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Moufang quadrangles and affine twin buildings of type $\widetilde{C}_2$

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In memory of Jacques Tits.

We prove a group-theoretic criterion in terms of root groups for two subquadrangles Γ_1, Γ_3 of a Moufang quadrangle Γ to arise from a Moufang twin building Δ of type $\tilde{\mathsf{C}}_2$ as two adjacent residues of distinct special type, naturally embedded in the building at infinity of Δ, which is contained in Γ.

1. Introduction

Twin buildings were introduced by Ronan and Tits in the late 1980s. Their definition was motivated by the theory of Kac–Moody groups over fields developed by Tits in [8]. Twin buildings generalize spherical buildings in a natural way. In view of the classification of irreducible spherical buildings of rank at least 3 in [7] there is the natural question about a possible classification of twin buildings. This question was discussed by Tits in his paper [9] and it turns out that there is only reasonable hope for such a classification in the 2-spherical case. In this case substantial progress has been made. In [5] it is proved that a 2-spherical building is uniquely determined by its local structure (if it is not too *fragile*, meaning, if the rank 2 residues are not too small; it usually suffices that all panels have size at least 5). This result implies that each 2-spherical twin building is Moufang and that its local structure is a Moufang foundation. The remaining problem is to decide whether a given Moufang foundation is integrable, that is, whether it is indeed the local structure of a twin building. The true difficulty is to show the existence of a twin building whose local structure is isomorphic to a given Moufang foundation. In [2] a technique of geometric descent was developed to show integrability of certain foundations. This is refined in [3] where also a strategy for a complete classification is outlined; see also [4]. This strategy, however, relies on a complete classification of all irreducible 2-spherical twin buildings of rank 3. For most of the rank 3

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diagrams, the classification is straightforward. For some other more involved cases the classification was accomplished in [12]. The only case remaining is the case $\widetilde{\mathsf{C}}_2$. The present paper provides an existence criterion for affine twin buildings of type $\widetilde{\mathsf{C}}_2$. We intend to apply this existence criterion for the classification of affine buildings of type $\widetilde{\mathsf{C}}_2$ in future work.

Let Δ be a twin building of type $\widetilde{\mathsf{C}}_2$. Then Δ is Moufang and therefore its building at infinity Δ^∞ is a Moufang quadrangle by a result of Van Maldeghem and Van Steen [11]. Moreover, each special residue of Δ can be realized as a subquadrangle of Δ^∞. More precisely, let $\Sigma = (\Sigma_+, \Sigma_-)$ be a twin apartment of $\Delta = (\Delta_+, \Delta_-)$, let v_1, v_3 be two adjacent special vertices in Σ_+ and let Γ_i denote the residue of v_i in Δ_+ for $i = 1, 3$. Then there is a canonical way to embed Γ_i in Δ^∞ by means of the twin apartment Σ (this follows from the arguments of [6], suitably adopted to type $\widetilde{\mathsf{C}}_2$). Viewed as subquadrangles of Δ^∞ they contain Σ^∞ and there is a vertex $v_\infty \in \Sigma^\infty$ such that its set of neighbors in Γ_1 coincides with its set of neighbors in Γ_3.

The goal of this paper is to analyze such a situation in an arbitrary Moufang quadrangle. That is, we start with a Moufang quadrangle Γ and two subquadrangles Γ_1, Γ_3 sharing an apartment Σ such that there exists a vertex $v \in \Sigma$ with $\Gamma_v \cap \Gamma_1 = \Gamma_v \cap \Gamma_3$ (with Γ_v the residue of v in Γ). We say that the pair (Γ_1, Γ_3) is *integrable* if it arises from a twin building Δ of type $\widetilde{\mathsf{C}}_2$ in the way described above. Our main result provides an integrability criterion for the pair (Γ_1, Γ_3). This criterion uses root groups. We label the apartment Σ by $i \in \mathbb{Z}/8\mathbb{Z}$ in a natural cyclic order such that v is labeled by 4 and for each i we let U_i denote the root group fixing the vertex i, but not the vertex $i-1$. The subquadrangles Γ_1 and Γ_3 yield root subgroups $X_i, Y_i \le U_i$ and we have $X_4 = Y_4$, $X_0 = Y_0$. Our criterion is the following.

Main result. *With the notation above, let* $A := \langle X_1, X_4, Y_7\rangle \le \mathrm{Aut}(\Gamma)$. *Then* (Γ_1, Γ_3) *is integrable if* $[\langle X_1, X_5\rangle, \langle Y_7, Y_3\rangle] = 1$ *and none of the groups* X_0, X_5, Y_3 *is contained in* A.

Let Δ be the affine twin building whose existence follows from the Main Result, and let Δ^∞ be its building at infinity. Then Δ^∞ is the subquadrangle of Γ generated by Γ_1 and Γ_3.

2. Preliminaries

2A. ***Root systems of type C_2 and $\widetilde{\mathsf{C}}_2$.*** Let $E = \mathbb{R}^2$ be the Euclidean plane and for each line L of E let s_L denote the Euclidean reflection about L.

Let $\overline{\Phi} \subseteq E$ be a crystallographic root system of type C_2 in E. Let $\overline{\Delta} = \{\eta_1, \eta_2\}$ be a root base of $\overline{\Phi}$ such that η_2 is long. Thus we have $\overline{\Phi}^+ := \{\eta_1, \eta_2, \eta_1+\eta_2, 2\eta_1+\eta_2\} \subseteq \overline{\Phi}$ and $\overline{\Phi} = \overline{\Phi}^+ \cup \overline{\Phi}^-$ where $\overline{\Phi}^- := -\overline{\Phi}^+$. For each $\alpha \in \overline{\Phi}$ we let s_α denote the reflection of E associated with α, that is, the reflection along the line through the origin which is perpendicular to the line $\mathbb{R}\alpha$. We put $s_i = s_{\eta_i}$ for $i = 1, 2$, $\bar{S} = \{s_1, s_2\}$ and $\overline{W} = \langle S \rangle$ and remark that we have a natural action $(w, \alpha) \mapsto w \cdot \alpha$ of $\overline{W}$ and $\overline{\Phi}$.

A *natural cyclic numbering* of $\overline{\Phi}$ is a map $\nu : \mathbb{Z}/8\mathbb{Z} \to \overline{\Phi}$ such that

$$\nu(z+4) = -\nu(z)$$

for all $z \in \mathbb{Z}$ and

$$((\nu(k), \nu(k+1), \nu(k+2), \nu(k+3)) \in \{(\eta_1, 2\eta_1 + \eta_2, \eta_1 + \eta_2, \eta_2), (\eta_2, \eta_1 + \eta_2, 2\eta_1 + \eta_2, \eta_1)\}$$

for some $k \in \mathbb{Z}$.

Let $\alpha \in \overline{\Phi}$ and $z \in \mathbb{Z}$. We denote the line in E perpendicular to $\mathbb{R}\alpha$ that passes through $z\alpha$ by $L_{[\alpha,z]}$. We put $s_{[a;z]} := s_{L_{[a;z]}} \in \mathrm{Isom}(E)$. The *affine root* $[\alpha; z]$ is the open half-plane of E such that $\partial(\alpha; z) = L_{[a;z]}$, where $\partial(\alpha; z)$ is the boundary of $[\alpha; z]$, and $(z-1)\alpha \in [a; z]$. (Note the semicolon in the notation for affine roots to avoid confusion with later notation of intervals of roots.)

The *root system of type* $\widetilde{\mathrm{C}}_2$ is the set $\Phi := \{[\alpha; z] \mid \alpha \in \overline{\Phi}, z \in \mathbb{Z}\}$. For $\gamma = [\alpha; z] \in \Phi$ we put $\bar{\gamma} := \alpha \in \overline{\Phi}$ and $-\gamma := [-\alpha; -z] \in \Phi$.

We put $\delta_1 := [\eta_1; 0]$, $\delta_2 := [\eta_2; 0]$, $\delta_3 := [-(\eta_1+\eta_2); 1]$ and $s_i := s_{\delta_i}$ for $1 \le i \le 3$. This is consistent with the previous definition of s_i for $i = 1, 2$. Furthermore, we define $\Delta := \{\delta_1, \delta_2, \delta_3\}$, $S := \{s_1, s_2, s_3\}$ and $W := \langle S \rangle \le \mathrm{Isom}(E)$. We observe that $o(s_2 s_i) = 4$ for $i = 1, 3$ and $s_1 s_3 = s_3 s_1$. It is a well known fact that (W, S) is the Coxeter system of type $\widetilde{\mathrm{C}}_2$ where the diagram is labeled in a linear order. Furthermore, we have a natural action $(w, \gamma) \mapsto w \cdot \gamma$ of W on Φ.

Denote $t_{12} := s_2 s_1 s_2$, $t_{13} := s_3$, $t_{21} := s_1 s_2 s_1$, $t_{23} := s_3 s_2 s_3$, $t_{31} := s_1$ and $t_{32} := s_2 s_3 s_2$. We leave the proof of the following observations to the reader.

Lemma 2.1. (a) Φ *is the disjoint union of the three subsets* $W \cdot \delta_i$ *for* $i = 1, 2, 3$.

(b) $\mathrm{Stab}_W(\delta_1) = \langle t_{12}, t_{13} \rangle$, $\mathrm{Stab}_W(\delta_2) = \langle t_{21}, t_{23} \rangle$ *and* $\mathrm{Stab}_W(\delta_3) := \langle t_{31}, t_{32} \rangle$.

Using the two observations of the previous lemma, we obtain the following.

Proposition 2.2. *Let* Ω *be a set*, $(w, \omega) \mapsto w \cdot \omega$ *be an action of* W *on* Ω *and* $\omega_1, \omega_2, \omega_3 \in \Omega$. *Then the following assertions are equivalent*:

(i) *There exists a* W*-equivariant map* $f : \Phi \to \Omega$ *such that* $f(\delta_i) = \omega_i$.

(ii) $t_{ij} \cdot \omega_i = \omega_i$ *for* $\{i, j\} \subseteq \{1, 2, 3\}$.

For a point $v \in E$ we put $\Phi_v := \{\gamma \in \Phi \mid v \in \partial\gamma\}$ and v is called a *vertex of* Φ if $|\Phi_v| \ge 2$, it is called *special vertex* if $|\Phi_v| = 4$. The set of vertices of Φ is denoted by $V(\Phi)$; it is just the (root) lattice spanned by $\overline{\Phi}$. We set $v_i := \partial\delta_j \cap \partial\delta_k$ for $\{i, j, k\} = \{1, 2, 3\}$. We have a natural action of W on $V(\Phi)$ and for each $v \in V(\Phi)$ there exist $w \in W$ and $1 \le i \le 3$ such that $w \cdot v_i = v$. We note also that $w \cdot \Phi_v = \Phi_{w \cdot v}$ for all $v \in V(\Phi)$ and $w \in W$.

A pair $\pi = (\alpha, \beta) \in \Phi^2$ of roots is called *prenilpotent* if $\alpha \neq \beta$ and if $\alpha \cap \beta \neq \varnothing \neq (-\alpha) \cap (-\beta)$. For a prenilpotent pair $(\alpha, \beta) \in \Phi^2$ we put

$$[\alpha, \beta] := \{\gamma \in \Phi \mid \alpha \cap \beta \subseteq \gamma \text{ and } (-\alpha) \cap (-\beta) \subseteq -\gamma\}$$

and $]\alpha, \beta[:= [\alpha, \beta] \setminus \{\alpha, \beta\}$.

Let $\alpha, \beta \in \Phi$, $\alpha \neq \beta$. If $\alpha, \beta \in \Phi_v$ for some vertex $v \in V(\Phi)$, then the pair (α, β) is prenilpotent as soon as $\beta \neq -\alpha$, and then $[\alpha, \beta] = \{\gamma \in \Phi_v \mid \alpha \cap \beta \subseteq \gamma\}$. If $\partial\alpha \cap \partial\beta = \varnothing$, then (α, β) is a prenilpotent pair if and only if $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$; moreover, if $\alpha \subseteq \beta$, then $[\alpha, \beta] = \{\gamma \in \Phi \mid \alpha \subseteq \gamma \subseteq \beta\}$ and we have $\bar{\gamma} = \bar{\alpha}$ for all $\gamma \in [\alpha, \beta]$.

For $\beta \neq \pm\alpha \in \overline{\Phi}$ we put

$$[\alpha, \beta] := \{\gamma \in \overline{\Phi} \mid [\alpha; 0] \cap [\beta; 0] \subseteq [\gamma; 0]\}$$

and $]\alpha, \beta[:= [\alpha, \beta] \setminus \{\alpha, \beta\}$. This conforms to the above definition for affine roots.

A *sector of* Φ is a pair of roots (α, β) such that $o(s_\alpha s_\beta) = 4 = |[\alpha, \beta]|$. Two roots $\alpha, \beta \in \Phi$ are called *parallel* if $\bar{\alpha} = \bar{\beta}$. Parallelism is an equivalence relation on Φ that is compatible with the action of W on Φ. We identify the set of parallel classes with $\overline{\Phi}$ and denote the unique epimorphism from W onto $\overline{W}$ fixing s_i for $i = 1, 2$ by π_W. We have $\pi_W(w) \cdot \bar{\gamma} = \overline{w \cdot \gamma}$ for all $w \in W$ and $\alpha \in \Phi$.

2B. *Commutator calculations.* Throughout this subsection G is a group. For $g, h \in G$ we set $h^g := g^{-1}hg$, $[h, g] := h^{-1}g^{-1}hg$ and observe that $[h, g]^{-1} = [g, h]$. (It will always be clear from the context whether $[\cdot, \cdot]$ means an interval of roots, or a commutator in a group.) For two subsets X and Y of G we put $[X, Y] := \langle [x, y] \mid x \in X, y \in Y\rangle$ and observe that $[X, Y] = [Y, X]$. In the following lemma we collect some commutator identities that can be verified by straightforward calculations.

Lemma 2.3. *Let* $a, c, d, x \in G$. *Then*:

(i) $[cd, a] = [c, a]^d [d, a]$.

(ii) *If* $[d, a] = 1$, *then* $[[d, c], a] = [d, [a, c^{-1}]]^c$.

(iii) *If* $[x, a] = 1 = [d, a]$, *then* $[[d, c]x, a] = [d, [a, c^{-1}]]^{cx}$.

2C. *RGD-systems of type* C_2. Throughout this subsection G is a group and for a subgroup of U of G we put $U^\sharp := U \setminus \{1\}$.

A *rank*-1-*system in* G is a triple (U_+, U_-, μ) such that U_+, U_- are subgroups of G and $\mu : U_+^\sharp \to U_- U_+ U_-$ is a map, such that the following hold:

(ROS1) $U_\pm \neq \{1\} = U_+ \cap U_-$.

(ROS2) $U_+^{\mu(u)} = U_-$ and $U_-^{\mu(u)} = U_+$ for all $u \in U_+^\sharp$.

The notion of a rank-1-system is closely related to the notion of a RGD-system of type A_1 (or a rank-1-group) as defined in Section 7.8.2 in [1]. A straightforward adaptation of the arguments given there yields the following proposition.

Proposition 2.4. *Let $\Sigma = (U_+, U_-, \mu)$ be a 1-system in G. Let $L := \langle U_+, U_- \rangle$, $e \in U_+^\sharp$, $r := \mu(e)$ and $H := \langle \mu(u)^{-1}\mu(v) \mid u, v \in U_+^\sharp \rangle$. Then the following hold:*

(a) $H = N_L(U_+) \cap N_L(U_-)$.

(b) *$r \in N_L(H)$ and $r^2 \in H$.*

(c) *$rH = \mu(u)H$ for all $u \in U_+^\sharp$.*

(d) *If $\mu' : U_+^\sharp \to U_-U_+U_-$ is such that (U_+, U_-, μ') is a rank-1-system, then $\mu = \mu'$.*

Let $\Sigma = (U_+, U_-, \mu)$ be a rank-1-system in G. By Assertion (d) of the previous proposition, the mapping μ is uniquely determined by the pair (U_+, U_-) and therefore it makes sense to talk about the rank-1-system (U_+, U_-).

Let $\Sigma = (U_+, U_-, \mu)$ be a rank-1-system in G. A *subsystem of* Σ is a pair $\Pi = (V_+, V_-)$ with V_+ (resp. V_-) a subgroup of U_+ (resp. U_-) such that $(V_+, V_-, \mu|_{V_+})$ is a rank-1-system.

General RGD-systems have been introduced by Tits [9] and a detailed account can be found in [1] (see Definition 7.82 therein). Here, we are especially interested in RGD-systems of type C_2. However, we provide a slightly modified set of axioms, which better suits our purposes. In Proposition 2.6, we comment on the equivalence of both definitions.

An *RGD-system of type* C_2 *in* G is a family $(U_\alpha)_{\alpha\in\overline{\Phi}}$ such that the following hold:

(RGD1′) $(U_\alpha, U_{-\alpha})$ is a rank-1-system in G (yielding a map μ_α) for all $\alpha \in \overline{\Phi}$.

(RGD2′) For all $\alpha, \beta \in \overline{\Phi}$ and all $u \in U_\alpha^\sharp$ we have

$$\mu_\alpha(u)U_\beta\mu_\alpha(u)^{-1} = U_{s_\alpha(\beta)}.$$

(RGD3′) For all $\alpha \neq \beta \neq -\alpha$ we have

$$[U_\alpha, U_\beta] \subseteq U_{]\alpha,\beta[} := \langle U_\gamma \mid \gamma \in]\alpha, \beta[\rangle.$$

(RGD4′) $U_{-\eta_i}$ is not contained in $U_+ := \langle U_\gamma \mid \gamma \in \overline{\Phi}^+ \rangle$ for $i = 1, 2$.

In the remainder of this subsection we fix the following setup.

Conventions 2.5. $\Sigma = (U_\alpha)_{\alpha\in\overline{\Phi}}$ is an RGD-system of type C_2 in G and

(1) $L_\alpha := \langle U_\alpha, U_{-\alpha} \rangle$ and $H_\alpha := \langle \mu_\alpha^{-1}(a)\mu_\alpha(b) \mid a, b \in U_\alpha^\sharp \rangle$ for each $\alpha \in \overline{\Phi}$;

(2) for $i = 1, 2$ we choose $e_i \in U_{\eta_i}^\sharp$ and put $r_i := \mu_{\eta_i}(e_i)$; furthermore, we set $L_i := L_{\eta_i}$ and $H_i := H_{\eta_i}$;

(3) $H := \langle H_\alpha \mid \alpha \in \overline{\Phi} \rangle$, $N := \langle r_1, r_2, H \rangle$ and $L := \langle U_\alpha \mid \alpha \in \overline{\Phi} \rangle$.

Proposition 2.6. *With the above conventions, the following hold:*

(a) $[H_1, H_2] \leq H_1 \cap H_2$, $H = H_1 H_2$ *and* $H = \cap_{\alpha \in \overline{\Phi}} N_L(U_\alpha)$.

(b) $r_i \in N_L(H)$ *and* $r_i^2 \in H$ *for* $i = 1, 2$.

(c) $r_1 r_2 r_1 r_2 = r_2 r_1 r_2 r_1$.

(d) H *is normal in* N *and the map* $s_i \mapsto r_i H$ *for* $i = 1, 2$ *extends to an isomorphism* $\varphi : \overline{W} \to N/H$.

(e) *For each* $x \in N$ *and each* $\alpha \in \overline{\Phi}$ *we have* $(xH)U_\alpha(xH)^{-1} = U_{\varphi^{-1}(xH)\cdot\alpha}$.

(f) $r_i r_j r_i$ *normalizes* U_j *for* $\{i, j\} = \{1, 2\}$.

Proof. We first observe that $(L, (U_\alpha)_{\alpha \in \overline{\Phi}}, H)$ is an RGD-system of type $(\overline{W}, \overline{S})$ in the sense of Definition 7.82 in [1]. Indeed (RGD0) and (RGD2) in [1] follow from (RGD1′), (RGD2′) and the definition of H; (RGD1) (resp. (RGD3)) in [1] corresponds to (RGD3′) (resp. (RGD4′)); (RGD4) in [1] follows from the definition of L, and (RGD5) in [1] follows from the fact that $\mu(x)\mu(y)^{-1}$ normalizes all subgroups U_β for any two $x, y \in U_\alpha^\sharp$ and $\alpha \in \overline{\Phi}$. Setting $B := HU_+$, it follows from Theorem 7.115 in [1] that (B, N) is a BN-pair of type $(\overline{W}, \overline{S})$ in L and, by Theorem 7.116 in [1], that the building $\Delta = \Delta(L, B)$ is a Moufang building of type $(\overline{W}, \overline{S})$. Moreover, the group H corresponds to the pointwise stabilizer in L of an apartment Σ in Δ and the family $(U_\alpha)_{\alpha \in \overline{\Phi}}$ corresponds to the root groups associated with Σ.

In view of the previous remarks, Δ is a Moufang quadrangle which enables us to use results from [10]. The proof of Assertions (a) and (b) can be extracted from the proof of (33.9) in [10]. Assertion (c) follows from (6.9) in [10]. Assertion (d) follows from the fact that the groups U_α are pairwise distinct which follows from 7.90 in [1]. Finally, Assertions (e) and (f) are consequences of Assertion (d). □

Proposition 2.7. *Let* $\nu : \mathbb{Z}/8\mathbb{Z} \to \overline{\Phi}$ *be a natural cyclic order on* $\overline{\Phi}$ *and put* $U_i := U_{\nu^{-1}(\alpha)}$ *for each* $\alpha \in \overline{\Phi}$. *Then the following hold for each* $k \in \mathbb{Z}/8\mathbb{Z}$:

(a) *The product map*

$$U_k \times U_{k+1} \times U_{k+2} \times U_{k+3} \to \langle U_{k+i} \mid 0 \leq i \leq 3 \rangle$$

is bijective.

(b) $XU_{k+1} = U_{k+1}X = U_{k+1}U_{k+2} = XU_{k+2} = U_{k+2}X$ *where* $X := \{[a, b] \mid a \in U_k, b \in U_{k+3}\}$.

(c) U_k *or* U_{k+1} *is abelian and if* U_k *is abelian, then* $U_k \leq Z(U_{k-1}U_k) \cap Z(U_k U_{k+1})$.

(d) *If* $x \in U_k, c \in U_{k+1}$ *and* $y \in U_k U_{k+1} U_{k+2}$ *are such that* $y^{cx} \in U_{k+2}$, *then* $y \in U_{k+2}$.

Proof. Assertion (a) is Proposition (5.6) in [10] and Assertion (b) follows from (6.4) in [10] (because $n = 4$). The first assertion in (c) is (21.26) in [10] and the second is a consequence of it because $[U_k, U_{k+1}] = 1 = [U_{k-1}, U_k]$. Assertion (d) follows from (a) and the fact that U_{k+2} normalizes the subgroup $U_k U_{k+1}$. □

Definition 2.8. A *subsystem of* Σ is an RGD-system $(X_\alpha)_{\alpha\in\overline{\Phi}}$ of type C_2 in G such that $(X_\alpha, X_{-\alpha})$ is a subsystem of the rank-1-system $(U_\alpha, U_{-\alpha})$ for each $\alpha \in \overline{\Phi}$.

Remark 2.9. *As explained in the proof of Proposition 2.6, the notion of a Moufang quadrangle is essentially equivalent to the notion of an RGD-system of type C_2 in a group G. More precisely, given a Moufang quadrangle Γ and an apartment Σ of Γ, then the family of root groups associated with the roots of Σ is an RGD-system Π of type C_2 in* $\mathrm{Aut}(\Gamma)$. *Now, each thick subquadrangle Γ' of Γ that contains Σ provides in a natural way a subsystem of Π in the sense of the previous definition.*

The following observation is obvious.

Lemma 2.10. *Let $\Sigma' = (X_\alpha)_{\alpha\in\overline{\Phi}}$ be a subsystem of Σ and let $e_i' \in X_{\eta_i}^\sharp$. Define H_1', H_2', H',r_1', r_2' and N' for the system Σ' as in Conventions 2.5 for Σ and let $\varphi' : \overline{W} \to N'/H'$ be the isomorphism from Assertion (d) in Proposition 2.6. Then $H_i' \leq H_i$, $H' \leq H$ and $(xH')U_\alpha(xH')^{-1} = U_{\varphi^{-1}(xH')\cdot\alpha}$ for all $x \in N'$ and $\alpha \in \overline{\Phi}$.*

3. Proof of the main result

Conventions 3.1. In this section G is a group, $\Sigma = (U_\alpha)_{\alpha\in\overline{\Phi}}$ is an RGD-system of type C_2 in G and $(X_\alpha)_{\alpha\in\overline{\Phi}}$,$(Y_\alpha)_{\alpha\in\overline{\Phi}}$ are subsystems of Σ such that the following conditions are satisfied:

(C1) $(X_{\eta_2}, X_{-\eta_2}) = (Y_{\eta_2}, Y_{-\eta_2})$.

(C2) $[\langle X_{\eta_1}, X_{-\eta_1}\rangle, \langle Y_{\eta_1+\eta_2}, Y_{-(\eta_1+\eta_2)}\rangle] = 1$.

We put $(V_1, V_{-1}) := (X_{\eta_1}, X_{-\eta_1})$, $(V_2, V_{-2}) := (X_{\eta_2}, X_{-\eta_2})$ and $(V_3, V_{-3}) := (Y_{-(\eta_1+\eta_2)}, Y_{\eta_1+\eta_2})$. Let $1 \leq i \leq 3$. We put $L_i := \langle V_i, V_{-i}\rangle$. We denote the μ-map of the rank-1-system (V_i, V_{-i}) by μ_i and put $H_i := \langle \mu_i(a)\mu_i(b) \mid a, b \in V_i^\sharp\rangle$. We choose $e_i \in V_i^\sharp$ and put $r_i := \mu_i(e_i)$. Finally, we set $H := \langle H_i \mid 1 \leq i \leq 3\rangle$ and $N := \langle r_1, r_2, r_3, H\rangle$.

Lemma 3.2. *With the above conventions, the following hold:*

(a) *For $1 \leq i \neq j \leq 3$ we have $[H_i, H_j] \leq H_i \cap H_j$ and in particular $H = H_1H_2H_3$.*

(b) *For $1 \leq i \leq 3$ we have $H \leq N_G(V_i)$, $r_iHr_i^{-1} = H$ and $r_i^2 \in H$.*

(c) *$r_2r_ir_2r_i = r_ir_2r_ir_2$ for $i = 1, 3$ and $r_1r_3 = r_3r_1$.*

(d) *$r_2r_1r_2, r_3 \in N_G(V_1)$, $r_1r_2r_1, r_3r_2r_3 \in N_G(V_2)$ and $r_1, r_2r_3r_2 \in N_G(V_3)$.*

Proof. Assertions (a) and (b) follow from Proposition 2.6(a), (b) and (C2). Assertion (c) follows from Proposition 2.6(c) and (C2). Assertion (d) follows from Proposition 2.6(f). □

Remark and notation. We set $\Omega := \{xV_ix^{-1} \mid x \in N, 1 \le i \le 3\}$. By Assertion (b) of the previous lemma, H is a normal subgroup of N that normalizes each V_i and therefore each element of Ω. Thus, we obtain an action of N/H on Ω. By Assertions (b) and (c) of the previous lemma there is a unique homomorphism $\varphi : W \to N/H$ such that $\varphi(s_i) = r_iH$ for each $1 \le i \le 3$. In this way we obtain an action $(w, V) \mapsto w \cdot V$ of W on Ω. By Assertion (d) and Proposition 2.2 we obtain a unique W-equivariant map $\alpha \mapsto V_\alpha$ from Φ onto Ω such that $V_{\delta_i} = V_i$ for $1 \le i \le 3$.

Lemma 3.3. (a) *Let $\overline{\Omega} := \{U_\alpha \mid \alpha \in \overline{\Phi}\}$. Then $H \le N_G(U)$ and $xUx^{-1} \in \overline{\Omega}$ for each $U \in \overline{\Omega}$ and each $x \in N$. In particular, N/H acts on $\overline{\Omega}$.*

(b) *For each $\alpha \in \Phi$ we have $V_\alpha \le U_{\bar{\alpha}}$.*

(c) *For each special vertex v of Φ the family $(V_\alpha)_{\alpha\in\Phi_v}$ is an RGD-system of type C_2 in G.*

Proof. Assertion (a) follows from Lemma 2.10.

We have $V_{\delta_1} = V_1 = X_{\eta_1} \le U_{\eta_1}$ and $\overline{\delta_1} = \eta_1$; $V_{\delta_2} = V_2 = X_{\eta_2} \le U_{\eta_2}$ and $\overline{\delta_2} = \eta_2$; $V_{\delta_3} = V_3 = Y_{-(\eta_1+\eta_2)} \le U_{-(\eta_1+\eta_2)}$ and $\overline{\delta_3} := -(\eta_1 + \eta_2)$. Thus (b) holds for $\alpha \in \Delta$. Let $\alpha \in \Phi$ be an arbitrary root. Then there exist $w \in W$ and $\delta \in \Delta$ such that $w \cdot \delta = \alpha$. Let $x \in \varphi(w) \in N/H$. Then $xV_\delta x^{-1} \le xU_{\bar{\delta}}x^{-1} \in \overline{\Omega}$. Since the action of W on Φ respects parallelism in Φ, it follows that $V_\alpha \le U_{\bar{\alpha}}$.

If $v = v_1$ then $(V_\alpha)_{\alpha\in\Phi_v} = (Y_\alpha)_{\alpha\in\overline{\Phi}}$, and if $v = v_3$, then $(V_\alpha)_{\alpha\in\Phi_v} = (X_\alpha)_{\alpha\in\overline{\Phi}}$. Thus Assertion (c) holds for $v \in \{v_1, v_3\}$. Let v be an arbitrary special vertex of Φ. Then there exist $i \in \{1, 3\}$ and $w \in W$ such that $v = w \cdot v_i$. Let $x \in \varphi(w)$, then $(V_\alpha)_{\alpha\in\Phi_v} = (xV_\beta x^{-1})_{\beta\in\Phi_{v_i}}$. Since conjugation by x is an automorphism of G, Assertion (c) holds for v as well. □

Corollary 3.4. *Let $(\alpha, \beta) \in \Phi^2$ be a prenilpotent pair such that $\partial\alpha$ and $\partial\beta$ intersect in a vertex v. Then*

$$[V_\alpha, V_\beta] \le V_{]\alpha,\beta[} := \langle V_\gamma \mid \gamma \in]\alpha, \beta[\rangle.$$

Proof. If v is a special vertex, then the assertion follows from Assertion (c) of the previous proposition. If v is not special, there exists $w \in W$ such that $\{w \cdot \delta_1, w \cdot \delta_3\} = \{\alpha, \beta\}$. Let $x \in \varphi(w)$. Then we have $[V_\alpha, V_\beta] = [xV_{\delta_1}x^{-1}, xV_{\delta_3}x^{-1}] = x[V_1, V_3]x^{-1} = 1$ and the assertion holds as well. □

Let $\alpha, \beta \in \Phi$ be parallel roots such that $\alpha \subseteq \beta$. Then there exist unique $m \le n \in \mathbb{Z}$ such that $\alpha = [\bar{\alpha}; m]$ and $\beta = [\bar{\alpha}; n]$. The pair (α, β) is called *even* (*odd*) if $n - m$ is even (odd). If (α, β) is even, we put $\mu(\alpha, \beta) := [\bar{\alpha}; (n + m)/2]$.

Proposition 3.5. *Let $\alpha \neq \beta \in \Phi$ be parallel roots such that $\alpha \subseteq \beta$. If $[V_\alpha, V_\beta] \neq 1$, then the pair (α, β) is even and $[V_\alpha, V_\beta] \leq V_{\mu(\alpha,\beta)}$.*

Proof. Let v be a special vertex on $\partial\beta$ and let $(\gamma, \delta) \in \Phi_v^2$ be a sector such that $\beta \in]\gamma, \delta[$ and such that $\partial\gamma$ is perpendicular to $\partial\beta$. Let $\xi \in]\gamma, \delta[$ be the unique root with $\xi \neq \beta$. Then $(\bar{\gamma}, \bar{\xi}, \bar{\alpha} = \bar{\beta}, \bar{\delta})$ are in a natural cyclic order. We put $U_1 := U_{\bar{\gamma}}$, $U_2 := U_{\bar{\xi}}$, $U_3 := U_{\bar{\alpha}}$ and $U_4 := U_{\bar{\delta}}$.

Suppose $a \in V_\alpha$ and $b \in V_\beta$ are such that $[b, a] \neq 1$. Since V_α and V_β are subgroups of U_3 (by Lemma 3.3(b)) the group U_3 is nonabelian and $[b, a] \in U_3$. Thus, by Proposition 2.7(c), U_2 is abelian and $U_2 \leq Z(U_2U_3)$. Thus $[x, a] = 1$ for each $x \in V_\xi$. For a similar reason we have $[d, a] = 1$ for each $d \in V_\delta$. By Proposition 2.7(b) there exist $x \in V_\xi$, $c \in V_\gamma$ and $d \in V_\delta$ such that $b = [d, c]x$. By Lemma 2.3(iii) we have $[b, a] = [[d, c]x, a] = [d, [a, c^{-1}]]^{cx}$. As $[b, a] \neq 1$, we have $[a, c^{-1}] \neq 1$ and hence $[V_\gamma, V_\alpha] \neq 1$. This implies that $\partial\gamma$ and $\partial\alpha$ intersect in a special vertex v'. As $\partial\gamma$ is perpendicular to $\partial\alpha$ we have $]\gamma, \alpha[= \{\epsilon\}$ for a root $\epsilon \in \Phi_{v'}$ and $1 \neq y := [a, c^{-1}] \in V_\epsilon$. As $[b, a] \neq 1$ we have $[d, y] \neq 1$ which implies that $\partial\epsilon$ intersects $\partial\delta$ in a special vertex. Now, $]\delta, \epsilon[= \{\mu\}$ for a unique root μ that is parallel with α and β. Furthermore, by elementary Euclidean geometry, $\partial\alpha$ and $\partial\beta$ are at the same distance from $\partial\epsilon$ and we conclude that (α, β) is even and $\mu = \mu(\alpha, \beta)$. As $[b, a] \in U_3$, $[d, y] \in V_\mu \leq U_3$, $c \in V_\gamma \leq U_1$ and $x \in V_\xi \leq U_2$ it follows from Proposition 2.7(d) that $[d, y] = [b, a]$ and hence $[b, a] \in V_\mu$. □

Corollary 3.6. *For each prenilpotent pair $(\alpha, \beta) \in \Phi^2$ we have*

$$[V_\alpha, V_\beta] \leq \langle V_\gamma \mid \gamma \in]\alpha, \beta[\rangle.$$

Proof. This follows from Corollary 3.4 and Proposition 3.5. □

Theorem 3.7. *Let $L := \langle V_\alpha \mid \alpha \in \Phi \rangle$ and $V_+ := \langle V_\alpha \mid \alpha \in \Phi^+ \rangle$. Then the following are equivalent*:

(i) *$\Pi = (L, (V_\alpha)_{\alpha\in\Phi}, H)$ is an RGD-system of type $\widetilde{C}_2$, that is, it satisfies the axioms* (RGDi) ($0 \leq i \leq 5$) *of Definition* 7.82 *in* [1].

(ii) *$V_{-i} = V_{-\delta_i}$ is not contained in $V_+ := \langle V_\alpha \mid \alpha \in \Phi^+ \rangle$ for $i = 1, 2, 3$.*

Proof. Condition (ii) coincides with Axiom (RGD3) and therefore (i) implies (ii). Thus it remains to show that (ii) implies (i).

Let $\alpha \in \Phi$. Then we have $w \in W$ and $1 \leq i \leq 3$ such that $w \cdot \delta_i = \alpha$. Let $x \in \varphi(w)$. Then $xV_\delta x^{-1} = V_\alpha$. Since $V_{\delta_i} \neq 1$, the system Π satisfies Axiom (RGD0). Axiom (RGD1) for Π follows from Corollary 3.6.

Let $1 \leq i \leq 3$. Then (V_i, V_{-i}, μ_i) is a rank-1-system in L and

$$H_i := \langle \mu_i(a)^{-1}\mu_i(b) \mid a, b \in V_i^\sharp \rangle \leq \langle H_i \mid 1 \leq i \leq 3 \rangle =: H.$$

Let $a \in V_i^\sharp$. Then $\mu_i(a)H = r_iH = \varphi(s_i)$ and therefore $\mu_i(a)V_\alpha\mu_i(a)^{-1} = V_{s_i\cdot\alpha}$ for each $\alpha \in \Phi$. Thus, Axiom (RGD2) holds with $m = \mu_i$.

As already mentioned above, Axiom (RGD3) is equivalent to (ii). Further, Π satisfies (RGD4) by the definition of L. Finally, it follows from Lemma 3.3 that Π satisfies (RGD5). $\square$

Our next aim is to establish the existence a Moufang twin building of type $\widetilde{\mathsf{C}}_2$ on which the group L acts in a natural way. This follows from general facts about Moufang twin buildings and RGD-systems as described in [1]. We recall these facts below. Throughout the discussion we assume that (X, R) is an irreducible Coxeter system of finite rank at least 2.

Let Δ be a Moufang twin building of type (X, R), let Σ be a twin apartment of Δ and let $\Phi(\Sigma)$ denote the set of twin roots of Σ. For each twin root α of Σ let $U_\alpha \leq \mathrm{Aut}(\Delta)$ be the root group associated with α. Let $Y \leq \mathrm{Aut}(\Delta)$ be a group of type-preserving automorphisms containing all root groups U_α and let T be the pointwise stabilizer of Σ in Y. Then, by Exercise 8.47 in [1], $(Y, (U_\alpha)_{\alpha\in\Phi(\Sigma)}, T)$ is an RGD-system of type (X, R) in the sense of Definition 7.82 in [1].

In the other direction, let $(Y, (U_\alpha)_{\alpha\in\Phi(X,R)}, T)$ be an RGD-system of type (X, R). Then it follows by Theorems 8.80 and 8.81 in [1] that the group Y acts on a Moufang twin building Δ of type (X, R) in such a way that the subgroups U_α map isomorphically onto the root groups associated with a suitable twin apartment Σ of Δ and such that T is the pointwise stabilizer of Σ in Y.

In view of these two general facts Theorem 3.7 has the following consequence:

Corollary 3.8. *Let $L := \langle V_\alpha \mid \alpha \in \Phi\rangle$ and suppose that $V_{-i} = V_{-\delta_i}$ is not contained in $V_+ := \langle V_\alpha \mid \alpha \in \Phi^+\rangle$ for $i = 1, 2, 3$.*

Then L acts on a Moufang twin building Δ of type $\widetilde{\mathsf{C}}_2$ in such a way that the subgroups $(V_\alpha)_{\alpha\in\Phi}$ can be identified with the set of root groups associated with a suitable apartment Σ of Δ.

Let Π be the RGD-system of Theorem 3.7 and let Δ be the Moufang twin building of type $\widetilde{\mathsf{C}}_2$ from Corollary 3.8. Let Σ be as in Corollary 3.8 and let v be a vertex of Σ. Then the residue of Δ corresponding to v is the Moufang quadrangle associated with the RGD-system $(V_\alpha)_{\alpha\in\Phi_v}$. Thus, by Lemma 3.3(c) they are all isomorphic to one of the Moufang quadrangles associated with $(X_\alpha)_{\alpha\in\overline{\Phi}}$ or $(Y_\alpha)_{\alpha\in\overline{\Phi}}$. This yields the following.

Proposition 3.9. *Let Π be the RGD-system of **Theorem 3.7** and let Δ be the Moufang twin building of type $\widetilde{\mathsf{C}}_2$ from Corollary 3.8. Then the residues of type $\{2, 3\}$ of Δ are isomorphic to the Moufang quadrangle associated with the RGD-system $(Y_\alpha)_{\alpha\in\overline{\Phi}}$ and the residues of type $\{1, 2\}$ of Δ are isomorphic to the Moufang quadrangle associated with the RGD-system $(X_\alpha)_{\alpha\in\overline{\Phi}}$.*

Conclusion of the proof of the main result. Let Γ, Σ, Γ_1 and Γ_3 be as in the statement of the main result, let $G := \mathrm{Aut}(\Gamma)$ and let $(U_\alpha)_{\alpha\in\overline{\Phi}}$ be the RGD-system of type C_2 in G associated with Γ and Σ. By Remark 2.9 the subquadrangles Γ_1 and Γ_3 yield subsystems $(X_\alpha)_{\alpha\in\overline{\Phi}}$ and $(Y_\alpha)_{\alpha\in\overline{\Phi}}$ of $(U_\alpha)_{\alpha\in\overline{\Phi}}$ in the sense of Definition 2.8. The condition $[\langle X_1, X_5\rangle, \langle Y_7, Y_3\rangle] = 1$ in the statement of the main result corresponds to the Convention (C2) in this section. Moreover, since it is assumed that the set of neighbors v in Γ_1 coincides with its set of neighbors in Γ_3, it follows that $X_4 = Y_4$ and $X_0 = Y_0$ which corresponds to Convention (C1) in this section. Thus we are in the position to apply the results obtained in this section. The condition in the main result that none of the groups X_0, X_5, Y_3 is contained in A coincides in the setup of this section with the condition that $V_{-i} = V_{-\delta_i}$ is not contained in $V_+ := \langle V_\alpha \mid \alpha \in \Phi^+\rangle$ for $i = 1, 2, 3$ in 3.7. Thus the assertion of the main result follows from Theorem 3.7, Corollary 3.8 and Proposition 3.9.

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