3 arises as an equator geometry of Δ_3 (see Definition 3.8). Once more, we first show that each full embedding of Γ_3 in Δ_3 is isometric.

Lemma 4.8. *Suppose* $\Gamma_3 \cong \mathsf{E}_{7,1}(\mathbb{K})$ *is fully embedded in* $\Delta_3 \cong \mathsf{E}_{8,8}(\mathbb{K})$. *Then the embedding of* Γ_3 *in* Δ_3 *is isometric.*

Proof. Let p, q be points of Γ_3. If $p \perp\!\!\!\perp_{\Gamma_3} q$ then also $p \perp\!\!\!\perp_{\Delta_3} q$ since each symp of Γ_3 embeds isometrically in a symp of Δ_3 (no singular subspace of Δ_3 is large enough to contain a symp of Γ_3). If p and q are special in Γ_3, say with $r = p \bowtie q$, then in $\mathrm{Res}_{\Gamma_3}(r) \cong \mathsf{D}_{6,6}(\mathbb{K})$, the points p and q are at distance 3, and hence by Lemma 4.6, p and q are also at distance 3 in $\mathrm{Res}_{\Delta_3}(r) \cong \mathsf{E}_{7,7}(\mathbb{K})$. We conclude that p and q are also special in Δ_3. Since Γ_3 and Δ_3 are both long root geometries, Fact A.23 now implies that opposite points in Γ_3 are also opposite in Δ_3. □

Proposition 4.9. *Suppose* $\Gamma_3 \cong \mathsf{E}_{7,1}(\mathbb{K})$ *is fully embedded in* $\Delta_3 \cong \mathsf{E}_{8,8}(\mathbb{K})$. *Then there are opposite points* x *and* x' *in* Δ_3 *such that* Γ_3 *coincides with the equator geometry* $E(x, x')$. *Moreover, if* y, y' *are opposite points of* Δ_3 *such that* $\Gamma_3 = E(y, y')$, *then* x, x', y, y' *are on an imaginary line.*

Proof. Let p, q be opposite points of Γ_3, which are hence also opposite in Δ_3 by Lemma 4.6. The same lemma allows us to speak about distances without referring to Γ_3 or Δ_3. In particular, we use $\perp$ to denote the collinearity relation.

Step 1. Determining poles x, x' *for the equator geometry.* Define Δ_2^p as the point-line geometry induced by the points of $p^\perp$ which are special to q and Γ_2^p as the subgeometry of Δ_2^p obtained by intersecting Δ_2^p with Γ_3. Then $\Gamma_2^p \cong \mathsf{D}_{6,6}(\mathbb{K})$ is fully embedded in $\Delta_2^p \cong \mathsf{E}_{7,7}(\mathbb{K})$. Likewise, we define Δ_2^q and Γ_2^q with respect to q. In this case, collinearity induces an isomorphism ρ between Δ_2^p and Δ_2^q, and its restriction to Γ_2^p gives an isomorphism between Γ_2^p and Γ_2^q. Proposition 4.7 yields opposite symps Σ_p and Σ'_p in Δ_2^p such that $\Gamma_2^p = E(\Sigma_p, \Sigma'_p)$; each point of Γ_2^p is collinear to a $5'$-space of Σ_p and Σ'_p. Let $\overline{\Sigma}_p$ and $\overline{\Sigma}'_p$ denote the corresponding symps of Δ_3 through p. According to the point–symp relations in Δ_3 (Fact A.22), q is symplectic to unique points x, x' of $\overline{\Sigma}_p$ and $\overline{\Sigma}'_p$, respectively. Let $\overline{\Sigma}_q$ and $\overline{\Sigma}'_q$ denote the symps determined by q and x and by q and x', respectively, and consider the induced symps Σ_q and Σ'_q in Δ_2^q, i.e., $\Sigma_q = x^\perp \cap q^\perp$ and $\Sigma'_q = x'^\perp \cap q^\perp$. Since Δ_2^p and Δ_2^q are disjoint, we have $\overline{\Sigma}_p \cap \overline{\Sigma}_q = \{x\}$. Since $\overline{\Sigma}_p \cup \overline{\Sigma}_q$ contains the opposite points p and q, $\overline{\Sigma}_p$ and $\overline{\Sigma}_q$ are locally opposite (see (iv) and (v) of Fact A.16). This implies that each point of Σ_p is collinear to a unique point of Σ_q (and this correspondence is an isomorphism), i.e., $\rho(\Sigma_p) = \Sigma_q$. Analogous statements hold for $\overline{\Sigma}'_p$ and $\overline{\Sigma}'_q$ and for Σ'_p and Σ'_q. Moreover, the symps $\overline{\Sigma}_p$ and $\overline{\Sigma}'_p$ are locally opposite since Σ_p and Σ'_p are opposite in Δ_2^p. Consequently, by Fact A.22(iv), it follows that x and x' are opposite.

Step 2. Showing that $\Gamma_3 = E(x, x')$. Firstly, p and q are symplectic to both x and x' by construction. Moreover, each point y of Γ_2^p is collinear to a $5'$-space W_y^p of Σ_p which is in turn collinear to x, and hence x and y are symplectic (they cannot be collinear since points collinear to both x and p belong to Σ_p, which is disjoint from Γ_2^p). Moreover, if y' is the unique point of Γ_2^q collinear to y, i.e., $y' = \rho(y)$, then y' is collinear to the $5'$-space $W_y^q := \rho(W_y^p)$, which belongs to $\Sigma_q = \rho(\Sigma_p)$. From this we deduce that $\Gamma_2^q = E(\Sigma_q, \Sigma_q')$ and that also the points of Γ_2^q are symplectic to both x and x'.

Let z be a point of Γ_3 collinear to q and opposite p. The points x and x' are the unique points of $\overline{\Sigma}_p$ and $\overline{\Sigma}'_p$ symplectic to q, respectively. Noting that x and x' are also symplectic to z (observe that $q^{\perp_{\Gamma_2}} = \langle q, \Gamma_2^q \rangle \subseteq E(x, x')$), and that the definition of $\overline{\Sigma}_p$ does not depend on q, that is, $z^{\bowtie} \cap p^{\perp} \cap \overline{\Sigma}_p$ and $z^{\bowtie} \cap p^{\perp} \cap \overline{\Sigma}'_p$ are poles for $z^{\bowtie} \cap p^{\perp} \cap \Gamma_2$, the foregoing implies that also $z^{\perp_{\Gamma_2}} \subseteq E(x, x')$. By connectedness of Γ_2, we obtain that $\Gamma_2 \subseteq E(x, x')$. Just like in Lemmas 4.4 and 4.7, we conclude that $\Gamma_3 = E(x, x')$.

The last statement follows from the construction and from the last statement of Proposition 4.7. □

This finishes the proof of Main Result 4.1 (see Propositions 4.9, 4.7, 4.4 and 4.2).

5. Uniqueness of equator geometry isomorphic to $\mathsf{E}_{6,2}(\mathbb{K})$ in $\mathsf{E}_{8,8}(\mathbb{K})$

The goal of this section is to show that a long root geometry isomorphic to $\mathsf{E}_{6,2}(\mathbb{K})$ has, up to projectivity, a unique full embedding in a long root geometry isomorphic to $\mathsf{E}_{8,8}(\mathbb{K})$. We accomplish this inductively, giving rise to a sequence of full embeddings as mentioned in the introduction.

Main Result 5.1. *Let Υ_1, Υ_2, Υ_3 be point-line geometries isomorphic to* $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$, $\mathsf{A}_{5,3}(\mathbb{K})$ *and* $\mathsf{E}_{6,2}(\mathbb{K})$, *respectively*; *and let Δ_1, Δ_2, Δ_3 be point-line geometries isomorphic to* $\mathsf{E}_{6,1}(\mathbb{K})$, $\mathsf{E}_{7,7}(\mathbb{K})$ *and* $\mathsf{E}_{8,8}(\mathbb{K})$, *respectively. Let $i \in \{1, 2, 3\}$ and suppose Υ_i is fully embedded in Δ_i. If $i = 1$, suppose additionally that this embedding is isometric. Then this embedding is unique up to a projectivity of Δ_i and arises as the intersection of two equator geometries of Δ_i isomorphic to* $\mathsf{A}_{5,2}(\mathbb{K})$ *if $i = 1$,* $\mathsf{D}_{6,6}(\mathbb{K})$ *if $i = 2$ and* $\mathsf{E}_{7,1}(\mathbb{K})$ *if $i = 3$.*

For the $i = 1$ case, we already remarked that there are also nonisometric full embeddings of Υ_1 in Δ_1. These will be discussed in fair detail in the next section.

5A. *Full embeddings of* $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$ *in* $\mathsf{E}_{6,1}(\mathbb{K})$.

We discuss all full embeddings of the point-line geometry $\Upsilon_1 \cong \mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$ in the point-line geometry $\Delta_1 \cong \mathsf{E}_{6,1}(\mathbb{K})$, giving rise to four additional cases in which the embedding is not isometric. We do not claim that we classify up to projectivity; we only classify up to distinction of some specific features. This will be enough to help us in proving that a full embedding of $\Upsilon_2 \cong \mathsf{A}_{5,3}(\mathbb{K})$ in $\Delta_2 \cong \mathsf{E}_{7,7}(\mathbb{K})$ is isometric (see Lemma 5.30).

We start with some examples of nonisometric full embeddings of Υ_1 into Δ_1. For this we will use the absolute universal embeddings of both geometries, which are given by the *Segre variety* $\mathscr{S}_{2,2}(\mathbb{K})$ in $\mathsf{PG}(8,\mathbb{K})$ and the *Cartan variety* $\mathscr{E}_{6,1}(\mathbb{K})$ in $\mathsf{PG}(26,\mathbb{K})$, respectively. Below we briefly give a coordinate description of both varieties; for more information we refer to Section A1 and [Van Maldeghem and Victoor 2022].

5A1. *Four classes of examples.* Recall, on the one hand, that the Segre variety $\mathscr{S}_{2,2}(\mathbb{K})$ over the field $\mathbb{K}$ is defined by the image of the Segre map

$$\sigma : \mathsf{PG}(2,\mathbb{K}) \times \mathsf{PG}(2,\mathbb{K}) \to \mathsf{PG}(8,\mathbb{K}),$$
$$((x,y,z),(a,b,c)) \mapsto (ax,ay,az,bx,by,bz,cx,cy,cz).$$

This image is given by the points $(a_{11},a_{12},a_{13},a_{21},a_{22},a_{23},a_{31},a_{32},a_{33})$ with $\mathrm{rk}(a_{ij})_{1\le i,j\le 3} = 1$, the rank 1 matrices, as is well known. Hence $\mathscr{S}_{2,2}(\mathbb{K})$ is the intersection of the quadrics with equation $x_{ij}x_{k\ell} - x_{i\ell}x_{kj} = 0$ (with self-explaining notation) for all $1 \le i < k \le 3$ and $1 \le j < \ell \le 3$.

Example 5.2. It is obvious that, with the above notation, $\mathscr{S}_{2,2}(\mathbb{K})$ is contained in the quadric of $\mathsf{PG}(8,\mathbb{K})$ with equation

$$x_{11}x_{22} + x_{22}x_{33} + x_{33}x_{11} = x_{12}x_{21} + x_{23}x_{32} + x_{31}x_{13},$$

which is easily shown to be a nondegenerate parabolic quadric (indeed, apply the coordinate transformation $x'_{11} = x_{11} + x_{33}$, $x'_{22} = x_{22} + x_{33}$ to see this). Since this parabolic quadric is contained in a hyperbolic quadric isomorphic to $\mathsf{D}_{5,1}(\mathbb{K})$ as a geometric hyperplane, we obtain an embedding of Υ_1 into $\mathsf{D}_{5,1}(\mathbb{K})$ and hence also in $\mathsf{E}_{6,1}(\mathbb{K})$.

On the other hand, an embedding $\mathscr{E}_{6,1}(\mathbb{K})$ of $\mathsf{E}_{6,1}(\mathbb{K})$ into $\mathsf{PG}(26,\mathbb{K})$ is given by the intersection of 27 quadrics as follows. Label the coordinates of a 27-dimensional vector space over $\mathbb{K}$ by $(u,v,w;U,V,W)$, where $U = (u_i)_{0\le i\le 7}$, $V = (v_i)_{0\le i\le 7}$, $W = (w_i)_{0\le i\le 7}$ belong to the *split octonion algebra over* $\mathbb{K}$, that is, an 8-dimensional algebra with multiplication defined by, using the above notation,

$$\begin{aligned} UV = (&u_0v_0 + u_4v_1 + u_5v_2 + u_6v_3, \quad u_1v_0 + u_7v_1 - u_5v_6 + u_6v_5,\\ &u_2v_0 + u_7v_2 + u_4v_6 - u_6v_4, \quad u_3v_0 + u_7v_3 - u_4v_5 + u_5v_4,\\ &u_0v_4 + u_4v_7 + u_2v_3 - u_3v_2, \quad u_0v_5 + u_5v_7 - u_1v_3 + u_3v_1,\\ &u_0v_6 + u_6v_7 + u_1v_2 - u_2v_1, \quad u_1v_4 + u_2v_5 + u_3v_6 + u_7v_7). \end{aligned}$$

We also define $\overline{U} = (u_7, -u_1, -u_2, -u_3, -u_4, -u_5, -u_6, u_0)$. Writing the (central) element $(k,0,0,0,0,0,0,k)$, $k \in \mathbb{K}$, of the octonion algebra briefly as k, the equations of the 27 quadrics are given in short hand notation (each of the equations on the second row represents eight equations over $\mathbb{K}$) by

$$vw = U\overline{U}, \quad wu = V\overline{V}, \quad uv = W\overline{W}, \tag{1}$$

$$VW = u\overline{U}, \quad WU = v\overline{V}, \quad UV = w\overline{W}. \tag{2}$$

The lines of $\mathscr{E}_{6,1}(\mathbb{K})$ are precisely the lines of $\mathsf{PG}(26, \mathbb{K})$ that are fully contained in $\mathscr{E}_{6,1}(\mathbb{K})$. Consequently, two points $(u, v, w; U, V, W)$ and $(u', v', w'; U', V', W')$ of $\mathscr{E}_{6,1}(\mathbb{K})$ are collinear if, and only if,

$$\begin{gathered} vw' + v'w = U\overline{U}' + U'\overline{U}, \quad wu' + w'u = V\overline{V}' + V'\overline{V}, \\ uv' + u'v = W\overline{W}' + W'\overline{W}, \end{gathered} \tag{3}$$

$$\begin{gathered} VW' + WV' = u\overline{U}' + u'\overline{U}, \quad WU' + W'U = v\overline{V}' + v'\overline{V}, \\ UV' + U'V = w\overline{W}' + w'\overline{W}. \end{gathered} \tag{4}$$

Denote by p, q, r, p_i, q_i, r_i the base points corresponding, for $0 \le i \le 7$, to the coordinate u, v, w, u_i, v_i, w_i, respectively. All base points belong to $\mathscr{E}_{6,1}(\mathbb{K})$ and for a base point b with corresponding coordinate c, the set $b^\perp$ of points of $\mathsf{E}_{6,1}(\mathbb{K})$ is given by intersecting $\mathscr{E}_{6,1}(\mathbb{K})$ with the subspace given by setting to zero all coordinates d such that $\pm cd$ appears as a term in some equation of (1) or (2). For instance, $p^\perp$ is obtained by intersecting $\mathscr{E}_{6,1}(\mathbb{K})$ with $v = w = u_0 = \cdots = u_7 = 0$.

Example 5.3. The following form of the Segre map embeds $\mathscr{S}_{2,2}(\mathbb{K})$ into $\mathscr{E}_{6,1}(\mathbb{K})$, more exactly in $p^\perp$:

$$\begin{aligned} &\rho_1 : \mathsf{PG}(2, \mathbb{K}) \times \mathsf{PG}(2, \mathbb{K}) \to \mathsf{PG}(26, \mathbb{K}), \\ &\quad ((x, y, z), (a, b, c)) \mapsto \big(0, 0, 0; \underbrace{0, \dots, 0}_{8 \text{ times}}, ax, bx, by, bz, 0, -az, ay, 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad 0, cx, cy, cz, 0, 0, 0, 0\big). \end{aligned}$$

This also defines an embedding of $\Upsilon_1 \cong \mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$ into $\mathsf{D}_{5,5}(\mathbb{K})$. We note that $\mathrm{Im}(\rho_1)$ is also contained in $q_0^\perp$ (note that q_0 belongs to Υ_1) and so all symps of Υ_1 through q_0 are singular. One can check that no other singular symp exists.

The base points p_1, p_2, p_3 are pairwise collinear in $\mathsf{E}_{6,1}(\mathbb{K})$. Their common perp is obtained by intersecting $\mathscr{E}_{6,1}(\mathbb{K})$ with the subspace given by

$$u = u_i = v_j = w_k = 0, \quad \text{for } i = 4, 5, 6;\ j = 0, \dots, 6;\ k = 1, \dots, 7.$$

Example 5.4. The following form of the Segre map embeds $\mathscr{S}_{2,2}(\mathbb{K})$ into $\mathscr{E}_{6,1}(\mathbb{K})$, more exactly in $\{p_1, p_2, p_3\}^\perp$:

$$\begin{aligned} &\rho_2 : \mathsf{PG}(2, \mathbb{K}) \times \mathsf{PG}(2, \mathbb{K}) \to \mathsf{PG}(26, \mathbb{K}), \\ &\quad ((x, y, z), (a, b, c)) \mapsto \big(0, bx, cz; cx, ax, ay, az, 0, 0, 0, bz, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \underbrace{0, \dots, 0}_{7 \text{ times}}, cy, by, \underbrace{0, \dots, 0}_{7 \text{ times}}\big). \end{aligned}$$

Note that each plane of the image disjoint from $\langle p_1, p_2, p_3\rangle$ generates, together with $\langle p_1, p_2, p_3\rangle$, a 5-space of $\mathscr{E}_{6,1}(\mathbb{K})$. This implies that every symp with a line in $\langle p_1, p_2, p_3\rangle$ is contained in a singular subspace of $\mathscr{E}_{6,1}(\mathbb{K})$. The fact that not all planes disjoint from $\langle p_1, p_2, p_3\rangle$ are contained in the same 5-space implies that every symp disjoint from $\langle p_1, p_2, p_3\rangle$ embeds in a (unique) symp of $\mathscr{E}_{6,1}(\mathbb{K})$. This is the situation obtained and described in Lemma 5.13.

We now describe an embedding of Υ_1 into a 5-dimensional projective space. This does not seem to work over an arbitrary field, though. However, we content ourselves with mentioning one example which works.

Example 5.5. Let $\mathbb{K}$ be a field of characteristic 3 and consider the field $\mathbb{K}(t)$ of rational functions in t over $\mathbb{K}$. Consider the following embedding ρ of $\mathscr{S}_{2,2}(\mathbb{K}(t))$ in $\mathsf{PG}(5, \mathbb{K}(t))$:

$$\begin{aligned}\rho_3 : \mathsf{PG}(2, \mathbb{K}(t)) \times \mathsf{PG}(2, \mathbb{K}(t)) &\to \mathsf{PG}(5, \mathbb{K}(t)),\\ ((x, y, z), (a, b, c)) &\mapsto \big(bx - taz, by - ax, bz - ay,\\ &\qquad cx - tay, cy - taz, cz - ax\big).\end{aligned}$$

This example arises from projecting the standard Segre variety defined by the 3×3 matrices over $\mathbb{K}$ of rank 1 from the subspace U defined by

$$\begin{pmatrix} \lambda & \mu & \nu \\ \nu t & \lambda & \mu \\ \mu t & \nu t & \lambda \end{pmatrix}, \qquad \lambda, \mu, \nu \in \mathbb{K}(t).$$

Every nonzero vector of U has determinant $\lambda^3 + t\mu^3 + t^2\nu^3$ and is hence invertible (as a matrix). Since the sum of two rank 1 matrices can never have rank 3, it follows that this projection is injective.

Finally, for completeness's sake, we describe an isometric embedding; it is obtained from the octonion representation above by restricting each octonion to the first and last coordinate.

Example 5.6. The following form of the Segre map embeds $\mathscr{S}_{2,2}(\mathbb{K})$ isometrically into $\mathscr{E}_{6,1}(\mathbb{K})$:

$$\begin{aligned}\rho_4 : \mathsf{PG}(2, \mathbb{K}) \times \mathsf{PG}(2, \mathbb{K}) &\to \mathsf{PG}(26, \mathbb{K}),\\ ((x, y, z), (a, b, c)) &\mapsto \big(ax, by, cz; bz, \underbrace{0, \dots, 0}_{6 \text{ times}}, cy,\\ &\qquad cx, \underbrace{0, \dots, 0}_{6 \text{ times}}, az, ay, \underbrace{0, \dots, 0}_{6 \text{ times}}, bx\big).\end{aligned}$$

5A2. *The main theorem.* Recall that we refer to a symp of Υ_1 which embeds isometrically in a symp of Δ_1 as an *isometric symp*, and to a symp which embeds in a singular subspace as a *singular symp*.

Proposition 5.7. *Suppose Υ_1 is a point-line geometry isomorphic to $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$, fully embedded in a point-line geometry Δ_1 isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$. Then one of the following occurs* (*and all options can occur*).

(i) *Each symp of Υ_1 is singular*, *in which case Υ_1 is contained in a singular subspace of Δ_1.*

(ii) *There is a unique plane π in Υ_1 such that Υ_1 is contained in the union of 5-spaces of Δ_1 containing π. A symp of Υ_1 is singular if and only if it contains a line of π. Moreover*, *two isometric symps of Υ_1 embed in the same symp of Δ_1 if and only if they share a line that is contained in a plane of Υ_1 disjoint from π.*

(iii) *There is a unique symp of Δ_1 containing Υ_1 and for each point p of Υ_1 there exist two isometric symps of Υ_1 that intersect in p only.*

(iv) *There is a unique point p in Υ_1 such that Υ_1 is contained in $p^{\perp}$. A symp of Υ_1 is singular if and only if it contains p.*

(v) *Each symp ξ of Υ_1 embeds isometrically in a symp Σ_ξ of Δ_1 and the map $\xi \mapsto \Sigma_\xi$ is injective and preserves the distance*: *$\xi \cap \xi'$ is a point if and only if $\Sigma_\xi \cap \Sigma_{\xi'}$ is a point. In this case*, *Υ_1 embeds isometrically in Δ_1 and arises as the intersection of equator geometries $E(U, U')$ and $E(V, V')$ where U and U' are opposite 5-spaces of Δ_1 and V and V' are opposite 5-spaces of Δ_1 such that the planes $U \cap V$ and $U' \cap V'$ are also opposite in Δ_1.*

Examples 5.2, 5.3, 5.4, 5.5 and 5.6 show that the respective cases (iii), (iv), (ii), (i) and (v) really do occur.

Structure of the proof of Proposition 5.7. In case each symp of Υ_1 embeds in a singular symp, it follows immediately that we are in case (i), because Υ_1 is a strong parapolar space of diameter 2: any pair of points of Υ_1 is contained in a symp of Υ_1 and therefore collinear in Δ_1. Also, if every symp is isometric, then the embedding is isometric and we deal with this situation in Section 5A6. Before we arrive there, we treat the mixed case (in which there are both isometric and singular symps), which leads to three distinct cases. To see how these three cases arise, we start with some general lemmas.

In Sections 5A3, 5A4 and 5A5, the standing hypothesis is that Υ_1 possesses at least one singular and at least one isometric symp. We will freely use the basic properties of Υ_1 mentioned in Section A1 of the Appendix. We also denote by Σ_ξ the unique symp of Δ_1 in which an isometric symp ξ of Υ_1 is embedded.

5A3. *General lemmas.* We start with an easy lemma.

Lemma 5.8. *If ξ is an isometric symp of Υ_1 and ξ' a singular one, with $\xi \cap \xi'$ a line, then $\xi' \subseteq \Sigma_\xi$.*

Proof. Let x' be an arbitrary point of $\xi' \setminus L$. Then x' is Υ_1-collinear to a unique point x_L of L and hence to the unique line L_x of ξ containing x_L and distinct from L. Now take a point x in $L_x \setminus \{x_L\}$. Then x' and x are collinear in Υ_1 and hence also in Δ_1. Moreover, x' is also Δ_1-collinear to all points of L. Taking a point $y_L \neq x_L$ on L, we hence obtain that $x' \in \Sigma(x, y_L) = \Sigma_\xi$. Since $x' \in \xi'$ was arbitrary, the lemma follows. □

Let p be any point of Υ_1. Then the singular lines of Υ_1 through p are contained in the union of two singular planes of Υ_1, say π_1^p and π_2^p; each symp of Υ_1 containing p has one line in each plane (see also Section A1 in the Appendix, in particular Fact A.6). The mutual position in Δ_1 of the planes π_1^p and π_2^p tells us a lot. For that, we introduce the following notion:

Notation. A line of Υ_1 with the property that each symp of Υ_1 containing that line is singular, will be called an S-line.

We study the S-lines through p in π_1^p and π_2^p.

Lemma 5.9. *Let p be a point of Υ_1 and let π_1^p and π_2^p be the unique planes of Υ_1 containing p. Then zero, one or all lines of π_i^p through p, with $i \in \{1, 2\}$, are S-lines. In case all lines of π_1^p through p are S-lines, also all lines of π_2^p through p are S-lines and then all symps through p are singular and hence $p \perp_{\Delta_1} \Upsilon_1$. In case π_1^p and π_2^p both contain a unique S-line through p, there is a unique symp Σ of Δ_1 which contains Υ_1.*

Proof. Note first that a symp ξ of Υ_1 containing p is determined by a line L_1 in π_1^p through p and a line L_2 in π_2^p through p (and L_1 and L_2 are not collinear in Υ_1). Therefore, ξ is singular if and only if $L_1 \perp_{\Delta_1} L_2$, and L_1 is an S-line if and only if L_1 is Δ_1-collinear with π_2^p (likewise for L_2 with respect to π_1^p).

Now suppose that π_1^p contains two S-lines L_1 and L_1' through p. Then each line L_2 of π_2^p containing p is Δ_1-collinear to both L_1 and L_1' and hence to the entire plane π_1^p, i.e., L_2 is an S-line. It follows that all lines through p are S-lines indeed, and hence all symps of Υ_1 through p are singular. Since each point of Υ_1 is contained in a symp together with p, we obtain that $p \perp_{\Delta_1} \Upsilon_1$.

Next, suppose that π_i^p contains a unique S-line L_i through p for $i = 1, 2$. Then the 3-spaces $\langle L_1, \pi_2^p \rangle$ and $\langle L_2, \pi_1^p \rangle$ of Δ_1 meet in the plane $\langle L_1, L_2 \rangle$ and are not Δ_1-collinear (otherwise all lines through p would be S-lines as above). Hence there is a unique symp Σ of Δ_1 containing $\pi_1^p \cup \pi_2^p$. Let ξ be a symp of Υ_1 opposite p (no point of ξ is Υ_1-collinear to p). Then there is an isomorphism between the pairs (M_1, M_2) of Υ_1-lines through p with $M_i \subseteq \pi_i^p$ for $i = 1, 2$, and the points of ξ in the sense that each symp $\xi(M_1, M_2)$ meets ξ in a unique point. If we restrict to the pairs (M_1, M_2) where $M_1 \neq L_1$ and $M_2 \neq L_2$ then the corresponding points of ξ constitute a subpolar space $G := \xi \setminus \{L'_1, L'_2\}$ where L'_1 and L'_2 are two intersecting lines of ξ. Let q be any point of G. Then $\xi(p, q) = \xi(M_1, M_2)$ is an isometric symp of Υ_1 since M_1 is not Δ_1-collinear to M_2 and therefore $\Sigma(p, q) = \Sigma$. So G and therefore $\langle G\rangle = \langle \xi\rangle$ is entirely contained in Σ. Since Υ_1 is generated by $p^{\perp_{\Upsilon_1}}$ and ξ, we conclude that $\Upsilon_1 \subseteq \Sigma$ indeed. □

We now turn our attention to the lines in a symp of Υ_1. Recall that a symp ξ of Υ_1 is a hyperbolic polar space of rank 2, and hence its generators (which are lines) come into two families: two lines belong to the same family if and only if they are disjoint.

Notation. We denote the two families of lines of ξ by $\mathscr{L}_1^\xi$ and $\mathscr{L}_2^\xi$. Each line L of Υ_1 is contained in a unique plane π_L of Υ_1. Consider the set Π_i^ξ of planes meeting ξ in a line belonging to the two respective families, i.e., $\Pi_i^\xi := \{\pi_L \mid L \in \mathscr{L}_i\}$ for $i = 1, 2$. Then, for each $i \in \{1, 2\}$, the union of the planes in Π_i^ξ induces a subgeometry of Υ_1 isomorphic to $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$. We refer to this as a *Segre subgeometry*, and denote it by $\hat{\xi}_i$. Note that $\hat{\xi}_1$ and $\hat{\xi}_2$ are the two unique (full and isometric) subgeometries of Υ_1 isomorphic to $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$ containing ξ.

Lemma 5.10. *Let ξ be a singular symp of Υ_1 and $\{i, j\} = \{1, 2\}$. With notation as above, either one or all lines of $\mathscr{L}_i^\xi$ are S-lines. In the first case, $\hat{\xi}_j$ is contained in a unique symp of Δ_1; in the second case, $\hat{\xi}_j$ is contained in and spans a singular 5-space.*

Proof. Suppose L is a line of ξ contained in an isometric symp ξ', say $L \in \mathscr{L}_1^\xi$. Note that the symps containing L induce a Segre subgeometry, which coincides with $\hat{\xi}_2$. By Lemma 5.8, ξ is contained in the unique symp Σ' of Δ_1 containing ξ'. Since ξ and ξ' generate, in Υ_1, the Segre subgeometry $\hat{\xi}_2$, the latter is contained in Σ' too.

Next, we claim there is a unique S-line in $\mathscr{L}_1^\xi$. Take a point $x' \in \xi' \setminus L$. As above, x' is Υ_1-collinear to a line K of ξ meeting L in a unique point x_L. Inside Σ', we hence obtain that x' is collinear to a plane of $\langle \xi\rangle$ through the line K, and therefore this plane contains a second line L^* of ξ, disjoint from L. Therefore, L^* is Δ_1-collinear to all points of the plane $\langle K, x'\rangle$ and hence it is an S-line. Suppose for a contradiction that there is a second S-line L^{**} in ξ. Then x' would be Δ_1-collinear to $\langle \xi\rangle = \langle L^*, L^{**}\rangle$, contradicting the fact that x' is not Δ_1-collinear to L. The claim follows.

Finally, suppose that all lines of $\mathscr{L}_1^\xi$ are S-lines. The symps through these lines are precisely all symps of the Segre subgeometry $\hat{\xi}_2$ and these symps are hence all singular, meaning that each pair of points in $\hat{\xi}_2$ is Δ_1-collinear, implying that it is contained in a singular 5-space. Since Segre subgeometries contain disjoint planes, $\hat{\xi}_2$ spans the singular 5-space in which it is contained. □

Next, we look at the particular situation in which two isometric symps, intersecting each other in a line, embed in the same symp.

Lemma 5.11. *Let ξ_1 and ξ_2 be two isometric symps of Υ_1, intersecting each other in a line L, and contained in the same symp Σ of Δ_1. Then the Segre subgeometry determined by ξ_1 and ξ_2 contains a unique maximal singular line which is an S-line. In particular, there is a singular symp ξ' meeting each of ξ_1 and ξ_2 in a line.*

Proof. A point q in $\xi_1 \setminus L$ is Υ_1-collinear to a line M of ξ_2 and, selecting a line M' of ξ_2 disjoint from M, it is (looking in Σ) Δ_1-collinear to a point m of M' and hence Δ_1-collinear to the line $L' \neq L$ of ξ_2 through m and distinct from M'. The symp ξ containing q and L' is singular. Now take a point r on L. Then r is Υ_1-collinear to a line M^* of ξ, and as above it follows that r is then also Δ_1-collinear to a second line L^* of ξ. The symp through L and L^* is singular, so L^* is Δ_1-collinear to L. We claim that each point x of L^* is Δ_1-collinear to all points of the Segre subgeometry $\mathscr{S}$ determined by ξ_1 and ξ_2. This follows from the fact that $\mathscr{S}$ is generated by L and ξ, and x is collinear to both since $x \in L^*$ and ξ is singular, respectively. Since each symp through L^* is contained in $\mathscr{S}$, it follows that L^* is an S-line. If there were a second maximal singular line K^* which is an S-line, then by Lemma 5.10, all maximal singular lines in $\mathscr{S}$ contained in the symp ξ^* determined by K^* and L^* would be S-lines, from which we deduce that each symp in $\mathscr{S}$ would be singular, a contradiction. □

We now consider two subcases, depending on whether or not there is a singular symp containing more than two S-lines.

5A4. *Subcase: there is a singular symp ξ^* containing more than two S-lines.* The assumptions in this section are according to the standing hypothesis and the title of this section.

Lemma 5.12. *Each singular symp of Υ_1 has a line which is not an S-line.*

Proof. Let ξ be a singular symp and suppose for a contradiction that each line of ξ is an S-line. Then the two Segre subgeometries $\hat{\xi}_1$ and $\hat{\xi}_2$ containing ξ (as introduced before) are contained in respective singular 5-spaces S_1 and S_2 by Lemma 5.10. Since $S_1 \cap S_2$ contains the 3-space $\langle \xi \rangle$, it follows from the 5–5 relations of Δ_1 that $S_1 = S_2$. Take any symp ξ'. If ξ' meets ξ in a line, then ξ' is singular as the intersection is an S-line; If ξ' meets ξ in a unique point p, then the two unique lines L_1 and L_2 of ξ' containing p are contained in $S_1 = S_2$ since they are contained in $\hat{\xi}_1$ and $\hat{\xi}_2$, respectively. But then $L_1 \perp L_2$ and hence ξ' is singular. We conclude that all symps of Υ_1 are singular, contradicting our assumption. □

Lemmas 5.10 and 5.12 imply, together with the assumption that ξ^* has more than two S-lines, that ξ^* has one full system of S-lines, and the other system contains a unique S-line. We can now show that this gives rise to case (ii) of Proposition 5.7.

Lemma 5.13. *Suppose Υ_1 has an isometric symp and a (singular) symp ξ^* with more than two S-lines. Then there is a unique plane π^* of Υ_1 such that:*

(i) *A symp of Υ_1 is singular if and only if it meets π^* in a line.*

(ii) *$\Upsilon_1 \subseteq \pi^{*\perp_{\Delta_1}}$, each Segre subgeometry of Υ_1 containing π^* is contained in and spans a unique 5-space containing π^*, and distinct such Segre subgeometries span distinct such 5-spaces.*

(iii) *Two isometric symps of Υ_1 embed in the same symp of Δ_1 if and only if they share a line that is contained in a plane of Υ_1 disjoint from π^*.*

Proof. As explained just before this lemma, we may assume (up to renumbering) that the family $\mathscr{L}_1^{\xi^*}$ has S-lines only and that the other family $\mathscr{L}_2^{\xi^*}$ has a unique S-line L^*. Let $\hat{\xi}_1^*$ and $\hat{\xi}_2^*$ be the corresponding Segre subgeometries containing ξ^* (see notation introduced above). By Lemma 5.10, $\hat{\xi}_2^*$ is contained in a singular 5-space S and $\hat{\xi}_1^*$ is contained in a unique symp Σ of Δ_1. Since S and Σ share the 3-space $\langle\xi^*\rangle$, it follows from the symp–5 relations (see Fact A.12) that $S\cap\Sigma$ is a $4'$-space U. Therefore, a line of $\hat{\xi}_2^*$ disjoint from ξ^* meets $S\cap\Sigma$ in a point u outside $\langle\xi^*\rangle$ and hence the unique plane π^* of $\hat{\xi}_2^*$ through u is contained in Σ. Set $L=\pi^*\cap\xi^*$. Take any point q on L and let M be the unique other line of ξ^* containing q (so M is an S-line). Consider the second plane π of Υ_1 containing q, the one containing M (so π is a plane of $\hat{\xi}_1^*$ and hence $\pi\subseteq\Sigma$). The fact that $\pi\cup\pi^*\subseteq\Sigma$ and that M is Δ_1-collinear to π^*, implies that π^* also contains a line through q that is Δ_1-collinear to π, that is, π^* contains an S-line. If L is not an S-line, we deduce from Lemma 5.9 that π and π^* each contain a unique S-line through q, and by the same lemma, we obtain $\Upsilon_1\subseteq\Sigma$. However, $\hat{\xi}_2^*$ generates the singular 5-space S, which is not contained in Σ, a contradiction. Hence, $L=L^*$, and by the arbitrariness of q, all lines of π^* are S-lines, and so are all lines through a point of L. We also conclude that the unique plane π^* of $\hat{\xi}_2^*$ contained in Σ contains L^*.

(i) Since every line of π^* is an S-line, each symp not disjoint from π^* is singular. Since each point of Υ_1 is contained in such a symp, we deduce $\Upsilon_1\subseteq\pi^{*\perp_{\Delta_1}}$. So, if a symp ξ disjoint from π^* were singular, then $\langle\pi^*,\xi\rangle$ would be a singular 6-space of Δ_1, a contradiction. This shows the assertion (i).

(ii) We already deduced that $\Upsilon_1\perp_{\Delta_1}\pi^*$. Any Segre subgeometry $\mathscr{S}$ containing π^* plays the same role as $\hat{\xi}_2^*$ since it arises from one of the symps containing L^*, and by the above it has one full system of S-lines (since each line meeting L^* is an S-line). So indeed, $\mathscr{S}$ is contained in a singular 5-space containing π^*. If such a

5-space contained two such Segre subgeometries, then Υ_1 would be contained in this 5-space, contradicting the hypothesis that Υ_1 has an isometric symp.

(iii) Let ξ_1 and ξ_2 be symps that embed isometrically (both are disjoint from π^*). Suppose first that they share a line that is contained in a plane of Υ_1 disjoint from π^*. Then ξ_1 and ξ_2 determine a Segre subgeometry $\mathscr{S}$ whose planes intersect π^* in points. It follows that $\mathscr{S}$ intersects π^* in a unique line N. Then N is Δ_1-collinear to ξ_1 and therefore N is contained in the unique symp Σ_1 of Δ_1 containing ξ_1. Since N and ξ_1 are disjoint, they generate $\mathscr{S}$. We obtain that $\xi_1 \cup \xi_2 \subseteq \mathscr{S} \subseteq \Sigma_1$.

Conversely, suppose ξ_1 and ξ_2 embed in the same symp Σ of Δ_1. Suppose first that $\xi_1 \cap \xi_2$ is a unique point, say p. Then one of the planes of Υ_1 containing p, say π_1^p, shares a point p' with π^*. By (i), the line pp' is an S-line. Let π_2^p be the other plane through p. Then pp' and π_2^p generate a singular 3-space and so, by properties of the polar space Σ, it follows that π_2^p contains a line through p which is Σ-collinear to π_1^p. This line is then an S-line. However, by Lemma 5.9, not all lines of π_1^p and π_2^p through p are S-lines, since ξ_1 and ξ_2 are isometric, so it then follows from the same lemma that $\Upsilon_1 \subseteq \Sigma$. As $\hat{\xi}_2^*$ generates a singular 5-space of Δ_1, this is a contradiction. Secondly, suppose ξ_1 and ξ_2 share a line that is contained in a plane of Υ_1 not disjoint from π^*. Then $\Upsilon_1 \subseteq \pi^{*\perp_{\Delta_1}}$ implies that $\pi^* \subseteq \Sigma$, contradicting the fact that π^* and each plane of the Segre geometry containing ξ_1 and ξ_2 generate a singular 5-space of Δ_1. □

We conclude that we are in case (ii) of Proposition 5.7.

5A5. *Subcase: each singular symp contains exactly two S-lines.* The assumptions in this section are according to the standing hypothesis and the title of this section. This will lead to case (iii) or case (iv) of Proposition 5.7, depending on whether there is an isometric symp through each point or not. We start by assuming that this is the case.

Lemma 5.14. *If there is an isometric symp through each point of Υ_1, then there is a unique symp Σ of Δ_1 containing Υ_1.*

Proof. Let ξ be a singular symp (which exists by assumption). Then ξ contains exactly two S-lines L and M, also by assumption. Let p be the point $L \cap M$ (these lines intersect by Lemma 5.10). As there is an isometric symp through p by assumption, it follows from Lemma 5.9 that L and M are the unique S-lines through p. Also by Lemma 5.9, there is a unique symp Σ containing Υ_1. □

Remark 5.15. We can show that in the above case, Υ_1 is actually contained in a subquadric of Σ, namely a parabolic quadric $Q(8, \mathbb{K})$ arising as the intersection of Σ with an 8-dimensional subspace. Such a quadric is given by an equation of the form $X_1X_2 + \cdots + X_7X_8 = X_0^2$. An example of a full embedding of the Segre variety $\mathscr{S} := \mathscr{S}_{2,2}(\mathbb{K})$ in $Q(8, \mathbb{K})$ is given in Example 5.2. In [De Schepper and Victoor 2023], the first author and Magali Victoor study this embedding and consider the geometry $(\mathscr{X}, \mathscr{L})$, where $\mathscr{L}$ is the set of S-lines and $\mathscr{X}$ is the set of points contained in an S-line, and show that this forms a nonthick generalised hexagon (which is a geometric hyperplane of $\mathscr{S}$).

Next, we assume that there is a point p in Υ_1 through which there is no isometric symp. We show that there is only one such point, and that a symp is singular if and only if it contains p.

Lemma 5.16. *Suppose there is a point p in Υ_1 such that each symp through p is singular. Then the symps of Υ_1 not through p are all isometric. If ξ_1 and ξ_2 are two isometric symps which either meet in a unique point or one of them is far from p, then the respective corresponding symps Σ_1 and Σ_2 of Δ_1 containing ξ_1 and ξ_2 are distinct and share a 4-space containing p.*

Proof. First note that our assumption on p implies that each line of Υ_1 through p is an S-line (see also Lemma 5.9). Let ξ be a symp close to p, i.e., ξ contains a unique line L which is Υ_1-collinear to p. Suppose ξ is singular. Then ξ contains an S-line M meeting L in a point q. Consider the symp determined by M and the line pq. This symp contains more than two S-lines: the two lines through p are S-lines, and so is M, a contradiction. Now suppose ξ is a singular symp far from p, so p is not collinear to any point of ξ. Let L be an S-line of ξ. Then each symp through L is singular, and since at least one of them is close to p by Fact A.4, we obtain a contradiction to the foregoing. So the singular symps of Υ_1 are precisely those containing p.

Now take two isometric symps ξ_1 and ξ_2 and suppose that they embed in the same symp Σ of Δ_1. Suppose first that $\xi_1 \cap \xi_2$ is a line L, and let $\mathscr{S}$ be the Segre subgeometry determined by ξ_1 and ξ_2. By Lemma 5.11, $\mathscr{S}$ contains a unique maximal singular line which is an S-line. By the above, this means that p is contained in this line. In particular, p is contained in a plane of $\mathscr{S}$ which meets ξ_1 and ξ_2 in a line; and hence p is close to both ξ_1 and ξ_2. Next, suppose that ξ_1 and ξ_2 intersect in a unique point q, and let L_i and L'_i be the unique lines of ξ_i, $i = 1, 2$, containing q, ordered such that $L_1 \perp_{\Upsilon_1} L'_2$ and $L_2 \perp_{\Upsilon_1} L'_1$. Let ξ be the symp of Υ_1 determined by L_1 and L_2. We claim that ξ is singular. Indeed, suppose not; then clearly $\xi \subseteq \Sigma$. Since ξ_1 and ξ are both contained in Σ and share a line, it follows from the beginning of this paragraph that p is contained in a plane of the Segre subgeometry determined by ξ_1 and ξ. The same holds for ξ and ξ_2 though, implying that $p \in \xi$, a contradiction. The claim follows. Hence $p \in \xi$ and, likewise, $p \in \xi'$, with ξ' the symp of Υ_1 determined by L'_1 and L'_2. But $\xi \cap \xi' = \{q\}$ and $q \neq p$ (since ξ_1 and ξ_2 are isometric). This contradiction shows that ξ_1 and ξ_2 cannot be contained in the same symp if they meet in a unique point. In any case, the respective symps Σ_1 and Σ_2 containing ξ_1 and ξ_2 are hence distinct, but meet each other in a 4-space since both also contain p. □

Remark 5.17. In this case, we can also show the existence of a unique line K of Δ_1 through p such that $\Upsilon_1 \subseteq K^{\perp_{\Delta_1}}$. Then K is not contained in Υ_1 and the projection of Υ_1 from any point $x \in K \setminus \{p\}$ is injective. So we obtain a geometry isomorphic to $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$ fully embedded in $\mathrm{Res}_{\Delta_1}(x) \cong \mathsf{D}_{5,5}(K)$. An example of this situation is given in Example 5.3. We conjecture that the conditions of Lemma 5.16 always give rise to an embedding projectively equivalent to Example 5.3.

We reached the conclusion of cases (iii) and (iv) in Proposition 5.7.

5A6. *Subcase: Each symp embeds isometrically.* Finally, we treat the case that the full embedding of $\Upsilon_1 \cong \mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$ in $\Delta_1 \cong \mathsf{E}_{6,1}(\mathbb{K})$ is such that each symp of Υ_1 is isometric, meaning that the embedding is isometric (since each pair of points of Υ_1 is contained in a symp). We will hence use the symbol $\perp$ for both Υ_1 and Δ_1. This situation will lead to case (v) of Proposition 5.7.

We recall that, when a symp ξ of Υ_1 embeds isometrically in a symp Σ_ξ of Δ_1, it arises as the intersection of the 3-space $\langle \xi \rangle$ with the quadric Σ_ξ. We first show that distinct symps embed in distinct symps.

Lemma 5.18. *The map $\xi \mapsto \Sigma_\xi$ is injective.*

Proof. Suppose for a contradiction that there are distinct symps ξ_1, ξ_2 with $\Sigma := \Sigma_{\xi_1} = \Sigma_{\xi_2}$. If $\xi_1 \cap \xi_2$ is a unique point p, then a symp meeting both ξ_1 and ξ_2 in a line through p will also embed in Σ. Lemma 5.11 now yields a singular symp, contradicting our assumption. □

We start with the mutual position in Δ_1 of disjoint planes of Υ_1. As mentioned above, we do not need to make a distinction between collinearity in Υ_1 and in Δ_1, so Fact A.6 immediately implies:

Lemma 5.19. *If π and π' are disjoint planes of Υ_1, then collinearity between π and π' is an isomorphism.*

Notation. We will call (disjoint) planes π, π' of Δ_1 in *Segre relation* if collinearity between them is an isomorphism. Indeed, the geometry that arises when taking the union of the lines meeting π and π' is a Segre subgeometry isomorphic to $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$, which we will denote by $\mathscr{S}(\pi, \pi')$.

Planes of Δ_1 which are in Segre relation bring along a unique plane which is collinear to both of them:

Lemma 5.20. *Let π and π' be two planes of Δ_1 in Segre relation. Then $\pi^\perp \cap \pi'^\perp$ is a plane α. If U and U' denote the 5-spaces $\langle \pi, \alpha \rangle$ and $\langle \pi', \alpha \rangle$, respectively, and V and V' are 5-spaces containing π and π', respectively, with $V \neq U$ and $V' \neq U'$, then V and V' are opposite.*

Proof. Let p_1, p_2, p_3 be a triangle in π (that is, three points not in a common line) and let p'_1, p'_2, p'_3 be the respective collinear points in π'. The symps ξ_2 and ξ_3 containing p_1, p'_2 and p_1, p'_3, respectively, have a 4-space U in common. By convexity of symps, $p_2^\perp \cap U = p_3^\perp \cap U$ and $p_2'^\perp \cap U = p_3'^\perp \cap U$; hence $\pi^\perp \cap \pi'^\perp \cap U$ is a plane α and clearly coincides with $\pi^\perp \cap \pi'^\perp$.

For the second assertion, let $v \in V \setminus \pi$ be arbitrary. By Fact A.10, it suffices to show that $|v^\perp \cap V'| = 1$. Suppose not, then Fact A.11 implies that $v^\perp \cap V'$ is a 3-space, which has some point v' in common with π'. Since $v^\perp \cap U$ is also a 3-space, which contains π, we find a point $a \in \alpha$ not in $v^\perp$. Then $\xi(a, v)$ contains π and $v' \in \pi'$, contradicting $|v'^\perp \cap \pi| = 1$. □

Next, we have a look at the mutual position of the Δ_1-symps in which the symps of Υ_1 embed.

Lemma 5.21. *If ξ, ξ' are distinct symps of Υ_1, then the symps Σ_ξ and $\Sigma_{\xi'}$ of Δ_1 in which they embed intersect each other in exactly a point if $\xi \cap \xi'$ is a point; and they intersect in a 4-space if $\xi \cap \xi'$ is a line. If p is a point of Υ_1 and ξ a symp of Υ_1 such that $p^\perp \cap \xi = \varnothing$, then $p^\perp \cap \Sigma_\xi = \varnothing$.*

Proof. Lemma 5.18 yields $\Sigma_\xi \neq \Sigma_{\xi'}$. Next, consider a point p and symp ξ with $p^\perp \cap \xi = \varnothing$ and suppose for a contradiction that $p^\perp \cap \Sigma_\xi$ is not empty. If $p \in \Sigma_\xi$, then $p^\perp \cap \xi \neq \varnothing$, a contradiction. So $p^\perp \cap \Sigma_\xi$ is a $4'$-space V. Since $\xi = \langle \xi \rangle \cap \Sigma_\xi$, we know that V and $\langle \xi \rangle$ are disjoint. Hence V contains a point q collinear to ξ. Obviously, $q \notin \Upsilon_1$. We claim that q is collinear to all points of Υ_1. Take two symps ξ_1 and ξ_2 containing p and meeting each other in a line L. Let Σ_1 and Σ_2 be the corresponding symps of Δ_1 containing ξ_1 and ξ_2, respectively. Since q is Δ_1-collinear to two noncollinear points of both ξ_1 and ξ_2, namely p and the points $\xi_1 \cap \xi$ and $\xi_2 \cap \xi$, we obtain that $q \in \Sigma_1 \cap \Sigma_2$. Since the latter is a singular subspace of Δ_1 also containing L, we obtain that $q \perp_{\Delta_1} L$. Since this holds for any line L of Υ_1 containing p, we obtain that q is Δ_1-collinear to $p^{\perp_{\Upsilon_1}}$ and ξ and hence to all of Υ_1. The claim follows.

We conclude that Υ_1 is fully and isometrically embedded in $\widetilde{\Delta}_1 := \mathrm{Res}_{\Delta_1}(q) \cong \mathsf{D}_{5,5}(\mathbb{K})$. We show that this is impossible. Indeed, let π_1 and π_2 be the two planes of Υ_1 through some point x of Υ_1. Then no point of $\pi_1 \setminus \{x\}$ is collinear to any point of $\pi_2 \setminus \{x\}$ in either Υ_1 or $\widetilde{\Delta}_1$. This yields two lines in $\mathrm{Res}_{\widetilde{\Delta}_1}(x) \cong \mathsf{A}_{4,2}(\mathbb{K})$ between which the collinearity relation is empty, contradicting the fact that each pair of planes in the 4-dimensional space $\mathsf{A}_{4,1}(\mathbb{K})$ intersects nontrivially.

Finally, suppose $\xi \cap \xi'$ is a point p. It is easily verified that a point $q \in \xi \setminus \{p^\perp\}$ is not collinear to any point of ξ'. The previous paragraph then implies that q is far from $\Sigma_{\xi'}$, which means that $\Sigma_\xi \cap \Sigma_{\xi'} = \{p\}$. By the first paragraph we have that, if $\xi \cap \xi'$ is a line, then $\Sigma_\xi \cap \Sigma_{\xi'}$ is a 4-space. □

Notation. We denote the two families of planes of Υ_1 by Π and $\overline{\Pi}$. Henceforth, we let π, π', π'' be three distinct planes of Π meeting a given plane $\bar{\pi} \in \overline{\Pi}$ in three points generating $\bar{\pi}$ (the three planes generate Υ_1). Also, let α denote the unique plane of Δ_1 collinear to both π' and π'', α' the plane collinear to both π and π'', and α'' the one collinear to both π and π' (see Lemma 5.20).

Lemma 5.22. *The planes α and π are opposite planes of Δ_1, i.e., the collinearity relation is empty between α and π (likewise for α' and π' and α'' and π'').*

Proof. Let p be a point of π. Take a symp ξ in the Segre subgeometry determined by the planes π' and π'' such that $p^\perp \cap \xi = \varnothing$. Clearly, $\alpha \subseteq \Sigma_\xi$. Since p is far from Σ_ξ by Lemma 5.21, p is not collinear to any point of α. □

It turns out that the mutual position between $\alpha, \alpha', \alpha''$ is the same as between the planes π, π', π''.

Lemma 5.23. *The planes α and α' are in Segre relation (likewise for α' and α'' and α'' and α). Moreover, if x is a point in α and x' and x'' are the respective points in α' and α'' collinear to x, then $\alpha_x := \langle x, x', x''\rangle$ is a singular plane of Δ_1.*

Proof. Observe that the planes α and α' are contained in the respective 5-spaces $U := \langle \alpha, \pi''\rangle$ and $U' := \langle \alpha', \pi''\rangle$. From the point–5 relations in Δ_1 (Fact A.11) it now follows easily that collinearity is a bijection between α and α', and by considering symps through noncollinear points of $\alpha \cup \alpha'$, it follows that this bijection is an isomorphism.

Next, consider x, x', x'' as in the statement. Suppose for a contradiction that the unique point $\bar{x}'$ of α'' collinear to x' is distinct from x''. Then the symp (of Δ_1) determined by x and $\bar{x}'$ contains the plane π' and the point $x' \in \alpha'$. Therefore x' is collinear to a line of π', contradicting Lemma 5.22. We conclude that $\langle x, x', x''\rangle$ is singular. Clearly, $\langle x, x'\rangle$ is a line since α and α' are disjoint. If $x'' \in \langle x, x'\rangle$, then x'' would be collinear to π'', again contradicting Lemma 5.22. □

We keep using the notation α_x, as introduced in the statement of the previous lemma, for the unique singular plane of Δ_1 containing $x \in \alpha$ and meeting α' and α'' in points. Then two such planes α_x and α_y are also in Segre relation, as we show below.

Lemma 5.24. *Let x, y be distinct points of α. Then α_x and α_y are in Segre relation.*

Proof. Let x, x', x'' be the intersection points of α_x and $\alpha, \alpha', \alpha''$, respectively; likewise for y, y', y''. Clearly, x is collinear to y and not collinear to y' and y''. So if $x^\perp \cap \alpha_y$ is more than just y, it is a line L. The symp Σ of Δ_1 containing x and y' then contains $\alpha_y = \langle L, y'\rangle$, and hence also $\alpha_x = \langle x, x', x''\rangle$. Let p be a point of π. Since π is collinear to α' and α'' by definition, p is collinear to $\langle x', y'\rangle$ and $\langle x'', y''\rangle$, yielding $p \in \Sigma$ (because x' and y'' are not collinear). Since $p \in \pi$ was arbitrary and π plays the same role as π', we obtain $\pi \cup \pi' \subseteq \Sigma$, contradicting Lemma 5.19. □

We actually showed more or less that the "α_x-planes" constitute a subgeometry Υ_1^α of Δ_1 isomorphic to Υ_1. We now repeat this "construction" to obtain yet another such geometry, say Υ_1^β, starting from Υ_1^α instead of from Υ_1. We then focus on a hexagon of 5-spaces determined by three planes of one family of the Υ_1^α geometry:

Notation. Put $\bar{\alpha} := \alpha_x$, $\bar{\alpha}' := \alpha_y$ and $\bar{\alpha}'' := \alpha_z$, with $\langle x, y, z\rangle = \alpha$. By Lemma 5.24 these planes are in Segre relation, and hence by Lemma 5.20, there are unique planes β, β', β'' such that β is collinear to $\bar{\alpha}'$ and $\bar{\alpha}''$; β' is collinear to $\bar{\alpha}$ and $\bar{\alpha}''$ and β'' is collinear to $\bar{\alpha}$ and $\bar{\alpha}'$. Recall that these planes are disjoint from the planes $\bar{\alpha}, \bar{\alpha}', \bar{\alpha}''$. We consider the hexagon of 5-spaces they determine: $U := \langle \bar{\alpha}, \beta''\rangle$, $V = \langle \beta'', \bar{\alpha}'\rangle$, $W = \langle \bar{\alpha}', \beta\rangle$, $U' = \langle \beta, \bar{\alpha}''\rangle$, $V' = \langle \bar{\alpha}'', \beta'\rangle$ and $W' = \langle \beta', \bar{\alpha}\rangle$.

We want to show that the above mentioned 5-spaces correspond to a hexagon in the geometry Δ_1^* isomorphic to $\mathsf{E}_{6,2}(\mathbb{K})$ associated to Δ_1, where "opposite" points in the hexagon are opposite in Δ_1^*. By looking in Δ_1^*, this is almost trivial:

Lemma 5.25. *The 5-spaces U and U' are opposite in Δ_1; likewise for V and V' and for W and W'.*

Proof. This is completely similar to the proof of Lemma 5.20. □

Recall the definition of the equator geometry $E(U, U') \cong \mathsf{A}_{5,2}(\mathbb{K})$ (Definition 3.4). We show that $\Upsilon_1 = E(U, U') \cap E(V, V') \cap E(W, W') = E(U, U') \cap E(V, V')$.

Proposition 5.26. *We have $\Upsilon_1 = E(U, U') \cap E(V, V') \cap E(W, W')$ and this point set coincides with the set of points which are simultaneously collinear to exactly a line of each of the planes $\bar{\alpha}, \beta'', \bar{\alpha}', \beta, \bar{\alpha}'', \beta'$.*

Proof. Let p be any point of Υ_1. Then p is contained in a unique plane $\bar{\pi}_p$ of Υ_1 meeting each of π, π', π'' in unique points q, q', q''. We claim that $\bar{\alpha}$ is contained in a symp of Δ_1 together with $\bar{\pi}_p$; likewise for $\bar{\alpha}'$ and $\bar{\alpha}''$. Recall that $\bar{\alpha} = \alpha_x = \langle x, x', x''\rangle$ with $x \in \alpha$, $x' \in \alpha'$ and $x'' \in \alpha''$. The points $x \in \alpha$ and $q \in \pi$ are not collinear by Lemma 5.22, so they determine a symp Σ_p. Since α is collinear to $\pi' \cup \pi''$ by definition, in particular x is collinear to q', q''; likewise, π is collinear to $\alpha' \cup \alpha''$ and hence q is collinear to x' and x''. Therefore, q', q'', x' and x'' all belong to $x^\perp \cap q^\perp \subseteq \Sigma_p$ and hence $\bar{\pi}_p \cup \bar{\alpha} \subseteq \Sigma_p$. The claim follows. Since $q^\perp \cap q'^\perp \cap q''^\perp \cap \bar{\alpha} = xx' \cap x'x'' \cap xx'' = \varnothing$, the planes $\bar{\pi}_p$ and $\bar{\alpha}$ are opposite in the polar space Σ_p. Consequently, p is collinear to a unique line of $\bar{\alpha}$.

By the point–5 relations in Δ_1 (Fact A.11), this means that p is collinear to a 3-space U_p of U (which meets $\bar{\alpha}$ in a line L_p). By the same token, p is collinear to a 3-space V_p of V meeting $\bar{\alpha}'$ in a line L'_p. By dimension, U_p meets $\beta'' = U \cap V$ in at least a point w, and w then also belongs to V_p. Since $\bar{\alpha}$ and $\bar{\alpha}'$ are in Segre relation, there exist noncollinear points $a \in L_p$ and $a' \in L'_p$. Clearly, the symp $\xi := \xi(a, a')$ contains p, w. We claim that ξ also contains L_p. Let b be the unique point of $\bar{\alpha}$ collinear to a'. Then $b \in \xi$. So, if $b \in L_p$, then $a'b = L_p \subseteq \xi$; if $b \notin L_p$ then p and b are not collinear and hence $L_p \subseteq \xi(b, p) = \xi$. The claim follows. Likewise, $L'_p \subseteq \xi$. Fact A.12 then implies that $U \cap \xi$ and $V \cap \xi$ are $4'$-spaces in ξ. Since those intersect in either a plane or a point, we either have $\beta'' \subseteq \xi$, or $U \cap \xi \cap \beta''$ and $V \cap \xi \cap \beta''$ are distinct lines, which contradicts the fact that both sets equal $\xi \cap \beta''$. We conclude that p is collinear to a unique line M_p of β''. By symmetry, we showed that p is collinear to a line of each of $\bar{\alpha}$, β'', $\bar{\alpha}'$, β, $\bar{\alpha}''$, β'. In particular, $p \in E(U, U') \cap E(V, V') \cap E(W, W')$. Note that M_p is collinear to all points of $\bar{\pi}_p$ since the line of β'' collinear to a point of $\bar{\pi}_p$ is necessarily contained in the symp Σ_p (which meets β'' in M_p).

Conversely, suppose p is a point of $E(U, U') \cap E(V, V') \cap E(W, W')$. Note that Υ_1 is fully and isometrically embedded in $E(U, U') \cong \mathsf{A}_{5,2}(\mathbb{K})$, and that the planes β' and $\bar{\alpha}'$ both are contained in $E(U, U')$. By Lemma 6.12 of [De Schepper et al. 2022], Υ_1 coincides with $E(\beta', \bar{\alpha}')$, which is by definition the set of points of $E(U, U')$ collinear to a line of β' and to a line of $\bar{\alpha}'$. An analogous argument as in the previous paragraph shows that p is collinear to a line of each of $\bar{\alpha}$, β'', $\bar{\alpha}'$, β, $\bar{\alpha}''$, β'. Therefore, p is also contained in $E(\beta', \bar{\alpha}') \cap E(U, U') = E(U, U') \cap E(V, V') \cap E(W, W')$ and hence $\Upsilon_1 = E(U, U') \cap E(V, V') \cap E(W, W')$. □

This finishes the proof of Proposition 5.7.

Remark 5.27. Consider the following geometry. Let $\mathscr{U}$ be the set of 5-spaces U of Δ_1 such that each point of Υ_1 is collinear to a 3-space of U, and for each plane A of Δ_1 occurring as the intersection of two such 5-spaces in $\mathscr{U}$, let $A_{\mathscr{U}}$ be the set of 5-spaces of Δ_1 containing A. It is easily verified that, for such a plane A, each point of Υ_1 is collinear to a line of A and hence the point–5-space relations of Δ_1 imply that each 5-space of Δ_1 containing A belongs to $\mathscr{U}$ (it actually implies that a plane occurs as the intersection of two such 5-spaces if and only if each point of Υ_1 is collinear to a line of A). Let $\mathscr{A}$ be the set of $A_{\mathscr{U}}$ for all planes A of Δ_1 as above. Then one could show that the point-line geometry $(\mathscr{U}, \mathscr{A})$ is a thin generalised hexagon (with only two lines per point), which can also be seen as a (full) subgeometry of the $\mathsf{E}_{6,2}(\mathbb{K})$ geometry Δ_1^* associated to Δ_1.

5B. ***Full embeddings of $\mathsf{A}_{5,3}(\mathbb{K})$ in $\mathsf{E}_{7,7}(\mathbb{K})$.*** Suppose $\Upsilon_2 := \mathsf{A}_{5,3}(\mathbb{K})$ is fully embedded in $\Delta_2 := \mathsf{E}_{7,7}(\mathbb{K})$. Take any point p in Υ_2. Then $\mathrm{Res}_{\Upsilon_2}(p) \cong \mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$ is fully embedded in $\mathrm{Res}_{\Delta_2}(p) \cong \mathsf{E}_{6,1}(\mathbb{K})$. By Proposition 5.7, there are five possibilities for the nature of this embedding, labelled by (i) up to (v); we will call this label the *type* of p. Our first goal is to show that every point of Υ_2 has type (v). For that, the following lemma will be useful.

By a *standard subgeometry of* Υ_2 *isomorphic to* $\mathsf{A}_{4,2}(\mathbb{K})$ we mean the subgeometry of Υ_2 arising from the residue of either a point or a hyperplane in the underlying projective space $\mathsf{PG}(5,\mathbb{K})$. Since residues in buildings are convex, such a subgeometry is always isometric.

Lemma 5.28. *Consider a standard subgeometry* Ω *of* Υ_2 *isomorphic to* $\mathsf{A}_{4,2}(\mathbb{K})$ *and let* p *be a point of* Ω. *Suppose* $\mathrm{Res}_\Omega(p) \cong \mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$ *is embedded in a singular* 5*-space of* $\mathrm{Res}_{\Delta_2}(p)$. *Then* Ω *is embedded in a singular* 6*-space of* Δ_2.

Proof. Note that $\mathrm{Res}_\Omega(p)$ generates a 5-dimensional space in $\mathrm{Res}_{\Delta_2}(p)$ and hence, lifted to Δ_2, the set $p^{\perp_\Omega}$ is contained in a 6-dimensional singular subspace S of Δ_2. Suppose for a contradiction that there is a symp ξ of Ω through p such that ξ is not contained in S, and let q be a point of $\xi \setminus S$. Since $\langle \xi \rangle$ meets S in the 4-space $\langle p^{\perp_\xi} \rangle$, it follows that $\langle \xi \rangle$ is a maximal singular 5-space of Δ_2 through p. In particular, $q^{\perp_{\Delta_2}} \cap S$ coincides with $\langle p^{\perp_\xi} \rangle$ by the point–5 relations of $\mathrm{Res}_{\Delta_2}(p)$ (Fact A.11).

Now let ξ' be another symp of Ω through p. Then $\xi \cap \xi'$ is a plane π. In Υ_2, we see that q is collinear to a point $q' \in \xi' \setminus \pi$, because q is collinear to a line of π and hence to a plane of ξ' distinct from π. In the ambient projective space $\mathsf{PG}(55,\mathbb{K})$ of the universal embedding of $\mathsf{E}_{7,7}(\mathbb{K})$, the subspace $\langle \xi, \xi' \rangle$ either has dimension 7 or 8: two symps in $\mathrm{Res}_\Omega(p)$ generate a 5-space, so $\langle p^{\perp_\xi}, p^{\perp_{\xi'}} \rangle = S$ and therefore, as $q \notin S$, we have $\dim\langle \xi, \xi' \rangle \geq 7$; on the other hand, $\dim\langle \xi, \xi' \rangle \leq 8$ because ξ and ξ' share a plane.

Suppose first that $\dim\langle \xi, \xi' \rangle = 8$. Then $q' \notin \langle S, \xi \rangle$ and q and q' play the same role. Note that q' is Δ_2-collinear to the 3-space $\langle q, \pi \rangle$ of the maximal 5-space $\langle \xi \rangle$. Fact A.20 implies that q' is Δ_2-collinear to a 4-space of $\langle \xi \rangle$. This implies that q' is Δ_2-collinear to a 5-space of $\langle p^{\perp_\xi}, p^{\perp_{\xi'}} \rangle = S$, a contradiction.

Next, suppose $\dim\langle \xi, \xi' \rangle = 7$. Then the Υ_2-line qq' shares a point q^* with S and the symp ξ^* of Υ_2 determined by p and q^* also contains π and therefore this symp also belongs to Ω. Since $q^* \in q^{\perp_{\Delta_2}} \cap S$, we obtain $q^* \in \langle p^{\perp_\xi} \rangle$. Let L be the unique line of π that is Υ_2-collinear to q and q^*. Then the singular plane $\langle L, q^* \rangle$ of Ω has a nontrivial intersection with any plane of ξ through p, a contradiction in Ω. The lemma follows. □

We can now exclude possibilities (i), (ii), (iii) and (iv).

Lemma 5.29. *Each point of* Υ_2 *has type* (*v*).

Proof. Let p be any point. Suppose first that p has type (v). If q is a point Υ_2-collinear to p, then each symp of Υ_2 through the line pq is isometric and hence in $\mathrm{Res}_{\Upsilon_2}(q)$, there is no singular symp through the point corresponding to pq. So q has type (v) too, because for all other types, there is a singular symp through each point of the residue (see Proposition 5.7). By connectedness, each point has type (v) then. We now exclude all other possibilities.

To that end, suppose first that p has type (iii), so in particular, there is a unique symp Σ of Δ_2 containing $p^{\perp_{\Upsilon_2}}$. Let q be any point Υ_2-collinear to p. According to Proposition 5.7, there are two isometric symps ξ_1 and ξ_2 of Υ_2 with $\xi_1 \cap \xi_2 = pq$. Since they embed isometrically in a symp of Δ_2 and since their p-residues embed in Σ, we obtain that ξ_1 and ξ_2 are contained in Σ. So in $\mathrm{Res}_{\Upsilon_2}(q)$, ξ_1 and ξ_2 correspond to isometric symps which embed in the same symp and which meet each other in a unique point (corresponding to the line pq). Considering the list of possibilities in Proposition 5.7, we see that the latter situation cannot occur if the point q has type (i), (ii) or (v). Also if q has type (iv), the situation does not occur, according to Lemma 5.16. We conclude that q also has type (iii). By connectedness, it again follows that all points of Υ_2 have type (iii). Let ξ be a singular symp of Υ_2 containing p. Recall that an S-line in $\mathrm{Res}_{\Upsilon_2}(p)$ is a line through which each symp of $\mathrm{Res}_{\Upsilon_2}(p)$ is singular, and hence it corresponds to a plane of Υ_2 through p through which each symp of Υ_2 is singular, and vice versa. We will refer to these planes as S-planes. Noting that Proposition 5.7(iii) arose from the situation in which every singular symp contains exactly two S-lines (see Section 5A5), we see that there are exactly two S-planes π_1 and π_2 through p in ξ, which intersect each other in a line, say pq. Now let r be a point of $\pi_1 \setminus \pi_2$. Then also through r there are exactly two S-planes in ξ, one of which is π_1, the other is a plane π_3 (necessarily also distinct from π_2 since it contains r). Then π_1 and π_3 share a line rq', and since pq and rq' are lines in π_1, they intersect. But then there are three S-planes in ξ through that intersection point, a contradiction.

Next, suppose p has type (ii). Then there is a unique 3-space Π_p of Υ_2 through p such that $p^{\perp_{\Upsilon_2}}$ is contained in the union of 6-spaces through Π_p. Take any such 6-space U. Then the corresponding 5-space in $\mathrm{Res}_{\Upsilon_2}(p)$ contains a Segre subgeometry, say $\mathscr{S}_U$, isomorphic to $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$. A straightforward verification in the projective 5-space $\mathsf{PG}(5, \mathbb{K})$ corresponding to Υ_2 now shows that the union of all symps of Υ_2 through p whose point residue at p is contained in $\mathscr{S}_U$ is a standard subgeometry Ω_U isomorphic to $\mathsf{A}_{4,2}(\mathbb{K})$. By Lemma 5.28, $\Omega_U \subseteq U$. Now let q be any point of Υ_2 in $p^{\perp_{\Upsilon_2}} \cap (U \setminus \Pi_p)$. Then through pq there is an isometric symp, and $\mathrm{Res}_{\Upsilon_2}(q)$ has a Segre subgeometry, namely $\mathrm{Res}_{\Omega_U}(q)$, contained in and spanning the 5-space corresponding to U. We claim that q has type (ii) too. Indeed, it cannot have type (i) because there are no isometric symps in this case, neither it can have type (v) as there are no singular symps in that case. Because of the Segre subgeometry $\mathscr{S}_U$ in $\mathrm{Res}_{\Upsilon_2}(q)$ contained in a singular subspace of $\mathrm{Res}_{\Delta_2}(q)$, it cannot have type (iii) since then $\mathrm{Res}_{\Upsilon_2}(q)$ is contained in a symp of $\mathrm{Res}_{\Delta_2}(q)$ and these do not contain singular 5-spaces; nor can it have type (iv) since all symps in $\mathscr{S}_U$ are singular but they do not have a common point. The claim follows. Since $\mathrm{Res}_{\Omega_U}(q)$ spans the 5-space corresponding to U, and since obviously each 6-spaces contained in the union of all 6-spaces through a given 3-space, contains that 3-space, the unique 3-space Π_q defined analogously as Π_p is contained in U. Consider a point q' of Υ_2 in a second 6-space U' containing Π_p, not in Π_p, and Υ_2-collinear to q. Since qq' belongs to $\mathrm{Res}_{\Upsilon_2}(q)$, we obtain that q' is Δ_2-collinear to Π_q; and obviously since $q' \in U'$, we also have that q' is Δ_2-collinear to Π_p. So q' is Δ_2-collinear to Π_p and Π_q. However, the latter are 3-spaces of the $\mathsf{A}_{4,2}(\mathbb{K})$-geometry Ω_U, and hence intersect in a unique point, i.e., they generate the maximal singular subspace U of Δ_2, a contradiction because $q' \notin U$.

Now suppose p has type (i): all symps through p are singular and $p^{\perp_{\Upsilon_2}}$ is contained in a singular 6-space U. Let q be a point of $p^{\perp_{\Upsilon_2}}$ distinct from p. Consider a Segre subgeometry $\mathscr{S}$ contained in $\mathrm{Res}_{\Upsilon_2}(p)$ and containing q. As in the previous paragraph, the union of all symps containing p whose point residue at p is contained in $\mathscr{S}$ gives a standard subgeometry Ω isomorphic to $\mathsf{A}_{4,2}(\mathbb{K})$. Again by Lemma 5.28, $\Omega \subseteq U$. Since $q^{\perp_{\Upsilon_2}}$ contains a Segre subgeometry, namely, $\mathrm{Res}_{\Omega}(q)$, in a singular 5-space (corresponding to U), it follows that q has type (i) too since (iv) was now the only alternative. By connectedness, all points of Υ_2 have type (i). In this case, $p^{\perp_{\Upsilon_2}}$ generates a maximal 6-space S of Δ_2. As in the previous case, it follows with the help of Lemma 5.28 that all symps of Υ_2 containing p are contained in S too. Now let q be a point of Υ_2 opposite p. Since q is Υ_2-symplectic to a unique point of each Υ_2-line through p, and q has type (i) too, it follows that q is Δ_2-collinear to a 5-space of S, implying that $q \in S$. We conclude that $\Upsilon_2 \subseteq S$, a contradiction because Υ_2 contains disjoint 3-spaces.

Finally, suppose p has type (iv). Then, by the foregoing, all points of Υ_2 have type (iv). This means that through each point x of Υ_2, there is a unique line L_x of Υ_2 such that all symps of Υ_2 through L_x are singular. Observe that, if y is any point on L_x, then $L_x = L_y$ (since the residue at y contains a unique point through which each symp is singular, and hence this point corresponds to the line L_x). Moreover, a symp through x is singular if and only if it contains L_x, so if ξ is a singular symp of Υ_1, the line L_x is contained in ξ. Now consider two singular symps ξ_1 and ξ_2 of Υ_2 intersecting each other in a plane π through L_p and let x be a point of $\pi \setminus L_p$. Then L_x is contained in $\xi_1 \cap \xi_2 = \pi$, implying that it meets L_x in a point y. But then $L_x = L_y = L_p$, a contradiction. □

Knowing that each point has type (v), we can show that the embedding is isometric.

Lemma 5.30. *The embedding of* $\Upsilon_2 = \mathsf{A}_{5,3}(\mathbb{K})$ *in* $\Delta_2 = \mathsf{E}_{7,7}(\mathbb{K})$ *is isometric.*

Proof. This can be proven analogously to Lemma 4.6, with two small changes. Firstly, to see that symps of Υ_2 embed isometrically in Δ_2, we now use Lemma 5.29 instead of a dimension argument. Secondly, given a point p which is Υ_2-collinear to the unique point r of a symp ξ of Υ_2, to deduce that p is also Δ_2-collinear only to the point r in the unique symp of Δ_2 containing ξ, we now rely on Lemma 5.21 instead of Corollary 4.5. □

It suffices to apply induction and hence exploit our result about the isometric embedding of corresponding point residues (see Proposition 5.26).

Proposition 5.31. *Suppose* $\Upsilon_2 = \mathsf{A}_{5,3}(\mathbb{K})$ *is fully embedded in* $\Delta_2 = \mathsf{E}_{7,7}(\mathbb{K})$. *Then* $\Upsilon_2 = E(\Sigma_1, \Sigma_4) \cap E(\Sigma_2, \Sigma_5) \cap E(\Sigma_3, \Sigma_6)$, *where* $\Sigma_1, \dots, \Sigma_6$ *are symps of* Δ_2 *with* Σ_i *and* Σ_{i+3} *opposite and* $U_i := \Sigma_i \cap \Sigma_{i+1}$ *a singular* 5*-space, with* $i \in \mathbb{Z}/6\mathbb{Z}$.

Proof. By Lemma 5.30, the Δ_2-distance between two points of Υ_2 is the same as their Υ_2-distance, so we make no distinction; in particular we write $\perp$ instead of $\perp_{\Upsilon_2}$ or $\perp_{\Delta_2}$. Let p and q be opposite points of Υ_2. As in the proof of Proposition 4.4, we let Δ_1^p and Υ_1^p denote, respectively, the set of points of Δ_2 and Υ_2 which are collinear to p and at distance 2 from q, likewise for q. Observe that $\Delta_1^p \cong \mathsf{E}_{6,1}(\mathbb{K})$ and $\Upsilon_1^p \cong \mathsf{A}_{2,2}(\mathbb{K}) \times \mathsf{A}_{2,2}(\mathbb{K})$ and recall from the proof of Proposition 4.4 that collinearity gives an isomorphism ρ between Δ_1^p and Δ_1^q mapping points to symps.

By Proposition 5.26, there are 5-spaces $V_1, \dots, V_6$ of Δ_1^p with V_i and V_{i+3} opposite and $\pi_i := V_i \cap V_{i+1}$ a plane, such that $\Upsilon_1^p = E(V_1, V_4) \cap E(V_2, V_5) \cap E(V_3, V_6)$; with $i \in [1, 6]$. Just like in the proof of Proposition 4.4, the 5-space V_i and its image $\rho(V_i)$ determine a symp Σ_i of Δ_2. Observe that $U_i := \Sigma_i \cap \Sigma_{i+1}$ is given by $\langle \pi_i, \rho(\pi_i) \rangle$ and hence is a 5-space. The fact that Σ_i and Σ_{i+3} are opposite is shown in the proof of Proposition 4.4.

One can show that $\Upsilon_2 = E(\Sigma_1, \Sigma_4) \cap E(\Sigma_2, \Sigma_5) \cap E(\Sigma_3, \Sigma_6)$ as follows. It is straightforward to see that $p^{\perp_{\Upsilon_2}} \cup q^{\perp_{\Upsilon_2}}$ is contained in $E(\Sigma_1, \Sigma_4) \cap E(\Sigma_2, \Sigma_5) \cap E(\Sigma_3, \Sigma_6)$. Since $p^{\perp_{\Upsilon_2}} \cup q^{\perp_{\Upsilon_2}}$ generates Υ_2 as a subspace of itself by [Blok and Brouwer 1998; Cooperstein and Shult 1997], and since equator geometries are subspaces, it already follows that $\Upsilon_2 \subseteq E(\Sigma_1, \Sigma_4) \cap E(\Sigma_2, \Sigma_5) \cap E(\Sigma_3, \Sigma_6)$. Equality then follows from the induction and Lemma 2.3. □

5C. *Full embedding of* $\mathsf{E}_{6,2}(\mathbb{K})$ *in* $\mathsf{E}_{8,8}(\mathbb{K})$. Suppose $\Upsilon_3 := \mathsf{E}_{6,2}(\mathbb{K})$ is fully embedded in $\Delta_3 := \mathsf{E}_{8,8}(\mathbb{K})$. Take any point p in Υ_3. Then $\mathrm{Res}_{\Upsilon_3}(p) \cong \mathsf{A}_{5,3}(\mathbb{K})$ is fully embedded in $\mathrm{Res}_{\Delta_3}(p) \cong \mathsf{E}_{7,7}(\mathbb{K})$. By Lemma 5.30, this embedding is unique up to projectivity, automatically isometric, and given by an intersection of equator geometries isomorphic to $\mathsf{D}_{6,6}(\mathbb{K})$. To show that Υ_3 embeds isometrically in Δ_3 is now easy.

Lemma 5.32. *The embedding of* $\Upsilon_3 = \mathsf{E}_{6,2}(\mathbb{K})$ *in* $\Delta_3 = \mathsf{E}_{8,8}(\mathbb{K})$ *is isometric.*

Proof. This can be proven analogously to Lemma 4.8, using Fact A.23 and noting that Υ_3 and Δ_3 are both long root geometries and that the embedding of the point residues $\mathrm{Res}_{\Upsilon_3}(p) \cong \mathsf{A}_{5,3}(\mathbb{K})$ in $\mathrm{Res}_{\Delta_3}(p) \cong \mathsf{E}_{7,7}(\mathbb{K})$, for each point p of Υ_3, is also isometric by Lemma 5.30. □

We can again apply our inductive method, as in the previous section.

Proposition 5.33. *Suppose* $\Upsilon_3 = \mathsf{E}_{6,2}(\mathbb{K})$ *is fully embedded in* $\Delta_3 = \mathsf{E}_{8,8}(\mathbb{K})$. *Then* $\Upsilon_3 = E(x_1, x_4) \cap E(x_2, x_5) \cap E(x_3, x_6)$, *where* $x_1, \ldots, x_6$ *are points of* Δ_3 *with* x_i *and* x_{i+3} *opposite and* x_i *and* x_{i+1} *collinear*, *with* $i \in [1, 6]$ (*indices modulo* 6).

Proof. By Lemma 5.32, the Δ_3-distance between two points of Υ_3 is the same as their Υ_3-distance, so we make no distinction; in particular we write $\perp$ instead of $\perp_{\Upsilon_3}$ or $\perp_{\Delta_3}$. As in the proof of Proposition 4.9, we let Δ_2^p and Υ_2^p denote, respectively, the set of points of Δ_3 and Υ_3 which are collinear to p and at distance 2 from q, likewise for q. Observe that $\Delta_2^p \cong \mathsf{E}_{7,7}(\mathbb{K})$ and $\Upsilon_2^p \cong \mathsf{A}_{5,3}(\mathbb{K})$ and recall from the proof of Proposition 4.9 that collinearity gives an isomorphism ρ between Δ_2^p and Δ_2^q.

By Proposition 5.31, there are symps $\Sigma_1, \ldots, \Sigma_6$ of Δ_2^p with Σ_i and Σ_{i+3} opposite and $U_i := \Sigma_i \cap \Sigma_{i+1}$ a 5-space, such that $\Upsilon_2^p = E(\Sigma_1, \Sigma_4) \cap E(\Sigma_2, \Sigma_5) \cap E(\Sigma_3, \Sigma_6)$; with $i \in [1, 6]$. Just like in the proof of Proposition 4.9, the symps of Δ_3 containing Σ_i and $\rho(\Sigma_i)$ meet each other in a unique point x_i and one can show that $\Upsilon_3 = E(x_1, x_4) \cap E(x_2, x_5) \cap E(x_3, x_6)$. The proof is really a multiple copy of the proof of Proposition 4.9, one for each of the equators $E(x_i, x_{i+3})$, $i = 1, 2, 3$, to deduce $\Upsilon_3 \subseteq E(x_1, x_4) \cap E(x_2, x_5) \cap E(x_3, x_6)$. Then we exploit the fact that $E(x_1, x_4) \cap E(x_2, x_5) \cap E(x_3, x_6)$ is a subspace which, endowed with the induced lines, is isomorphic to Υ_3. Indeed, this follows from the fact that the set of symps through a line through p corresponds to a para of $E(p, q)$ isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$, and the equator of a pair of opposite such paras in $\mathsf{E}_{7,1}(\mathbb{K})$ is a geometry isomorphic to $\mathsf{E}_{6,2}(\mathbb{K})$, see Definition 6.6 of [De Schepper et al. 2022]. Lemma 2.3 then completes the proof.

Note that the mutual position of the symps Σ_i corresponds to the position of the points x_i: the fact that $\Sigma_i \cap \Sigma_{i+1}$ is a 5-space translates to x_i and x_{i+1} being collinear, and Σ_i and Σ_{i+3} opposite translates to x_i and x_{i+1} opposite, as can easily be verified. □

Remark 5.34. Alternatively, the above proof could be ended when we deduced that Υ_3 is contained in one equator geometry Γ_3 isomorphic to $\mathsf{E}_{7,1}(\mathbb{K})$. Indeed, by Proposition 6.14 of [De Schepper et al. 2022] the embedding of Υ_3 in Γ_3 is projectively unique and the embedding of Γ_3 in Δ_3 is also projectively unique by Main Result 4.1. Combining these two facts, we also obtain that the embedding of Υ_3 in Δ_3 is projectively unique and therefore also given as an intersection of equator geometries.

This finishes the proof of Main Result 5.1 (see Propositions 5.33, 5.31 and 5.26).

Appendix: Properties of the parapolar spaces under consideration

In the following paragraphs we review some incidence and distance-related properties of the parapolar spaces occurring in this paper. Most of them also occur in

[De Schepper et al. 2022], but we include it for ease of reference. Everything in this section serves as reference material for later use.

A1. ***The direct product of two projective spaces (Segre geometries).*** Let ℓ and k be natural numbers with $\ell, k \geq 1$ and $\mathbb{K}$ a field. Consider the direct product $\mathsf{A}_{\ell,1}(\mathbb{K}) \times \mathsf{A}_{k,1}(\mathbb{K})$ of two projective spaces over $\mathbb{K}$ of respective dimensions ℓ and k. Abstractly, this gives a point-line geometry: the points are the pairs $p = (p_1, p_2)$ with p_1 a point of $\mathsf{PG}(\ell, \mathbb{K})$ and p_2 a point of $\mathsf{PG}(k, \mathbb{K})$, the lines have the form $\{p_1\} \times L_2 := \{(p_1, p_2) \mid p_2 \in L_2\}$ with p_1 a point of $\mathsf{PG}(\ell, \mathbb{K})$ and L_2 a line of $\mathsf{PG}(k, \mathbb{K})$; or, likewise defined, $L_1 \times \{p_2\}$. If $k = \ell = 1$, this geometry is isomorphic to a hyperbolic polar space of rank 2. If $k\ell > 1$, then this geometry is a strong parapolar space of diameter 2, whose symps are hyperbolic polar spaces of rank 2 given by $L_1 \times L_2$, where L_1 is a line of $\mathsf{PG}(\ell, \mathbb{K})$ and L_2 a line of $\mathsf{PG}(k, \mathbb{K})$.

If both k and ℓ are at least 2, then the universal embedding of the $\mathsf{A}_{\ell,1}(\mathbb{K}) \times \mathsf{A}_{k,1}(\mathbb{K})$ geometry is given by the Segre variety $\mathscr{S}_{k,\ell}(\mathbb{K})$ [Zanella 1996], which lives in $\mathsf{PG}(m, \mathbb{K})$ for $m := (\ell+1)(k+1) - 1$. It is the set of points in the image of the *Segre map*

$$\sigma : \mathsf{PG}(\ell, \mathbb{K}) \times \mathsf{PG}(k, \mathbb{K}) \to \mathsf{PG}(m, \mathbb{K}),$$

$$((x_0, \dots, x_\ell), (y_0, \dots, y_k)) \mapsto (x_i y_j)_{0 \leq i \leq \ell, 0 \leq j \leq k}.$$

If k or ℓ equals 1, say $\ell = 1$, then an embedding in projective space of the $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{k,1}(\mathbb{K})$ geometry requires dimension $m = 2k+1$ because it contains two disjoint k-spaces. Although technically speaking, there is no absolutely universal embedding, the global image of the Segre map is unique.

In this paper, we encountered $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{3,1}(\mathbb{K})$ and $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$. The former can be constructed by two disjoint projective 3-spaces in a projective 7-space, and a projectivity ρ between those. Two points between these 3-spaces are joined by a (maximal singular) line if they are each other's image under ρ. We now state some (elementary and well known) properties of the point-line geometry Γ associated to $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$. Then Γ has points, lines, planes as singular subspaces. Recall from the above that the symps of Γ are hyperbolic polar spaces of rank 2.

The following facts can easily be deduced by reasoning in the two projective planes π_1, π_2 associated to Γ. Note that the singular planes are given by $p_1 \times \pi_2$ or $\pi_1 \times p_2$, with p_i a point in π_i for $i \in \{1, 2\}$.

Fact A.1 (point–point relations). *Let x, y be two points of Γ. Then $\delta_\Gamma(x, y) \leq 2$, and if $\delta_\Gamma(x, y) = 2$, then x and y are contained in a unique symp.*

Fact A.2 (point–symp relations). *Let p be a point and ξ a symp of Γ with $p \notin \xi$. Then p is either collinear to no point of ξ (in which case we say that p is **far** from ξ), or p is collinear to a line of ξ (p is **close** to ξ). Hence p is never collinear to a unique point of ξ.*

Fact A.3 (symp–symp relations). *Let ξ and ξ' be distinct symps of Γ. Then $\xi \cap \xi'$ is either a unique point or a line.*

Fact A.4. *Given a point p and a line $L \not\ni p$, there is always at least one symp close to p and containing L.*

Fact A.5 (point–plane relations). *Given a point p and a singular plane π of Γ with $p \notin \pi$, the point p is collinear to a unique point of π.*

Fact A.6 (plane–plane relations). *Let π, π' be distinct singular planes of Γ. Then π and π' are either disjoint, in which case collinearity gives an isomorphism between them, or $\pi \cap \pi'$ is a unique point, in which case all singular lines containing this point belong to $\pi \cup \pi'$.*

The above fact divides the singular planes of Γ into two natural families: two planes belonging to the same family are disjoint; two planes of distinct families intersect each other in a unique point. Given two planes π and α of distinct families, the geometry Γ can be represented as the direct product of π and α, since each point p of $\Gamma \setminus (\pi \cup \alpha)$ is collinear to unique points of $p_\pi \in \pi$ and $p_\alpha \in \alpha$, and the unique planes of Γ containing p_π and p_α meet each other in precisely p.

Suppose π is a singular plane of Γ generated by three points p, p', p'', and $\alpha, \alpha', \alpha''$ are the unique singular planes of Γ distinct from π containing p, p', p'' respectively. Then $\alpha \cup \alpha' \cup \alpha''$ generates Γ since each point x of Γ is on a unique plane π_x generated by the unique points of $\alpha, \alpha', \alpha''$ collinear to x.

A2. ***Parapolar spaces of type $\mathsf{E}_{6,1}$.*** Let Δ be a geometry isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$, for some field $\mathbb{K}$. Then Δ is a strong parapolar space of diameter 2, each symp of which is isomorphic to $\mathsf{D}_{5,1}(\mathbb{K})$ and each point residue of which is isomorphic to $\mathsf{D}_{5,5}(\mathbb{K})$. The maximal singular subspaces have dimension 4 and 5. A symp and a 5-space are called incident when they share a 4-dimensional subspace. We refer to those as $4'$-spaces. A $4'$-space is contained in a unique symp and in a unique 5-space and hence it corresponds to a flag of type $\{2, 6\}$. The generators of a symp of Δ come into two natural families; one of them consists of 4-spaces of Δ and the other family consists of $4'$-spaces. Incidence between other elements of Δ is given by containment. The opposition relation on Δ interchanges types 1 and 6 and types 3 and 5, and preserves type 2 and type 4. We restrict our overview of the properties of Δ to those that we will rely on.

Fact A.7 (point–symp relations). *Let p be a point and Σ a symp of Γ with $p \notin \Sigma$. Then either $p^\perp \cap \Sigma$ is empty (in which case we say that p is **far** from Σ) or $p^\perp \cap \Sigma$ is a $4'$-space of Σ (p and Σ are **close**).*

Fact A.8 (symp–symp relations). *Two symps Σ and Σ' of Γ either intersect in a point or in a 4-space.*

Fact A.9. *Let L and π be a line and a plane, respectively, disjoint from a symp Σ, but all of whose points are close to Σ. Then there exist a unique plane π' and a unique line L' in Σ such that $L \perp \pi'$ and $\pi \perp L'$. Also, $\langle L, \pi' \rangle$ is a 4-space whereas $\langle L', \pi \rangle$ is a $4'$-space.*

Fact A.10 (5–5 relations). *Two 5-spaces U, V either intersect in a plane, a point, or in the empty subspace. In the latter case, there are two options. Either there is a unique 5-space intersecting both U and V in respective planes π_U and π_V, in which case each point of $U \setminus \pi_U$ is collinear to a unique point of V, which lies in π_V; or U and V are* ***opposite****, in which case collinearity between U and V gives a type-preserving isomorphism (each point of U is collinear to a unique point of V and vice versa).*

Fact A.11 (point–5 relations). *Let p be a point and U be a 5-space with $p \notin U$. Then $p^\perp \cap U$ is either a point or 3-space. It is a point if, and only if, p is contained in an opposite 5-space.*

Fact A.12 (symp–5 relations). *Let Σ be a symp and U be a 5-space. Then $\Sigma \cap U$ is either empty, a line, or a $4'$-space.*

Fact A.13. *Let p be a point and Σ a symp with p far from Σ. Then each line through p contains a unique point close to Σ and the set $H_{p,\Sigma}$ of these points, equipped with the lines it naturally contains, is isomorphic to $\mathsf{D}_{5,5}(\mathbb{K})$. Moreover, collinearity between $H_{p,\Sigma}$ and Σ follows the natural correspondence between $\mathsf{D}_{5,5}(\mathbb{K})$ and $\mathsf{D}_{5,1}(\mathbb{K})$: a point of $H_{p,\Sigma}$ corresponds to a $4'$-space of Σ, a point of Σ corresponds to a symp of $H_{p,\Sigma}$.*

Fact A.14. *Every singular 3-space is contained in a unique 5-space and a unique 4-space.*

A3. ***Strong parapolar spaces of type $\mathsf{E}_{7,7}$.*** Let Δ be isomorphic to $\mathsf{E}_{7,7}(\mathbb{K})$, for some field $\mathbb{K}$. Then Δ is a strong parapolar space of diameter 3; points at distance 3 are called *opposite*. The opposition relation is type preserving. The symps of Δ are isomorphic to $\mathsf{D}_{6,1}(\mathbb{K})$ and a point residue is isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$. The maximal singular subspaces have dimension 5 and 6. A symp and a 6-space are called incident when they share a 5-dimensional subspace. We refer to those as $5'$-spaces. A $5'$-space is contained in a unique 6-space and in a unique symp, i.e., it is a flag of type $\{1, 2\}$. Other incidences between pairs of elements of Δ are given by containment. A 5-space on the other hand is contained in at least two symps of Δ. The two families of maximal singular subspaces of a symp of Δ consist of 5-spaces and $5'$-spaces, respectively.

We now review the point–symp and symp–symp relations. As in the previous section, they can be deduced by considering an appropriate model of an apartment of a building of type E_7, as given in [Van Maldeghem and Victoor 2019]; they can sometimes be deduced from the previous section by considering an appropriate residue.

Fact A.15 (point–symp relations). *If p is a point and Σ a symp of Δ with $p \notin \Sigma$, then precisely one of the following occurs*:

(i) *p is collinear to a unique point $q \in \Sigma$. In this case, p and x are symplectic if $x \in \Sigma \cap (q^{\perp} \setminus \{q\})$ and $\delta(p, x) = 3$ for $x \in \Sigma \setminus q^{\perp}$.*

(ii) *p is collinear to a $5'$-space U of Σ. In this case, x and p are symplectic if $x \in \Sigma \setminus U$.*

Fact A.16 (symp–symp relations). *If Σ and Σ' are two symps of Δ, then precisely one of the following occurs*:

(i) $\Sigma = \Sigma'$.

(ii) *$\Sigma \cap \Sigma'$ is a 5-space.*

(iii) *$\Sigma \cap \Sigma'$ is a line L. Then points $x \in \Sigma \setminus L$ and $x' \in \Sigma' \setminus L$ are collinear only if $x, x' \in L^{\perp}$, and $\delta(x, x') = 3$ if, and only if, $x^{\perp} \cap L$ is disjoint from $x'^{\perp} \cap L$.*

(iv) *$\Sigma \cap \Sigma' = \varnothing$ and there is a unique symp Σ'' intersecting Σ in a 5-space U and intersecting Σ' in a 5-space U', with U and U' opposite in Σ''. Then each point of U is collinear to a $5'$-space of Σ' (intersecting U' in a 4-space) and each point of $\Sigma \setminus U$ is collinear to a unique point of Σ' (contained in U').*

(v) *$\Sigma \cap \Sigma' = \varnothing$ and every point of Σ is collinear to a unique point of Σ'. In this situation, Σ and Σ' are* ***opposite***.

Using the above relations, one can show that:

Fact A.17. *Let p, q be opposite points of Δ. Then each line through p contains a unique point symplectic to q and each symp through q contains a unique point collinear to p. For each point $x \in p^{\perp}$ at distance 2 from q, let $S_x = x^{\perp} \cap q^{\perp}$. This $x \mapsto S_x$ induces an isomorphism between $\mathrm{Res}_{\Delta}(p)$ and $\mathrm{Res}_{\Delta}(q)$ mapping points to symps.*

Fact A.18. *Let Σ and Σ' be two opposite symps. Then collinearity between the points of Σ and Σ' defines an isomorphism between Σ and Σ'.*

Fact A.19 (symp–6 relations). *Let Σ be a symp and U be a 6-space. Then $\Sigma \cap U$ is either empty, a point, a plane, or a $5'$-space.*

Fact A.20. *If a point is collinear to a 3-space of a maximal 5-space, then it is collinear to a 4-space of it.*

A4. ***Nonstrong parapolar spaces of type $\mathsf{E}_{8,8}$.*** Let Δ be the long root geometry $\mathsf{E}_{8,8}(\mathbb{K})$, for some field $\mathbb{K}$. Then Δ is a parapolar space, which has diameter 3 and is nonstrong. The elements of the corresponding building of types 1, 2, 3, 4, 5, 6, 7, 8, are the *symps*, 7-*spaces*, 6-*spaces*, 5-*spaces*, 3-*spaces*, *planes*, *lines and points*, respectively. The symps are isomorphic to polar spaces $\mathsf{D}_{7,1}(\mathbb{K})$. The other types are singular (projective) subspaces of Δ. A symp and a 7-space of Δ are incident when they share a subspace of dimension 6. These are referred to as $6'$-spaces and do not correspond to a type, but to a flag of type $\{1, 2\}$; each $6'$-space is contained in a unique symp and a unique 7-space. The two families of 6-dimensional subspaces of a symp of Δ are then given by 6-spaces and $6'$-spaces, respectively. All other incidence relations between elements of Δ are given by containment. One can deduce the possible mutual position of points, symps, etc., by considering an appropriate model of an apartment of a building of type E_8. Such models are given in [Van Maldeghem and Victoor 2019]. The point–point relations are as usual in a long root geometry of diameter 3.

Fact A.21 (point–point relations). *Let x and y be two points of Δ. Then $\delta_\Delta(x, y) \leq 3$ (and distance 3 occurs and corresponds to opposite points) and if $\delta_\Delta(x, y) = 2$, then either x and y are contained in a unique symp, or there is a unique point $x \bowtie y$ collinear with both x and y.*

Fact A.22 (point–symp relations). *Let p be a point and Σ be a symp of Δ with $p \notin \Sigma$. Then precisely one of the following occurs:*

(i) *p is collinear to a $6'$-space U of Σ. In this case, p and x are symplectic for all $x \in \Sigma \setminus U$.*

(ii) *p is collinear to a unique line L of Σ. In this case, p and x are symplectic if $x \in \Sigma \cap L^\perp$ and p and x are special if $x \in \Sigma \setminus L^\perp$ (and $p \bowtie x \in L$).*

(iii) *p is symplectic to each point of a 6-space U of Σ. In this case, p and x are special if $x \in \Sigma \setminus U$ (and $p \bowtie x \notin \Sigma$).*

(iv) *p is symplectic to a unique point q of Σ. In this case, p and x are special if $x \in \Sigma \cap q^\perp \setminus \{q\}$ and p and x are opposite if $x \in \Sigma \setminus q^\perp$.*

Finally, we record the following property of Δ, which in fact holds for all long root geometries related to spherical buildings, and also for all thick metasymplectic spaces. It is proved in [Cohen and Ivanyos 2007], see Lemma 2(v) therein.

Fact A.23 [De Schepper et al. 2022, Fact 3.16]. *Let $p \perp x \perp y \perp q$ be a path in Δ with (p, y) and (q, x) special. Then p and q are opposite, that is, $\delta(p, q) = 3$.*

Acknowledgement. The authors are very grateful to the referee for very helpful remarks that improved the paper.

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Received 9 Aug 2022. Revised 8 Nov 2022.

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