Suzuki-Tits ovoids through the years

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Abstract. We review the impact of Jacques Tits' paper, "Ovoïdes et groupes de Suzuki," *Arch. Math.* **13** (1962), 187–198. The paper turned out to be a milestone for both geometry (incidence geometry, finite geometry, Galois geometry) and algebra (group theory).

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1. Introduction

One of the main achievements of Jacques Tits' work is a successful geometric interpretation of the so-called *groups of Lie type*, among which semisimple algebraic groups, classical and mixed type groups, Chevalley groups and their twisted variants. If the isotropic rank of these groups is at least 2, then they act in a very explicit and canonical way on a geometric structure called *Moufang spherical Tits building of rank at least* 2. These structures and the action of their corresponding automorphism groups are—globally—well understood thanks to the groundbreaking seminal work of Tits [34] and Tits & Weiss [35]. Special mentioning deserves the case where the ground field has a valuation, since in this case a second and for certain purposes more efficient geometrical structure called a *Bruhat-Tits building* or a non-discrete variant exists by the work of Bruhat & Tits [5]. However, if the isotropic rank of the group in question is equal to 1, then the canonical definition of the corresponding geometric structure only gives a set without obvious additional geometric structure besides a rather precise and restricted kind of action of the group—these sets are called *Moufang sets*. In many cases, for instance in the case where the group is a non-split simple algebraic group of relative rank 1, it suffices to extend the ground field to see this set embedded—in some specific way—in a more elaborate geometry. For example, quadrics of Witt index 1 turn up here. This trick does not work so well for the twisted Chevalley groups of types ${}^2\mathsf{B}_2$ (the *Suzuki groups* discovered by Suzuki [26]) and ${}^2\mathsf{G}_2$

(the *(small) Ree groups* discovered by Ree [21]). And that is precisely the reason why these groups were first discovered as permutation groups seemingly unrelated to the nice geometries that were already studied by Tits at the time of their discovery, and which would later be unified by the theory of buildings. It is precisely in Tits' paper [32] titled *Groupes et ovoïdes de Suzuki*, which appeared in *Archiv der Mathematik*, volume **13** (1962), pages 187–198, that the connection is made with the standard geometries for Chevalley groups. Slightly before that, Tits [30] already generalized the constructions of Suzuki and Ree to non-perfect fields and exhibited, with modern terminology, a split BN-pair inside it, providing for instance concrete descriptions of the unipotent subgroups showing, again with modern terminology, that these groups defined Moufang sets. Already there, the geometries were present in an implicit way; of course this was much more explicit in Tits' mind, and revealed to the mathematical community precisely in [32]. Here is perhaps also a good place to note that the observation of Tits that the Suzuki and Ree groups can be defined over imperfect fields led him to discover the so-called *groups of mixed type* introduced in [34], Section 10.3.2.

A few years earlier, the Italian geometer Beniamino Segre developed his ideas about fundamental questions in finite geometry. Segre thereby laid the foundations of Galois geometry, that is, the study of interesting objects in projective spaces defined over finite fields—called Galois spaces and Galois fields, respectively. Central in his approach were point sets of projective space with precise intersection behaviour with respect to subspaces, in particular with lines. This led for instance to the following two fundamental theorems, using modern standard notation.

Theorem 1.1 (Segre [22]). *A set of $q+1$ points in the Desarguesian projective plane $\mathsf{PG}(2,q)$ over the Galois field $\mathsf{GF}(q) = \mathbb{F}_q$ of q elements, with q odd, meeting every line in at most two points, is a conic.*

Theorem 1.2 (Barlotti [2], Panella [16]). *A set of q^2+1 points in the 3-dimensional Desarguesian projective space $\mathsf{PG}(3,q)$ over the Galois field $\mathsf{GF}(q) = F_q$ of q elements, with q odd, meeting every line in at most two points, is a quadric.*

A crucial assumption in the previous theorems is the oddness of q. Indeed, in the even case many counter examples to Theorem 1.1 were known, and Segre constructed a counter example to Theorem 1.2 in $\mathsf{PG}(3,8)$. That counter example was precisely the smallest nontrivial protagonist of Tits' paper [32]. By Segre, a *k-cap* $\mathscr{K}$ of $\mathsf{PG}(3,q)$ is a set of k points no three of which are collinear. If k is maximal then $\mathscr{K}$ is called an *ovaloid* [23]. For $q>2$, the ovaloids of $\mathsf{PG}(3,q)$ are exactly the ovoids, whose definition is due to Tits [31] (and also given below). For $q=2$, an ovaloid has eight points, while an ovoid has five points. The 5-caps of $\mathsf{PG}(3,2)$ which are not contained in a larger cap are exactly the ovoids.

Whereas Segre approached ovoids from a combinatorial point of view, Tits approached them from a group theoretic point of view. The latter allows to produce a definition where the case $q = 2$ above is included without extra condition, and which also makes sense in the infinite case over any field (even skew field). Let us explicitly state this definition; it was first formulated by Tits, though not in [32], but in [31].

Definition 1.3 (Tits [31]**).** Let O be a set of points of some projective space Σ. Then O is called an *ovoid* if every line of Σ intersects O in at most two points, and if, for each point $x \in O$, the union of the set of lines of Σ intersecting O in precisely $\{x\}$, is a hyperplane of Σ.

The crucial observation made by Tits in [32] is that the groups found by Suzuki in [26], generalized by Tits in [30], act 2-transitively on ovoids of a suitable 3-dimensional projective space (hence providing a geometric description of Suzuki's set of $q^2 + 1$ points upon which the Suzuki group $\mathsf{Suz}(q)$ acts as a permutaton group). Moreover, again using modern terminology, that ovoid can be constructed as the set of absolute points of a polarity of a symplectic quadrangle (in the perfect case), or what is called a *Suzuki quadrangle* in the imperfect case. Hence, Tits not only found an infinite class of new ovoids of $\mathsf{PG}(3, \mathbb{K})$, where $\mathbb{K}$ is a field in characteristic 2 whose Frobenius $x \mapsto x^2$ admits a square root, but he also showed that this ovoid is contained in a symplectic generalized quadrangle $\mathsf{W}(\mathbb{K})$ in such a way that each line of $\mathsf{W}(\mathbb{K})$ has exactly one point in common with the ovoid. Such sets of generalized quadrangles are nowadays also called *ovoids (of generalized quadrangles)*. The origin and the connection between these two seemingly totally different meanings of the notion of an "ovoid" is contained, discovered and explained in [32].

In contemporary mathematics we know ovoids in all kinds of (mostly building-like) structures, and their definition is either based on that of an ovoid in projective space, or on that of an ovoid of a generalized quadrangle, or a mixture. We mention ovoids of polar spaces, of generalized polygons and of the natural geometry related to the 27-dimensional module of an exceptional group of type E_6.

We now describe the impact of Tits' paper [32] on modern mathematics. First on finite geometry, where the impact is immeasurable. Then we review some other directions.

2. Impact on finite geometry

In this section ovoids, in particular Suziki-Tits ovoids, in finite projective spaces $\mathsf{PG}(3, q)$ are considered.

2.1. Comparing definitions and small cases

By Segre [23] a set $\mathscr{K}$ of k points of $\mathsf{PG}(n, q)$ no three of which are collinear is called a *k-arc* for $n = 2$ and a *k-cap* for $n \geq 3$. For k maximal Segre calls $\mathscr{K}$ an *oval* for $n = 2$ and an *ovaloid* for $n \geq 3$. For q even such an oval has $q + 2$ points and for q odd an oval has $q + 1$ points by [4], see also [20] and Theorem 8.5 of [10]; for $q \neq 2$ an ovaloid has $q^2 + 1$ points, and for $q = 2$ it has 8 points by [4] for q odd and by [20] for q even, see also Theorem 16.1.5 of [9]. In modern terminology the $(q+1)$-arcs of $\mathsf{PG}(2,q)$ are called *ovals*, and the $(q+2)$-arcs of $\mathsf{PG}(2,q)$ (then q is even) are called *hyperovals*.

The ovals of $\mathsf{PG}(2,q)$ and, for $q \neq 2$, the ovaloids of $\mathsf{PG}(3,q)$ are exactly the ovoids of Tits [31, 32] in these dimensions; in $\mathsf{PG}(3,2)$ the ovoids are the 5-caps which are not contained in a larger cap.

The definitions of Segre are very useful for finite fields, but for general fields we need the ovoids in the sense of Tits.

For q even, the only known ovoids of $\mathsf{PG}(3,q)$ are the elliptic quadrics and the Suzuki-Tits ovoids. Let us now look at small values of q.

(a) For $q = 4$, an ovoid of $\mathsf{PG}(3,4)$ is an elliptic quadric, see [2].
(b) For $q = 8$, an ovoid of $\mathsf{PG}(3,8)$ is either an elliptic quadric or a Suzuki-Tits ovoid; see [7, 24].
(c) For $q = 16$, all ovoids of $\mathsf{PG}(3,16)$ are elliptic quadrics; see [13, 14].
(d) For $q = 32$, an ovoid of $\mathsf{PG}(3,32)$ is either an elliptic quadric or a Suzuki-Tits ovoid; see [15].
(e) For $q = 64$, all ovoids of $\mathsf{PG}(3,64)$ are elliptic quadrics, see [19].

Hence up to (even) $q = 64$ each ovoid is either an elliptic quadric or a Suzuki-Tits ovoid. We recall from the introduction that the first ovoid of $\mathsf{PG}(3,q)$ which was not an elliptic quadric was discovered in 1959 by Segre [24] for $q = 8$. It was shown by Fellegara [7] that this ovoid of $\mathsf{PG}(3,8)$ is nothing else than the smallest Suzuki-Tits ovoid.

Finally, let us remark that in the finite case ovoids do not exist in dimension $n > 3$, see [28].

2.2. Plane intersections of Suzuki-Tits ovoids and translations

Let O be a Suzuki-Tits ovoid in $\mathsf{PG}(3,q)$. If π is a plane of $\mathsf{PG}(3,q)$, which is not tangent to O, then $\pi \cap O$ is an oval of π. Such an oval is called by Tits a *θ-conic* (here, θ refers to the Tits endomorphism, see Subsection 3.1). He proves that all θ-conics are projectively equivalent [32].

Let C be the θ-conic of π. Then all projectivities of π stabilizing C fix some point x of C (this is ∂C of Subsection 3.2) and also the tangent $L \subseteq \pi$ of C at x. The group G of these projectivities acts 2-transitively on $C \setminus \{x\}$ [32]. Also, the projectivity of π fixing precisely all points of L (a *translation* of π with axis L) and mapping $y \in C \setminus \{x\}$ onto $z \in C \setminus \{x\}$, stabilizes the θ-conic C. The line L defined by the θ-conic C will be called a *θ-absolute* line. Any oval D admitting this property for some point $u \in D$ is nowadays called a *translation oval*.

All translation ovals of $\mathsf{PG}(2,q)$ were determined by Payne [17] in 1971.

Theorem 2.1 (Payne [17]). *In $\mathsf{PG}(2,2^h)$, the oval D is a translation oval if and only if, in a suitable reference system, it consists of the points $(x^\gamma, x, 1)$, with $\gamma : t \mapsto t^{2^n}$ and $(n,h) = 1$, together with the point $(0,1,0)$.*

In the case of the Suzuki-Tits ovoid with Tits endomorphism θ we clearly have $\gamma = \theta$, with $h = 2r+1$ and $n = r+1$.

The following definition is due to Tits [31].

Definition 2.2 (Tits [31]**).** Let O be any oval or ovoid of some projective plane or higher dimensional space Σ, respectively, over any field. Let x be any point of O and let π be the tangent hyperplane of O at x. Then O is called an *oval* or *ovoid with translations* (*ovoïde à translations*) if for any two points $y, z \in O \setminus \{x\}$ the translation with axis π which maps y onto z, stabilizes O.

Suzuki-Tots ovoids are not translation ovoids, though. In the finite case ovals and ovoids with translations admit a 2-transitive group of projectivities. In the 2-dimensional case the oval is always a conic; see [31]. The following important result is also due to Tits.

Theorem 2.3 (Tits [31, 32, 33]**).** *In* $\mathsf{PG}(3, q)$*, q even, every ovoid having a 2-transitive group of projectivities is either an elliptic quadric or a Suzuki-Tits ovoid. The group is 3-transitive if and only if the ovoid is a quadric.*

For characterizations of the many incidence structures where ovoids of $\mathsf{PG}(3, q)$ are involved Theorem 2.3 plays a key role. The next result led to a new class of translation planes.

Theorem 2.4 (Tits [32]**).** *Let O be a Suzuki-Tits ovoid of* $\mathsf{PG}(3, q)$*, $q = 2^{2r+1}$ and $r \geq 1$. Then the $q^2 + 1$ θ-absolute lines form a 1-spread of* $\mathsf{PG}(3, q)$*, that is, they form a partition of* $\mathsf{PG}(3, q)$*.*

The impact of this theorem will be further elaborated in Section 2.4.

2.3. Inversive planes

Definition 2.5. An *inversive plane* or *Möbius plane* is an incidence structure (finite or infinite) $\Gamma = (\mathscr{P}, \mathscr{C})$, with $\mathscr{P}$ the set of *points* and $\mathscr{C}$ the set of blocks called *circles*, satisfying the following axioms.

(*i*) Any three distinct points are contained in exactly one circle.

(*ii*) If $x, y \in \mathscr{P}$ and $C \in \mathscr{C}$ is such that $x \in C$ and $y \notin C$, then there is a unique circle $D \in \mathscr{C}$ with $x, y \in D$ and $C \cap D = \{x\}$.

(*iii*) $|\mathscr{P}| \geq 4$, there exist $x \in \mathscr{P}$ and $C \in \mathscr{C}$ with $x \notin C$, and $|D| > 0$ for every $D \in \mathscr{C}$.

In the finite case it can be shown that Γ is equivalent to a set $\mathscr{P}$ of $n^2 + 1$ points, together with a set $\mathscr{C}$ of subsets of $\mathscr{P}$, each of size $n + 1$, such that any three points are contained in exactly one element of $\mathscr{C}$, for some $n \in \mathbb{N} \setminus \{0, 1\}$. That is, Γ is a $3 - (n^2 + 1, n + 1, 1)$ design. The integer n is called the *order* of the inversive plane Γ.

Let O be an ovoid of $\mathsf{PG}(3, q)$. If we set $\mathscr{P} = O$, and $\mathscr{C}$ the set of intersections $O \cap \pi$ with π any non-tangent plane of O, then $\Gamma = (\mathscr{P}, \mathscr{C})$ is an inversive plane of order n. Such an inversive plane is called *egglike.* If O is an elliptic quadric, then the corresponding inversive plane is called *classical* or *Miquelian.* By Theorem 1.2 each egglike inversive plane of odd order q is classical. But it is not known whether or not each inversive plane of odd order is egglike. On the other hand we have the following important theorem.

Theorem 2.6 (Dembowski [6]**).** *Each inversive plane of even order is egglike.*

Hence finite inversive planes of even order q are equivalent to ovoids of $\mathsf{PG}(3, q)$, and so the Suzuki-Tits ovoids play a key role in the theory of finite inversive planes. There is a large amount of literature on classifications and characterizations of inversive planes; see for instance [6]. In many of these results Suzuki-Tits ovoids appear. There is a famous classification of inversive planes due to Hering [8], see also [6], and the inversive plane arising from the Suzuki-Tits ovoid is of Hering type VI.1. In fact, by Tits [33] the Suzuki-Tits ovoids are the only egglike inversive planes of this type (recall that each egglike inversive plane of odd order is classical and that each inversive plane of even order is egglike).

2.4. Translation planes

Let Γ be a projective plane with point set $\mathscr{P}$ and suppose for some line L of Γ there is a group of collineations of Γ fixing L pointwise and acting sharply transitively on $\mathscr{P} \setminus L$. Then Γ is a *translation plane* with *translation line* L.

Let $\mathscr{S}$ be a 1-spread of a 3-dimensional projective space Σ (finite or infinite), that is, a partition of the point set of Σ into lines. Then it is well-known that $\mathscr{S}$ defines a translation plane Γ as follows. Consider Σ as a hyperplane of a 4-dimensional projective space $\overline{\Sigma}$. The points of Γ are the points of $\overline{\Sigma} \setminus \Sigma$, together with the members of $\mathscr{S}$; the lines of Γ are the planes of $\overline{\Sigma}$ which have precisely a member of $\mathscr{S}$ in common with Σ, together with Σ itself—incidence is natural containment. If $\Sigma = \mathsf{PG}(3, q)$ the translation plane has order q^2. By Theorem 2.4 the $q^2 + 1$ θ-absolute lines of a Suzuki-Tits ovoid O form a 1-spread of $\mathsf{PG}(3, q)$. So to O corresponds a translation plane of order q^2. In [11], Lüneburg also constructs $\mathscr{S}$, but his approach is different. Let us call these translation planes the *Lüneburg-Tits planes.*

Many papers and several books are on translation planes; in particular, see [3, 11, 12]. The book [11] by Lüneburg is entirely on Suzuki-Tits ovoids, Suzuki groups and Lüneburg-Tits planes. In all these books, Lüneburg-Tits planes play an important role.

Influenced by [32] the first author proved in 1972 [27] that each ovoid of $\mathsf{PG}(3, q)$, $q = 2^h$ and $h > 1$, corresponds to a 1-spread of $\mathsf{PG}(3, q)$ which belongs to a linear complex of lines of $\mathsf{PG}(3, q)$, and conversely. The translation plane defined by such a 1-spread was called an *ovoidal translation plane* by the first author. Also, the generalized quadrangle $\mathsf{W}(q)$ arising from a linear complex of lines of $\mathsf{PG}(3, q)$ is self-dual if and only if q is even. We remark that Tits, who refereed the paper [27], provided several suggestions.

2.5. Other impact, in particular on generalized quadrangles

Ovoids are used to construct many other geometric objects. From ovoids, in particular from Suzuki-Tits ovoids, arise strongly regular graphs, linear projective 2-weight codes, cap-codes, maximal arcs, partial geometries, generalized quadrangles, partial quadrangles, unitals, ovoids in polar spaces, ...

Much is written on the so called *generalized quadrangles* $\mathsf{T}_3(O)$ *of order* (q, q^2) *of Tits*, arising from an ovoid O of $\mathsf{PG}(3, q)$, see [18]. This generalized quadrangle is classical, that is, it is isomorphic to a generalized quadrangle arising from an elliptic quadric of $\mathsf{PG}(5,q)$, if and only if O is an elliptic quadric of $\mathsf{PG}(3,q)$. Hence for q odd $\mathsf{T}_3(O)$ is always classical by Theorem 1.2. For q even, $q > 2$, the $\mathsf{T}_3(O)$, with O a Suzuki-Tits ovoid, is the only known non-classical $\mathsf{T}_3(O)$. In fact, this non-classical $\mathsf{T}_3(O)$ is the only known non-classical *translation generalized quadrangle* of order (s, s^2), with s even, see [29].

3. Other impact

In this final section, we review some other impact of [32]. As Tits' œuvres are situated on the cross roads of geometry and group theory, it seems appropriate to mention some examples in incidence geometry and group theory (and their interaction). We start with the description of a special family of quadrangles and ovoids that emerged from [32].

3.1. Moufang quadrangles of mixed type

As alluded to before, one of the crucial basic observations of Jacques Tits in [32] is that the set of absolute points of a polarity in a generalized quadrangle forms an ovoid of that quadrangle. And when the quadrangle is a symplectic quadrangle $\mathsf{W}(\mathbb{K})$ that admits a polarity, then this ovoid—which is the Suzuki-Tits ovoid—is also an ovoid of $\mathsf{PG}(3,\mathbb{K})$. A necessary and sufficient condition for the existence of a polarity in $\mathsf{W}(\mathbb{K})$ is that $\mathbb{K}$ is perfect and the Frobenius admits a square root, that is, a map $x \mapsto x^\theta$ such that $(x^\theta)^\theta = x^2$. The second author called θ in [37] a *Tits endomorphism*, and since then this name is used by many authors. When $\mathbb{K}$ is not perfect, then the Frobenius $x \mapsto x^2$ is not bijective (it is injective, though), and so it does not admit an inverse, which translates into the fact that there is no polarity possible (not even a duality). However, if a Tits endomorphism exists (not bijective if the Frobenius is not bijective!), then one can restrict the line pencils in such a way that a polarity exists in the resulting subquadrangle. The latter quadrangle is called a *Suzuki* quadrangle in [37]. Also, the corresponding ovoid is an ovoid of $\mathsf{PG}(3,\mathbb{K})$, which is slightly surprising since it is *not* an ovoid of the ambient symplectic quadrangle!

In case the underlying field $\mathbb{K}$ is not perfect, and the dimension of $\mathbb{K}$ over its subfield $\mathbb{K}^2$ of all squares is large, then there are all sorts of mutually nonisomorphic subquadrangles of $\mathsf{W}(\mathbb{K})$ which potentially admit polarities, and so the Suzuki-Tits ovoid defined above can have a lot of subovoids!

3.2. Circle geometries: inversive planes, Möbius planes

An ovoid in $\mathsf{PG}(3,q)$ gives rise to an inversive plane, see Definition 2.5. This also holds in the infinite case, with exactly the same set of axioms. As in the finite case, there is no determination of all inversive planes. However, under some suitable conditions one can easily isolate all inversive planes that arise from the Suzuki-Tits ovoids. These conditions are the following.

First we make sure that we deal with the case of characteristic 2 by the following two axioms. Let $\Gamma = (\mathscr{P}, \mathscr{C})$ be an inversive plane.

(CH1) For every circle $C \in \mathscr{C}$ and every pair of points x, y both not in C, either there is a unique circle containing both x and y and touching C, or all circles through x and y touch C.

(CH2) There do not exist three circles two by two touching each other in distinct points.

And here is the additional axiom for the Suzuki-Tits ovoids (over a perfect field).

(ST) The inversive plane Γ is furnished with a map $\partial : \mathscr{C} \to \mathscr{P} : C \mapsto \partial C \in C$ such that:

(1) For every pair of points $x, y \in \mathscr{P}$ there is a unique circle $C \in \mathscr{C}$ containing x and y and such that $\partial C = x$.

(2) For every circle C and every point $x \notin C$, there is at most one circle D containing x and ∂C and such that $\partial D \in C$.

The only inversive planes satisfying (CH1), (CH2) and (ST) are obtained from the Suzuki-Tits ovoids by declaring the circle C through three points the intersection of the ovoid with the unique plane of the ambient projective space through these points. That plane contains the perp of a unique point x of the symplectic quadrangle and ∂C is by definition the intersection of C with the image of x under the polarity.

Struyve and the second author [25] also axiomatize the ovoids in the non-perfect case, and even the subovoids mentioned at the end of Section 3.1.

3.3. Tits webs from generalized Suzuki-Tits ovoids

A programme initiated by Jacques Tits himself in the second half of the nineties consists of characterizing the rank one Chevalley groups and simple algebraic groups of relative rank one as the automorphism groups of a geometry attached to them, in a precise way defined by Tits (using the orbits of the centres of the unipotent subgroups). These geometries are called *Tits webs*. Also the Suzuki groups and their generalizations to imperfect fields by Tits admit Tits webs. These Tits webs just happen to be the Suzuki-Tits ovoids and their subovoids as defined at the end of Section 3.1, furnished with the circles (but not with the map ∂). It was shown in [38] that the automorphism groups of these Tits webs are exactly the Suzuki groups and their generalizations by Tits. Of course, Tits' paper [32] was instrumental for such result, as it introduced the core geometry in this case!

3.4. Intersections of Suzuki-Tits ovoids and maximal subgroups of $\mathsf{Suz}(q)$

The ovoids studied by Tits in [32] were also used to complete a geometric description of all maximal subgroups of the finite Suzuki groups. Indeed, in [1], Bagchi & Sastry show that, within the same generalized quadrangle $\mathsf{W}(q)$, the size of the intersection of a Suzuki-Tits ovoid with a classical ovoid, is either $q+\sqrt{2q}+1$, or $q-\sqrt{2q}+1$. Moreover, there is always a cyclic group of the same respective order stabilizing the Suzuki-Tits ovoid and acting sharply transitively on that intersection. The full stabilizer of such an intersection of size s is a semi-direct product of the cyclic group of order s with a cyclic group of order 4, and is a maximal subgroup of $\mathsf{Suz}(q)$. All maximal subgroups of $\mathsf{Suz}(q)$ can hence be described as follows.

Theorem 3.1. *A subgroup of* $\mathsf{Suz}(q)$*, with associated Suzuki-Tits ovoid* O*, is a maximal subgroup if and only if one of the following occurs.*

(*i*) *It is the full stabilizer of a point of* O*;*
(*ii*) *it is the full stabilizer of a circle of the inversive plane defined by* O*;*
(*iii*) *it is the full stabilizer of a maximal subovoid of* O*;*
(*iv*) *it is the full stabilizer of an intersection of* O *with a classical ovoid.*

The previous theorem was noted at the end of Section 7.6 of [37] and follows from the description of the maximal subgroups of $\mathsf{Suz}(q)$ as given in [26], together with Tits' definition of the Suzuki-Tits ovoid in [32].

3.5. Wilson's construction of $\mathsf{Suz}(q)$

In [39], Robert Wilson gives a new and quite elementary explicit construction of the Suzuki groups $\mathsf{Suz}(q)$. At the same time he constructs the Suzuki-Tits ovoid. Let us do the latter here.

Let a symplectic (alternating bilinear) form on a four-dimensional vector space V over the perfect field $\mathbb{K}$, with $\mathrm{char}\,\mathbb{K} = 2$, be given by $x_{-1}y_1 + x_1y_{-1} + x_{-2}y_2 + x_2y_{-2}$, and let $\{e_{-2}, e_{-1}, e_1, e_2\}$ be the standard basis (with self-explaining notation). Suppose $\mathbb{K}$ admits an automorphism θ with the property that $((x^\theta)^\theta)^2 = x$, for all $x \in \mathbb{K}$ (then θ is a square root of the inverse of the Frobenius). Define a symmetric product $\bullet$ on the points of V as follows:

$$\begin{cases} e_i \bullet e_{\pm i} = 0, & i \in \{-2,-1,1,2\}, \\ e_{-2} \bullet e_{-1} = e_{-2}, & \\ e_{-2} \bullet e_1 = e_{-1}, & \\ e_{-1} \bullet e_2 = e_1, & \\ e_1 \bullet e_2 = e_2, & \\ v \bullet \lambda w = \lambda^\theta (v \bullet w), & \text{for all } v, w \in V, \lambda \in \mathbb{K}, \\ (u+v) \bullet w = (u \bullet w) + (v \bullet w), & \text{for all } u, v, w \in V. \end{cases}$$

Then the Suzuki-Tits ovoid consists of the projective points $\langle v \rangle$ such that there exists a vector w, with v and w not orthogonal with respect to the above symplectic form, with $v \bullet w = v$.

Wilson's paper appeared in 2009, well 47 years after Tits' paper appeared. Still, after all this time, a new construction was found of one of Tits' many creations—perhaps among the most beautiful and simple ones—the Suzuki-Tits ovoids, originating and studied in [32]. A paper with a lot of impact on mathematics as the present paper witnesses!

References

[1] B. Bagchi & N. S. N. Sastry, Intersection pattern of the classical ovoids in symplectic 3-space of even order, *J. Algebra* **126** (1989), 147–160.

[2] A. Barlotti, Un'estensione del teorema di Segre-Kustaanheimo, *Boll. Un. Mat. Ital.* (3) **10** (1955), 498–506.

[3] M.Billiotti, V. Jha & N. L. Johnson, *Foundations of Translation Planes*, Dekker, New York, 2001.

[4] R. C. Bose, Mathematical theory of the symmetrical factorial design, *Sankhyã* **8** (1947), 107–166.

[5] F. Bruhat & J. Tits, Groupes réductifs sur un corps local, I. Données radicielles valuées, *Inst. Hautes Études Sci. Publ. Math.* **41** (1972), 5–252.

[6] P. Dembowski, *Finite Geometries*, Springer, New York, 1968.

[7] G. Fellegara, Gli ovaloidi di uno spazio tridimensionale di Galois di ordini 8, *Atti Accad. Naz. Lincei Rend.* **32** (1962), 170–176.

[8] C. Hering, Eine Klassifikation der Möbius-Ebenen, *Math. Z.* **87** (1965), 252–262.

[9] J. W. P. Hirschfeld, *Finite Projective Spaces of Three Dimensions*, Oxford University Press, Oxford, 1985.

[10] J. W. P. Hirschfeld, *Projective Geometries over Finite Fields*, Second Edition, Oxford University Press, Oxford,1998.

[11] H. Lüneburg, Die Suzukigruppen und ihre Geometrien, *Lect. Notes Math.* **10**, Springer, Berlin, 1965.

[12] H. Lüneburg, *Translation Planes*, Springer, Berlin, 1980.

[13] C. M. O'Keefe & T. Penttila, Ovoids of $\mathsf{PG}(3,16)$ are elliptic quadrics, *J. Geom.* **38** (1990), 95–106.

[14] C. M. O'Keefe & T. Penttila, Ovoids of $\mathsf{PG}(3,16)$ are elliptic quadrics, II,*J. Geom.* **44** (1992), 140–159.

[15] C. M. O'Keefe, T. Penttila & G. F. Royle, Classification of ovoids in $\mathsf{PG}(3,32)$, *J. Geom.* **50** (1994), 143-150.

[16] G. Panella, Caratterizzazione delle quadriche di uno spazio (tridimensionale) lineare sopra un corpo finito, *Boll. Un. Mat. Ital.* **10** (1955), 507–513.

[17] S. E. Payne, A complete determination of translation ovoids in finite Desarguesian planes, *Atti Accad. Naz. Lincei Rend.* **51** (1971), 328–331.

[18] S. E. Payne & J. A. Thas, *Finite Generalized Quadrangles*, Second ed., *Ser. Lect. Math.*, Europ. Math. Soc., Zürich, 2009.

[19] T. Penttila, Uniqueness of the inversive plane of order sixty-four, *Des. Codes Cryptogr.* **90** (2022), 827–834.

[20] B. Qvist, Some remarks concerning curves of the second degree in a finite plane, *Ann. Acad. Sci. Fenn. Ser. A* **134** (1952), 1–27.

[21] R. Ree, A family of simple groups associated with the Lie algebra of type (G_2), *Amer. J. Math.* **83** (1961), 797–462.

[22] B. Segre, Ovals in a finite projective plane, *Canad. J. Math.* **7** (1955), 414–416.

[23] B. Segre, Le geometrie di Galois, *Ann. Mat. Pura Appl.* **48** (1959), 1–96.

[24] B. Segre, On complete caps and ovaloids in three-dimensional Galois spaces of characteristic two, *Acta Arith.* **5** (1959), 315–332.

[25] K. Struyve & H. Van Maldeghem, Moufang quadrangles of mixed type, *Glasgow Math. J.* **50** (2008), 143–161.

[26] H. Suzuki, A new type of simple groups of finite order, *Proc. Mat. Acad. Sci. U. S. A.* **46** (1960), 868–870.

[27] J. A. Thas, Ovoidal translation planes, *Arch. Math.* **23** (1972), 110–112.

[28] J. A. Thas, On semi ovals and semi ovoids, *Geom. Dedicata* **3** (1974), 229–231.

[29] J. A. Thas, K. Thas & H. Van Maldeghem, *Translation Generalized Quadrangles*, World Scientific Publishing, Hackensack, 2006.

[30] J. Tits, Les groupes simples de Suzuki et de Ree, *Séminaire Bourbaki* **13** (1960/61), no. 210, 1–18.

[31] J. Tits, Ovoïdes à translations, *Rend. Mat. Appl.* (5) **21** (1962) 37–59.

[32] J. Tits, Ovoïdes et groupes de Suzuki, *Arch. Math.* **13** (1962), 187–198.

[33] J. Tits, Une propriété caractéristique des ovoïdes associés aux groupes de Suzuki, *Arch. Math.* **17** (1966), 136–153.

[34] J. Tits, *Buildings of Spherical Type and Finite BN-Pairs*, Springer Lecture Notes Series **386**, Springer-Verlag, 1974.

[35] J. Tits & R. Weiss, *Moufang Polygons*, Springer Monographs in Mathematics, Springer, 2002.

[36] H. Van Maldeghem, A geometric characterization of the perfect Suzuki-Tits ovoids, *J. Geom.* **58** (1997), 192–202.

[37] H. Van Maldeghem, *Generalized Polygons*, Modern Birkhäuser Classics, Birkhäuser, 1998.

[38] H. Van Maldeghem, Moufang lines defined by (generalized) Suzuki groups, *European J. Combin.* **28** (2007), 1878–1889.

[39] R. A. Wilson, A new approach to the Suzuki groups, *Math. Proc. Camb. Phil. Soc.* **148** (2009), 425–428.

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