A uniform characterisation of the varieties of the second row of the Freudenthal-Tits magic square over arbitrary fields

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Abstract

We characterize the projective varieties related to the second row of the Freudenthal-Tits magic square, for both the split and the non-split form, using a common, simple and short geometric axiom system. A special case of our result simultaneously captures the analogues over arbitrary fields of the Severi varieties (comprising the 27-dimensional E_6 module and some of its subvarieties), as well as the Veronese representations of projective planes over composition division algebras (most notably the Cayley plane). It is the culmination of almost four decades of work since the original 1984 result by Mazzocca and Melone who characterised the quadric Veronese variety over a finite field of odd order. The latter result is a finite counterpart to the characterisation of the complex quadric Veronese surface by Severi from 1901.

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1 Introduction

In [21], the second and third author of this paper obtained a classification of the split varieties corresponding to the second row of the Freudenthal-Tits magic square over arbitrary fields. The method used starts from an axiomatic geometric approach directly inspired by common basic properties of these varieties: the existence of an abundance of split quadrics, the smoothness of the varieties and the boundedness of the dimension (via the tangent space at each point) in terms of the dimension of the aforementioned quadrics (see below for the precise axioms). The lack of any assumption on the dimension of the whole space implied a slightly longer list in the conclusion; basically also some specific subvarieties of these split varieties satisfy the axioms. Over algebraically closed fields of characteristic 0 these split varieties are known as Severi varieties, and this classification recovers Zak's classification [28] of Severi varieties which was proved using different methods (algebraic geometry). Zak's result has its origins in Severi's 1901 characterization of the complex quadric surface [22].

On the other side of the spectrum, namely when the Witt index is minimal, in [16] the same axiomatic setup for quadrics without lines was used to characterize the Veronese representations of projective planes over quadratic alternative division algebras. Now, these Veronese representations and the analogues of the Severi varieties over arbitrary fields are closely related: generically they correspond to non-split and split, respectively, forms of the same algebraic groups, namely those of types A_2 (only the split form), $\mathsf{A}_2 \times \mathsf{A}_2$, A_5 and E_6. Whence the need to check whether or not other forms of those groups give rise to varieties with similar behaviour. "Similar" means in a global setting encompassing the two separate ones.

An obvious way to achieve this global setting is to omit the assumption that the quadrics are split, or non-ruled, respectively. Intuitively, possible additional examples are expected to satisfy the property that all quadrics are isomorphic. However, we here consider the most general situation in which the quadrics not only can be non-isomorphic, they also need not have the same Witt index (but inherent to the axioms is the property that all quadrics span a subspace of equal dimension). In this most general setup, we show that only the aforementioned varieties occur. This yields a very neat and complete geometric characterization of the varieties of the second row of the Freudenthal-Tits Magic Square. It is also an example of how simple geometric axioms give rise to a class of more advanced algebraic objects with a large symmetry group, notably containing (isotropic forms of) algebraic groups of exceptional type.

There are two reasons why we are now able to prove the current Main Theorem although it was already stated as a conjecture in [21]. The first one is that an approach to include degenerate quadrics in the picture in [10] generated a new technique, which seems to work particularly well in our setting. Roughly, it is demonstrated in Lemmas 5.1 and 5.2. The second reason is that we now have at our disposal a classification of parapolar spaces which are so-called 0-*lacunary* [8, 9], see Definition 4.6. We use that result in a crucial way.

In Section 3 we illustrate the power of combinatorial methods in algebra by providing a geometric explanation for the well-known fact that the stabilizer of $\mathscr{D}_{5,5}(\mathbb{K})$ or $\mathscr{E}_{6,1}(\mathbb{K})$ in $\mathbb{P}^{15}(\mathbb{K})$ or $\mathbb{P}^{26}(\mathbb{K})$, respectively, acts with two or three orbits, respectively, on the points (and likewise the hyperplanes) of the projective space. This extends work of Cooperstein and Shult [5]. Although not logically needed for the rest of our paper these results are highly related and interesting in their own right.

Below we outline the axiomatic setup and we discuss the Main Theorem in some greater detail.

1.1 Axiomatic setup

Projective quadrics and ovoids. For a (commutative) field $\mathbb{K}$ and a non-zero cardinal number n, we denote by $\mathbb{P}^n(\mathbb{K})$ the n-dimensional projective space over $\mathbb{K}$. The subspace generated by a family $\mathscr{F}$ of subsets of points is denoted by $\langle S \mid S \in \mathscr{F} \rangle$. A *non-degenerate quadric* Q in $\mathbb{P}^n(\mathbb{K})$, n finite, is the null set of an irreducible quadratic homogeneous polynomial in the (homogeneous) coordinates of points of $\mathbb{P}^n(\mathbb{K})$. The *projective index* of Q is the (common) dimension of the maximal subspaces of $\mathbb{P}^n(\mathbb{K})$ entirely contained in Q (in the literature, one finds more commonly the *Witt index*, which is the projective index plus one; we prefer to work in a projective setting and hence express all dimensions projectively instead of in the underlying vector space). A *tangent line to Q (at a point $x \in Q$)* is a line which has either only x or all its points in Q. The union of the set of tangent lines to Q at one of its points x is a hyperplane of $\mathbb{P}^n(\mathbb{K})$, denoted by $T_x(Q)$. An *ovoid* O of $\mathbb{P}^n(\mathbb{K})$ is a set of points which behaves like a quadric of projective index 0: each line of $\mathbb{P}^n(\mathbb{K})$ intersects O in at most two points, and the union of the set of tangent lines (defined as above) at each point is a hyperplane of $\mathbb{P}^n(\mathbb{K})$.

Axiomatic Veronese varieties. Let (X, Ξ) be a pair, where X is a spanning point set of a projective space $\mathbb{P}^N(\mathbb{K})$ over some field $\mathbb{K}$ and with $N \in \mathbb{N} \cup \{\infty\}$, and where Ξ is a collection of at least two different $(d+1)$-dimensional subspaces of $\mathbb{P}^N(\mathbb{K})$, where $1 \leq d < \infty$, such that for each $\xi \in \Xi$, the set $X(\xi) := X \cap \xi$ is a non-degenerate quadric or ovoid generating ξ. We denote $T_x(X(\xi))$ also by $T_x(\xi)$. A subspace of $\mathbb{P}^N(\mathbb{K})$ is called *singular* if it has all its points in X; the set of singular lines is denoted by $\mathscr{L}$.

The *tangent space at $x \in X$ to X* is the subspace T_x generated by the sets $\{T_x(\xi) \mid x \in \xi \in \Xi\}$ and $\{L \mid x \in L \in \mathscr{L}\}$. Usually only the former set is used to define T_x, as in view of (MM1) below, the latter set is automatically contained in what is generated by the former set. The reason that we use both sets is that our inductive approach leads to structures in which (MM1) holds in a weaker form (see Section 4).

Definition 1.1. We say that the pair (X,Ξ) is an *axiomatic Veronese variety of type d* (or, briefly, an AVV of type d) if it satisfies the following axioms:

(MM1) Any pair of points $x_1, x_2 \in X$ lies in at least one element of Ξ;
(MM2) if $\xi_1, \xi_2 \in \Xi$ are distinct, then $\xi_1 \cap \xi_2 \subseteq X$;
(MM3) for each $x \in X$, $\dim T_x \leq 2d$.

The letters MM refer to Mazzocca and Melone, as they introduced these axioms in 1984 ([14]) in their most simplified form, i.e., for quadrics which are finite conics, to characterise the quadric Veronese variety in $\mathbb{P}^5(\mathbb{K})$ for finite fields $\mathbb{K}$. We refer to Section 2 for an overview of the evolution of a problem in finite geometry to the ultimate general setting introduced in the current paper. In that section we also provide explicit descriptions of some of the examples, and explain the context of the Freudenthal-Tits magic square.

1.2 Main Result

In [16] it has been shown that AVVs of type d such that all members of Ξ are ovoids, exist precisely if d is a power of 2; if char$(\mathbb{K}) \neq 2$, then $d \leq 8$. In these cases we call the AVV a *Veronese cap*, since the examples arise as the image of a projective plane over a quadratic alternative division algebra under the standard Veronese map. Moreover, it is shown in [21] that AVVs of type d such that all members of Ξ are *split* quadrics, that is, quadrics with projective index $\lfloor \frac{d}{2} \rfloor$, exist precisely for $d = 1,2,4,6,8$, and a complete classification is obtained. These AVVs are called *split*.

In the present paper, we show that there are no AVVs of type d other than these:

Main Theorem. *An axiomatic Veronese variety (AVV) of type d is either split or a Veronese cap, i.e., either the quadrics are split (of projective index $\lfloor \frac{d}{2} \rfloor$) or the quadrics are ovoids.*

Using the main results of [16] and [21], we can formulate the Main Theorem more explicitly. For the definitions and descriptions of the varieties we refer to Section 2.

Theorem 1.2. *An axiomatic Veronese variety (AVV) of type d in $\mathbb{P}^N(\mathbb{K})$ is projectively equivalent to one of the following:*

$d = 1$. *The quadric Veronese variety $\mathcal{V}_2(\mathbb{K})$, and then $N = 5$;*

$d = 2$. *the Segre variety* $\mathscr{S}_{1,2}(\mathbb{K})$ *($N = 5$),* $\mathscr{S}_{1,3}(\mathbb{K})$ *($N = 7$) or* $\mathscr{S}_{2,2}(\mathbb{K})$ *($N = 8$);*
$d = 4$. *the line Grassmannian variety* $\mathscr{G}_{4,1}(\mathbb{K})$ *($N = 9$) or* $\mathscr{G}_{5,1}(\mathbb{K})$ *($N = 14$);*
$d = 6$. *the half-spin variety* $\mathscr{D}_{5,5}(\mathbb{K})$*, and then* $N = 15$*;*
$d = 8$. *the Cartan variety* $\mathscr{E}_{6,1}(\mathbb{K})$*, and then* $N = 26$*;*
$d = 2^\ell$ *the Veronese variety* $\mathscr{V}_2(\mathbb{K},\mathbb{A})$*, for some d-dimensional quadratic alternative division algebra* $\mathbb{A}$ *over* $\mathbb{K}$*. Moreover, if the characteristic of the underlying field is not* 2*, then* $d \in \{1,2,4,8\}$*. Here,* $N = 3d+2$*.*

Note that the case $d = 1$ is also included in the last case, $d = 2^\ell$. We repeat it though, as the quadric Veronese variety is both split and a Veronese cap.

1.3 Structure of the proof

Let (X,Ξ) be an AVV of type d. For each $\xi \in \Xi$, its *index* w_ξ is the projective index of $X(\xi)$, if $X(\xi)$ is a quadric, and 0 if $X(\xi)$ is an ovoid. For each point $x \in X$, the set $W_x := \{w_\xi \mid x \in \xi \in \Xi\}$ is called a *local index set* of (X,Ξ); the *global index set* W is the union of all these W_x. We will distinguish cases depending on these index sets.

Our main technique uses an inductive argument to reduce both d and the index set, based on the local structure of the AVVs. Indeed, we derive conditions under which the point-residue of (X,Ξ) at a point $x \in X$ is an AVV (the main problem being Axiom (MM3)), which then necessarily is of type $d-2$ and has index set $\{w-1 \mid w \in W_x, w \geq 1\}$. In the cases where this technique fails, we require a totally different and more inventive approach. More precisely, we proceed as follows.

If $W = \{0\}$, then (X,Ξ) is a Veronese cap (as was proved in [16]), so we will assume that there is a point $x \in X$ contained in at least one member of Ξ of index at least 1. Our first aim is to show that there are no points $x \in X$ contained in exactly one member of Ξ of index at least 1 (cf. **Section 6**), which guarantees that all point-residues are sufficiently rich in order to deduce properties. Knowing this, we continue systematically:

- **Case 1:** Suppose first that there is a point $x \in X$ with $\max(W_x) = 1$, and so $d \geq 2$. In this case, a rather general argument using normal rational cubic scrolls excludes values of d exceeding 3. The case $d = 3$ can be ruled out by relying on a result of the second and third author ([20]). The case where $d = 2$ leads to the existing cases where all quadrics have index 1 are split, and then the main result of [21] says that (X,Ξ) is one of the Segre varieties $\mathscr{S}_{1,2}(\mathbb{K})$, $\mathscr{S}_{2,2,}(\mathbb{K})$ and $\mathscr{S}_{1,3}(\mathbb{K})$ (cf. Section 2).

- **Case 2:** Secondly, suppose that there is a point $x \in X$ with $\max(W_x) = 2$, and so $d \geq 4$. As in the previous case, a rather general argument rules out the cases $d \geq 6$. The case where $d = 5$ does not require much additional effort. The case where $d = 4$ leads to existing cases, this time its quadrics are all split and of projective index 2. Again, the main result of [21] says that (X, Ξ) is a line Grassmannian $\mathscr{G}_{n,1}(\mathbb{K})$ for $n \in \{4,5\}$ (cf. Section 2).

- **Case 3:** Finally, we may assume that for each point $x \in X$ holds that either $W_x = \{0\}$ or $\max(W_x) \geq 3$ and the latter option occurs at least once, so $d \geq 6$. We consider a point $x \in X$ such that $w^* := \max(W) \in W_x$ (note that $w^* \geq 3$). The corresponding point-residue is a (possibly weak) AVV, and the induction hypothesis then reveals that all members of Ξ through x are split and of the same index w^*. From this, we will deduce that each member of Ξ is either of index w^* and split, or has index 0. Our final task is to get rid of the index 0 members. When this is accomplished, once again the main result of [21] implies that (X, Ξ) is either the half spin variety $\mathscr{D}_{5,5}(\mathbb{K})$ (in which case $d = 6$ and the quadrics have index 3) or the Cartan variety $\mathscr{E}_{6,1}(\mathbb{K})$ (in which case $d = 8$ and the quadrics have index 4) (cf. Section 2).

Before embarking on the proof, we give an overview of the involved varieties and provide more motivation and background of the problem in **Section 2**. Afterwards, in **Section 4**, we fix notation and show some general properties of AVVs. In **Section 5** we gather technical properties of some specific varieties that we will encounter later on.

2 History, motivation and examples

In 1984, Mazzocca & Melone [14] introduced the axioms (MM1), (MM2) and (MM3) for $d = 1$, $N = 5$ and merely in the finite case, that is, for sets of points in a finite projective space of dimension 5. Using our present terminology, they show in [14] that finite AVVs of type 1 in Galois spaces of dimension 5 are quadric Veronese varieties. As noted by Hirschfeld & Thas [11], their proof for the case of even characteristic contains a flaw and this was corrected in [11]. Cooperstein, Thas & Van Maldeghem [6] introduced Hermitian Veronese caps over finite fields and, with the current terminology, classified finite AVVs of type 2 which are Veronese caps. Then the second and third author classified in [17] all AVVs of type 1, for the first time including in general the infinite case. The same authors also classified in [18] all Veronese caps of type 2. Thus far only Veronese caps had been classified. The first paper dealing with ruled quadrics is [19], where the authors classified all AVVs of type 2, even including a generalization using degenerate quadrics. Meanwhile Krauss [15] classified Veronese caps of type 4 over fields admitting exactly two quadratic residue classes, showcasing the hardness of the problem in general. Using some ideas of Krauss' thesis, and some additional ones, Krauss, Schillewaert & Van Maldeghem managed to classify all Veronese caps of arbitrary type (including the infinite-dimensional case rewording (MM3) slightly). Around the same time, the second and third author [21] classified all split AVVs, explicitly conjecturing the Main Result of the present paper.

One of the main reasons why the split AVVs were considered in the first place was because it became clear in [19] that this case has a link with the *Freudenthal-Tits Magic Square (FTMS)*. The split AVVs of type 1 and 2 are exactly the varieties appearing in the first two cells of the second row of Tits' geometric version of the FTMS, see page 142 in [25], hinting at the fact that the other varieties of the second row also qualify as split AVVs. The eventual classification [21] revealed that certain subvarieties of those are also split AVVs. On top of that, the so-called *non-split* geometric version of the FTMS contains, in the second row, the Veronese representations of the projective planes over quadratic alternative division algebras. Since both the split AVVs and Veronese caps are strongly linked to the second row of the FTMS, it is highly desirable to find a unified form of the axiom systems. This is done in the present paper.

Now we introduce the varieties mentioned in Theorem 1.2. Let $\mathbb{K}$ be an arbitrary field.

Quadric Veronese varieties — The *quadric Veronese variety* $\mathscr{V}_n(\mathbb{K})$, $n \geq 1$, is the set of points in $\mathbb{P}^{\binom{n+2}{2}-1}(\mathbb{K})$ obtained by taking the images of all points of $\mathbb{P}^n(\mathbb{K})$ under the Veronese mapping, which maps the point $(x_0,\ldots,x_n)$ of $\mathbb{P}^n(\mathbb{K})$ to the point $(x_ix_j)_{0\leq i\leq j\leq n}$ of $\mathbb{P}^{\binom{n+2}{2}-1}(\mathbb{K})$. If $n=2$, then it is an AVV of type 1, and all AVVs of type 1 arise this way.

Segre varieties — The *Segre variety* $\mathscr{S}_{k,\ell}(\mathbb{K})$ of $\mathbb{P}^k(\mathbb{K})$ and $\mathbb{P}^\ell(\mathbb{K})$ is the set of points of $\mathbb{P}^{k\ell+k+\ell}(\mathbb{K})$ obtained by taking the images of all pairs of points, one in $\mathbb{P}^k(\mathbb{K})$ and one in $\mathbb{P}^\ell(\mathbb{K})$, under the Segre map

$$\sigma(\langle(x_0,x_1,\ldots,x_k),(y_0,y_1,\ldots,y_\ell)\rangle) = (x_iy_j)_{0\leq i\leq k;0\leq j\leq \ell}.$$

If $(k,\ell)\in\{(1,2),(1,3),(2,2)\}$, then $\mathscr{S}_{k,\ell}(\mathbb{K})$ is a split AVV of type 2, and all split AVVs of type 2 arise this way.

Line Grassmannian varieties — The *line Grassmannian variety* $\mathscr{G}_{m,1}(\mathbb{K})$, $m\geq 2$, of $\mathbb{P}^m(\mathbb{K})$ is the set of points of $\mathbb{P}^{\frac{m^2+m-2}{2}}(\mathbb{K})$ obtained by taking the images of all lines of $\mathbb{P}^m(\mathbb{K})$ under the Plücker map

$$\rho(\langle(x_0,x_1,\ldots,x_m),(y_0,y_1,\ldots,y_m)\rangle) = \left(\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}\right)_{0\leq i<j\leq m}.$$

If $m \in \{4,5\}$, then $\mathscr{G}_{m,1}(\mathbb{K})$ is a split AVV of type 4, and every split AVV of type 4 arises this way.

Half-spin varieties — An algebraic description of half-spin varieties in full generality is due to Chevalley [3], see also the recent reference [13]. A geometric approach was taken in [27]. Since we only need the case of type D_5 it is more convenient to follow the latter approach.

Let U_1 and U_2 be two disjoint 7-dimensional subspaces in $\mathbb{P}^{15}(\mathbb{K})$, respectively containing hyperbolic (projective index 3) quadrics Q_1 and Q_2. Let τ be a triality of type I_{id} (with the terminology of [24]) of Q_1 and let ι be a linear isomorphism from Q_1 to Q_2, and set $\varphi = \tau\iota$. Note that, for each point $p \in Q_1$, the image p^{φ} is a 3-space belonging to one natural system of generators of Q_2.

The half-spin variety $\mathscr{D}_{5,5}(\mathbb{K})$ consists of all points of $\mathbb{P}^{15}(\mathbb{K})$ that contained in a line which intersects U_1 in a point $p \in Q_1$ and U_2 in a point $q \in p^{\varphi}$. These varieties are the only split AVVs of type 6.

The Cartan variety — Since we will not need the precise definition of the variety $\mathscr{E}_{6,1}(\mathbb{K})$, which is the projective version of the well known 27-dimensional module of the (split) exceptional group of Lie type E_6, we simply refer to the literature here. Aschbacher [1] provides an algebraic description, Cohen [4] provides a construction using intersections of quadrics (with explicit equations).

The varieties

$\mathscr{V}_2(\mathbb{K})$	$\mathscr{S}_{2,2}(\mathbb{K})$	$\mathscr{G}_{5,1}(\mathbb{K})$	$\mathscr{E}_{6,1}(\mathbb{K})$

form the second row of the FTMS, split version.

The Veronese varieties — These varieties will be of little importance in the rest of the paper. Let us limit ourselves by mentioning that each finite-dimensional quadratic alternative division algebra $\mathbb{A}$ over $\mathbb{K}$, say $\dim_{\mathbb{K}} \mathbb{A} = j$, defines a unique Veronese variety $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ in $\mathbb{P}^{3j+2}(\mathbb{K})$ using the standard Veronese map. Also, $\mathscr{V}_2(\mathbb{K},\mathbb{K}) = \mathscr{V}_2(\mathbb{K})$, and if $\mathbb{L}$ is a quadratic Galois extension of $\mathbb{K}$, $\mathbb{H}$ a quaternion division ring with center $\mathbb{K}$ containing $\mathbb{L}$, and $\mathbb{O}$ a Cayley algebra over $\mathbb{K}$ containing $\mathbb{H}$, then the Veronese varieties

$\mathscr{V}_2(\mathbb{K},\mathbb{K})$	$\mathscr{V}_2(\mathbb{K},\mathbb{L})$	$\mathscr{V}_2(\mathbb{K},\mathbb{H})$	$\mathscr{V}_2(\mathbb{K},\mathbb{O})$

form the second row of the FTMS, non-split version.

As can easily been observed, all examples of AVVs of type d in $\mathbb{P}^N(\mathbb{K})$ satisfy $N \leq 3d+2$. In fact the parameters of all Veronese caps satisfy the equality $N = 3d+2$, as do most examples in the split case, except for the Segre varieties $\mathscr{S}_{1,2}(\mathbb{K})$ and $\mathscr{S}_{1,3}(\mathbb{K})$, the line Grassmannian $\mathscr{G}_{4,1}(\mathbb{K})$, and the half-spin variety

$\mathscr{D}_{5,5}(\mathbb{K})$. This is related to the theory of Severi varieties, from which we derive that, if $\mathbb{K}$ is algebraically closed, then the inequality $N < 3d+2$ readily implies that every point of $\mathbb{P}^n(\mathbb{K})$ is contained in a secant line of the variety (a *secant line*, in our case, is a line of $\mathbb{P}^N(\mathbb{K})$ intersecting the variety in exactly two points). However, we will need this property for arbitrary fields. For the most involved variety, namely $\mathscr{D}_{5,5}(\mathbb{K})$, it follows from the fact that the automorphism group has only two orbits on the projective points, as is shown in [12]. However, we present a more or less unified and purely geometric proof allowing for an interesting digression afterwards.

Proposition 2.1. *Let (X,Ξ) be one of the following AVVs: $\mathscr{S}_{1,2}(\mathbb{K})$, $\mathscr{S}_{1,3}(\mathbb{K})$, $\mathscr{G}_{4,1}(\mathbb{K})$, $\mathscr{D}_{5,5}(\mathbb{K})$. Then every point of the ambient projective space $\mathbb{P}$ is contained in a secant, that is, a line of $\mathbb{P}$ intersecting X in exactly two points.*

Proof. (*i*) If $(X,\Xi) \cong \mathscr{S}_{1,n}(\mathbb{K})$, then X contains two disjoint singular n-spaces. It follows immediately that each point of $\mathbb{P}^{2n+1}(\mathbb{K})$ not in X is contained in a unique line meeting both planes in a point. This holds for all $n \geq 1$.

(*ii*) If $(X,\Xi) \cong \mathscr{G}_{4,1}(\mathbb{K})$, then we can select a member $\xi \in \Xi$ and a disjoint singular 3-space Σ. If we identify $\mathscr{G}_{4,1}(\mathbb{K})$ with the line Grassmannian of the projective space $\mathbb{P}^4(\mathbb{K})$, then $Q := X(\xi)$ corresponds to all lines in a 3-space U of $\mathbb{P}^4(\mathbb{K})$, whereas Σ corresponds to the set of lines through a point p of $\mathbb{P}^4(\mathbb{K})$ not in U. Each line L of $\mathbb{P}^4(\mathbb{K})$ not in U and not through p is contained in a unique planar line pencil with vertex $x := L \cap U$ and containing the lines $\langle x,p\rangle$ and $\langle p,L\rangle \cap U$. It follows that, if $q \in Q$ corresponds to the line L_q in U, and if M is the singular line in Σ corresponding to the planar line pencil in $\mathbb{P}^4(\mathbb{K})$ with vertex p in the plane $\langle p, L_q\rangle$, then the plane $\langle q, M\rangle$ is entirely contained in X.

Now let z be any point of $\mathbb{P}$ (and we may assume $z \notin X$). If $z \in \langle Q\rangle$, then clearly z is on a secant of Q. If $z \notin \langle Q\rangle$, then it is contained in a unique line K intersecting $\langle Q\rangle$ in a point z_Q and Σ in a point z_Σ. If $z_Q \in Q$, then we are done. If not, then z_Q is on some secant S of Q; let $u,v \in S \cap Q$, $u \neq v$. By the previous paragraph, there are planes π_u and π_v containing u,v, respectively, intersecting Σ in lines L_u, L_v, respectively. Note that L_u and L_v do not intersect as u and v are not collinear on Q. It follows that there exists a line K containing z_Σ and intersecting both L_u and L_v non-trivially, say in the points p_u and p_v, respectively. Hence there is a line through z intersecting the lines $\langle u, p_u\rangle$ and $\langle v, p_v\rangle$ non-trivially (as z and these lines are contained in the 3-space spanned by S and K).

(*iii*) Let $(X,\Xi) \cong \mathscr{D}_{5,5}(\mathbb{K})$. This case is treated similarly as the previous one, now using the construction above with the quadrics Q_1, Q_2. Each point x of Q_1 defines a unique 4-space $U_x = \langle x, x^\varphi\rangle$ intersecting Q_2 in the singular 3-space $\langle x^\varphi\rangle$. A point $z \notin (U_1 \cup U_2)$ is contained in a line $\langle z_1, z_2\rangle$, with $z_i \in \langle Q_i\rangle$, $i = 1,2$. The point z_1 is on a secant $\langle u,v\rangle$, with $u,v \in Q_1$ (possibly $z \in \{u,v\}$), and z_2 is on a secant $\langle p_u, p_v\rangle$, with $p_u \in U_u$ and $p_v \in U_v$. The point z is contained in a secant intersecting $\langle u, p_u\rangle$ and $\langle v, p_v\rangle$ non-trivially. $\square$

3 Digression: Geometric hyperplanes of $\mathscr{D}_{5,5}(\mathbb{K})$ and $\mathscr{E}_{6,1}(\mathbb{K})$

In general, a *(proper) geometric hyperplane* of a geometry with non-empty point and line set is a (proper) subset of the point set such that every line intersects the point set either in a unique point or is fully contained in it. The main result of [5] states that every proper geometric hyperplane of the varieties $\mathscr{D}_{5,5}(\mathbb{K})$ and $\mathscr{E}_{6,1}(\mathbb{K})$ in $\mathbb{P}^{15}(\mathbb{K})$ or $\mathbb{P}^{26}(\mathbb{K})$, respectively, arises as the intersection of the variety with a hyperplane of the projective space. In this section we complement the geometric approach initiated by Cooperstein and Shult in [5] by giving an *intrinsic* description of these geometric hyperplanes, i.e., within the geometry itself and not needing the ambient projective space. Therefore we will mostly work with abstract geometries of type $\mathsf{D}_{5,5}(\mathbb{K})$ or of type $\mathsf{E}_{6,1}(\mathbb{K})$ instead of the varieties $\mathscr{D}_{5,5}(\mathbb{K})$ and $\mathscr{E}_{6,1}(\mathbb{K})$ which are embedded in projective space.

Since we do not need this part in the sequel, we will be brief and skip uninteresting details, only focusing on the beautiful arguments which provide deeper geometric insight. We assume the reader is familiar with the basic notions of point-line geometries (collinearity, singular subspaces, distance) and refer to [4] for the definitions.

3.1 The geometric hyperplanes of $\mathscr{D}_{5,5}(\mathbb{K})$

Let $\Gamma = (X, \mathscr{L})$ be a geometry of type $\mathsf{D}_{5,5}(\mathbb{K})$, where X denotes its point set and $\mathscr{L}$ its line set (where each line is viewed as the set of points incident with it). Let Γ^* denote the associated hyperbolic polar space of rank 5, i.e., Γ^* is of type $\mathsf{D}_{5,1}(\mathbb{K})$. Denote the two natural families of maximal singular subspaces of Γ^* by Ψ_1 and Ψ_2. Without loss of generality, X corresponds to Ψ_1, and then the set of maximal singular subspaces of Γ corresponds to Ψ_2, and the point set of a line $L \in \mathscr{L}$ corresponds to the subset of 4-spaces of Ψ_1 containing a singular plane of Γ^*.

A first type of geometric hyperplane of Γ—Let U be a maximal singular subspace of Γ of dimension 4. Define H_U as the set of points which are collinear to at least one point of U (alternatively, one could picture H_U as the union of lines sharing at least one point with U). The set H_U is a proper geometric hyperplane of Γ. This can be proved in an elementary way, for instance by using the correspondence with Γ^*. We omit the proof but describe the correspondence anyway, for future use: Translated to Γ^*, where U corresponds to a subspace $\overline{U} \in \Psi_2$, the set H_U is the set of 4-spaces of Ψ_1 having a non-empty intersection with $\overline{U}$ (that is, intersecting it in either a line or a singular 3-space). It is easily verified that, if U, U' are 4-spaces of Γ, then $H_U \subseteq H_{U'}$ implies $U = U'$.

We now prepare for the description of the second type of geometric hyperplane. To that end, we note that the set of 4-spaces of Ψ_1 containing a given point of Γ^* corresponds to a subgeometry of Γ isomorphic to a polar space of type $\mathsf{D}_{4,1}(\mathbb{K})$, as can be seen on the diagram. In the language of parapolar spaces, this subgeometry is called a *symp* and each symp of Γ arises in this way (see Definition 4.4 for a general definition of symp). Let Q_1 and Q_2 be disjoint symps of Γ (these correspond to non-collinear points of Γ^*). Then collinearity induces a map ρ between the points of Q_2 and the 3-spaces of Q_1 of one type, preserving incidence (i.e., collinear points go to 3-spaces sharing a line), and the union of all 4-spaces $\langle q_2, \rho(q_2)\rangle$ with $q_2 \in Q_2$ is precisely X (this is the abstract version—and explanation—of the construction of $\mathscr{D}_{5,5}(\mathbb{K})$ encountered above in Section 2).

A second type of geometric hyperplane of Γ—Let K_2 be a non-degenerate subquadric of Q_2 of type $\mathsf{B}_{3,1}(\mathbb{K})$, i.e., a parabolic quadric of rank 3. Then K_2 is a geometric hyperplane of Q_2 and $K_1^* = \rho(q_2 \mid q_2 \in K_2\}$ also has the structure of a quadric of type $\mathsf{B}_{3,1}(\mathbb{K})$ by triality. Moreover, each point of Q_1 is contained in a member of K_1^* since it is collinear to a 3-space of Q_2 which shares at least a plane with K_2. We define $H_{K_1^*} \subseteq X$ as the union of all 4-spaces $\langle q_2, \rho(q_2)\rangle$ with $q_2 \in K_2$, or equivalently the union of all 4-spaces meeting Q_1 in a member of K_1 (each 3-space of Γ being contained in a unique 4-space). One can verify that $H_{K_1^*}$ is a proper geometric hyperplane of Γ by relying on the correspondence with Γ^*, but to make this conceivable, we note that in $\mathscr{D}_{5,5}(\mathbb{K})$, it follows that $\langle H_{K_1^*}\rangle = \langle Q_1, K_2\rangle$ and the latter is a hyperplane of $\mathsf{PG}(15,\mathbb{K})$. Finally we mention that Q_1 is the unique symp of Γ fully contained in $H_{K_1^*}$ (if $Q_1' \neq Q_1$ would also meet each 4-space $\langle q_2, \rho(q_2)\rangle$ with $q_2 \in K_2$ in a 3-space, then $Q_1 \cap Q_1'$ is a 3-space incident with each 3-space of K_1^*, a contradiction). Therefore the second type of geometric hyperplanes is in one-to-one correspondence with the subquadrics of type $\mathsf{B}_{3,1}(\mathbb{K})$ on symps of Γ.

Different behavior of the hyperplanes with respect to symps—The difference between these two types of geometric hyperplanes can be seen from the intersection with symps of Γ: a hyperplane H_U of type 1 contains all symps Q with $U \subseteq Q$ and shares a degenerate quadric with a symp Q if $U \cap Q$ is a unique point p (and then $H_U \cap Q = p^\perp \cap Q$); a hyperplane $H_{K_1^*}$ of type 2 contains a unique symp Q_1 (namely the unique symp containing K_1^*), meets the symps sharing a 3-space with Q_1 in a degenerate quadric and the symps disjoint from Q_1 in a quadric of type $\mathsf{B}_{3,1}(\mathbb{K})$. Note that none of these geometric hyperplanes contains two disjoint symps (in accordance with the given construction where two disjoint symps determine Γ).

Conclusion for the variety $\mathscr{D}_{5,5}(\mathbb{K})$—By the above, the hyperplanes of type 1 of Γ are in one-to-one correspondence with the 4-spaces U of Γ, or equivalently, the members of Ψ_2 of Γ^*. So, considering $\mathscr{D}_{5,5}(\mathbb{K})$, we see that the set of hyperplanes $\langle H_U \rangle$ of $\mathbb{P}^{15}(\mathbb{K})$ with U a 4-space of Γ form, in the dual of $\mathbb{P}^{15}(\mathbb{K})$, a point set isomorphic to that of $\mathscr{D}_{5,5}(\mathbb{K})$. Hence, since the stabilizer of $\mathscr{D}_{5,5}(\mathbb{K})$ has two orbits on the points of $\mathbb{P}^{15}(\mathbb{K})$ (the points on and off the variety), the same holds for the (geometric) hyperplanes. This geometrically shows that the stabilizer of $\mathscr{D}_{5,5}(\mathbb{K})$ in $\mathbb{P}^{15}(\mathbb{K})$ acts with two orbits on the hyperplanes of $\mathbb{P}^{15}(\mathbb{K})$, and the two types of (geometric) hyperplanes are as described above.

3.2 The geometric hyperplanes of $\mathscr{E}_{6,1}(\mathbb{K})$

Now consider the variety $\mathscr{E}_{6,1}(\mathbb{K})$ in $\mathbb{P}^{26}(\mathbb{K})$. We denote its point set by X and its set of elements of type 6 (each of which is isomorphic to a quadric of type $\mathsf{D}_{5,1}(\mathbb{K})$) by Ξ, and we refer to the members of Ξ as symps (cf. Definition 4.4). For each point $p \in X$, we denote the *point-residue* at p by $p^\perp$ as it is induced by the singular lines of $\mathscr{E}_{6,1}(\mathbb{K})$ containing p, and we note that $p^\perp$ is isomorphic to $\mathscr{D}_{5,5}(\mathbb{K})$.

Let H be a geometric hyperplane of $\mathscr{E}_{6,1}(\mathbb{K})$ and let Ω be the corresponding hyperplane of $\mathbb{P}^{26}(\mathbb{K})$. Using the colorful terminology for geometric hyperplanes given in [5], below we will arrive at the following intrinsic descriptions for the three different kinds of hyperplanes.

- H is the set of points collinear to at least one point of a given symp $\xi \in \Xi$ (H is called a white hyperplane);
- H is the union of a set of symps Σ through a point $p \in X$ such that, in $p^\perp$ the symps corresponding to the members of Σ is the point set of a quadric of type $\mathsf{B}_{4,1}(\mathbb{K})$ (recall that a symp in $\mathscr{D}_{5,5}(\mathbb{K})$ corresponds to a point of a quadric of type $\mathsf{D}_{5,1}(\mathbb{K})$) ($H$ is called a grey hyperplane);
- H arises as the fixed point structure of a symplectic polarity of $\mathscr{E}_{6,1}(\mathbb{K})$ and has the structure of a geometry of type $\mathsf{F}_{4,4}(\mathbb{K})$ (H is called a black hyperplane).

We study the possibilities for H through its intersections with the symps and point-residues of $\mathscr{E}_{6,1}(\mathbb{K})$. So let $\xi \in \Xi$ be any symp. Since ξ only has two types of proper geometric hyperplanes, the following three situations could occur:

(C) The symp ξ is contained in H (ξ has *H-type* C);
(N) $\xi \cap H$ is a non-degenerate quadric of type $\mathsf{B}_{4,1}(\mathbb{K})$ (ξ has *H-type* N);
(D) $\xi \cap H$ is a degenerate quadric, i.e., $T_p(\xi)$ for some point $x \in \xi \cap H$ (ξ has *H-type* D).

For an arbitrary point $p \in H$, the two types of geometric hyperplanes of the point-residue $p^\perp$ (see previous subsection) lead to the following possible intersections.

(0) The point residue $p^\perp$ is entirely contained in H (p has *H-type* 0);
(1) The lines through p in H define a geometric hyperplane of $p^\perp$ of type 1 (p has *H-type* 1).
(2) The lines through p in H define a geometric hyperplane of $p^\perp$ of type 2 (p has *H-type* 2).

We aim at showing (without going into the details) that only the following possibilities occur, and each form a single orbit under the automorphism group of (X,Ξ):

Type of H	H-types of symps	H-types of points
White	C,D	0, 1
Grey	C, D, N	0, 1, 2
Black	D, N	2

We distinguish two cases, the first of which leading to white and grey hyperplanes, the second leading to black hyperplanes.

Case 1: Suppose first that H contains a point p of H-type 0.
Let $\xi \in \Xi$ be a symp opposite p (which means that p is not collinear to any point of ξ). Then X is the union of all symps $\xi(p,x)$ containing p, with x ranging over the points of ξ, and hence H is the union of the symps $\xi(p,x)$ where x ranges over $H \cap \xi$. So ξ is of H-type D or N (and any other symp ξ' opposite p has the same H-type as ξ).

- *Case 1(a): Suppose ξ has H-type D.* Let $q \in \xi$ be such that $q^\perp \cap \xi \subseteq H$. Then q also has H-type 0 (since $q^\perp$ contains the disjoint symps corresponding to ξ and $\xi(p,q)$). One verifies that all points of $\xi(p,q)$ have H-type 0 and that H is a white geometric hyperplane.

The white geometric hyperplanes are in one-to-one correspondence with the symps, and the corresponding hyperplanes of $\mathbb{P}^{26}(\mathbb{K})$ define a variety isomorphic to $\mathscr{E}_{6,1}(\mathbb{K})$. Like in the previous section, this gives a geometric proof that the number of point orbits equals the number of hyperplane orbits under the automorphism group G of $\mathscr{E}_{6,1}(\mathbb{K})$. Also, since G acts transitively on Ξ, the white geometric hyperplanes form a single orbit under G. It is easy to see that every symp intersecting $\xi(p,q)$ in a maximal singular subspace has H-type C, while every other symp, intersecting $\xi(p,q)$ in a unique point, has H-type D. One can verify that every other point of H not in $\xi(p,q)$ has H-type 1 (use the fact that the geometric hyperplane induced in the residue contains at least two symps).

- *Case 1(b): Suppose ξ has H-type N.* Noting that the map taking a symp through p to the unique intersection point with ξ is an isomorphism of $p^\perp$ to ξ, and recalling that $\xi \cap H$ has the structure of a quadric of type $\mathsf{B}_{4,1}(\mathbb{K})$, it follows that H is a grey geometric hyperplane (and the set Σ is the set of symps $\xi(p,x)$ with $x \in \xi \cap H$).

Since G acts transitively on the points, and the stabilizer in G of a point acts transitively on the above mentioned B_{4}, subquadrics, we see that the grey geometric hyperplanes form a single orbit. Naturally, every member of Σ has H-type C, every symp through p not belonging to Σ has H-type D, every symp not through p but not disjoint from $p^\perp$ has H-type D, and every symp disjoint from $p^\perp$ has H-type N. Moreover, p is the only point that has H-type 0; every point of $p^\perp \setminus \{p\}$ has H-type 1 (because the geometric hyperplane induced in the residue contains at least two symps) and every point of $H \setminus p^\perp$ has H-type 2.

Case 2: Suppose that H contains no points of H-type 0.
Let ξ be any symp. We first claim that ξ is of H-type D or N. Suppose for a contradiction that ξ has H-type C. Since H is not a white hyperplane, H contains a point p opposite ξ (i.e., with $p^\perp \cap \xi = \emptyset$). By assumption, p has H-type 1 or 2. The geometric hyperplane induced in $p^\perp$ contains at least one symp, which extends to a symp $\zeta \in \Xi$. Since $\zeta \cap \xi$ is a point q, and since $p^\perp \cap \zeta \subseteq H$, we deduce $\zeta \subseteq H$. However, this implies that q has H-type 0 (the geometric hyperplane induced in the residue at q contains two disjoint symps), a contradiction. The claim follows.

Next, we claim that all points are of H-type 2. Indeed, suppose for a contradiction that $p \in H$ has H-type 1. Let ξ be a symp of Ξ disjoint from $p^\perp$. From the definition of type 1 geometric hyperplane of $\mathscr{D}_{5,5}(\mathbb{K})$ and the fact that the mapping defined by intersecting a given member of Ξ through p with ξ induces an isomorphism of buildings, we deduce that there is a unique maximal singular subspace $U \subseteq \xi$ such that $\xi(p,u) \cap p^\perp \subseteq H$ for all points $u \in U$. As in the previous paragraph, a point in $U \cap H$ (which is non-empty) is of H-type 0, a contradiction. The claim is proved.

So, to every point $p \in H$ we can associate a unique symp $\xi_p \ni p$ with the property that $p^\perp \cap \xi_p \subseteq H$. Now let $q \in X \setminus H$. Then $q^\perp \cap H$ with induced lines is a geometry isomorphic to $\mathscr{D}_{5,5}(\mathbb{K})$. Consider the set of points $x \in X$ such that $x^\perp \cap q^\perp \subseteq H$ (so $x \notin q^\perp$ since $q \notin H$). One shows (in general, that is, for every subgeometry of $q^\perp$ isomorphic to $\mathscr{D}_{5,5}(\mathbb{K})$ having exactly one point on each line through q) that this set of points forms a symp ξ_q (which is opposite q). If ξ_q had H-type D, then ξ_q would contain a point r contained in at least two symps (ξ_q and $\xi(q,r)$) with the property that their residue at r belongs to the geometric hyperplane induced in the residue of r, so r would have H-type 0, a contradiction. By the first claim, ξ_q

had H-type N.

Now it takes some (long but elementary) work to show that the mapping $x \to \xi_x$, $x \in X$, defines an isomorphism of $\mathscr{E}_{6,1}(\mathbb{K})$ to its dual, and that it induces a duality. Since either $x \in \xi_x$ or $x^\perp \cap \xi_x = \emptyset$ (that is, x and ξ_x are opposite), Main Result 2.1 of [26] implies that the duality is a symplectic polarity. Particularly nice is now that [7] shows in a geometric way that all such polarities are conjugate and hence we deduce that H, which is called a black geometric hyperplane in [5], defines a subvariety of type F_4 and all black geometric hyperplanes form a single orbit under the action of G. They are in one-to-one correspondence with the symplectic polarities or, equivalently, with the subvarieties of type F_4 on (X,Ξ). The geometric homogeneity in the points of H (all have H-type 2) translates into the algebraic property of the stabilizer G_H acting transitively on H.

This concludes our geometric approach, proving that only white, grey and black geometric hyperplanes exist, each of them forming a single orbit under G. A similar, though simpler, analysis holds for the geometric hyperplanes of the line Grassmannian $\mathscr{G}_{5,1}(\mathbb{K})$.

4 Preliminaries

Let (X,Ξ) and d be as in the introduction. We start by introducing a more general version of AVVs by omitting axiom (MM3) and/or considering the following, weaker version of (MM1):

(MM1$'$) *Any pair of non-collinear points $x_1,x_2 \in X$ lies in at least one element of Ξ.*

Definition 4.1. We say that a pair (X,Ξ) is a *pre-AVV of type d* if it satisfies Axioms (MM1) and (MM2); we call it a *weak AVV of type d* if it satisfies Axioms (MM1$'$), (MM2) and (MM3). A *weak pre-AVV of type d* is then a pair (X,Ξ) which satisfies Axioms (MM1$'$) and (MM2).

Henceforth, let (X,Ξ) be a weak pre-AVV of type d in $\mathbb{P}^N(\mathbb{K})$.

4.1 Collinearity relations

Recall that a subspace of $\mathbb{P}^N(\mathbb{K})$ is called *singular* if it has all its points in X. Two points x,y of X are called *collinear* if they are on a common singular line L, in which case we write $x \perp y$ and, if $x \neq y$, we also write $L = xy$; moreover, $x^\perp$ denotes the set of points collinear to x.

Lemma 4.2. *Let (X,Ξ) be a weak pre-AVV of type d. Then each line of $\mathbb{P}^N(\mathbb{K})$ containing at least three points of X is singular. Secondly, if $x,y \in X$ are non-collinear points then there is a* unique *member of Ξ through them, denoted by $[x,y]$.*

Proof. Let L be a line of $\mathbb{P}$ with $|L \cap X| \geq 3$. Let x_1,x_2 be two points in $L \cap X$. If L is not singular, (MM1′) yields a $\xi \in \Xi$ containing x_1,x_2. Since $X(\xi)$ is a quadric, L has to be singular after all. As for the second statement, (MM1′) implies that there is at least one member of Ξ containing x and y; uniqueness follows from (MM2). □

The next lemma should be compared to Lemmas 4.1 and 4.2 in [21].

Lemma 4.3. *Let (X,Ξ) be a weak pre-AVV of type d. Let L_1 and L_2 be singular lines, meeting each other in a point x. Then either L_1 and L_2 are contained in a singular plane, or there is a unique $\xi \in \Xi$ (which we denote by $[L_1,L_2]$) containing $L_1 \cup L_2$. Consequently, if $x \in X$ and $\xi \in \Xi$ with $x \notin \Xi$, then $x^\perp \cap X(\xi)$ is a singular subspace (possibly empty).*

Proof. Let x_1,x_2 be points on $L_1 \setminus \{x\}$ and $L_2 \setminus \{x\}$, respectively, and suppose that they are not collinear. Let x_1',x_2' be points on $L_1 \setminus \{x,x_1\}$ and $L_2 \setminus \{x,x_2\}$. Then the line $\langle x_1',x_2' \rangle$ meets the line $\langle x_1,x_2 \rangle$ in a point z not on $L_1 \cup L_2$. By Lemma 4.2, $z \notin X$, and by the same lemma x_1' and x_2' are not collinear. By (MM1′) $[x_1,x_2],[x_1',x_2'] \in \Xi$ and since they both contain z, (MM2) implies that they are equal. So if $L_1 \cup L_2$ contains a pair (x_1,x_2) of non-collinear points, then $L_1 \cup L_2 \subseteq [x_1,x_2]$. If not, then clearly, the plane $\langle L_1,L_2 \rangle$ is singular.

Now consider $x \in X$ and $\xi \in \Xi$ with $x \notin \xi$. Suppose for a contradiction that x is collinear to two non-collinear points x_1,x_2 in ξ. Set $L_i = \langle x,x_i \rangle$, $i = 1,2$. The previous paragraph implies that $[x_1,x_2]$ contains $L_1 \cup L_2$, in particular $x \in [x_1,x_2] = \xi$, a contradiction. □

4.2 The point-line geometry associated to (X,Ξ)

The set of singular lines of X is denoted by $\mathscr{L}$. In case $\mathscr{L}$ is non-empty (which is not necessarily the case, for instance if Ξ has only quadrics of index 0), then the pair $(X,\mathscr{L})$, equipped with containment as incidence, is the natural *point-line geometry* associated to (X,Ξ). Considering this point-line geometry carries a lot of information on (X,Ξ), especially when we can invoke the theory of parapolar spaces.

In general a *point-line geometry* Γ is a pair $\Gamma = (Y,\mathscr{M})$ where Y is a set of points and $\mathscr{M}$ a non-empty set of lines, each of which is a subset of X. A *subspace* S is a subset with the property that each line not contained in S intersects S in at most one point. *Collinearity* between points again corresponds to being contained in a common line (not necessarily unique), and we also denote this by the symbol $\perp$. The *collinearity graph* of Γ is the graph on Y with collinearity as adjacency relation. The *distance* $\delta(x,y)$ between two points $x,y \in Y$ is the distance between x and y in the collinearity graph (possibly $\delta(x,y) = \infty$ if there is no path between them). A path between x and y of length $\delta(x,y)$ is called a *shortest path*. The diameter of Γ is the diameter of its collinearity graph. We say that Γ is *connected* if for every two points x,y of Y, $\delta(x,y) < \infty$. A subspace $S \subseteq Y$ is called *convex* if all shortest paths between points $x,y \in S$ are contained in S. The *convex subspace closure* of a set $S \subseteq Y$ is the intersection of all convex subspaces containing S (this is well defined since Y is a convex subspace itself).

Before moving on to the viewpoint of parapolar spaces, we need to consider each member of Ξ of index at least 1 as a convex subspace of $(X,\mathscr{L})$ isomorphic to a so-called *polar space* (for a precise definition and background see Section 7.4 of [2]). Indeed, for each $\xi \in \Xi$ with $w_\xi \geq 1$, $X(\xi)$ is an instance of a polar space, that is, a point-line geometry $(X',\mathscr{L}')$ in which, apart from three non-degeneracy axioms, the *one-or-all axiom* holds: *Each point $x \in X'$ is collinear to either exactly one or all points of any given line*. Still assuming $w_\xi \geq 1$, we also have that $X(\xi)$ is a convex subspace: Obviously, for any pair of distinct collinear points $x,x' \in X(\xi)$, the line xx' belongs to $X(\xi)$, and for any pair of non-collinear points $x,x' \in X(\xi)$, Lemma 4.3 implies that $x^\perp \cap x'^\perp$ belongs to $X(\xi)$ and hence so do the shortest paths between x and x' in the collinearity graph of $(X,\mathscr{L})$. Observe that $X(\xi)$ is the convex subspace closure of any pair of non-collinear points $x,x' \in X(\xi)$, since $X(\xi)$ contains no convex subspaces other than singular subspaces and itself.

Definition 4.4. A connected point-line geometry $\Gamma = (X,\mathscr{L})$ is a *parapolar space* if for every pair of non-collinear points p and q in $\mathscr{P}$, with $|p^\perp \cap q^\perp| > 1$, the convex subspace closure of $\{p,q\}$ is a polar space, called a *symplecton* (a *symp* for short); moreover, each line of $\mathscr{L}$ has to be contained in a symplecton and no symplecton contains all points of X.

The parapolar space is called *strong* if there are no pairs of points p,q with $|p^\perp \cap q^\perp| = 1$.

Lemma 4.5. *Suppose (X,Ξ) is a weak pre-AVV of type d. Then each connected component of the point-line geometry $(X,\mathscr{L})$ associated to (X,Ξ) is one of the following:*

(i) *A singular subspace of dimension at least* 0 *(no point of which is contained in member of Ξ of index ≥ 1);*

(ii) *A quadric* $\xi \in \Xi$ *of index at least* 1 *(and all members of* Ξ *meeting* ξ *nontrivially have index* 0*);*
(iii) *A strong parapolar space (which moreover has diameter* 2 *if* $\min(W) \geq 1$*).*

Proof. Suppose x belongs to the connected component C. If all points of C are collinear with x, then all points of C are mutually collinear since otherwise Lemma 4.3(1) yields a member of Ξ through x of projective index ≥ 1, which contains points (automatically in C) not collinear to x. Hence we are in Case (i). If there is a point y in C not collinear with x, then by Lemma 4.3 there is a member $\xi \in \Xi$ of index at least 1 containing x. If $C = \xi$ then we are in Case (ii).

If C strictly contains ξ, then we wish to show that C is a strong parapolar space. Let p, q be points of C at distance 2, i.e., there are lines L_p and L_q through p, q, respectively, meeting each other in a point. From Lemma 4.3, it follows that $L_p \cup L_q$ is contained in a unique member of Ξ, which, as noted before Definition 4.4, is the convex closure of p and q. In particular, $|p^\perp \cap q^\perp| \neq 1$, showing strongness. Finally, suppose L is a line in C. If L belongs to ξ there is nothing to prove; if L intersects ξ in a point, then by Lemma 4.3, L is contained in a member of Ξ together with a line of ξ. By connectivity we can repeat this argument to conclude that each line is contained in a member of Ξ. By assumption, C does not coincide with a member of Ξ. We conclude that C is a strong parapolar space indeed. The claim about the diameter is obvious. □

Definition 4.6. Let $k \in \mathbb{Z}_{\geq -1}$. A parapolar space is called *k-lacunary* if k-dimensional singular subspaces never occur as the intersection of two symplecta, and all symplecta do possess k-dimensional singular subspaces.

In [8] and [9], k-lacunary parapolar spaces have been classified for $k = -1$ and $k \geq 0$, respectively. At several points in the proof we will use the classification of 0-lacunary parapolar spaces, and also once that of (-1)-lacunary parapolar spaces. We extract from the Main Result of [9] the results that we will need, restricting our attention to strong parapolar spaces embedded in a projective space over a field $\mathbb{K}$.

Fact 4.7. *Let* $\Gamma = (X, \mathscr{L})$ *be a strong* (-1)*-lacunary parapolar space whose points are points of a projective space* $\mathbb{P}$ *over a field* $\mathbb{K}$*, whose lines are lines of* $\mathbb{P}$ *and whose symplecta are all isomorphic to each other. Then* $\Gamma = (X, \mathscr{L})$ *is, as a point-line geometry, isomorphic to either a Segre variety* $\mathscr{S}_{n,2}(\mathbb{K})$ *with* $n \in \{1, 2\}$*, a line Grassmannian variety* $\mathscr{G}_{n,1}(\mathbb{K})$ *with* $n \in \{4, 5\}$*, or to the Cartan variety* $\mathscr{E}_{6,1}(\mathbb{K})$*. In particular, the symps of* Γ *are all hyperbolic quadrics.*

Fact 4.8. *Let* $\Gamma = (X, \mathscr{L})$ *be a strong* 0*-lacunary parapolar space whose points are points of a projective space* $\mathbb{P}$ *over a field* $\mathbb{K}$*, whose lines are lines of* $\mathbb{P}$ *and* *whose symplecta are all isomorphic to each other. Then the symps of* Γ *are all hyperbolic quadrics. Moreover, if these quadrics all have projective index* 1*, then* $\Gamma = (X, \mathscr{L})$ *is, as a point-line geometry, isomorphic to a Segre variety* $\mathscr{S}_{1,n}(\mathbb{K})$*, for some* $n \in \mathbb{N}$ *with* $n \geq 2$*, or the direct product of a line and a hyperbolic quadric of projective index n, for some* $n \in \mathbb{N}$ *with* $n \geq 2$*.*

4.3 Point-residues of (X, Ξ)

Our main technique involves the use of local information coming from the point-residues, which are defined as follows.

Definition 4.9. Suppose (X, Ξ) is an AVV. Let $x \in X$ be arbitrary and consider a subspace C_x of T_x of dimension $\dim T_x - 1$ not containing x. Consider the set X_x of points of C_x which are contained in a singular line of X with x. Let Ξ_x be the collection of subspaces of C_x obtained by intersecting C_x with all $T_x(\xi)$, with ξ running through all members ξ of Ξ containing x together with at least two points of X_x. Note that the members of Ξ_x correspond precisely to the members of Ξ through x of index at least 1.

The next lemma is the counterpart of Lemma 4.6 in [21].

Lemma 4.10. *Suppose* (X, Ξ) *is an AVV of type d,* $d > 2$*, and with global index set W. Then for each* $x \in X$*, the pair* (X_x, Ξ_x)*, with* $X_x \subseteq C_x$ *as above, is a weak pre-AVV of type* $d-2$ *and with global index set* $\{w-1 \mid w \in W_x, w \geq 1\}$*, in the subspace* C_x *of dimension* $N_x \leq 2d-1$ *whose isomorphism type is independent of* C_x*.*

Proof. By construction, a member ξ of Ξ_x has dimension $d-1$ and the quadric $X(\xi)$ has index $w_\xi - 1$.

Let p_1 and p_2 be two non-collinear points of X_x. In X, they correspond to two non-collinear lines L_1 and L_2 through x, which are contained in a member of Ξ through x by Lemma 4.3, hence (MM1′) holds.

For (MM2), let ξ and ξ' in Ξ_x and suppose that $y \in \xi \cap \xi'$. Then y is contained in $T_x(\sigma) \cap T_x(\sigma')$, where σ and σ' are two members of Ξ containing x together with at least two points of X_x. Hence in particular $y \in \sigma \cap \sigma'$ and so by (MM2) for (X, Ξ) we obtain $y \in X_x$. Hence (MM2) holds in (X_x, Ξ_x).

If C'_x is another hyperplane of T_x, and if we denote by X'_x the set of points of C'_x on a singular line with x, then the projection from x of C_x onto C'_x yields an isomorphism from (X_x, Ξ_x) to (X'_x, Ξ'_x), where Ξ'_x is the collection of subspaces of C'_x obtained by intersecting C'_x with all $T_x(\xi)$, with ξ running through all quads ξ of Ξ containing x together with at least two points of X_x. □

Henceforth we denote by W_x' the index set $\{w-1 \mid w \in W_x, w \geq 1\}$. In case x satisfies $\min(W_x \setminus \{0\}) \geq 2$, we can prove that (MM1) holds. This relies on the following lemma.

Lemma 4.11. *Let (X,Ξ) be a weak pre-AVV of type d and let $y \in X$ be arbitrary. Then the local index set W_y is non-empty (and hence $\max(W_y)$ is well-defined). Moreover, if $1 \notin W_y$ and $\max(W_y) \geq 2$, then each singular plane that contains y is contained in a member of Ξ.*

Proof. We have to show that there is at least one member of Ξ containing y. Suppose the contrary. By assumption, we can pick $\xi \in \Xi$. Lemma 4.3 yields a point $y' \in \xi$ not collinear to y, and then (MM1′) yields $\xi' \in \Xi$ containing y and y'.

Next, suppose $1 \notin W_y$ and $\max(W_y) \geq 2$ and let π be a singular plane through y. Let ξ be any member of Ξ through y, with $w_\xi \geq 2$. If $\pi \subseteq X(\xi)$ we are done, so suppose there is a point $z \in \pi \setminus \xi$. We applying Lemma 4.3 several times. Firstly, it implies that there is a point $z' \in X(\xi)$ not collinear to z, but collinear to y. Then (MM1′) yields $[z,z']$, which contains the line $L = \langle y,z \rangle$. Note that our assumptions imply that $w_{[z,z']} \geq 2$. Let u be a point in $\pi \setminus L$. Then u is collinear to a singular subspace of $[z,z']$, so there is a plane π' in $X([z,z'])$ through L not all points of which are collinear to u. For a point $u' \in \pi' \setminus L$, we then have $\pi \cup \pi' \subseteq [u,u']$. □

Corollary 4.12. *Suppose (X,Ξ) is an AVV of type d. Then for each $x \in X$ with $\min(W_x \setminus \{0\}) \geq 2$, the pair (X_x, Ξ_x) is a pre-AVV of type $d-2$ with global index set W_x' in the projective space C_x of dimension $N_x \leq 2d-1$.*

Proof. Suppose x is a point with $\dim W_x \geq 2$. Note that this implies that $d \geq 5$. By Lemma 4.10, we only still need that each pair of collinear points of X_x is contained in a member of Ξ_x. By Lemma 4.11 and $\min(W_x) \geq 2$, this is the case. □

4.4 Basic general properties of weak pre-AVVs

Many of the following properties are similar to the split case in [21]. However, since we want to include weak pre-AVVs (which were not defined in [21]), some proofs must be modified. Hence we provide detailed proofs of all statements for completeness.

The next lemma generalizes Lemmas 4.9 and 4.10 of [21] from split quadrics to arbitrary ones.

Lemma 4.13. *Let Q be a non-degenerate quadric in $\mathbb{P}^{d+1}(\mathbb{K})$ of projective index w. Consider a subspace D of $\mathbb{P}^{d+1}(\mathbb{K})$, with $\dim D = d+1-w$. Then the following hold.*

(*i*) *The subspace D contains at least two non-collinear points of Q.*
(*ii*) *The intersection $D \cap Q$ spans D. Equivalently, for each hyperplane H of D, the complement $D \setminus H$ contains a point of Q.*

Proof. (*i*) We prove this by induction on w, the result for $w = 0$ being trivial, since D coincides with $\mathbb{P}^{d+1}(\mathbb{K})$ in this case. Suppose now that $w > 0$. Notice that $Q \cap D \neq \emptyset$ since a dimension argument implies that D intersects every singular w-space of Q non-trivially. Select $x \in D \cap Q$. If some line in D through x has exactly two points in common with Q, then we find a pair of non-collinear points of Q in D. So assume that any line in D through x either intersects Q in a unique point or is entirely contained in Q. Then D belongs to the tangent space $T_x(Q)$ at x to Q. In the residue at x we obtain a quadric Q' in $\mathbb{P}^{d-1}(\mathbb{K})$ of projective index $w-1$ and a subspace D' of D with $\dim D' = d - w = (d-1) - (w-1)$ which, by induction, contains two non-collinear points y', z' of Q'. These points correspond to two singular lines of Q through x and in D, not contained in a singular plane of Q. This shows the assertion.

(*ii*) This follows from the fact that quadrics containing two non-collinear points span the ambient projective space of their corresponding quadratic form. An explicit geometric proof goes as follows. Let H be a hyperplane of D and suppose that $D \cap Q \subseteq H$. By (*i*), H contains two non-collinear points y and z of Q. Let α be a plane in D through y and z with $\alpha \nsubseteq H$. Then $T_z(Q) \cap \alpha$ is precisely one line L, as y is not collinear to z. Then each line L' in α through z distinct from L contains a second point of Q. Taking $L' \neq \langle y, z \rangle$, this yields a point in $(D \cap Q) \setminus H$. □

The following lemma generalizes Lemma 4.12 of [21].

Lemma 4.14. *Suppose (X, Ξ) is an AVV of type d. If (distinct) $\xi_1, \xi_2 \in \Xi$ share a point $x \in X$, then $\langle T_x(\xi_1), T_x(\xi_2) \rangle \cap X \subseteq x^{\perp}$.*

Proof. Suppose for a contradiction that there are (distinct) ξ_1, ξ_2 through x such that $\langle T_x(\xi_1), T_x(\xi_2) \rangle$ contains a point $y \in X \setminus x^{\perp}$. A dimension argument yields points $a_i \in T_x(\xi_i)$, $i \in \{1, 2\}$, such that $y \in \langle a_1, a_2 \rangle$. If $a_1 \in X$, then there exists a member of Ξ through y and a_1, hence by (MM2) $a_2 \in X$ too, and so the plane $\langle x, a_1, a_2 \rangle$—containing two singular lines and an extra point $y \in X$—must be singular, contradicting the fact that y is not collinear to x. Hence we may assume $a_1, a_2 \notin X$. We claim that we can (re)choose the points y and a_1 in such a way that $a_1 \in X$.

Put $w_i := w_{\xi_i}$ for $i = 1, 2$. Without loss of generality, $w_1 \geq w_2$. If $w_2 = 0$, a dimension argument implies that $\xi_2 \cap [x, y]$ contains a line through x, which has to be singular by (MM2), a contradiction. So we may assume $w_2 \geq 1$. Put $U :=$

$\xi_1 \cap \xi_2$ and $\ell := \dim U$. Then $0 \leq \ell \leq w_2$. Since $\langle T_x(\xi_1), T_x(\xi_2)\rangle$ and $T_x([x,y])$ are subspaces of respective dimensions $2d-\ell$ and d in the $2d$-space T_x, we get that $\dim(T_x([x,y]) \cap \langle T_x(\xi_1), T_x(\xi_2)\rangle) \geq d-\ell$. Note that, for $i=1,2$, $T_x([x,y]) \cap T_x(\xi_i)$ has dimension at most w_1 (recall $w_1 \geq w_2$), so there is a (not necessarily singular) subspace Z in $\langle T_x(\xi_1), T_x(\xi_2)\rangle$ of dimension $d-\ell-w_1$ through x in $T_x([x,y])$ that intersects $T_x(\xi_1) \cup T_x(\xi_2)$ exactly in $\{x\}$. We consider the subspace $Z^* = \langle Z, y\rangle$, and since $y \notin Z$ we have $\dim Z^* = d-\ell-w_1+1$. Every line in $Z^* \subseteq [x,y]$ through x outside Z contains a unique point of $(T_x \cap X) \setminus x^\perp$. Together with $Z \cap \xi_i = \{x\}$, it then follows by (MM2) that $Z^* \cap \xi_i = \{x\}$, $i \in \{1,2\}$. A dimension argument yields unique $(d-w_1+1)$-spaces $U_i \subseteq T_x(\xi_i)$ containing U, $i=1,2$ such that $Z^* \subseteq \langle U_1, U_2\rangle$. Let U_1' be the $(d-w_1)$-space obtained by intersecting $\langle U_2, Z\rangle$ with U_1. By Lemma 4.13(2), there exists a point $a_1 \in (X(\xi_1) \cap U_1) \setminus U_1' \subseteq (X \cap U_1) \setminus U_1'$. Since U_2 and Z^* meet in only x, and $a_1 \in \langle U_2, Z^*\rangle$, there is a unique plane π containing x, a_1 and intersecting both Z^* and U_2 in (distinct) lines. By our choice of a_1 outside U_1', the line $\pi \cap Z^*$ is not contained in Z and intersects $[x,y]$ in a point y' not collinear to x. Hence $y' \in \langle T_x(\xi_1), T_x(\xi_2)\rangle \cap (X \setminus x^\perp)$ and the claim follows. The lemma is proved. □

The following lemma should be compared with Lemma 4.13 of [21].

Lemma 4.15. *Suppose (X,Ξ) is an AVV of type d and with global index set W. Suppose $x \in X$ is such that* $\min(W_x) \leq 1$. *Then $T_x \cap X \subset x^\perp$.*

Proof. Suppose for a contradiction that there exists $z \in (T_x \cap X) \setminus x^\perp$. We claim that we can find ξ_1, ξ_2 in Ξ through x such that $\langle T_x(\xi_1), T_x(\xi_2)\rangle \cap X$ contains a point non-collinear to X, which contradicts Lemma 4.14 and proves the assertion.

Suppose first that $\min(W_x) = 0$. Consider an element $\xi_1 \in \Xi$ through x of index 0, and an element $\xi_2 \in \Xi$ containing x. Then by (MM2) and (MM3) we obtain $z \in T_x = \langle T_x(\xi_1), T_x(\xi_2)\rangle$, showing the claim in this case. So suppose that $\min(W_x) = 1$. Note that in this case T_x is generated by all singular lines through x.

Let $x \in \xi_1 \in \Xi$ with $w_{\xi_1} = 1$. Put $\xi^* := [x,z]$ and note that $\xi^* = \langle T_x(\xi^*), z\rangle \subseteq T_x$. Since $\dim T_x \leq 2d$, the intersection $T_x(\xi_1) \cap \xi^*$ is at least a line. By (MM2), $T_x(\xi_1) \cap \xi^*$ is singular and as ξ_1 has index 1, it is a line L (through x). In particular, $\dim T_x = 2d$ and $\langle T_x(\xi_1), T_x(\xi^*)\rangle = 2d-1$. This means that there is a point $u \in (T_x \cap X) \setminus \langle T_x(\xi_1), T_x(\xi^*)\rangle$ with $\langle x,u\rangle$ singular. As $\dim\langle \xi^*, u\rangle = d+2$, we obtain that $\langle \xi^*, u\rangle \cap T_x(\xi_1)$ is a plane π through L. Let $v \in \pi \setminus L$ be a point. Inside the $(d+2)$-space $\langle \xi^*, u\rangle$, the line $\langle u,v\rangle$ meets ξ^* in a point y'. Since $u \notin \langle T_x(\xi_1), T_x(\xi^*)\rangle$ we have $y' \in \xi^* \setminus T_x(\xi^*)$ and hence the line $\langle x, y'\rangle$ contains a unique point y on $X(\xi^*) \setminus \{x\}$. Clearly, $y \in X \cap \langle T_x(\xi_1), \langle x,u\rangle\rangle$. Hence, for an arbitrary member ξ_2 through $\langle x,u\rangle$ holds $y \in \langle T_x(\xi_1), T_x(\xi_2)\rangle$ and $y \notin x^\perp$. The lemma is proved. □

4.5 Projections of (X,Ξ) from a member $\xi \in \Xi$

Projection from a member of Ξ is a successful tool in the proof of the classification of the case $W = \{0\}$, see [16]. Here, we extend its use to members with index ≥ 1.

Definition 4.16. If (X,Ξ) is a (possibly weak) pre-AVV, we can consider the projection ρ_ξ from some $\xi \in \Xi$ onto a subspace Π of $\mathbb{P}^N(\mathbb{K})$ complementary to ξ, i.e.,

$$\rho_\xi : \mathbb{P}^N(\mathbb{K}) \setminus \xi \to \Pi : z \mapsto \langle \xi, z \rangle \cap \Pi.$$

For any set $Z \subseteq \mathbb{P}^N(\mathbb{K})$, we write Z^{ρ_ξ} instead of $(Z \setminus \xi)^{\rho_\xi}$ for ease of notation.

Lemma 4.17. *Suppose (X,Ξ) is a (possibly weak) pre-AVV and $\xi \in \Xi$ arbitrary. If $p,q \in X \setminus \xi$ have the same image under ρ_ξ, then $\langle p,q \rangle$ is a singular line meeting $X(\xi)$ in a point.*

Proof. Put $\rho = \rho_\xi$. Then $p^\rho = q^\rho$ implies that ξ is a hyperplane of the subspace $\langle \xi, p, q \rangle$. If $\langle p,q \rangle$ is singular, then clearly it intersects ξ in a point of X. Suppose p and q are not collinear. Then, by (MM1′) and (MM2), $\langle p,q \rangle \cap \xi$ is a point of X, which by Lemma 4.2 implies that $\langle p,q \rangle$ is singular after all. □

The following properties of ρ_ξ will be used several times, mainly for $s \in \{0,1\}$.

Lemma 4.18. *Suppose (X,Ξ) is a (possibly weak) pre-AVV and $\xi \in \Xi$ arbitrary. Suppose $\xi' \in \Xi$ meets ξ in a singular subspace S of dimension $s \geq 0$. Then:*

- (i) *The image of ξ' under ρ_ξ is a $(d-s)$-space $\Pi_{\xi'}$, in which $T_S(\xi')^{\rho_\xi}$ is a subspace $H_{\xi'}$ of dimension $d-2s-1$;*
- (ii) *For any point q in $X(\xi')^{\rho_\xi}$, there is a point p on $X(\xi') \setminus S$ such that $\rho_\xi^{-1}(q) \cap X(\xi') = \langle p, p^\perp \cap S \rangle \setminus S$, and $p \in S^\perp$ if and only if $q \in H_{\xi'}$;*
- (iii) *For any point $q \in \Pi_{\xi'} \setminus X(\xi')^{\rho_\xi}$, the set $\rho_\xi^{-1}(q) \cap \xi'$ is an $(s+1)$-space through S inside $T_S(\xi')$ which only has S in X (in particular, $\Pi_{\xi'} \setminus H_{\xi'} \subseteq X(\xi')^{\rho_\xi}$);*
- (iv) *If $s = 0$ and L is any line in $\Pi_{\xi'}$ containing a unique point z in $H_{\xi'}$, then the union of $\rho_\xi^{-1}(L) \cap X(\xi')$ with S is one of the following:*
  - (a) *A conic C through S if $z \notin X(\xi')^{\rho_\xi}$. The image under ρ_ξ of the tangent line $T_S(C)$ is z;*
  - (b) *The union of two intersecting non-collinear singular lines if $z \in X(\xi')^{\rho_\xi}$. Exactly one of these lines contains S and is projected by ρ_ξ onto z.*

Proof. (i) Since $\dim(\xi') = d+1$ and $\dim(\xi \cap \xi') = s$, we get that $\Pi_{\xi'}$ has dimension $(d+1)-s-1 = d-s$ indeed. The dimension of the tangent space $T_S(\xi')$

is $d-s$, and $T_S(\xi') \cap \xi = S$, so likewise $H_{\xi'}$ has dimension $(d-s)-s-1 = d-2s-1$.

(ii) Let $q \in X(\xi')^{\rho_\xi}$ and let p be a point in $X(\xi')$ with $\rho_\xi(p) = q$. By definition of ρ_ξ and the choice of p, we have $\rho_\xi^{-1}(q) \cap X(\xi') = \langle p, \xi \rangle \cap X(\xi')$. Looking inside $X(\xi')$, it follows that the latter set coincides with $\langle p, p^\perp \cap S \rangle$. Moreover, $p \in S^\perp$ if and only if $p \in T_S(\xi')$ if and only if $q \in H_{\xi'}$.

(iii) This follows from the fact that an $(s+1)$-space of ξ' through S contains a point of $X(\xi') \setminus S$ if and only if it does not belong to $T_S(\xi')$. In particular we obtain that each point of $\Pi_{\xi'} \setminus H_{\xi'}$ is the image of some point of $X(\xi') \setminus S^\perp$.

(iv) Now let $s = 0$ and take a line L in $\Pi_{\xi'}$ containing a unique point z in $H_{\xi'}$. Then $\rho_\xi^{-1}(L) \cap \xi'$ is a plane π through S, and by the above, each point $q \in L \setminus \{z\}$ corresponds to a point p in $(\pi \cap X) \setminus S$ not collinear to the point S. Hence the intersection of π with the quadric $X(\xi')$ contains at least three points not on a line, and therefore it is either a conic or the union of two intersecting singular lines. Note that in the latter case, each point of L belongs to $X(\xi')^{\rho_\xi}$, i.e. $z \in X(\xi')$ too. Conversely, $z \in X(\xi')$ implies by (ii) that $\rho_\xi^{-1}(z) \cap X(\xi')$ is a singular line through S. So $z \in X(\xi')$ corresponds to case (b) indeed. Now, if $z \notin X(\xi')$, then $(\rho_\xi^{-1}(L) \cap X(\xi')) \cup S$ is a conic C through S, and the tangent line $T_S(C)$ is mapped onto z by ρ_ξ (cf. assertion (iii)). $\square$

5 Technical lemmas concerning specific situations

Some rather technical work needed for the later sections is done here. The main common goal is often to construct additional singular lines joining members of Ξ (Lemmas 5.1, 5.2 and 5.4) or, in one case, even prove that there are members with large enough index (Lemma 5.5). We put this in a separate section since all of these results will be used in quite different situations. However, the reader may wish to skip this section during a first reading as it is very technical and will seemingly be out of context. It is probably easier to refer back to the results here when they are used in subsequent sections.

In order to state the first lemma, we need the following concept. In $\mathbb{P}^4(\mathbb{K})$, consider a line L and a conic C in a plane complementary to L, and suppose $\varphi : L \to C$ is a bijection preserving the cross-ratio. Then the union of the *transversal lines* $\langle x, \varphi(x) \rangle$, with $x \in L$ is called a *normal rational cubic scroll* , denoted $\mathscr{N}_{1,2}(\mathbb{K})$, and L is called the *axis*. Then for each two points not on L which are on distinct transversal lines of $\mathscr{N}_{1,2}(\mathbb{K})$, there is a unique conic through them intersecting all transversal lines (also in points not on L). Every pair C_1, C_2 of such conics intersect in precisely one point p and $\langle C_1 \rangle \cap \langle C_2 \rangle = \{p\}$. Conversely, given two arbitrary conics C_1 and C_2 in $\mathbb{P}^4(\mathbb{K})$ intersecting in a unique point p, with $\langle C_1 \rangle \cap \langle C_2 \rangle = \{p\}$, and given a bijection $\psi : C_1 \to C_2$ fixing p and preserving the cross-ratio, there is a unique normal rational cubic scroll $\mathscr{N}$ containing all transversal lines $\langle x, \psi(x) \rangle$, with $x \in C_1 \setminus \{p\}$. In particular, there exists a unique line L intersecting all said transversal lines. Naturally, the line L and the conic C_1 determine $\mathscr{N}$, and the latter is defined by the map $\varphi : C_1 \to L$ taking a point $x \in C_1 \setminus \{p\}$ to $\langle x, \psi(x) \rangle \cap L$, and taking p to the unique 'remaining' point of L.

Lemma 5.1. *Let (X, Ξ) be a weak pre-AVV of type d. Suppose ξ_1 and ξ_2 are two members of Ξ of index 0, meeting each other in a point p and meeting some $\xi \in \Xi$ not through p in distinct points p_1 and p_2. If there is a singular line K meeting ξ_1, ξ_2 and ξ in three distinct points, then for $i \in \{1,2\}$, there is a conic C_i on $X(\xi_i)$ through p and p_i such that C_1 and C_2 are on a normal rational cubic scroll, and if $|\mathbb{K}| > 4$, all transversal lines except possibly the one through p are singular, as is the axis of the scroll.*

Proof. We consider the projection $\rho = \rho_\xi$ of (X, Ξ) from ξ onto a complementary subspace Π. By assumption, the respective images of ξ_1 and ξ_2 under ρ share at least two points: p^ρ and K^ρ (which are distinct by Lemma 4.17). Let L be the projective line $\langle p^\rho, K^\rho \rangle$. Then L contains exactly one point t_i contained in $T_{p_i}(\xi_i)^\rho$, $i = 1,2$, which does not belong to $X(\xi_i)^\rho$ since ξ_i has index 0. According to Lemma 4.18(iv), L corresponds to a conic C_i on $X(\xi_i)$ through the points p_i and p, for $i = 1,2$.

For $i = 1,2$, let $\mathscr{S}(p_i)$ denote the planar line pencil through p_i in $\langle C_i \rangle$ and let σ_i be the projectivity taking a line $M \in \mathscr{S}(p_i)$ to p_i if M is tangent to C_i, and to the unique point of M on $C_i \setminus \{p\}$ if M is a secant of C_i. Each line of $\mathscr{S}(p_i)$ corresponds to a unique point of L via ρ and hence we can consider the bijection τ taking a line of $\mathscr{S}(p_1)$ to the unique line of $\mathscr{S}(p_2)$ with the same image under ρ. Since $\langle \mathscr{S}(p_i) \rangle \cap \xi = \{p_i\}$, τ is a projectivity. As such, $\sigma_2 \circ \tau \circ \sigma_1^{-1}$ is a projectivity too, i.e., it preserves the cross-ratio.

We conclude that C_1 and C_2 are on a normal rational cubic scroll indeed. Let R denote the unique line intersecting all its transversal lines. Since the transversal lines $z_1 z_2$, with $z_i \in C_i \setminus \{p_i, p\}$ for $i = 1,2$, are such that $\rho(z_1) = \rho(z_2)$, Lemma 4.17 implies that $\langle z_1, z_2 \rangle$ is singular. Note that this excludes at most three of the transversal lines, namely the ones through p, p_1, p_2, say T, T_1, T_2 (possibly $T_1 = T_2$). Hence, if $|\mathbb{K}| > 4$, we obtain at least three singular transversals that meet R in three distinct points. Consequently R is singular. But then both T_1 and T_2 contain at least three points of X and are also singular. □

The previous lemma assumes the existence of a singular line meeting three members of Ξ. The next lemma creates a possibility of finding such a line.

Lemma 5.2. *Let* (X,Ξ) *be a weak pre-AVV of type* d *with* $d \geq 2$ *and, if* $d = 2$*, we also require* $|\mathbb{K}| > 2$*. Let* ξ, ξ_1, ξ_2 *be three distinct members of* Ξ *with* $\dim\langle \xi, \xi_1, \xi_2 \rangle \leq 2d+3$*,* $\xi_1 \cap \xi_2 = \{p\}$ *and* $\xi \cap \xi_i = \{p_i\}$*,* $i = 1,2$*, where* p_1, p_2, p *are three distinct points of* X *with* $p \notin p_1^\perp \cup p_2^\perp$*. Then there exists a singular line meeting* ξ, ξ_1, ξ_2 *in three distinct points* z, z_1, z_2*, respectively, with* z_i *and* p_i *non-collinear, for* $i = 1,2$*.*

Proof. We again consider the projection $\rho = \rho_\xi$ of (X,Ξ) from ξ onto a subspace Π in $\langle \xi, \xi_1, \xi_2 \rangle$ complementary to ξ. By Lemma 4.18, the respective images Π_{ξ_1} and Π_{ξ_2} of ξ_1 and ξ_2 under ρ are d-spaces of Π, which share the point p^ρ. Since $\dim\langle \xi, \xi_1, \xi_2 \rangle \leq 2d+3$, we have $\dim \Pi \leq d+1$, and hence $\Pi_{\xi_1} \cap \Pi_{\xi_2}$ has dimension at least $d-1 \geq 1$. Recall that $T_{p_i}(\xi_i)^\rho$ is a hyperplane of Π_{ξ_i} and that p^ρ is not contained in it since $p \notin p_i^\perp$, $i = 1,2$. This means that any line L in $\Pi_{\xi_1} \cap \Pi_{\xi_2}$ through p^ρ contains at most one point t_i of $T_{p_i}(\xi_i)^\rho$ for $i = 1,2$. Since $T_{p_1}(\xi_1)^\rho \cap T_{p_2}(\xi_2)^\rho$ has dimension at least $d-3$, we can choose L in such a way that $t_1 = t_2$ if $d \geq 3$. Note that, if $d = 2$ and $|\mathbb{K}| = 2$, it might be that t_1, t_2, p^ρ are the only points of L.

Let q be a point in $L \setminus \{p^\rho, t_1, t_2\}$ (which is non-empty by our assumptions on d and $|\mathbb{K}|$). Lemma 4.18(ii) yields points z_1, z_2 on $X(\xi_1), X(\xi_2)$, respectively, which are not collinear to p_1 and p_2, respectively (recall $q \notin \{t_1, t_2\}$) and with $z_1^\rho = z_2^\rho = q$. By Lemma 4.17, the latter implies that $\langle z_1, z_2 \rangle$ is a singular line meeting $X(\xi)$ in a point z. □

Here is an example of how Lemma 5.2 can be used to make an application of Lemma 5.1 possible.

Lemma 5.3. *Let* (X,Ξ) *be an AVV of type* 2 *containing a connected component* $\mathscr{C}$ *of* $(X,\mathscr{L})$ *isomorphic as a point-line geometry to* $\mathscr{S}_{1,1,1}(\mathbb{K})$*. Then, for any two points* $x, y \in \mathscr{C}$ *at distance* 3 *in* $(X,\mathscr{L})$*, the member* $[x,y] \in \Xi$ *is not contained in the subspace* $\langle \mathscr{C} \rangle$*.*

Proof. First note that, if $a, b \in \mathscr{C}$ are at distance 2 in C, then $[a,b] \in \Xi$ has index 1 and $X([a,b]) \subseteq \mathscr{C}$ is a hyperbolic quadric of rank 2 which we will refer to as a *grid*. Also, $\mathscr{C}$, being isomorphic to $\mathscr{S}_{1,1,1}(\mathbb{K})$, contains disjoint grids, and so, by (MM2), $\dim\langle \mathscr{C} \rangle = 7$.

Now let $x, y \in \mathscr{C}$ be points at distance 3. Then $\xi := [x,y] \in \Xi$ has index 0. Suppose for a contradiction that $[x,y]$ belongs to $\langle \mathscr{C} \rangle$. Consider any grid G of $\mathscr{C}$ through x. Then by (MM3), $T_x = \langle T_x(\xi), T_x(G) \rangle \subseteq \langle \mathscr{C} \rangle$. Suppose first that $|\mathbb{K}| = q < \infty$. Let $G' \subseteq \mathscr{C}$ be a grid not through x. Then G' contains q^2 points at distance 3 from x, and for each such point z, we have that $[x,z] \in \Xi$ has index 0. Noting that

$[x,z] \cap G' = \{z\}$ by (MM2), each point of G' at distance 3 from x determines a different member of Ξ, which results in q^2 tangent planes that pairwise intersect each other in x. In addition, there are the three tangent planes of the grids of $\mathscr{C}$ through x (which intersect each other pairwise in a line and the other tangent planes in only x). This yields $q^3+q^2+3q > q^3+q^2+q+1$ distinct lines through x in the 4-space T_x, a contradiction.

So suppose that $|\mathbb{K}| = \infty$. Let z be a point of $\mathscr{C} \setminus \xi$ at distance 3 from both x and y. Then $\xi_1 := [x,z]$ and $\xi_2 := [y,z]$ are members of Ξ of index 0. Recalling that $T_x \subseteq \langle \mathscr{C} \rangle$, we get $\xi_1 = \langle T_x(\xi_1), z\rangle \subseteq \langle \mathscr{C} \rangle$; likewise for ξ_2, from which it follows that $\dim\langle \xi, \xi_1, \xi_2\rangle \leq 7$. Therefore (X,Ξ) and the triple ξ, ξ_1, ξ_2 meet the conditions of Lemma 5.2, and hence also those of Lemma 5.1. The latter lemma implies that there are conics C_1 and C_2 on ξ_1 through x,z and on ξ_2 through y,z, respectively, such that C_1 and C_2 are on a normal rational cubic scroll, and each transversal line joining a point $C_1 \setminus \{z\}$ with its image on C_2 is singular. The line R meeting all these transversal lines, containing at least three points in X, is also singular (cf. Lemma 4.2). As a consequence all points of C_1 belong to the same connected component as x, hence to $\mathscr{C}$. We now show that this is not possible.

Let p_1, p_2, p_3, p_4 be four distinct points of C_1, which are pairwise at distance 3. Take grids G_i through p_i, $i = 1,2,3$, with G_1, G_2, G_3 pairwise intersecting in a line. Then $G_1 \cap G_2 \cap G_3$ is a unique point p. We claim that $\langle G_1, G_2, G_3, p_4\rangle = \langle \mathscr{C} \rangle$. Indeed, clearly any line of $\mathscr{C}$ through p_4 intersects one of G_1, G_2, G_3 and so is contained in $\langle G_1, G_2, G_3, p_4\rangle$. By connectivity of $\mathscr{C} \setminus (G_1 \cup G_2 \cup G_3)$, the claim follows. However, $\dim\langle G_1, G_2, G_3, p_4\rangle = 6$ as $p_4 \in \langle p_1, p_2, p_3\rangle$, contradicting $\dim\langle \mathscr{C} \rangle = 7$. □

We collect a further application of Lemma 5.2.

Lemma 5.4. *Let (X,Ξ) be a pre-AVV of type d with $d \geq 2$. If $d = 2$, we also require $|\mathbb{K}| > 2$. Suppose $\langle X \rangle \subseteq \mathbb{P}^{2d+3}(\mathbb{K})$. If ξ, ξ_1 are two members of Ξ intersecting each other in precisely a point p_1, then there is a point z_1 in $X(\xi_1) \setminus p_1^{\perp}$ collinear to a point z of $X(\xi) \setminus p_1^{\perp}$.*

Proof. Suppose that there are no such points z, z_1. Let p be any point in $X(\xi_1) \setminus p_1^{\perp}$, and p_2 any point in $X(\xi) \setminus p_1^{\perp}$. By our assumption, p and p_2 are non-collinear points. Moreover, $\xi_2 := [p, p_2]$ meets $X(\xi)$ in p_2 only: if $\xi_2 \cap \xi$ is at least a line, then p is collinear with at least a point p' of it, and $p' \in p_1^{\perp}$ by our assumption, but then $p' \in \xi_1$, a contradiction. Likewise, ξ_2 meets $X(\xi_1)$ in p only. Lemma 5.2 yields a singular line meeting ξ, ξ_1, ξ_2 in three distinct points z, z_1, z_2, respectively, with z_i and p_i non-collinear for $i = 1,2$. If z would be collinear to p_1, then $z \in \xi_1$, which is not the case as $\xi \cap \xi_1 = \{p_1\} \neq \{z\}$. Hence we found a pair of points as described in the statement of the lemma after all, a contradiction. □

Lemma 5.5. *Let (X,Ξ) be a pre-AVV with $d \geq 3$ and $\langle X\rangle \subseteq \mathbb{P}^{2d+3}(\mathbb{K})$. Suppose that $\xi,\xi_1,\xi_2 \in \Xi$ are such that $w_\xi = 1$, $w_{\xi_1} \leq 1$; $\xi \cap \xi_1$ is a point p_1, $\xi \cap \xi_2$ is a line L_2 and $\xi_1 \cap \xi_2$ contains a point p with $p \notin p_1^\perp \cap L_2^\perp$. If either $d \geq 4$, or $d = 3$, $|\mathbb{K}| > 2$ and $\xi_1 \cap \xi_2 = \{p\}$, then $w_{\xi_2} \geq 2$ and not all members of Ξ of index at least 2 contain a common point.*

Proof. Again let $\rho := \rho_\xi$ be the projection operator from ξ onto a complementary $(d+1)$-space Π. We claim that there exists a point q in $U := \xi_1^\rho \cap \xi_2^\rho$ neither contained in $(\xi_1 \cap \xi_2)^\rho$ nor in $T_{p_1}(\xi_1)^\rho \cup T_{L_2}(\xi_2)^\rho$.

Indeed, by Lemma 4.18(i), ξ_1^ρ is a d-space and ξ_2^ρ is a $(d-1)$-space. Hence $\dim U \geq d-2 \geq 1$. Obviously U contains $(\xi_1 \cap \xi_2)^\rho$ and $0 \leq \dim(\xi_1 \cap \xi_2) \leq w_{\xi_1} \leq 1$. Since $p \notin p_1^\perp \cup L_2^\perp$, we have $p^\rho \notin T_{p_1}(\xi_1)^\rho \cup T_{L_2}(\xi_2)^\rho$ and hence $H := U \cap (T_{p_1}(\xi_1)^\rho \cup T_{L_2}(\xi_2)^\rho)$ is contained in the union of two hyperplanes of U. Now, we can find a point q in the complement of $H' := H \cup (\xi_1 \cap \xi_2)^\rho$, since if $|\mathbb{K}| > 2$, the set H' is contained in the union of three hyperplanes of $\xi_1^\rho \cap \xi_2^\rho$, and otherwise our conditions imply either that H' is the union of H and a subspace of codimension at least 2—and the claim follows—or $d = 4$. In the latter case the only situation in which no such point q can be found is when U is a plane and $(\xi_1 \cap \xi_2)^\rho$, $U \cap T_{p_1}(\xi_1)^\rho$ and $U \cap T_{L_2}(\xi_2)^\rho$ are three distinct lines in U through a common point u. But then, if $u' \in X(\xi_1) \cap X(\xi_2)$ is such that $u'^\rho = u$, then $u' \in p_1^\perp \cap L_2^\perp$, contradicting $u' \notin \xi$. The claim is proved.

Now, Lemma 4.18(ii) yields a point $q_1 \in X(\xi_1) \setminus (\xi_1 \cap \xi_2)$ with $q_1^\rho = q$ and q_1 not collinear to q, and a line L in $X(\xi_2)$ intersecting L_2 in a unique point p_2, not collinear to L_2 and disjoint from $\xi_1 \cap \xi_2$, with $L^\rho = q$. By Lemma 4.17, $\langle q_1, L\rangle$ is a singular plane. Let q_2 be a point on $L \cap p^\perp$. Then $q_2 \in q_1^\perp \cap p^\perp$, from which we deduce that $q_1 \perp p$ (otherwise $q_2 \in \xi_1$, a contradiction). Now let q_2' be any point of $L \setminus \{q_2\}$. Then, likewise, $p \perp q_2'$, for otherwise $q_1 \in p^\perp \cap q_2'^\perp \in \xi_2$, contradicting our choice of q. We conclude that $\langle p, L\rangle$ is a singular plane π in ξ_2, collinear to q_1. This already implies that ξ_2 has index at least 2.

Finally, suppose for a contradiction that all members of Ξ of index ≥ 2 contain a certain point x (which hence belongs to ξ_2). Let q_1' be a point in $X(\xi_2)$ which is not contained in the singular subspace $q_1'^\perp \cap X(\xi_2)$ and not collinear to x. Then $[q_1, q_1']$ does not contain x and has index at least 2 since $q_1'^\perp \cap X(\xi_2)$ contains π, the sought contradiction. $\square$

The previous lemmas assume the existence of certain members of Ξ intersecting precisely in one point. In order to meet this condition, the next lemma, applied in a residue, will be helpful. It will also be crucial in proving Proposition 6.2 below.

Lemma 5.6. *Let (X,Ξ) be a weak pre-AVV of type d and suppose Ξ contains a unique member ξ^* of index at least 1. Then there exist two disjoint members of Ξ intersecting ξ^* non-trivially.*

Proof. Suppose for a contradiction that every pair of members of Ξ intersecting ξ^* non-trivially mutually intersect non-trivially. We will use the observation that no singular line in X intersects ξ^* in a point, for this would yield a second member of Ξ with index > 0.

If $\mathbb{K} = \mathbb{F}_q$ is finite, then $d = 2$ since quadrics of projective index 0 only exist in dimensions $d+1 = 2$ and $d+1 = 3$, and quadrics of projective index at least 1 require $d \geq 2$. Hence ξ^* has exactly $(q+1)^2$ points, and every other member of Ξ has exactly q^2+1 points. Now pick $\xi \in \Xi$ intersecting ξ^* in a point x and let $p \in X \setminus (\xi \cup \xi^*)$. Note that $p^\perp \cap \xi^* = \emptyset$ by the above observation. Hence the mapping $X(\xi^*) \setminus \{x\} \to X(\xi) \setminus \{x\}$ taking z to $[p,z] \cap \xi$ is well defined and clearly injective. But $|X(\xi^*) \setminus \{x\}| = q^2 + 2q > q^2 = |X(\xi) \setminus \{x\}|$, a contradiction.

So suppose $\mathbb{K}$ is infinite. Let L be a singular line of ξ^*. Using the above observation and our assumption, it is easy to see that there are three members ξ, ξ_1, ξ_2 of $\Xi \setminus \{\xi^*\}$ intersecting L in three distinct points and pairwise intersecting in distinct points. Then Lemma 5.1 and $\mathbb{K}$ being infinite yield the existence of a singular line intersecting ξ^* in a point—either the axis of the normal rational cubic scroll guaranteed by Lemma 5.1, or if the axis were contained in ξ^*, any singular transversal of the scroll distinct from L—contradicting our observation above. □

6 Connectivity

In this section we generate some arguments that will be crucial to show that certain geometries are connected, or certain connected components are large enough. In particular, they will enable us to conclude that appropriate point-residues are connected when a member of index at least 2 exists in Ξ. When the maximal index is 1, they will imply that there are connected components of $(X,\mathscr{L})$ that are induced by at least two members of index 1 of Ξ. Recall that we assume that the global index set W of (X,Ξ) is not $\{0\}$.

The first result will be used in point-residues and creates members of index at least 1 therein.

Lemma 6.1. *Let (X,Ξ) be a weak pre-AVV of type d, with $d \geq 2$. Suppose $\langle X \rangle \subseteq \mathbb{P}^{2d+3}(\mathbb{K})$. Let ξ be a member of Ξ of index at least 1 and suppose that $z \in X \setminus \xi$ is such that $T_z \cap X(\xi) = \emptyset$ and $\dim(T_z) \leq 2d+1$. Then there is a member of Ξ of index at least 1 containing z and intersecting ξ non-trivially.*

Proof. Suppose for a contradiction that $\xi_p := [z,p] \in \Xi$ has index 0 for each $p \in X(\xi)$ (note that $p \notin z^\perp$ indeed since $T_z \cap X(\xi) = \emptyset$). Let $p \in X(\xi)$, and let $\rho := \rho_{\xi_p}$ be the projection from ξ_p onto a subspace Π complementary to ξ_p in $\mathbb{P}^{2d+3}(\mathbb{K})$; so $\dim \Pi = d+1$. Moreover, $\dim T_z^\rho \in \{d-1,d\}$ (because $0 \in W_z$ implies $\dim T_z \geq 2d$) and $\dim \xi^\rho = \dim \xi_u^\rho = d$ for all $u \in X(\xi) \setminus \{p\}$ (cf. Lemma 4.18). For each $u \in X(\xi) \setminus \{p\}$, we have that ξ_u^ρ is determined by the $(d-1)$-space $T_z(\xi_u)^\rho \subseteq T_z^\rho$ and the point $u^\rho \in X(\xi)^\rho$.

Claim 1: T_z^ρ is disjoint from $(p^\perp \cap \xi)^\rho$.
By way of contradiction, let r be a point in $(p^\perp \cap \xi)^\rho \cap T_z^\rho$. Then r corresponds to a singular line L of ξ through p (cf. Lemma 4.18(ii)) and $L \subseteq \langle T_z, \xi \rangle$. However, T_z is at least a hyperplane of $\langle T_z, \xi \rangle$ and hence contains at least a point of $L \subseteq X$, contradicting $T_z \cap X(\xi) = \emptyset$. This shows the claim.

Claim 2: For each point $u \in X(\xi) \setminus \{p\}$, the subspace ξ_u^ρ contains no points of $(p^\perp \cap \xi)^\rho \setminus \{u^\rho\}$.
Let $u \in X(\xi) \setminus \{p\}$ be arbitrary and suppose that ξ_u^ρ contains a point $r \in (p^\perp \cap \xi)^\rho$ with $r \neq u^\rho$. By Claim 1, $r \notin T_z^\rho$, so Lemma 4.18(ii) implies that there is a point r_u on $X(\xi_u) \setminus \{z,u\}$ with $r_u^\rho = r$. By the same token, there is a point r' on $X(\xi)$ collinear to p with $r'^\rho = r$. By Lemma 4.16, the line $\langle r_u, r' \rangle$ is singular and meets $X(\xi_p)$ in a point, say p'. Observe that $r_u \neq u$ because $r \neq u^\rho$; in particular, $r_u \notin \xi$ and hence $p \neq p'$. This however means that r' is collinear to both p and p', contradicting $w_{\xi_p} = 0$. The claim follows.

Claim 3: For each point $u \in X(\xi) \setminus \{p\}$, the subspace $T_z(\xi_u)^\rho$ is disjoint from $X(\xi)^\rho \setminus \{u^\rho\}$.
Let $u \in X(\xi) \setminus \{p\}$ be arbitrary and suppose that $T_z(\xi_u)^\rho$ contains a point $r \in X(\xi)^\rho$ with $r \neq u^\rho$. Then, on the one hand, r corresponds to a line L through z in $T_z(\xi_u)$ (cf. Lemma 4.18(iii); note $L \cap X = \{z\}$ since $w_{\xi_u} = 0$), and on the other hand, r corresponds to a point r' on $X(\xi) \setminus \{p\}$ (cf. Lemma 4.18(ii)). Since $L^\rho = r'^\rho = r$, the plane $\langle L, r' \rangle$ meets ξ_p in a line M through z. If $M \subseteq T_z(\xi_p)$, then we obtain $r' \in \langle M, L \rangle \subseteq T_z$, contradicting the assumption $T_z \cap X(\xi) = \emptyset$; if $M \nsubseteq T_z(\xi_p)$, then M contains a point z' in X other than z, and using (MM2) we deduce that the point $\langle z', r' \rangle \cap L$ belongs to X, contradicting $L \subseteq T_z(\xi_{r'})$. This shows the claim.

Now let u be a point of $X(\xi)$ collinear to p. Then, since $\dim \Pi = d+1$, the $(d-1)$-space $T_z(\xi_u)^\rho$ has a subspace T of dimension at least $d-2$ in common with ξ^ρ. By Claim 3, T contains no points of $X(\xi)^\rho$, which by Lemma 4.18(iii) means that $T \subseteq T_p(\xi)^\rho$. Moreover, since $u \perp p$, we have $u^\rho \in T_p(\xi)^\rho$ as well, so $U := \langle T, u^\rho \rangle = T_p(\xi)^\rho$, where the equality follows from $\dim U = \dim T_p(\xi)^\rho = d-1$ (cf. Lemma 4.18(i)). Consequently, $U \subseteq \xi_u^\rho$ contains points of $(p^\perp \cap \xi)^\rho \setminus \{u^\rho\}$, violating Claim 2. This contradiction shows the lemma. □

The next proposition will be used globally, but also locally, in residues. It shows that we may often assume that there are at least two members of index at least 1, be it in (X,Ξ) or in a residue.

Proposition 6.2. *Let (X,Ξ) be a weak AVV of type d. Then each point of X is contained in either zero or at least two members of Ξ having index greater than* 0*.*

We will show this proposition in a series of lemmas. Suppose for a contradiction that (X,Ξ) is a weak AVV of type d such that there exists a point $x \in X$ contained in a unique member $\xi^* \in \Xi$ of index $w > 0$.

Lemma 6.3. *No singular line meets ξ^* in a unique point, and consequently for any point p in $X(\xi^*)$, ξ^* is the unique member of Ξ of index $w > 0$ containing p.*

Proof. If there were a singular line L_x through x not in ξ^*, then by Lemma 4.3, there exists a singular line L'_x of ξ^* through x such that the plane spanned by L_x and L'_x is not singular. The same lemma then implies that L_x and L'_x are contained in a unique member of $\Xi \setminus \{\xi^*\}$ through x, which is of index at least 1, contradicting our assumption on x. Now let z be a point in $X(\xi^*)$ collinear to x and suppose that L_z is a singular line meeting ξ^* in precisely z. Again by Lemma 4.3, we either have that the lines L_z and $\langle x,z\rangle$ determine a member of Ξ (which is of index at least 1) or a singular plane. Both options yield a singular line through x not in ξ^*, a possibility we already ruled out. Hence no lines through z outside ξ^* exists; consequently ξ^* is the unique member of Ξ through z. Now a connectivity argument completes the proof of the lemma. □

We first get rid of the finite case.

Lemma 6.4. *The field $\mathbb{K}$ is infinite.*

Proof. Suppose for a contradiction that $\mathbb{K}$ is finite and has order q. As in the proof of Lemma 5.6, this implies $d = 2$ and $|X(\xi^*)| = (q+1)^2$. Let p be a point of $X \setminus \xi^*$. By Lemma 6.3, for each point z of $X(\xi^*)$, the points p and z determine a unique member of Ξ, which has index 0. This yields $(q+1)^2$ tangent planes at p which intersect each other pairwise in p (by (MM2)). Hence they account for $(q+1)^2(q^2+q)+1 > q^4+q^3+q^2+q+1$ points of the 4-dimensional subspace T_p, a contradiction. □

The next three lemmas and corollary are generalisations of three lemmas and a proposition in [16] (there, all members of Ξ had index 0).

Lemma 6.5. *For each point $p \in X$ with $p \notin \xi^*$, the subspaces T_p and ξ^* are disjoint.*

Proof. Suppose for a contradiction that some point z belongs to both T_p and ξ^*. Pick two distinct points $r, q \in X(\xi^*)$ (and we can assume that $\dim\langle r, q, z\rangle = 2$). Then $[r, p]$ and $[q, p]$ have index 0 by Lemma 6.3 and so they intersect only in p, implying by (MM3) that $T_p = \langle T_p([r, p]), T_p([q, p])\rangle$. Hence there is a line L through z intersecting $T_p([r, p])$ in a point u and intersecting $T_p([q, p])$ in a point v. The line $\langle u, p\rangle$ intersects $T_r([r, p])$ in a point a and the line $\langle v, p\rangle$ intersects $T_q([q, p])$ in a point b. By Lemma 6.4, $|\mathbb{K}| > 2$, so we find two points $a' \in \langle u, p\rangle$ and $b' \in \langle v, p\rangle$ such that $z \in \langle a', b'\rangle$ and $a' \neq a$, $b' \neq b$. Since $a' \notin T_r([r, p])$, there is a point $a'' \in X \setminus \{r\}$ on $\langle r, a'\rangle$ and a point $b'' \in X \setminus \{q\}$ on $\langle q, b'\rangle$. The line $\langle a'', b''\rangle$ belongs to the 3-space $\langle r, q, z, a'\rangle$, hence it intersects the plane $\langle r, q, z\rangle$ in some point z', which consequently belongs to ξ^*. The line $\langle a'', b''\rangle$ is not singular by Lemma 6.3. Then (MM1$'$) and (MM2) yield $z' \in \xi^* \cap [a'', b''] \subseteq X$, implying that $\langle a'', b''\rangle$ is singular after all, a contradiction. □

For the rest of this section, set $\rho := \rho_{\xi^*}$ (see Definition 4.16).

Corollary 6.6. *The projection ρ is injective on $X \setminus \xi^*$.*

Proof. Suppose for a contradiction that $x, y \in X \setminus \xi^*$ have the same image under ρ. Then, by Lemma 4.17, the line $\langle x, y\rangle$ is singular and intersects $X(\xi^*)$ non-trivially, contradicting Lemma 6.3. □

Lemma 6.7. *Let $z \in X(\xi^*)$ be arbitrary. Then the subspace $\langle \xi^*, T_z\rangle$ does not contain any point of $X \setminus \xi^*$.*

Proof. Put $S = \langle \xi^*, T_z\rangle$. Suppose for a contradiction that there is a point $u \in (S \cap X) \setminus \xi^*$. Since u is not collinear with z by Lemma 6.3, we see that $[z, u] = \langle T_z[z, u], u\rangle \subseteq S$. By (MM3), $\dim S \leq 2d + 1$, so $[z, u] \cap \xi^*$ contains a line. This contradicts Lemma 6.3. □

Lemma 6.8. *The geometry (X, Ξ) is a projective plane and all members of $\Xi \setminus \{\xi^*\}$ have index 0.*

Proof. Take any $\xi \in \Xi \setminus \{\xi^*\}$ meeting ξ^* non-trivially. By Lemma 6.3, ξ has index 0 and intersects ξ^* in a unique point $z \in X$. We will use the following notation. By Lemma 4.18, $X(\xi)^\rho$ is an affine d-space α_ξ. By (MM3), its $(d-1)$-space at infinity only depends on z, and we denote it by Π_z. Finally, we set $\Pi_\xi := \alpha_\xi \cup \Pi_z$.

Now let z_1 and z_2 be two non-collinear points of $X(\xi^*)$ and fix an arbitrary point $p \in X \setminus \xi^*$. For $i = 1, 2$, set $\xi_i := [p, z_i]$. By Lemma 6.5, we may assume that $T_p \subseteq \Pi$, the target space of ρ. Then the subspace Π_{ξ_i} coincides with $T_p(\xi_i)$.

By Corollary 6.6, $\alpha_{\xi_1} \cap \alpha_{\xi_2} = \{p\}$, in particular $\Pi_{z_1} \cap \Pi_{z_2} = \emptyset$. We denote $\Sigma = \langle \Pi_{z_1}, \Pi_{z_2} \rangle$ and note that this is a hyperplane in the subspace T_p. Also, Π_{z_1} and Π_{ξ_2} are complementary subspaces in T_p.

Let q be an arbitrary point of $T_p \setminus (\Pi_{\xi_1} \cup \Pi_{\xi_2} \cup \Sigma)$, which is indeed always non-empty. Then the subspace $\langle q, \Pi_{z_1} \rangle$ intersects Π_{ξ_2} in a point $q_2 \in \alpha_{\xi_2} \setminus \{p\}$. Let $u_2 \in X$ be the inverse image under ρ of q_2 (cf. Lemma 6.6). Then the projection of $[z_1, u_2]$ clearly coincides with $\langle \Pi_{z_1}, q_2 \rangle$, and so q can be written as u^ρ with $u \in X([z_1, u_2])$. We claim that $[p, u]$ intersects ξ^* non-trivially. Indeed, suppose for a contradiction that $[p, u] \cap \xi^* = \emptyset$. Then ρ induces an isomorphism between $[p, u]$ and $[p, u]^\rho$, and hence $[p, u]^\rho = \langle T_p([p, u]), u \rangle^\rho = \langle T_p([p, u])^\rho, u^\rho \rangle = \langle T_p([p, u]), q \rangle \subseteq T_p$. This implies that $[p, u]^\rho$ and Π_{ξ_1} intersect in a line L containing p. This line is not contained in $T_p([p, u])$, as $T_p([p, u])$ and $\Pi_{\xi_1} = T_p(\xi_1)$ intersect precisely in p by (MM2). Hence L contains a second point y of $X([p, u])^\rho$, $y \neq p$. By Corollary 6.6, $\{y\} = L \cap \Pi_{z_1}$. This, however, contradicts Lemma 6.7. The claim is proved.

It follows that q is contained in $T_p([p, u]) = [p, u]^\rho$, and so every point of $T_p \setminus (\Pi_{\xi_1} \cup \Pi_{\xi_2} \cup \Sigma)$, and hence every point of $T_p \setminus \Sigma$, is contained in a tangent subspace at p to some member of Ξ containing p and intersecting ξ^* in a point. Axioms (MM2) and (MM3) imply that there is no room for additional tangent spaces. We conclude that every member of Ξ through p meets ξ^* non-trivially. Since $p \in X \setminus \xi^*$ was arbitrary, this shows that every member of $\Xi \setminus \{\xi^*\}$ intersects ξ^* in a point. This also implies that every point of $X \setminus \xi^*$ is projected into T_p and so T_p coincides with Π. From that we then deduce, by a dimension argument, that also each pair of members of $\Xi \setminus \{\xi^*\}$ has a non-trivial intersection. The proposition follows. □

Proof of Proposition 6.2. By Lemma 6.8, (X, Ξ) is a projective plane in which ξ^* is the unique member of Ξ having index greater than 0. This contradicts Lemma 5.6. □

7 Case 1: there is an $x \in X$ with $\max(W_x) = 1$

Suppose (X, Ξ) is an AVV of type d and with global index set W in $\mathbb{P}^N(\mathbb{K})$, with $\max(W_x) = 1$ for some $x \in X$. Our aim is to show the following proposition.

Proposition 7.1. *Let (X, Ξ) be an AVV of type d with global index set W in $\mathbb{P}^N(\mathbb{K})$, containing a point $x \in X$ with $\max(W_x) = 1$. Then $d = 2$ and $W = \{1\}$, and hence (X, Ξ) is isomorphic to $\mathscr{S}_{1,2}(\mathbb{K})$, $\mathscr{S}_{2,2}(\mathbb{K})$ or $\mathscr{S}_{1,3}(\mathbb{K})$.*

Note that the existence of $\xi \in \Xi$ through x with $w_\xi = 1$ implies $d \geq 2$. If $d > 2$, we can consider the residue (X_x, Ξ_x) (cf. Lemma 4.10); if $d = 2$, this makes no sense (a member $\xi \in \Xi$ with $w_\xi = 1$ would correspond to two points in the residue). Our technique for general d will work for $d \geq 4$, so we treat the cases $d = 2$ and $d = 3$ separately. The case $d = 2$ takes quite some effort, compared to the other cases; however, it is precisely this case that leads to the actual examples, so we begin with it.

7.1 The case $d = 2$

Note that $d = 2$ implies that $W \subseteq \{0,1\}$. If $W = \{1\}$ we are in the split case and we reach our desired conclusion, so assume that $W = \{0,1\}$. Hence we may assume that $W_x = \{0,1\}$. By Proposition 6.2, there are at least two members of Ξ through x of index 1. Henceforth, $\mathscr{C}_x$ is the connected component in $(X,\mathscr{L})$ containing x.

The approach we take is inspired by [21], where the case in which all members of Ξ are split quadrics was treated.

Our first goal is to show that there are no singular planes in $\mathscr{C}_x$.

Lemma 7.2. *Suppose π is a singular plane in $\mathscr{C}_x$ and let $z \in \pi$. If there are three singular lines L_1, L_2, L_3 through z not in π, then a pair of them is contained in a singular plane π'. Moreover, either $L_1 \cup L_2 \cup L_3 \subseteq \pi'$, or $\langle \pi, \pi' \rangle$ is a singular 3-space.*

Proof. Set $\Sigma := \langle L_1, L_2, L_3 \rangle$. If Σ is a plane, then, by Lemma 4.2, it is a singular plane and the assertion is proved. So we assume henceforth that $\dim \Sigma = 3$. By (MM3), $\dim T_z \leq 4$ and hence the 3-space Σ has a line L_4 in common with π. The planes $\langle L_1, L_2 \rangle$ and $\langle L_3, L_4 \rangle$ are distinct and hence meet in a line L_5. Using Lemma 4.3 and (MM2), we deduce that L_5 is singular. If $L_5 \notin \{L_1, L_2\}$, then the plane $\langle L_1, L_2 \rangle$ is singular. Else, $\langle L_3, L_5 \rangle$ is singular and we may renumber subscripts so that $\langle L_1, L_2 \rangle$ is singular again. Set $\pi' := \langle L_1, L_2 \rangle$.

Now suppose that $\pi \cap \pi' = \{z\}$. We claim that $L_3 \subseteq \pi'$. Indeed, if not, then L_3 is contained in a unique plane π_3 intersecting π and π' in respective lines L and L' through z. The plane π_3, containing three singular lines, is singular too. But then $\langle \pi, \pi_3 \rangle$ is a singular 3-space Π: If not, then (MM1) and Lemma 4.3 imply that $\pi \cup \pi_3$ is contained in a member of Ξ, which violates the assumption $\max(W) \leq 1$. Likewise, $\langle \pi', \pi_3 \rangle$ is a singular 3-space Π' and, again likewise, $\langle \Pi, \Pi' \rangle$ is a singular 4-space. Since $\langle \Pi, \Pi' \rangle = T_z$, this is not possible (no tangent plane to a member of Ξ is singular). We conclude that $\pi \cap \pi'$ is a line, and as before, this means that $\langle \pi, \pi' \rangle$ is a singular 3-space. $\square$

Lemma 7.3. *The connected component $\mathscr{C}_x$ contains no singular planes.*

Proof. Suppose first for a contradiction that π is a singular plane through x. For any line L of π through x, let ξ' be a member of Ξ through it (which exists by (MM1)). Denote by M the unique singular line of ξ' through x distinct from L. Inside T_x (which has dimension 4 by (MM3) and since $0 \in W_x$), a dimension argument implies that there is a plane π' through M meeting both planes π and $T_x(\xi)$ in respective lines. Lemma 4.3(1) implies that $\pi' \cap T_x(\xi)$ is a singular line, contradicting $w_\xi = 0$. We conclude that x is not contained in a singular plane.

Now let y be an arbitrary point collinear to x and suppose for a contradiction that there is a singular plane π through y. Denote by L the line $\langle x, y\rangle$. We establish three singular lines through y not in π. To that end, take three points y_1, y_2, y_3 on a line M in π with $y \notin M$. By the above, there are no singular planes through x, so we can consider $\xi_i := [x, y_i] \in \Xi$, $i = 1,2,3$. Note that ξ_i contains the lines L and $\langle y, y_i\rangle$. Let L_i be the unique singular line of ξ_i through y_i distinct from $\langle y, y_i\rangle$; let L'_i be the unique singular line of ξ_i through x distinct from L, $i = 1,2,3$. For each $i \in \{1,2,3\}$, Lemma 4.3(2) implies that $\langle M, L_i\rangle$ is not singular. So we can put $\xi'_i := [L_i, M]$. Since there are no singular planes through x, no pair of lines in $\{L'_1, L'_2, L'_3\}$ spans a singular plane. In particular, the points $p_1 := L_1 \cap L'_1$ and $p_2 := L_2 \cap L'_2$ are not collinear. So, if ξ'_1 coincided with ξ'_2, then $M \subseteq \xi'_1 = [p_1, p_2] \ni x$, a contradiction, since x is then collinear to some point of M, yielding a singular plane through x. Similarly, $\xi'_2 \neq \xi'_3 \neq \xi'_1$. Let M'_i be the unique singular line in ξ'_i through y_1 distinct from M, $i = 1,2,3$. Then $M'_1 = L_1, M'_2, M'_3$ are three distinct singular lines through y_1, not any belonging to π. If y were collinear to M'_i, then $y \in \xi'_i$ and hence ξ'_i contains the singular plane π, a contradiction to $w_{\xi'_i} = 1$. So we can consider $\xi''_i := [yy_1, M'_i]$, which yields three distinct members of Ξ. For each $i \in \{1,2,3\}$, we now take the unique singular line M_i in ξ''_i through y distinct from $\langle y, y_2\rangle$. We obtain three distinct singular lines M_1, M_2, M_3 through y not in π.

Renumbering if necessary, we may assume that $L \notin \{M_1, M_2\}$. Applying Lemma 7.2 to the triple $\{L, M_1, M_2\}$ and using that L is not contained in a singular plane, we obtain that $\langle M_1, M_2, \pi\rangle$ is a singular 3-space. This however contradicts the fact that M_1 and M_2 are not collinear with y_1. We conclude that there is no singular plane through y.

Now by connectivity, the lemma follows. □

The following proposition, which we will prove in a series of lemmas, is slightly more general, for it allows $W_x = \{1\}$ (this will be useful in the next section). It assumes that there are no singular planes, which is something we have already proved in case $W_x = \{0, 1\}$.

Proposition 7.4. *Let* (X, Ξ) *be an AVV of type* 2 *with global index set W and such that* $\max(W) = 1$. *Then each connected component of* $(X, \mathscr{L})$ *not containing* *singular planes, is either a point (if there are only members of index* 0 *through it) or isomorphic to* $\mathscr{S}_{1,1,1}(\mathbb{K})$.

We will prove this proposition in a series of lemmas. From now on, we let (X,Ξ) be an AVV of type 2 with $\max(W)=1$. We select an arbitrary point $x\in X$, consider its connected component $\mathscr{C}_x$, and assume that it does not contain singular planes. If all members of Ξ through x have index 0, then $\mathscr{C}_x=\{x\}$. By Proposition 6.2, we may henceforth assume that there are at least two members of Ξ through x of index 1 (and every such member intersects X in a non-thick quadrangle).

Lemma 7.5. *If every pair of members of* Ξ *of index* 1*, inside* $\mathscr{C}_x$*, which share a point, have a line in common, then* $\mathscr{C}_x\cong\mathscr{S}_{1,1,1}(\mathbb{K})$.

Proof. Our assumption implies that $\mathscr{C}_x$ is a 0-lacunary parapolar space whose symps are quadrics of projective index 1 (see Definition 4.6). Since the symps of $\mathscr{C}_x$ are all hyperbolic quadrics in dimension 3, Fact 4.8 implies that $\mathscr{C}_x\cong\mathscr{S}_{1,1,1}(\mathbb{K})$. □

Henceforth we may assume that $\mathscr{C}_x$ contains a pair of members of Ξ of index 1 sharing exactly a point.

Lemma 7.6. *Let* y *be a point of* $\mathscr{C}_x$*. Then there are four singular lines through* y*. Moreover, any four singular lines through* y *span a* 4*-space, which coincides with* T_y.

Proof. Let $z\in\mathscr{C}_x$ be a point contained in two members ξ_1,ξ_2 of Ξ intersecting in precisely $\{z\}$. This yields four singular lines L_1,L_2,L_3,L_4 through z. If $y=z$, the first assertion is proved. Now suppose that y is collinear to z, and put $L=\langle y,z\rangle$. Renumbering if necessary, we may assume that $L\notin\{L_1,L_2,L_3\}$. As there are no singular planes, $[L,L_i]\in\Xi$ for $i\in\{1,2,3\}$. Considering the singular lines M_i in $[L,L_i]$ through y distinct from L, $i=1,2,3$, we obtain four singular lines through y, too. By connectivity, the first assertion follows.

Next let K_1,K_2,K_3,K_4 be four singular lines through y. The absence of singular planes implies that $[K_1,K_2]$ and $[K_3,K_4]$ belong to Ξ, and hence (MM2) implies that $\langle K_1,K_2\rangle\cap\langle K_3,K_4\rangle\subseteq[K_1,K_2]\cap[K_3,K_4]=\{y\}$. So $\dim\langle K_1,K_2,K_3,K_4\rangle=4$. According to (MM3), $T_y=\langle K_1,K_2,K_3,K_4\rangle$. □

Henceforth we fix an index 1 member ξ of Ξ contained in $\mathscr{C}_x$.

The following lemmas will be helpful to study the projection ρ_ξ of $X\setminus\xi$ from ξ onto a complementary subspace.

Lemma 7.7. *Let L_1 and L_2 be two distinct singular lines of X meeting ξ exactly in (not necessarily distinct) points x_1, x_2, respectively. Then* $\dim\langle\xi, L_1, L_2\rangle = 5$.

Proof. If $x_1 = x_2$, then this follows from Lemma 7.6. So assume $x_1 \neq x_2$. Suppose for a contradiction that $\dim\langle\xi, L_1, L_2\rangle = 4$. Then we claim that L_1 and L_2 do not intersect. Indeed, if they do, say in a point p, then, if x_1 and x_2 are collinear then we get a singular plane $\langle p, x_1, x_2\rangle$, contradicting our assumption; if not, then $\xi = [x_1, x_2]$ contains L_1 and L_2 by Lemma 4.3, a contradiction. This shows the claim. Hence $\langle L_1, L_2\rangle$ is a 3-space, which hence intersects ξ in a plane π. Let y be a point on $\pi \setminus \langle x_1, x_2\rangle$ not in X. Then y lies on a line M meeting both L_1 and L_2 in points, say z_1, z_2, respectively. Since $y \notin \langle x_1, x_2\rangle$, we have $\langle z_1, z_2\rangle \not\subseteq \xi$. This means that $y \in [z_1, z_2] \cap \xi$, with $[z_1, z_2] \neq \xi$, contradicting (MM2). □

Lemma 7.8. *Suppose ξ_1, ξ_2 are distinct members of $\Xi \setminus \{\xi\}$ with $\xi_1 \cap \xi_2 \cap \xi$ a singular line L. Then $W := \langle\xi, \xi_1, \xi_2\rangle$ has dimension* 7.

Proof. Since ξ, ξ_1, ξ_2 share L, we already have $\dim W \leq 7$. Suppose for a contradiction that $\dim W \leq 6$. For $i = 1, 2$, put $W_i := \langle\xi, \xi_i\rangle$ and note that $\dim W_i = 5$. So either $W = W_1 = W_2$ or $W_1 \cap W_2$ has dimension 4. In the first case, we take any 4-space U in W through ξ; in the second case we put $U = W_1 \cap W_2$. In both cases, U is a hyperplane in W_i. We now take any singular line M_i on ξ_i disjoint from L. By choice of U, the line M_i has exactly one point m_i in common with U. For $i \in \{1, 2\}$, we denote by R_i the unique singular line of ξ_i through m_i distinct from M_i, and note that R_i intersects L, and hence ξ, in a point. Lemma 7.7 implies that $R_1 = R_2$. However, then $\xi_1 = [L, R_1] = [L, R_2] = \xi_2$, a contradiction. □

Lemma 7.9. *Let x_1 and x_2 be two distinct collinear points on $X(\xi)$. Then the* 5-*spaces $U_1 = \langle\xi, T_{x_1}\rangle$ and $U_2 = \langle\xi, T_{x_2}\rangle$ meet exactly in ξ and hence* $\dim\langle\xi, T_{x_1}, T_{x_2}\rangle = 7$.

Proof. Let L_1 and L_1' be two singular lines through x_1 not in ξ. By Lemma 7.6, T_{x_1} is generated by L_1, L_1' and the two singular lines of ξ passing through x_1. Set $L := \langle x_1, x_2\rangle$. As there are no singular planes, $\xi_1 := [L, L_1]$ and $\xi_1' := [L, L_1']$ belong to Ξ. Let L_2 and L_2' be the respective singular lines of ξ_1 and ξ_1' through x_2 distinct from L. Then T_{x_2} is generated by L_2, L_2' and the two singular lines of ξ through x_2. As such, $\langle\xi, \xi_1, \xi_1'\rangle = \langle U_1, U_2\rangle$. By Lemma 7.8, the latter is 7-dimensional, from which it follows that $U_1 \cap U_2$ is 3-dimensional, and hence coincides with ξ. □

Lemma 7.10. *Let $p \in X(\xi)$ be arbitrary. Then each point of $\langle\xi, T_p\rangle \cap X$ either belongs to ξ, or is on a singular line together with p.*

Proof. Suppose by way of contradiction that some point y not collinear to p is contained in $\langle \xi, T_p \rangle \setminus \xi$. Put $\xi' := [p,y]$. Then $\xi' = \langle T_p(\xi'), y \rangle \subseteq \langle \xi, T_p \rangle$. Since the latter has dimension 5, (MM2) implies that $\xi \cap \xi'$ is a singular line L through p. Let M be the singular line of ξ' through y meeting L in a point, say z. By assumption, $z \neq p$. So, recalling $y \in \langle \xi, T_p \rangle$ we get $M \subseteq \langle \xi, T_x \rangle \cap T_z \subseteq \langle \xi, T_x \rangle \cap \langle \xi, T_z \rangle$, and as $M \not\subseteq \xi$, this contradicts Lemma 7.9. $\square$

We are now ready to use to projection ρ_ξ from ξ onto some subspace Π complementary to ξ in the subspace S generated by the points of $\mathscr{C}_x$.

Lemma 7.11. *The subspace S generated by the points of $\mathscr{C}_x$ is 8-dimensional.*

Proof. Let x_1, x_2 be distinct points on a singular line L of ξ. Let $i = 1,2$. By Lemma 7.6, there are two singular lines L_i, L_i' through x_i outside ξ. Recalling that there are no singular planes, $\xi_i := [L_i, L_i']$ is well defined. By Lemma 4.18, the image of $X(\xi_i) \setminus (L_i \cup L_i')$ under ρ_ξ is an affine plane π_i^* in Π with projective extension π_i. By the same lemma, $T_i := \pi_i \setminus \pi_i^*$ is the image of $T_{x_i}(\xi_i)$, which coincides with $T_{x_i}^{\rho_\xi}$ by (MM3). According to Lemma 7.9, $W := \langle \xi, T_{x_1}, T_{x_2} \rangle$ is 7-dimensional, so T_1 and T_2 are skew lines and $\dim \Pi \geq 3$.

First suppose for a contradiction that $\dim \Pi = 3$. In this case, the plane π_1 has a point z in common with the line T_2, and $z \notin T_1$ by the above (so $z \in \pi_1^*$). Let y be the unique inverse image in ξ_1 of z in X; then $y \in X(\xi_1) \setminus x_1^\perp$ and $y \in \langle \xi, T_{x_2} \rangle$. By Lemma 7.10, $y \perp x_2$, implying that $x_2 \in [x_1, y] = \xi_1$, a contradiction. Hence $\dim \Pi \geq 4$. We need to show that $\dim \Pi = 4$, so suppose now for a contradiction that $\dim \Pi > 4$ (this means that there are projective lines in $\langle \mathscr{C}_x \rangle$ skew to W).

We distinguish two cases. First we assume that there is some singular line R disjoint from W. In particular, R is disjoint from T_{x_i}, and so no point of R is collinear to x_i, $i = 1,2$. We claim that R contains a point v with $v^\perp \cap L = \emptyset$. Indeed, if not then let y_1, y_2 be two distinct points of R and let $z_i \perp y_i$, with $z_i \in L$, $i = 1,2$. Since there are no singular planes z_i is unique, $i = 1,2$, and $z_1 \neq z_2$. Hence $[y_1, z_2]$ contains L and R and so x_1 is collinear to some point of R after all. This shows the claim.

So let $v \in R$ be such that $v^\perp \cap L = \emptyset$. The members $[v, x_1]$ and $[v, x_2]$ of Ξ do not contain L. Let M_i denote the line $T_v([v, x_i]) \cap T_{x_i}([v, x_i])$, $i = 1,2$. Then $M_1 \cap M_2 \subseteq T_{x_1} \cap T_{x_2}$ and Lemma 7.9 implies $M_1 \cap M_2 \subseteq L$. Since v is not collinear to any point of L, we deduce that M_1 and M_2 are disjoint. This implies that T_v contains a 3-dimensional subspace of W, namely $\langle M_1, M_2 \rangle$. Since, by (MM3), $\dim T_v \leq 4$, $R \subseteq T_v$ intersects W non-trivially, a contradiction.

Secondly, assume that no singular line is disjoint from W. Let $x_1^*, x_2^* \in X$ be such that $\langle x_1^*, x_2^* \rangle$ is disjoint from W. Put $\xi^* := [x_1^*, x_2^*]$. If ξ^* has index 1, then, as

R is skew to W we see that ξ^* meets W in at most a line, and hence we find a singular line R^* on ξ^* skew to W, a contradiction. So ξ^* has index 0. Since $x_i^* \notin W$, in particular x_i^* is not collinear to x_i, we can consider $\xi_i' := [x_i, x_i^*] \in \Xi$. Then $T_{x_1^*}(\xi_1') \cap T_{x_2^*}(\xi_2')$ is empty, for if it contained a point z, then by (MM2), $z \in X$ and hence $\langle x_1^*, z\rangle \cup \langle x_2^*, z\rangle$ would be two singular lines in ξ^* by Lemma 4.3, a contradiction to $w_{\xi^*} = 0$. By Lemma 7.6, there are four singular lines through x_i^* spanning $T_{x_i^*}$, and by assumption, each of these lines has a point in common with W. This implies that $T_{x_i^*}$ is a singular 4-space sharing a 3-space $W_{x_i^*}$ with W, $i = 1,2$. Clearly, $W_{x_1^*}$ and $W_{x_2^*}$ share the line $T_{x_1^*}(\xi^*) \cap T_{x_2^*}(\xi^*)$, so using $T_{x_1^*}(\xi_1') \cap T_{x_2^*}(\xi_2') = \emptyset$, we obtain $\dim\langle W_{x_1^*}, W_{x_2^*}\rangle = 5$. This means that $W^* := \langle T_{x_1^*}, T_{x_2^*}, \xi^*\rangle = \langle W_{x_1^*}, W_{x_2^*}, x_1^*, x_2^*\rangle$ has dimension 7; and hence $\langle T_{x_1^*}, \xi^*\rangle \cap \langle T_{x_2^*}, \xi^*\rangle = \xi^*$.

In what follows we interchange the roles of ξ, x_i, W and ξ^*, x_i^*, W^*, respectively. Since ξ and ξ^* have different index, the arguments are not entirely identical, hence we present them in detail.

Let R^* be a singular line of ξ disjoint from L. Let v^* be an arbitrary point on R^* and note that $v^* \notin y_i^\perp$ since $v^* \notin T_{y_i}$, $i = 1,2$. Let M_i^* denote the line $T_{v^*}([v^*, x_i^*]) \cap T_{x_i^*}([v^*, x_i^*])$, $i = 1,2$. There are two cases: Suppose first that $M_1^* \cap M_2^* \neq \emptyset$. Then $M_1^* \cap M_2^* \subseteq T_{x_1^*} \cap T_{x_2^*}$ and since by the previous paragraph $\langle T_{x_1^*}, \xi^*\rangle \cap \langle T_{x_2^*}, \xi^*\rangle = \xi^*$, we obtain $M_1^* \cap M_2^* \subseteq \xi^*$. By (MM2), the intersection $M_1^* \cap M_2^*$ belongs to X and hence to $x_1^{*\perp} \cap x_2^{*\perp}$, contradicting $w_{\xi^*} = 0$. Suppose now that $M_1^* \cap M_2^* = \emptyset$. This implies that T_{v^*} contains a 3-dimensional subspace of W^*, namely $\langle M_1^*, M_2^*\rangle$. Since, by (MM3), $\dim T_{v^*} \leq 4$, $R^* \subseteq T_{v^*}$ intersects W^* non-trivially, a contradiction. □

We are now ready to prove Proposition 7.4.

Proof of Proposition 7.4. Again, let x_1, x_2 be two points on ξ on a common singular line L. Let L_1, L_1' be two distinct singular lines through x_1, not inside ξ. Put $\overline{\xi} := [L, L_1]$ (recall that there are no singular planes) and let L_2 be the singular line of $\overline{\xi}$ through x_2 distinct from L. Finally, let L_2' be an arbitrary singular line through x_2, distinct from L_2 and not in ξ. Put $\xi_i' := [L_i, L_i']$ for $i = 1,2$.

By Lemma 7.11, there is a 4-dimensional subspace Π in $\langle \mathscr{C}_x\rangle$ complementary to ξ. Then the image of $X(\xi_i') \setminus (L_i \cup L_i')$ under ρ_ξ is the set of points of an affine plane π_i^* in Π, with projective completion π_i, and the line $T_i := \pi_i \setminus \pi_i^*$ is the projection of T_{x_i}, $i = 1,2$. The projective planes π_1 and π_2 meet non-trivially by a dimension argument. According to Lemma 7.9, the lines T_1 and T_2 are skew; also, the arguments of the second paragraph of the proof of Lemma 7.11 imply that T_1 does not meet π_2, and T_2 does not meet π_1. Hence the affine planes π_1^* and π_2^* meet in a unique point z and so we have points z_1 in $X(\xi_1') \setminus (L_1 \cup L_1')$ and z_2 in $X(\xi_2') \setminus (L_2 \cup L_2')$ lying in a common 4-space U with ξ. We claim that $z_1 = z_2$, so suppose for a contradiction that $z_1 \neq z_2$. Let ξ^* be a member of Ξ containing z_1, z_2.

Considering $\xi^* \cap \xi$ and (MM2), we see that $\langle z_1, z_2 \rangle$ is a singular line meeting ξ in some point $u \in X$. Note that $u \notin L$ because otherwise $L \subseteq \xi_1'$ by Lemma 4.3, a contradiction. So, possibly interchanging the roles of x_1 and x_2, we may assume that $\langle u, x_2 \rangle$ is not a singular line; let $\{v\} = u^\perp \cap M_2$, with M_2 the singular line of ξ through x_2 distinct from L. Recall that $z_1 \notin T_{x_2}$, as $z \notin T_2$, so we can consider $\xi_{12} := [z_1, x_2]$. Then we show $\xi_{12} \cap \xi = \{x_2\}$. Indeed, if $L \subseteq \xi_{12}$, then $x_1 \in \xi_{12}$ and hence $\xi_{12} = \xi_1'$, a contradiction; if M_2 belongs to ξ_{12}, then $u \in v^\perp \cap z_1^\perp$ (note $v \notin z_1^\perp$ by absence of singular planes), and then $\xi_{12} = \xi$, a contradiction. Thus we obtain that the image under ρ_ξ of $X(\xi_{12}) \setminus x_2^\perp$ coincides with the plane π_2^* (for the projection is determined by $z_1^{\rho_\xi} = z$ and $T_{x_2}(\xi_{12})^{\rho_\xi} = T_{x_2}^{\rho_\xi} = T_2$). Noting that $T_{z_1}(\xi_1')^{\rho_\xi} = \pi_1$ and $T_{z_1}(\xi_{12})^{\rho_\xi} = \pi_2$, and recalling that $\langle \pi_1, \pi_2 \rangle = \Pi$, we obtain that T_{z_1} is a 4-space disjoint from ξ. However, $u \in T_{z_1} \cap \xi$, a contradiction. The claim follows.

Now let M_i be the singular line in ξ_i' through z_i meeting L_i, $i = 1, 2$. Let m_i denote the point $M_i \cap L_i$. Remember that L_1, L, L_2 are contained in $\overline{\xi}$; let L' be the singular line of $\overline{\xi}$ through m_1. Note that $M_1 \neq M_2$, for otherwise $M_1 = L' = M_2$ and hence $\xi_1' = \overline{\xi} = \xi_2'$, a contradiction. Moreover, m_1 and m_2 are not collinear, for otherwise $\langle z_1, m_1, m_2 \rangle$ would be a singular plane, a possibility we excluded by assumption. However, this means that $z_1 \in m_1^\perp \cap m_2^\perp \subseteq \overline{\xi}$, a contradiction, recalling $\xi_1' \cap \overline{\xi} = L_1$ and $z_1 \notin L_1$.

We conclude that our initial assumption that $\mathscr{C}_x$ contains a pair of members of Ξ of index 1 sharing exactly one point is false. Hence by Lemma 7.5, $\mathscr{C}_x$ is isomorphic to $\mathscr{S}_{1,1,1}(\mathbb{K})$. $\square$

Finally, we rule out the existence of connected components isomorphic to $\mathscr{S}_{1,1,1}(\mathbb{K})$.

Proof of Proposition 7.1 in case $d = 2$. Suppose for a contradiction that $W = \{0, 1\}$. By assumption, there is a $\xi \in \Xi$ through x of index 1, so the connected component $\mathscr{C}_x$ of x in $(X, \mathscr{L})$ is more than a point. By Lemma 7.3, $\mathscr{C}_x$ does not contain singular planes and hence Proposition 7.4 then implies that $\mathscr{C}_x$ is isomorphic to $\mathscr{S}_{1,1,1}(\mathbb{K})$. Pick two points x and y at distance 3 in $\mathscr{C}_x$ and let $\Sigma = \langle \mathscr{C}_x \rangle$. Then $[x, y]$ is a member of Ξ of index 0. Note that the presence of $[x, y]$ implies (by (MM3)) that $\dim T_x = \dim T_y = 4$. Inside $[x, y]$, we see that the tangent planes $T_x([x, y])$ and $T_y([x, y])$ intersect each other in a line L. On the other hand, inside $\mathscr{C}_x \cong \mathscr{S}_{1,1,1}(\mathbb{K})$, the tangent spaces of x and y are disjoint 3-spaces (as indeed they span the 7-space, which follows from (P1) in Section 17.2 in [23]), which consequently both contain a unique point of L. Thus L belongs to Σ, and hence so does $[x, y] = \langle x, y, L \rangle$. However, this contradicts Lemma 5.3. We conclude that

$W = \{1\}$. The main results from [21] then reveal that $(X, \mathscr{L})$ is isomorphic to one of $\mathscr{S}_{1,2}(\mathbb{K})$, $\mathscr{S}_{2,2}(\mathbb{K})$, $\mathscr{S}_{1,3}(\mathbb{K})$ indeed. $\square$

7.2 The case $d = 3$

Proof of Proposition 7.1 in case $d = 3$. Let x be a point with $\max(W_x) = 1$. Then let $\Xi(\mathscr{C}_x)$ be the set of members $\xi \in \Xi$ of index 1 contained in $\mathscr{C}_x$, and let $\mathscr{L}(\mathscr{C}_x)$ be the set of singular lines of $\mathscr{C}_x$. Due to Proposition 6.2, $|\Xi(\mathscr{C}_x)| \geq 2$. Now we observe that $(\mathscr{C}_x, \Xi(\mathscr{C}_x))$ is a so-called *Lagrangian Grassmannian set*, as introduced in [20]. Indeed, the members of $\Xi(\mathscr{C}_x)$ are projective 4-spaces intersecting $\mathscr{C}_x$ in a non-singular parabolic quadric (and all such quadrics are isomorphic). Moreover, two points x, y at distance at most 2 in the point-line geometry $(\mathscr{C}_x, \mathscr{L}(\mathscr{C}_x))$ are contained in at least one member of $\Xi(\mathscr{C}_x)$; indeed, by (MM1) there is a member ξ of Ξ containing x and y, which obviously belongs to $\Xi(\mathscr{C}_x)$ if $x \perp y$. If x is at distance 2 from y, then $\emptyset \neq x^\perp \cap y^\perp \subseteq X(\xi)$, so ξ has index 1. Furthermore, Axioms (MM2) and (MM3) hold in $(\mathscr{C}_x, \Xi(\mathscr{C}_x))$. This now implies by Main Result 2 of [20] that $(\mathscr{C}_x, \Xi(\mathscr{C}_x))$ is isomorphic to the Lagrangian Grassmannian $\mathsf{LG}(3,6)(\mathbb{K})$, which, as a point-line geometry, is isomorphic to the dual polar space $\mathsf{C}_{3,3}(\mathbb{K})$, and which lives in projective 13-space (hence $\dim\langle\mathscr{C}_x\rangle = 13$).

This also implies that $(\mathscr{C}_x, \mathscr{L}(\mathscr{C}_x))$ has diameter 3. Hence there exist two points $y, z \in \mathscr{C}_x$ at distance 3. Then $[y, z]$ is a member of Ξ of index 0. Clearly, $T_y([y,z]) \cap T_z([y,z])$ is a plane. However, in $\mathsf{LG}(3,6)(\mathbb{K})$, the (6-dimensional) tangent spaces of points at distance 3 are disjoint (due to the fact that they span the 13-space $\langle\mathscr{C}_x\rangle$, which follows from (P1) in Section 17.2 of [23]), so $T_y \cap T_z = \emptyset$, a contradiction. $\square$

7.3 The case $d \geq 4$

We now aim to show Proposition 7.1 for $d \geq 4$. Let $x \in X$ be a point with $\max(W_x) = 1$. By Corollary 4.10, (X_x, Ξ_x) is a weak pre-AVV (use also Proposition 6.2 to see that $|\Xi_x| \geq 2$).

Lemma 7.12. *The maximal singular subspaces of* (X_x, Ξ_x) *are pairwise disjoint.*

Proof. Suppose for a contradiction that two distinct maximal singular subspaces $\mathscr{M}_1$ and $\mathscr{M}_2$ of (X_x, Ξ_x) have a point p in common. Note that this implies $\dim \mathscr{M}_i \geq 1$, $i = 1, 2$. Let L_1 and L_2 be lines in $\mathscr{M}_1$ and $\mathscr{M}_2$, respectively, through p. In (X, Ξ), this corresponds to planes π_1 and π_2 intersecting each other in a line M through x. Let p_i be a point of $\pi_i \setminus M$, for $i = 1, 2$. If p_1 were not collinear with p_2, then Axiom (MM1') together with Lemma 4.3 would imply that there is a member of Ξ containing $\pi_1 \cup \pi_2$, contradicting $\max(W_x) = 1$. Hence $p_1 \perp p_2$, which means that L_1 and L_2 span a singular plane. Varying L_1 and L_2 yields the singular subspace $\langle \mathscr{M}_1, \mathscr{M}_2 \rangle$, contradicting the maximality of $\mathscr{M}_i$, $i = 1, 2$. □

It is now convenient to distinguish between the finite and infinite case, noting that in the infinite case, we really only need the field $\mathbb{K}$ to have at least 5 elements, but the counting arguments for $|\mathbb{K}| \leq 4$ are uniform and hold for all finite fields.

7.3.1 The infinite case

Proof of Proposition 7.1 for $d \geq 4$ and $\mathbb{K}$ infinite. Take $\xi \in \Xi_x$ arbitrary. Since $|\Xi_x| \geq 2$, there exists $p \in X_x \setminus \xi$. By Lemma 4.3, $p^\perp \cap \xi$ is a singular subspace and hence there are distinct points $p_1, p_2 \in X(\xi)$ not collinear to p. Set $\xi_i := [p, p_i]$, $i = 1, 2$. By Lemma 5.2 (applied to (X_x, Ξ_x), recalling $\dim T_x \leq 2d$ and $|\mathbb{K}| > 2$) there is a singular line meeting ξ_1, ξ_2 and ξ in three distinct points. Lemma 5.1 then yields conics $C_1 \subseteq X(\xi_1)$ and $C_2 \subseteq X(\xi_2)$ through p on a common normal rational cubic scroll. Moreover, since $|\mathbb{K}| > 4$, all its transversal lines, except possibly the one through p, are singular; and so is the unique line M meeting all these transversal lines. Let L_1 and L_2 be two such singular transversal lines. Since both of them intersect M in a point, Lemma 7.12 implies that they are collinear with M and, repeating this argument, $\langle L_1, L_2 \rangle$ is a singular 3-space. This however contradicts the fact that the points $L_1 \cap C_1$ and $L_2 \cap C_1$ are not collinear. □

7.3.2 Finite case

In this subsection, we assume that $\mathbb{K}$ is the finite field $\mathbb{F}_q$. Since over a finite field quadrics of index 0 only exist in dimensions 2 and 3, we deduce $d = 4$. Hence, by (MM3), $N_x := \dim\langle X_x \rangle \leq 7$.

Lemma 7.13. *The maximal singular subspaces of (X_x, Ξ_x) have dimension at most 1, and at least one singular line in X_x exists.*

Proof. Let M be a maximal singular subspace of (X_x, Ξ_x). Let $p \in X_x$ be a point outside M (which exists, as there are non-collinear points in $(X_x, \mathscr{L}_x)$). For each point z of M, it follows from Lemma 7.12 that the points p and z are non-collinear, and hence they define a unique member ξ_z of Ξ_x. Let ρ be the projection of X_x from M onto a complementary subspace Π in $\langle X_x \rangle$. This projection is injective, since points with the same image are necessarily collinear to a point of M, contradicting Lemma 7.12. For each member ξ_z of Ξ_x, $z \in M$, the projection of $X(\xi_z) \setminus \{z\}$ is an affine plane $\pi_z^* \ni p^\rho$ with projective completion π_z and we have $L_z := \pi_z \setminus \pi_z^* = T_z(\xi_z)^\rho$. We claim that, for $z_1, z_2 \in M$, $z_1 \neq z_2$, $\pi_{z_1} \cap \pi_{z_2} = \{p^\rho\}$. Indeed, suppose for a contradiction that $p^\rho \neq u \in \pi_{z_1} \cap \pi_{z_2}$. By possibly considering $\langle p^\rho, u \rangle \cap L_{z_1}$, we may assume $u \in L_{z_1}$. If $u \in L_{z_2}$, then there exists a point

$v \in \langle p^\rho, u\rangle$, with $p^\rho \neq v \in \pi^*_{z_1} \cap \pi^*_{z_2}$, contradicting injectivity of ρ. So $u \notin L_{z_2}$ and hence there exists $u_2 \in X(\xi_{z_2})$ with $u_2^\rho = u$. Since $u \in L_{z_1}$, there exists a line U_1 in $T_{z_1}(\xi_{z_1})$ through z_1 with $U_1^\rho = u$. This implies that the plane $\langle U_1, u_2\rangle$ contains a point $u' \in M \setminus \{z_1\}$. Then $[u', u_2]$ exists and intersects ξ_{z_1} in a point of $U_1 \setminus \{z_1\}$ not belonging to X_x, contradicting (MM2). The claim is proved.

Suppose for a contradiction that $\dim(M) \geq 2$. Then $\dim(\Pi) \leq 4$ (recall $N_x \leq 7$). Then the number of points in the union of the $n := |M|$ planes π_z, $z \in M$, is at least $n(q^2+q)+1$. Since $n \geq q^2+q+1$ by assumption, this exceeds the number of points of Π. Hence $\dim(M) \leq 1$.

The first assertion follows.

For the second assertion, suppose for a contradiction that there are no singular lines in X_x. Let $\xi \in \Xi_x$ and let $\rho := \rho_\xi$ be the projection onto Π (recall that $N_x \leq 7$ so $\dim(\Pi) \leq 3$). For any $\xi' \in \Xi_x$ meeting ξ in a point $p \in X_x$, the q^2 points of $X_x(\xi') \setminus \{p\}$ determine distinct members of Ξ_x with any point $p' \in X_x(\xi) \setminus \{p\}$. The number of points of X_x on these q^2 members of Ξ_x distinct from p' is q^4, whereas Π contains at most q^3+q^2+q+1 points. By Lemma 4.17, this gives rise to a singular line in (X_x, Ξ_x) after all, a contradiction. □

Lemma 7.14. *Each point of X_x is contained in precisely one singular line and in at least q^2+q members of Ξ_x. Also, $|X_x| \geq q^4+q^3+q+1$.*

Proof. Suppose for a contradiction that there is a point p of X_x through which there are no singular lines. By Lemmas 7.12 and 7.13, there is a point $r \in X_x$ contained in a unique singular line. Let α_p and α_r be the respective numbers of members of Ξ_x through p and r. Note that members of Ξ_x have q^2+1 points. Then $|X_x| = \alpha_p q^2 + 1 = \alpha_r q^2 + (q+1)$. It follows that $(\alpha_p - \alpha_r)q = 1$, a contradiction.

Hence each point $p \in X_x$ is contained in a unique singular line. This means that $|X_x| = |\mathscr{L}_x| \cdot (q+1)$. Since $|X_x| = \alpha_p q^2 + q + 1$, it then follows that $q+1$ divides α_p. Taking a member of Ξ_x not through p, we also see that $\alpha_p \geq q^2$. Combined, this implies $\alpha_p \geq q^2+q$. It now also follows that $|X_x| = \alpha_p q^2 + q + 1 \geq q^4+q^3+q+1$. □

Proof of Proposition 7.1 for $d \geq 4$ and $\mathbb{K}$ finite. We consider the projection $\rho := \rho_\xi$ from any $\xi \in \Xi_x$ onto a complementary subspace Π (which has dimension at most 3). By Lemma 4.17, two points of $X_x \setminus \xi$ have the same image under ρ precisely if they are on a singular line meeting $X(\xi)$. Hence the number of points in $\rho(X_x)$ is, by Lemma 7.14, at least $(q^4+q^3+q+1) - (q^2+1)q = q^4+1$. This is strictly more than the number of points in Π however, a contradiction. □

8 Case 2: there is a point $x \in X$ with $\max(W_x) = 2$

Suppose (X,Ξ) is an AVV of type d with global index set W in $\mathbb{P}^N(\mathbb{K})$ and with $\max(W_x) = 2$ for some $x \in X$. The existence of $\xi \in \Xi$ through x with $w_\xi = 2$ implies $d \geq 4$. We show the following proposition.

Proposition 8.1. *Let (X,Ξ) be an AVV of type d with global index set W in $\mathbb{P}^N(\mathbb{K})$ containing a point $x \in X$ with $\max(W_x) = 2$. Then $W = \{2\}$, $d = 4$ and hence (X,Ξ) is isomorphic to $\mathscr{G}_{n,1}(\mathbb{K})$ for $n \in \{4,5\}$.*

As in the previous case, we need a different approach for the case $d = 4$. We start with the generic case $d \geq 5$. Henceforth let $x \in X$ be such that $\max(W_x) = 2$.

8.1 The case $d \geq 5$

Lemma 8.2. *Suppose ξ and ξ_1 are members of Ξ of index 2 and index at most 2, respectively, intersecting each other in precisely a line $L \ni x$. Then $d = 5$ and ξ_1 has index 2.*

Proof. In the residue (X_x,Ξ_x), the members $\xi,\xi_1 \in \Xi$ correspond to members $\xi',\xi_1' \in \Xi_x$ of index 1 and index at most 1, respectively, intersecting each other in precisely a point p_1. By Lemma 5.4 (note that $d-2 \geq 3$), we may assume that there is a singular line $\langle z_1, z\rangle$ with $z_1 \in X(\xi_1') \setminus p_1^\perp$ and $z \in X(\xi') \setminus p_1^\perp$. As ξ' has index 1, we can take a singular line M through z in ξ' that is not collinear to z_1. Then M and z_1 determine a unique member ξ_2' of Ξ_x.

Suppose first $d \geq 6$. Then Lemma 5.5 implies that ξ_2' has index at least 2, and hence the corresponding member ξ_2 of Ξ has index at least 3, which contradicts $\max(W_x) \leq 2$. Next suppose $d = 5$, $|\mathbb{K}| > 2$ and the index of ξ_1' is equal to 0 (the latter implies $\xi_1' \cap \xi_2'$ is exactly z_1). Then Lemma 5.5 yields the same contradiction as just above.

Finally, suppose $d = 5$ and $|\mathbb{K}| = 2$. Then $W = \{2\}$, as the only non-degenerate quadrics in finite 6-dimensional projective space are split. This possibility is excluded by the Main Result of [21]. $\square$

This has the following corollary.

Corollary 8.3. *We have $2 \in W_x \subseteq \{0,2\}$.*

Proof. Suppose for a contradiction that Ξ_x has a member ξ^0 of index 0. Recall that by assumption $\max(W_x) = 2$, and hence Ξ_x has at least one member ξ^1 of index 1 too. By Lemma 8.2, $\xi^0 \cap \xi^1$ is empty. We take a pair of non-collinear points

$p^0 \in X(\xi^0)$ and $p^1 \in X(\xi^1)$ and obtain $[p^0, p^1] \in \Xi_x$ with $\xi^0 \cap [p^0, p^1] = \{p^0\}$. By Lemma 8.2, the index of $[p^0, p^1]$ is 0. But then $\xi^1 \cap [p^0, p^1] = \{p^1\}$, contradicting Lemma 8.2. $\square$

Proof of Proposition 8.1 in the case $d = 5$. We proceed by showing some claims.

Claim 1. The residue (X_x, Ξ_x) is a pre-AVV of type 3 and global index set $\{1\}$.
Indeed, by Corollary 8.3, Ξ_x only contains members of index 1. By Proposition 6.2, $|\Xi_x| \geq 2$. The claim now follows from Corollary 4.12.

Claim 2. For each point $p \perp x$, we have $T_p \cap X \subseteq p^\perp$.
Indeed, this follows immediately from Lemma 4.14 if there exist two members of Ξ which intersect each other precisely in p, in particular if $0 \in W_p$. So we may assume that $(X_p, \mathscr{L}_p)$ is a (-1)-lacunary parapolar space. By Fact 4.7, all members of Ξ_p are split, a contradiction. This shows the claim.

Claim 3. For each point $p \perp x$, there exists $\xi \in \Xi$ with $x \in \xi$, $p \notin \xi$ and $w_\xi = 2$.
Indeed, by Claim 1, there exist two members $\xi_1, \xi_2 \in X$ of index 2 containing x. Suppose they both contain p. In ξ_1 we select a singular line L_1 through x not collinear to p. In $\xi_2 \setminus \xi_1$, we select a singular line L_2 through x not in a plane with L_1. Then the unique member of Ξ determined by L_1 and L_2 has index 2 (by Claim 1) and does not contain p, showing the claim.

Claim 4. For each point $p \perp x$, we have $\dim(T_x \cap T_p) \leq 8$.
The following argument is inspired by the proof of Corollary 4.15 of [21]. By Claim 3, there exists $\xi \in \Xi$ of index 2 with $p \notin \Xi \ni x$. Set $W_p := T_p \cap \xi$. By Claim 2, $W_p \cap X(\xi) = p^\perp \cap \xi$, so $W_p \subseteq T_x(\xi)$ and also $\dim W_p \geq 3$. As such there exists a line L in $T_x(\xi)$ disjoint from W_p. Noting that T_p and T_x are at most 10-dimensional, and that $L \subseteq T_x \setminus T_p$, the claim follows.

Now, the content of Section 6.1 of [21] is to prove non-existence of pre-AVVs of type 3 with global index set $\{1\}$ for which each tangent space has dimension at most 7. By Claims 1 and 4, this completes the proof of Proposition 8.1 in the case $d = 5$.

Proof of Proposition 8.1 in the case $d \geq 6$. By Corollary 8.3, all members of Ξ_x have index 1 and, by Lemma 8.2 and $d \geq 6$, no two of them share precisely a point. Note that $|\Xi_x| \geq 2$, according to Proposition 6.2, and the absence of members of index 0 in Ξ_x implies that (X_x, Ξ_x) is connected. We conclude that $(X_x, \mathscr{L}_x)$ is a strong 0-lacunary parapolar space of diameter 2 whose symps are quadrics of projective index 1. By Fact 4.8, $(X_x, \mathscr{L}_x)$ is the direct product of a line and a projective n-space. This however implies that the members of Ξ_x are hyperbolic quadrics in 3 dimensions, i.e. that $d - 1 = 3$, a contradiction. $\square$

8.2 The case $d = 4$

Recall that $d = 4$ implies that $\max(W) \leq 2$. If $W = \{2\}$ then the Main Result of [21] proves Proposition 8.1, hence we assume for a contradiction that $W \neq \{2\}$. Let $z \in X$ be a point with $\max(W_z) = 2$. If $W_z = \{2\}$, then our assumption $W \neq \{2\}$ implies that there exists $y \in X \setminus \{z\}$ with $\min W_y < 2$. Using (MM1) on the pair y, z, we have $2 \in W_y$, and so by $d = 4$ we obtain $\max(W_y) = 2$. We may hence assume that there are points $x \in X$ with $m_x := \min(W_x) < \max(W_x) = 2$.

Lemma 8.4. *For each $x \in X$ with $m_x < \max(W_x) = 2$, the residue (X_x, Ξ_x) contains no singular subspaces of dimension $2 + m_x$.*

Proof. Let ξ_1 and ξ_2 be two members of Ξ through x of index m_x and 2, respectively. Suppose for a contradiction that there is a singular subspace S of dimension $3 + m_x$ through x. Since $S \cap \xi_2$ is contained in a plane of ξ_2 and $\xi_1 \cap \xi_2$ is contained in a line of ξ_2, we can select a singular plane $\pi \subseteq \xi_2$ through x which intersects S and ξ_1 in precisely x. Inside T_x (which has dimension at most 8 by (MM3)), the subspace $\langle \pi, S \rangle$ has dimension $5 + m_x$ and therefore it intersects the 4-space $T_x(\xi_1)$ in a subspace S' of dimension at least $1 + m_x$. Since each point of $S' \setminus \{x\}$ is on a line meeting both S and π, (MM2) implies that S' is singular. However, $\dim S' = 1 + m_x > w_{\xi_1}$, a contradiction. □

Lemma 8.5. *Let $x \in X$ be a point with $m_x < \max(W_x) = 2$. Then the point-line geometry $(X_x, \mathscr{L}_x)$ is a strong parapolar space whose symps are quadrics of projective index 1.*

Proof. By assumption on x, there is at least one member $\xi^* \in \Xi$ of index 2 through x. In $(X_x, \mathscr{L}_x)$, ξ^* corresponds to a member ξ of Ξ_x of index 1. Observe that X_x contains a point $z \notin X_x(\xi)$, because either there is a second member of index 2 containing x, or, if not, then Proposition 6.2 yields at least one member of Ξ of index 1 containing x.

We show that z belongs to the connected component $\mathscr{C}_\xi$ of ξ in $(X_x, \mathscr{L}_x)$. Suppose not. Then $z^\perp \cap \xi = \emptyset$ and $[z, p] \in \Xi_x$ has index 0 for each $p \in X_x(\xi)$. Moreover, we claim that z satisfies the following two conditions: $T_z \cap X_x(\xi) = \emptyset$ and $\dim T_z \leq 5$. To that end, we consider the situation in (X, Ξ), where ξ corresponds to ξ^* and z to a singular line L containing x, on which we select a point $z' \neq x$. Then, since $1 \in W_{z'}$ by the above, Lemma 4.15 implies that $T_{z'} \cap X \subseteq z'^\perp$ and hence $T_{z'} \cap X(\xi^*) \subseteq z'^\perp \cap X(\xi^*) = \{x\}$. This shows the first part of the claim. It also implies that $T_{z'} \cap \xi$ is contained in $T_x(\xi)$ and is at most 2-dimensional for it contains no points of X other than x. Hence there is a singular line in $T_x(\xi)$ disjoint from $T_{z'}$, which means that $\dim(T_{z'} \cap T_x) \leq 2d - 2 = 6$. The claim follows.

Lemma 6.1 now implies that there is a member of Ξ_x of index at least 1 through z meeting ξ non-trivially, and hence $z \in \mathscr{C}_\xi$ after all, a contradiction. We conclude that $(X_x, \mathscr{L}_x)$ is connected and non-trivial (i.e., not a single point or a single member of Ξ_x). The lemma now follows from Lemma 4.5. $\square$

Lemma 8.6. *Let $x \in X$ be a point with $m_x < \max(W_x) = 2$. Then either $(X_x, \mathscr{L}_x)$ is isomorphic to $\mathscr{S}_{1,1,1}(\mathbb{K})$ or each point $p \in X_x$ is contained in four singular lines L_1, L_2, L_3, L_4 of $\mathscr{L}_x$ not in a common singular plane. In the latter case,*

- (i) *either* $\dim\langle L_1, L_2, L_3, L_4\rangle = 4$ *and* $\{L_1, L_2, L_3, L_4\}$ *contains at most two (necessarily disjoint) pairs of collinear lines;*
- (ii) *or* $\dim\langle L_1, L_2, L_3, L_4\rangle = 3$ *and three lines of* $\{L_1, L_2, L_3, L_4\}$ *lie in a common singular plane.*

Proof. By Lemma 8.5, $(X_x, \mathscr{L}_x)$ is a strong parapolar space whose symps are quadrics of projective index 1. Suppose first that $(X_x, \mathscr{L}_x)$ is 0-lacunary. By Fact 4.8, and the fact that X_x does not contain singular 3-spaces by Lemma 8.4, $(X_x, \mathscr{L}_x)$ is then isomorphic to either the direct product $\mathscr{S}_{1,1,1}(\mathbb{K})$, or to $\mathscr{S}_{1,2}(\mathbb{K})$. In the latter case, each point $p \in X_x$ is contained in four singular lines L_1, L_2, L_3, L_4 satisfying (ii).

So next, we suppose that there is a point $p \in X_x$ contained in two index 1 members of Ξ_x that intersect each other in p only. Hence there are four singular lines through p not all in one singular plane. Let q be a point of X_x collinear to p. Let L_1, L_2, L_3 be three singular lines through p distinct from pq. For each $i \in \{1,2,3\}$, the lines pq and L_i determine either a member of Ξ_x in X_x or a singular plane. Hence it is clear that there are at least four singular lines, not in a common plane, through q as well. By connectivity, there are four singular lines not in a common plane through each point of X_x.

Now assume that $p \in X_x$ is contained in four singular lines L_1, L_2, L_3, L_4, not in one singular plane. First suppose that $\dim\langle L_1, L_2, L_3, L_4\rangle = 3$. Then the planes $\langle L_1, L_2\rangle$ and $\langle L_3, L_4\rangle$ have a line in common. By (MM2), at least one of these planes, say $\langle L_1, L_2\rangle$ is singular, and one of L_3, L_4 is contained in $\langle L_1, L_2\rangle$. Hence (ii) holds. Next, suppose $\dim\langle L_1, L_2, L_3, L_4\rangle = 4$. Then L_1 is collinear with at most one member of $\{L_2, L_3, L_4\}$, as otherwise we obtain two singular planes sharing a line, and by absence of quadrics of index higher than 1, this yields a singular 3-space, violating Lemma 8.4. Hence (i) holds. $\square$

Lemma 8.7. *Let $x \in X$ be a point with $m_x < \max(W_x) = 2$. Then $(X_x, \mathscr{L}_x)$ is isomorphic to $\mathscr{S}_{1,1,1}(\mathbb{K})$.*

Proof. Recall that, by Lemma 4.10, (X_x, Ξ_x) is a weak pre-AVV of type 2 with global index set W'_x in $\mathbb{P}^N(\mathbb{K})$ with $N \leq 7$. We claim that Axioms (MM1) and

(MM3) hold in (X_x, Ξ_x). Indeed, (MM1$'$) holds, and the same argument that we used to show Axiom (PPS3) in the proof of Lemma 4.5(*iii*), completes the proof of (MM1). Suppose now for a contradiction that there is a point $p \in X_x$ with $\dim\langle T_p(\xi) \mid p \in \xi \in \Xi_x\rangle \geq 5$.

Claim: there exist $\xi \in \Xi_x$ containing p, and three singular lines of $\mathscr{L}_x$, also containing p and generating a 3-space S, such that $S \cap \xi = \{p\}$.
We may assume that $(X_x, \mathscr{L}_x)$ is not isomorphic to $\mathscr{S}_{1,1,1}(\mathbb{K})$, and hence Lemma 8.6 implies there are four singular lines $L_1, L_2, L_3, L_4 \in \mathscr{L}_x$ through p. Set $\Pi := \langle L_1, L_2, L_3, L_4\rangle$. Also by Lemma 8.6 either $\dim(\Pi) = 4$ and, up to renumbering, $[L_1, L_2], [L_3, L_4] \in \Xi_x$ (case (*i*)); or $\dim(\Pi) = 3$ and, up to renumbering, L_2, L_3, L_4 are in a singular plane π (case (*ii*)). By assumption on p, there exists $\xi^* \in \Xi_x$ through p such that $T_p(\xi^*)$ has at most a line in common with Π. We distinguish two cases.

Case 1: Suppose that ξ^* has index 1. Then we obtain that $T_p(\xi^*)$ contains a singular line L_5 with $L_5 \nsubseteq \Pi$. In case (*ii*), the lines L_1, L_2, L_3, L_5 span a 4-space and as $\max(W_x) \leq 2$ and there are no singular 3-spaces by Lemma 8.4, we deduce that $[L_1, L_2], [L_3, L_5] \in \Xi_x$, which brings us to case (*i*). So let us consider case (*i*) now, where we assume that $[L_1, L_2], [L_3, L_4] \in \Xi_x$. Since $L_5 \nsubseteq \Pi$, the lines L_1, L_2, L_3, L_4, L_5 generate a 5-space. We consider the 3-space $S := \langle L_3, L_4, L_5\rangle$ and $\xi := [L_1, L_2] \in \Xi_x$. Clearly, $\dim\langle \xi, S\rangle \geq \dim\langle L_1, \ldots, L_5\rangle = 5$. If $\dim\langle \xi, S\rangle = 6$, then (L_3, L_4, L_5) and ξ are as required by the claim. If $\dim\langle \xi, S\rangle = 5$, then $\xi \cap S$ is a line L. Since L does not belong to $\langle L_1, L_2\rangle = T_p(\xi)$, it contains a unique point $p' \in X_x(\xi) \setminus \{p\}$ (note that in particular, $L \notin \{L_3, L_4, L_5\}$). Since $p \notin p'^{\perp}$, there is a point $p_5 \in L_5 \setminus \{p\}$ with $p_5 \notin p'^{\perp}$. Using (MM1) and (MM2), we deduce that $\langle p', p_5\rangle$ meets $\langle L_3, L_4\rangle$ in a point of X_x. This however implies that the line $\langle p', p_5\rangle$ is singular, contradicting our choice of $p_5 \notin p'^{\perp}$. This concludes Case 1.

Case 2: Suppose now that ξ^* has index 0. In Case (*ii*), we immediately obtain that ξ^* cannot share a (necessarily non-singular) line L with Π, for $\langle L_1, L\rangle \cap \pi$ would be a singular line L' through p and hence $L \subseteq \xi \cap [L_1, L']$, contradicting (MM2). As such, L_1, L_2, L_3 and ξ^* are as required by the claim. So we may assume Case (*i*). If $\xi^* \cap \Pi = \{p\}$, we are done, so suppose $\dim(\xi^* \cap \Pi) \geq 1$. If $\xi^* \cap \Pi$ is exactly a line L, then renumbering if necessary, we have $L \nsubseteq \langle L_1, L_2, L_3\rangle$ and then the pair (L_1, L_2, L_3), ξ^* does the trick. So finally, suppose $\xi^* \cap \Pi$ is a plane α. Note that, by assumption on ξ^*, α contains precisely one line T of $T_p(\xi^*)$. By (MM2) and $w_{\xi^*} = 0$, α meets $\langle L_1, L_2\rangle$ and $\langle L_3, L_4\rangle$ in p only. A dimension argument then implies that there is a unique plane α_i through L_i that meets both $\langle L_3, L_4\rangle$ and α in respective lines M_i and M'_i, for $i = 1, 2$. Clearly, $M'_1 \neq M'_2$, so we may assume that $M'_1 \neq T$. Since $M'_1 \nsubseteq T_p(\xi^*)$, it contains a unique point $p' \in X_x \setminus \{p\}$. Let p_1 be a point on $L_1 \setminus \{p\}$ and considering the line $\langle p', p_1\rangle$ and its intersection with M_1, we deduce as in the previous paragraph, that the plane $\langle L_1, M_1\rangle$ is singular, contradicting the fact that M_1' is not. This concludes Case 2 and the claim follows.

Henceforth, let $L_1, L_2, L_3 \in \mathscr{L}_x$ be three lines containing p such that $S := \langle L_1, L_2, L_3 \rangle$ has dimension 3 and such that there exists $\xi \in \Xi_x$ with $S \cap \xi = \{p\}$. Since $\max(W_x) \leq 2$ and by absence of singular 3-spaces, we obtain that, up to renumbering, $\xi_1 := [L_1, L_2]$ and $\xi_3 := [L_2, L_3]$ are members of Ξ_x. Since $S \cap \xi = \{p\}$, also $\xi \cap \xi_i = \{p\}$ for $i = 1, 3$. Consequently the subspaces $\langle \xi, \xi_1 \rangle$ and $\langle \xi, \xi_3 \rangle$ are 6-dimensional, and recalling that $\dim\langle X_x \rangle \leq 7$, they share a 5-space Σ containing $\langle \xi, L_2 \rangle$. It follows that Σ meets ξ_i in a plane π_i containing L_2 and hence at least one other singular line R_i of ξ_i, for $i = 1, 3$.

Suppose first that $\langle R_1, R_3 \rangle$ is a 3-space (equivalently, $R_1 \cap L_2 \neq R_3 \cap L_2$). Then $\langle R_1, R_3 \rangle$ meets ξ in a line R containing p. Let r be any point of $R \setminus \{p\}$ and consider the unique line R' through r meeting R_1 and R_3 non-trivially, say in points r_1 and r_3, respectively. By (MM2), R' is singular. Note that $r \neq p$ implies that $r_1, r_3 \notin L_2$. However, r_3 is now collinear to the non-collinear points $L_2 \cap R_3$ and r_1 of $X(\xi_1)$, so $r_3 \in \xi_1 \cap \xi_3 = L_2$, a contradiction.

Therefore $R_1 \cap L_2 = R_3 \cap L_2 =: y$. Note that $y \neq p$, for otherwise $R_1 = L_1$ and $R_3 = L_3$, violating the fact that $\langle R_1, L_2, R_3, \xi \rangle = 5$. Then the plane $\langle R_1, R_3 \rangle$ meets ξ in a point z. Clearly $z \neq p$ (otherwise $R_3 \in \langle R_1, p \rangle \subseteq \xi_1$) and $z \notin R_1 \cup R_3$ (since $\xi \cap \xi_i = \{p\}$ and $p \notin R_i$ for $i = 1, 3$). Using (MM2) if $\langle R_1, R_3 \rangle$ is non-singular, we obtain that $z \in X(\xi)$ and hence $\langle R_1, R_3 \rangle$ is singular anyway. If p and z are not collinear, then y, being collinear to both p and z, belongs to ξ, a contradiction. Hence $\langle p, z \rangle$ is singular and as a consequence, $\langle L_2, z \rangle$ is a singular plane intersecting the singular plane $\langle R_1, R_3 \rangle$ in line. Since $\max(W_x) \leq 2$ this yields a singular 3-space $\langle R_1, L_2, R_3 \rangle$, contradicting Lemma 8.4.

We conclude (X_x, Ξ_x) is an AVV of type 2 whose global index set W' has $\max(W') = 1$. Proposition 7.1 yields $W' = \{1\}$ and hence $W_x = \{0, 2\}$. By Lemma 8.4, there are no singular planes in X_x. Recalling that $(X_x, \mathscr{L}_x)$ is connected by Lemma 8.5, it now follows from Proposition 7.4 that $(X_x, \mathscr{L}_x)$ is isomorphic to $\mathscr{S}_{1,1,1}(\mathbb{K})$ after all. $\square$

Proof of Proposition 8.1 in the case $d = 4$. We already noted that, if $W = \{2\}$, the proposition follows from the Main Result of [21], and that we therefore may assume that there is a point $x \in X$ with $\min\{W_x\} < \max\{W_x\} = 2$. By Lemma 8.7, $(X_x, \mathscr{L}_x)$ is isomorphic to $\mathscr{S}_{1,1,1}(\mathbb{K})$. Noting that, for any $y, z \in X_x$ at distance 3 from each other measured in $(X_x, \mathscr{L}_x)$, we have $[y, z] \in \Xi_x$ is contained in $\langle X_x \rangle$, this contradicts Lemma 5.3. $\square$

9 Case 3: For each $x \in X$, either $W_x = \{0\}$ or $\max(W_x) \geq 3$

Proposition 9.1. *Let (X, Ξ) be an AVV of type d with global index set W such that, for each $x \in X$, either $W_x = \{0\}$ or $\max(W_x) \geq 3$. Then W is a singleton $\{w^*\}$ and one of the following occurs.*

- (i) *$w^* = 0$, $d \in \{2^a \mid a \in \mathbb{N}\} \cup \{\infty\}$ and (X, Ξ) is the standard Veronese representation of a projective plane over a quadratic alternative division ring;*
- (ii) *$w^* = 3$, $d = 6$ and (X, Ξ) is the half spin variety $\mathscr{D}_{5,5}(\mathbb{K})$;*
- (iii) *$w^* = 4$, $d = 8$ and (X, Ξ) is the Cartan variety $\mathscr{E}_{6,1}(\mathbb{K})$.*

Again, we show this in a series of lemmas. Throughout, let w^* be the maximum of W, which is well defined as $|W|$ is bounded above by $\lceil \frac{d+1}{2} \rceil$. If $w^* = 0$, then by the Main Result of [16], (i) of Proposition 9.1 holds. So we assume from now on that $w^* \geq 3$ (and hence $d \geq 6$).

Lemma 9.2. *Let $x \in X$ be a point with $\max(W_x) \geq 3$ and let $p \in X$ be a point collinear to x. Then there exists $\xi \in \Xi$ of index at least 3 going through p and not through x.*

Proof. Suppose for a contradiction that all members of Ξ through p of index at least 3 also contain x. First note that (MM1) assures that there is at least one member of Ξ, necessarily of index at least 1, through the singular line $\langle p, x \rangle$; hence our assumptions imply $\max(W_p) \geq 3$ and so there is at least one member of Ξ of index at least 3 through p.

By Corollary 4.12 and Lemma 6.2, (X_p, Ξ_p) is a pre-AVV. The line $\langle p, x \rangle$ corresponds to a point $q \in X_p$ which, by the previous paragraph, has the property that it is contained in all members of Ξ_p of index at least 2.

We claim that there is a pair ξ_1 and ξ_2 in Ξ_p of index at most 1 and index 1, respectively, intersecting each other in exactly one point. Let ξ^* be a member of Ξ_p of index at least 2 containing q (which exists by the above). Let p_1 be a point in $X(\xi^*)$ not collinear to q. Then all members of Ξ_p through p_1, except for $\xi^* = [p_1, q]$, have index at most 1. There are three cases:

1. *There are ξ_1, ξ_2 in $\Xi_p \setminus \{\xi^*\}$ through p_1 of index* 0 *and* 1*, respectively.* In this case, it is clear that $\xi_1 \cap \xi_2 = \{p_1\}$.
2. *All members of $\Xi_p \setminus \{\xi^*\}$ through p_1 have index* 1. By Lemma 5.6 applied in (X_{pp_1}, Ξ_{pp_1}), we find a pair of members of Ξ_p with index 1 intersecting in precisely p_1.
3. *All members of $\Xi_p \setminus \{\xi^*\}$ through p_1 have index* 0. Let ξ_1 be any such quadric, and let $r \in X(\xi_1) \setminus \{p_1\}$ be a point not collinear to q. Let p_2 be a

point in $X(\xi^*)$ collinear to q but not contained in $[r,q]$ (in particular, r and p_2 are not collinear). By (MM1), there is a $\xi_2 \in \Xi_p$ through r and p_2, which does not contain q by the choice of p_2, and hence has index at most 1. If ξ_2 has index 1, then ξ_1 and ξ_2 satisfy our requirements, so suppose ξ_2 has index 0. Applying Lemma 5.2 (note that $d-2 \geq 4$) on the triple ξ^*, ξ_1, ξ_2 yields a singular line L meeting these three quadrics in three distinct points z, z_1 and z_2, respectively, with z_2 and p_2 non-collinear. This implies that $z \neq q$: otherwise, $p_2 \perp q = z \perp z_2$, and hence $q \in \xi_2$, a contradiction. Hence we can take a line L' through z in $X(\xi^*)$ collinear to neither q, nor z_1. The unique member ξ_2' of Ξ_p through L and L' then does not contain q; hence it has index at most 1. As it contains the singular line L, it has precisely index 1. The pair ξ_1, ξ_2' qualifies.

This shows the claim. Let ξ_1, ξ_2 be such members of Ξ_p, intersecting in a unique point p'. By Lemma 5.4, we may assume that there is a singular line $\langle z_1, z_2 \rangle$ with $z_1 \in X(\xi_1) \setminus p'^{\perp}$ and $z_2 \in X(\xi_2) \setminus p'^{\perp}$.

As ξ_2 has index 1, there exists a line L through z_2 in ξ_2 that is not collinear to z_1. Then L and z_1 determine a unique member ξ of Ξ_p. According to Lemma 5.5, there would be members of Ξ_p of index at least 2 not going through q, a contradiction. The lemma follows. □

Lemma 9.3. *Suppose $x \in X$ has $\max(W_x) \geq 3$. Then either $T_x \cap X \subseteq x^{\perp}$ or $(X_x, \mathscr{L}_x)$ is isomorphic to one of the following: $\mathscr{G}_{n,1}(\mathbb{K})$ for $n \in \{4,5\}$ or $\mathscr{E}_{6,1}(\mathbb{K})$.*

Proof. Suppose that $T_x \cap X$ contains a point y not contained in $x^{\perp}$. Then, by Lemma 4.15, $\min(W_x) \geq 2$, in which case $(X_x, \mathscr{L}_x)$ is a strong parapolar space of diameter 2 (cf. Corollary 4.12 and Lemma 4.5). If there are two members $\xi_1, \xi_2 \in \Xi$ intersecting in x only, then $T_x = \langle T_x(\xi_1), T_x(\xi_2) \rangle$ by (MM3) and Lemma 4.14 leads to a contradiction. So $(X_x, \mathscr{L}_x)$ is (-1)-lacunary. Since $\max(W_x) \geq 3$, the only possibilities are, according to Fact 4.7, $\mathscr{G}_{n,1}(\mathbb{K})$ for $n \in \{4,5\}$ or $\mathscr{E}_{6,1}(\mathbb{K})$. □

Lemma 9.4. *Let x be a point of X with $\max(W_x) \geq 3$. Then (X_x, Ξ_x) is an AVV of type $d-2$ with global index set W_x' and $\dim\langle X_x \rangle \leq 2d-1$.*

Proof. By Lemma 9.3, we may assume that $T_x \cap X \subseteq x^{\perp}$, as otherwise $(X_x, \mathscr{L}_x)$ is isomorphic to $\mathscr{G}_{n,1}(\mathbb{K})$ for $n \in \{4,5\}$ or to $\mathscr{E}_{6,1}(\mathbb{K})$, which are AVVs indeed.

Claim 1: for each $z \in X_x$, $\dim T_z \leq 2d-4$.
We consider the situation in (X, Ξ), in which z corresponds to a singular line L containing x. Let p be a point of $L \setminus \{x\}$. Since $\max(W_x) \geq 3$, Lemma 9.2 yields a member $\xi \in \Xi$ of index at least 3 containing p and not containing x. The fact that $T_x \cap X \subseteq x^{\perp}$ implies that $T_x \cap X(\xi)$ is a singular subspace S of $X(\xi)$. Then

$S \subseteq T_x \cap \xi \subseteq T_p(\xi)$ and $\dim(T_x \cap \xi) \leq d - w_\xi$ by Lemma 4.13. Consequently there is a subspace S' in $T_p(\xi) \setminus T_x$ of dimension $w_\xi - 1 \geq 2$. Since S' is not contained in T_x, we obtain that $\dim(T_p \cap T_x) \leq 2d - 3$. The claim follows.

Claim 2: $(X_x, \mathscr{L}_x)$ is connected. Let ξ be a member of Ξ through x with $w_\xi \geq 3$. Suppose for a contradiction that there is a point $z \in X_x$ not contained in the connected component $\mathscr{C}_\xi$ of ξ in $(X_x, \mathscr{L}_x)$. Then $z^\perp \cap \xi = \emptyset$ and $[z, p] \in \Xi_x$ has index 0 for each $p \in X_x(\xi)$. In exactly the same manner as in the proof of Lemma 8.5, we obtain $T_z \cap X_x(\xi) = \emptyset$ and, by Lemma 6.1, this implies that $z \in \mathscr{C}_\xi$ after all. This contradiction shows the claim.

By Lemma 4.10 and Claim 1, (X_x, Ξ_x) is a weak AVV of type $d-2$ with global index set W'_x and $\dim\langle X_x\rangle \leq 2d-1$. We show that (X_x, Ξ_x) satisfies (MM1). Let $\xi \in \Xi_x$ be of index at least 2. Let L be a line of $\mathscr{L}_x$ intersecting ξ in a unique point p. Let M be a singular line in $\xi \setminus L^\perp$, then we obtain that $[L, M]$ is a member of Ξ_x containing L. Claim 2 now implies that (MM1) holds in (X_x, Ξ_x). Proposition 6.2 implies that $|\Xi_x| \geq 2$. The lemma follows. □

We are ready to show that W has to be a singleton.

Proof of Proposition 9.1. We show this by induction on w^*. If $w^* = 0$, then clearly, $W = \{0\}$ and the main result of [16] leads us to possibility (i) of Proposition 9.1. So assume $w^* \geq 0$ and take an arbitrary $x \in X$ with $w^* = \max W_x$. By Lemma 9.4, the residue (X_x, Ξ_x) is an AVV of type $d-2$ with global index set W'_x and with $\dim\langle X_x\rangle \leq 2d-1$.

First, suppose that $w^* = 3$. Then (X_x, Ξ_x) contains a point z with $\max(W'_z) = 2$ and hence, by Proposition 8.1, (X_x, Ξ_x) is isomorphic to $\mathscr{G}_{n,1}(\mathbb{K})$ for $n \in \{4, 5\}$; in particular, $d = 6$. Since $\dim\langle X_x\rangle \leq 2d-1 = 11$ and $\dim\langle \mathscr{G}_{5,1}(\mathbb{K})\rangle = 14$, we deduce that (X_x, Ξ_x) is isomorphic to $\mathscr{G}_{4,1}(\mathbb{K})$ (which lives in dimension 9). Since the latter's diameter is 2, all members of Ξ_x have index 2 and consequently, $W_x \subseteq \{0, 3\}$. Suppose for a contradiction that there exists $\xi \in \Xi$ with $x \in \xi$ and $w_\xi = 0$. Then $T_x(\xi)$ is a 6-space in T_x which has at least a 3-space Π in common with $\langle X_x\rangle \subseteq T_x \setminus \{x\}$. Proposition 2.1$(ii)$ implies that there are points $x_1, x_2 \in X_x$ such that $\langle x_1, x_2\rangle$ intersects Π non-trivially, and hence $[x_1, x_2] \cap T_x(\xi)$ is non-empty, contradicting (MM2). We conclude that $W_x = \{3\}$. Now let $y \in X \setminus \{x\}$ be arbitrary. By (MM1), there is a member of Ξ containing x and y, which necessarily has index 3 as $W_x = \{3\}$. Therefore, $\max(W_y) = 3$ and we may apply the above arguments to y as well to obtain $W_y = \{3\}$. We conclude that $W = \{w^*\} = \{3\}$ indeed. It follows from the main result of [21] that (X, Ξ) is isomorphic to $\mathscr{D}_{5,5}(\mathbb{K})$.

Next, suppose that $w^* \geq 4$. By induction, all members of Ξ_x have index $w^* - 1 \geq 3$, i.e., $W_x \subseteq \{0, w^*\}$. In particular, the main result of [21] implies that (X_x, Ξ_x) is isomorphic to either $\mathscr{D}_{5,5}(\mathbb{K})$ or $\mathscr{E}_{6,1}(\mathbb{K})$; in particular $d = 8$ and $w^* = 4$. Since

$\dim\langle X_x\rangle \leq 2d-1=1$ and $\dim\langle \mathscr{E}_{6,1}(\mathbb{K})\rangle = 26$, we conclude that (X_x,Ξ_x) is isomorphic to $\mathscr{D}_{5,5}(\mathbb{K})$ (and $\dim(\mathscr{D}_{5,5}(\mathbb{K}))=15$).

We show that $0\notin W_x$. Indeed, suppose that there is a member $\xi\in\Xi$ with $x\in\xi$ and $w_\xi=0$. Then $T_x(\xi)$ is an 8-space in T_x sharing a 7-space Π with the 15-space $\langle X_x\rangle\subseteq T_x$. As above, Proposition 2.1(iii) leads to a contradiction. We conclude that $W_x=\{w^*\}$. Just like in the previous case, we deduce that $W_y=\{w^*\}$ for any $y\in X\setminus\{x\}$ as well, and hence $W=\{w^*\}=\{4\}$. The main result of [21] now yields that (X,Ξ) is isomorphic to $\mathscr{E}_{6,1}(\mathbb{K})$. $\square$

10 Conclusion

Proof of the Main Theorem. Let (X,Ξ) be an AVV of type d with index set W.

(1) If for some $x\in X$, $\max(W_x)=1$, then Proposition 7.1 implies that (X,Ξ) is split, and we have Case $d=2$ of Theorem 1.2.

(2) If for some $X\in X$, $\max(W_x)=2$, then Proposition 8.1 implies that (X,Ξ) is split, and we have Case $d=4$ of Theorem 1.2.

$(0,\geq 3)$ If for all $x\in X$, $\max(W_x)\geq 3$ or $W_x=\{0\}$, then by Proposition 9.1

- (0) either $W=\{0\}$ (and we have a Veronese cap), and we have the case $d=2^\ell$ (including $\ell=0$ giving rise to the case $d=1$) of Theorem 1.2,
- (3) or $W=\{3\}$, (X,Ξ) is split and we have the case $d=6$ of Theorem 1.2,
- (4) or $W=\{4\}$, (X,Ξ) is split and we have the case $d=8$ of Theorem 1.2.

This covers all cases and proves Theorem 1.2, in particular the Main Theorem.

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