3-UNIFORM HYPERGRAPHS FROM VECTOR SPACES

JEROEN MEULEWAETER AND HENDRIK VAN MALDEGHEM

Abstract. The Fundamental Theorem of Projective Geometry states that, in a vector space, a permutation of vector lines preserving triples that span a vector plane is induced by a semi-linear automorphism. We consider a generalisation to triples of subspaces, not necessarily of the same dimension, spanning, or being contained in a subspace of fixed dimension. We determine all cases in which the permutation is necessarily induced by a semi-linear automorphism.

1. Introduction

Let $\mathbb{L}$ be any skew field and let V be a finite-dimensional vector space over $\mathbb{L}$, say of dimension $n+1$. Let $\mathsf{PG}(n,\mathbb{L})$ be the corresponding n-dimensional projective space over $\mathbb{L}$. Let $\mathcal{P}$ be the set of 1-spaces of V; hence $\mathcal{P}$ is the set of points of $\mathsf{PG}(n,\mathbb{L})$. Let $\mathcal{T}$ be the set of triples $\{a,b,c\}$ of $\mathcal{P}$ such that a,b,c are contained in a 2-space of V, or, equivalently, a,b,c span a line of $\mathsf{PG}(n,\mathbb{K})$. Then $(\mathcal{P},\mathcal{T})$ is a so-called 3-*uniform hypergraph*, i.e., a set with a given collection of 3-subsets, called the *triangles. The Fundamental Theorem of Projective Geometry* (see e.g. [1]) states that every automorphism of $(\mathcal{P},\mathcal{T})$ is induced by a semi-linear automorphism of V. We shall call 3-uniform hypergraphs simply 3-graphs for brevity. We shall call an edge of a 3-graph a triangle.

In the present paper, we consider the following problem, which gives rise to a direct generalisation of the aforementioned fundamental theorem. Let i,j,k,ℓ be natural numbers, $0 \le i \le j \le k \le n-1$, $k \le \ell \le n$. Let V_i, V_j, V_k be copies of the set of i-spaces, j-spaces and k-spaces, respectively, of $\mathsf{PG}(n,\mathbb{L})$. Put $\mathcal{V} = V_i \sqcup V_j \sqcup V_k$ and let $\mathcal{T}_{i,j,k;\ell}$ or $\mathcal{T}_{i,j,k;\le\ell}$ consist of the triples $\{I,J,K\}$, with $I \in V_i$, $J \in V_j$, $K \in V_k$, such that I,J,K span an ℓ-space, or are contained in an ℓ-space, respectively. Then the question is whether the 3-graphs $\Gamma_{i,j,k;\ell} = (\mathcal{V},\mathcal{T}_{i,j,k;\ell})$ and $\Gamma_{i,j,k;\le\ell} = (\mathcal{V},\mathcal{T}_{i,j,k;\le\ell})$ uniquely determine the structure of V, or, in other words, determine the projective space $\mathsf{PG}(n,\mathbb{L})$. If this is the case, then the automorphism group of the 3-graph coincides with that of the projective space. So we determine the automorphism group of such a 3-graph. Note that every such 3-graph is *tripartite*, i.e., every triangle has exactly one vertex in each of the sets V_i, V_j, V_k, whose union is the complete vertex set.

If two or more of the parameters i,j,k are equal, say $i=j$, then we assume that V_i and V_j are different copies of the same set without introducing new notation. Formally, we conceive V_j as a set V_i', but we shall not be so formal: the use of the subscripts i and j suffices to distinguish them in our statements and arguments, and this will rather be convenient than confusing. Also, in this case, the vertices of

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V_i and V_j corresponding to the same subspace of $\mathsf{PG}(n,\mathbb{L})$ of projective dimension $i=j$ will be called *copies.*

In case two or more of i,j,k coincide, we can also consider a smaller graph. The following possibilities occur. If $i=j=k$, then we consider the 3-graphs $\Gamma_{i;\ell}=(V_i,\mathcal{T}_{i;\ell})$ and $\Gamma_{i;\leq\ell}=(V_i,\mathcal{T}_{i;\leq\ell})$, where a triple of (distinct) i-spaces forms a triangle if they generate an ℓ-space or are contained in an ℓ-space, respectively. If $i=j<k$, then we consider the 3-graphs $\Gamma_{i,\{i,k\};\ell}=(V_i\sqcup V_k,\mathcal{T}_{i,\{i,k\};\ell})$ and $\Gamma_{i,\{i,k\};\leq\ell}=(V_i\sqcup V_k,\mathcal{T}_{i,\{i,k\};\leq\ell})$, where a triple $\{I_1,I_2,K\}$ consisting of two (distinct) i-spaces and a k-space forms a triangle if I_1,I_2,K generate an ℓ-space, or are contained in an ℓ-space, respectively. Finally, if $i<j=k$, then we define the 3-graphs $\Gamma_{k,\{i,k\};\ell}$ and $\Gamma_{k,\{i,k\};\leq\ell}$ similarly. The cases $i=j<k$ and $i<j=k$ are called *bipartite* since the vertex set can be split into two sets in such a way that one vertex of every triangle is contained in the first of those sets, while the other two vertices are always contained in the second one.

Our proofs reduce the 3-graphs to ordinary graphs. Then we have to our disposal the results of [3], where it is proved that the automorphism group of the following graphs, if not trivial or not a matching, are induced by the semi-linear automorphisms or possibly dualities of the given vector space: the bipartite graph with vertices the i-dimensional subspaces and j-dimensional subspaces (two distinct copies if $i=j$) of a vector space of dimension at least $\max\{i,j\}+1$, an i-space being adjacent to a j-space if they intersect in a subspace of given dimension k, $0\leq k\leq\min\{i,j\}$, or if they intersect in a subspace of dimension at least k, respectively. If $i=j$, then we may also consider the non-partite graphs of subspaces of dimension i, adjacent when intersecting in a k-subspace, or when their intersection contains a k-subspace, respectively. It is the latter graphs that were earlier treated by Lim [5]. Chow [2] already proved the non-bipartite case with $k=i-1$ in 1949. For the precise and detailed statements we refer to Theorems 3.1 and 3.2 below.

2. Statements of the results

Obviously, every semi-linear automorphism of the vector space V (or, equivalently, every collineation of the projective space $\mathsf{PG}(n,\mathbb{L})$), induces a unique automorphism of any of the graphs $\mathcal{T}_{i,j,k;\ell}$, $\mathcal{T}_{i,j,k;\leq\ell}$, $\Gamma_{i,\{i,k\};\ell}$, $\Gamma_{i,\{i,k\};\leq\ell}$, $\Gamma_{k,\{i,k\};\ell}$, $\Gamma_{k,\{i,k\};\leq\ell}$, $\Gamma_{i;\ell}$ and $\Gamma_{i;\leq\ell}$ preserving the different classes V_i,V_j,V_k (in the cases where these exist). We briefly talk about *naturally induced* automorphisms in these cases. In this paper, we will prove the following results.

Main Result 2.1.

- *Let $0\leq i\leq j\leq k\leq n-1$, $k\leq\ell\leq n$ with $n\geq 2$. Then every automorphism of the tripartite 3-graph $\Gamma_{i,j,k;\leq\ell}$ is naturally induced , except only in the following cases.*
  - *$\ell=n$; in this case $\Gamma_{i,j,k;\leq\ell}$ is a complete tripartite 3-graph.*
  - *$i+j+k+2\leq\ell$; in this case $\Gamma_{i,j,k;\leq\ell}$ is also a complete tripartite 3-graph.*
  - *$i=j=k=\ell$; in this case $\Gamma_{i,j,k;\leq\ell}$ is a 3-matching, i.e., every vertex is contained in exactly one triangle.*
  - *$i=j<k$, $\ell<n$, $2i+k+2>\ell$ and the automorphism interchanges the classes V_i and V_j; composing it with the automorphism fixing every member of V_k and interchanging each member of V_i with its copy contained in V_j, we obtain a naturally induced automorphism. (Here, the normal subgroup of naturally induced automorphisms has index 2 in the full group of automorphisms.)*

- *$i < j = k < \ell < n$, $i + 2k + 2 > \ell$ and the automorphism interchanges the classes V_j and V_k; composing with the automorphism fixing every member of V_i and interchanging each member of V_j with its copy contained in V_k, we obtain a naturally induced automorphism. (Here, the normal subgroup of naturally induced automorphisms has index 2 in the full group of automorphisms.)*
- *$i = j = k < \ell < n$, $3i + 2 > \ell$ and the automorphism permutes the classes V_i, V_j and V_k nontrivially (say, it maps V_a to V_{a^*}); composing with the automorphism mapping each member of V_{a^*} to its copy contained in V_a, $a \in \{i, j, k\}$, we obtain a naturally induced automorphism. (Here, the normal subgroup of naturally induced automorphisms has index 6 in the full group of automorphisms and the corresponding quotient is the symmetric group on 3 letters.)*
- *$i < j = k = \ell < n$, the automorphism preserves $V_j \cup V_k$ but not V_j itself, and it also preserves the set of pairs consisting of copies in V_j and V_k. The map that fixes every member of V_i and interchanges every element J of V_j with its copy in V_k if J got mapped to a member of V_k by the automorphism, is itself an automorphism of $\Gamma_{i,k,k;\leq\ell}$. The composition of these two commuting automorphisms is naturally induced.*

- *Let $0 \leq i, k \leq \ell \leq n$ with $n \geq 2$. Then every automorphism of the bipartite 3-graph $\Gamma_{i,\{i,k\};\leq\ell}$ is induced by a semi-linear automorphism of the underlying vector space V, except only in the following cases.*
  - *$\ell = n$; in this case $\Gamma_{i,\{i,k\};\leq\ell}$ is a complete bipartite 3-graph.*
  - *$2i + k + 2 \leq \ell$; in this case $\Gamma_{i,\{i,k\};\leq\ell}$ is also a complete bipartite 3-graph.*
  - *$k \leq i = \ell$; in this case $\Gamma_{i,\{i,k\};\leq\ell}$ has no triangles.*
- *Let $0 \leq i \leq \ell \leq n$ with $n \geq 2$. Then every automorphism of the 3-graph $\Gamma_{i;\leq\ell}$ is induced by a semi-linear automorphism of the underlying vector space V, except only in the following cases.*
  - *$\ell = n$; in this case $\Gamma_{i;\leq\ell}$ is a complete 3-graph.*
  - *$3i + 2 \leq \ell$; in this case $\Gamma_{i;\leq\ell}$ is also a complete 3-graph.*
  - *$i = \ell$; in this case $\Gamma_{i;\leq\ell}$ has no triangles.*

Now we note that the 3-graphs $\Gamma_{i,j,k;k}$ and $\Gamma_{i,j,k;\leq k}$ are isomorphic, hence we may assume in Main Result 2.2 that $\ell > k$. Similar considerations hold for the bipartite 3-graphs and the graph $\Gamma_{i;i}$; the latter being isomorphic to $\Gamma_{i;\leq i}$ has no triangles anyway.

Main Result 2.2.
- *Let $0 \leq i \leq j \leq k < \ell \leq n$, with $n \geq 2$. Then every automorphism of the tripartite 3-graph $\Gamma_{i,j,k;\ell}$ is induced by a semi-linear automorphism of the underlying vector space V, except only in the following cases.*
  - *$i + j + k + 3 \leq \ell$; in this case $\Gamma_{i,j,k;\ell}$ has no triangles.*
  - *$i = j = k = n - 1$ and $\ell = n$; in this case $\Gamma_{i,j,k;\ell}$ is the tripartite complement of a 3-matching, i.e., a complete tripartite graph minus a 3-matching.*
  - *$i = j < k$, $\ell < n$, $2i + k + 3 > \ell$ and the automorphism interchanges the classes V_i and V_j; composing with the automorphism fixing every member of V_k and interchanging each member of V_i with its copy contained in V_j, we obtain a naturally induced automorphism. (Here, the normal subgroup of naturally induced automorphisms has index 2 in the full group of automorphisms.)*

- ◦ *$i < j = k < \ell \leq n$, $i + 2k + 3 > \ell$ and the automorphism interchanges the classes V_j and V_k; composing with the automorphism fixing every member of V_i and interchanging each member of V_j with its copy contained in V_k, we obtain a naturally induced automorphism. (Here, the normal subgroup of naturally induced automorphisms has index 2 in the full group of automorphisms.)*
- ◦ *$i = j = k < \ell < n$, $3i + 3 > \ell$ and the automorphism permutes the classes V_i, V_j and V_k nontrivially (say, it maps V_a to V_{a^*}); composing with the automorphism mapping each member of V_{a^*} to its copy contained in V_a, $a \in \{i, j, k\}$, we obtain a naturally induced automorphism. (Here, the normal subgroup of naturally induced automorphisms has index 6 in the full group of automorphisms and the corresponding quotient is the symmetric group on 3 letters.)*

- *Let $0 \leq i, k < \ell \leq n$, with $n \geq 2$. Then every automorphism of the bipartite 3-graph $\Gamma_{i,\{i,k\};\ell}$ is induced by a semi-linear automorphism of the underlying vector space V, except in the following cases.*
  - ◦ *$2i + k + 3 \leq \ell$; in this case $\Gamma_{i,\{i,k\};\ell}$ has no triangles.*
  - ◦ *$i + 1 = n = \ell$; in this case $\Gamma_{i,\{i,k\};\ell}$ is a complete bipartite 3-graph.*
- *Let $0 \leq i < \ell \leq n$ with $n \geq 2$. Then every automorphism of the 3-graph $\Gamma_{i;\ell}$ is induced by a semi-linear automorphism of the underlying vector space V, except in the following cases.*
  - ◦ *$3i + 3 \leq \ell$; in this case $\Gamma_{i;\ell}$ has no triangles.*
  - ◦ *$i + 1 = n = \ell$; in this case $\Gamma_{i;\ell}$ is a complete 3-graph.*

In the special case that $\mathbb{L}$ is a finite field, one can use a group theoretic result of Liebeck, Preager and Saxl [4] to determine the automorphism groups. However, we treat the problem in a purely geometric way since this is necessary for general skew fields anyway, and since it gives more insight into the problem.

In all cases, we will be able to reduce the situation to an ordinary (perhaps bipartite) graph, and then use [2], [3] or [5].

It is most convenient to work in the projective space $\mathsf{PG}(n, \mathbb{L})$, and that is what we are going to do. We now introduce some notation.

Notation. For a set S of subspaces (possibly just points) of $\mathsf{PG}(n, \mathbb{L})$, we define $\langle S \rangle$ to be the subspace of $\mathsf{PG}(n, \mathbb{L})$ generated by all members of S. If S consists of two distinct points p_1, p_2, then we also denote the unique line passing through these points by p_1p_2. We use projective dimension. In particular, the dimension of the empty space (this corresponds to the trivial subspace of V) is -1. Finally, for a k-subspace K, we denote by $\mathrm{Res}(K)$ the projective space of dimension $n - k - 1$ obtained from the underlying vector space by factoring out K, and we call it the residue of K. Hence the i-spaces of $\mathrm{Res}(K)$, $-1 \leq i \leq n - k - 1$, are the quotients W/K, where W is an $(i + k + 1)$-space of $\mathsf{PG}(n, \mathbb{L})$ containing K.

3. Proofs—Reduction

3.1. **Main tools and Strategy. Tools**—Our first important tool consists of the main results of [3], which we now recall.

We say that an ordinary bipartite graph is *trivial* if it is a graph without edges, a complete bipartite graph, a matching, or the bipartite complement of a matching (the latter is a complete bipartite graph minus a matching).

Theorem 3.1 (Main Results 2.1 and 2.2 of [3]). *Let $\Gamma^n_{a,b;\geq c}(\mathbb{L})$ and $\Gamma^n_{a,b;c}(\mathbb{L})$ be the ordinary bipartite graphs consisting of the a-spaces and the b-spaces of a projective space of dimension n over the skew field $\mathbb{L}$, with $-1 \leq c \leq a \leq b \leq n-1$, $0 \leq a$, and an a-space A is adjacent to a b-space B if $\dim(A \cap B) \geq c$ and $\dim(A \cap B) = c$, respectively. Then*

(1) $\Gamma^n_{a,b;\geq c}(\mathbb{L})$ *is trivial if and only if one of the following occurs:*
- $a + b \geq n + c$;
- $c = -1$;
- $a = b = c$.

(2) $\Gamma^n_{a,b;c}(\mathbb{L})$ *is trivial if and only if one of the following occurs:*
- $a = b = c$;
- $a = b = 0$ *and* $c = -1$;
- $a = b = n-1$ *and* $c = n-2$;
- $n + c < a + b$.

(3) *If $\Gamma^n_{a,b;\geq c}(\mathbb{L})$ or $\Gamma^n_{a,b;c}(\mathbb{L})$ is not trivial, then its class preserving automorphism group is the collineation group of $\mathsf{PG}(n, \mathbb{L})$, except if $a = b = \frac{n-1}{2}$, in which case every class preserving automorphism which is not a collineation of $\mathsf{PG}(n, \mathbb{L})$ is a duality of $\mathsf{PG}(n, \mathbb{L})$.*

We say that an ordinary (not necessarily bipartite) graph is *trivial* if either it is a complete graph, or it does not contain any edges.

Theorem 3.2 (Corollaries 3.12 and 3.16 of [3]). *Let $\Gamma^n_{a;\geq c}(\mathbb{L})$ and $\Gamma^n_{a;c}(\mathbb{L})$ be the ordinary graphs consisting of the a-spaces of a projective space of dimension n over the skew field $\mathbb{L}$, with $-1 \leq c \leq a \leq n-1$, $0 \leq a$, and an a-space A is adjacent to another a-space A' if $\dim(A \cap A') \geq c$ and $\dim(A \cap A') = c$, respectively. Then*

(1) $\Gamma^n_{a;\geq c}(\mathbb{L})$ *is trivial if and only if one of the following occurs:*
- $2a \geq n + c$;
- $c = -1$;
- $c = a$.

(2) $\Gamma^n_{a;c}(\mathbb{L})$ *is trivial if and only if one of the following occurs:*
- $a = c$;
- $a = 0$ *and* $c = -1$;
- $a = n-1$ *and* $c = n-2$;
- $n + c < 2a$.

(3) *If $\Gamma^n_{a;\geq c}(\mathbb{L})$ or $\Gamma^n_{a;c}(\mathbb{L})$ is not trivial, then its automorphism group is the collineation group of $\mathsf{PG}(n, \mathbb{L})$, except if $a = \frac{n-1}{2}$, in which case every automorphism which is not a collineation of $\mathsf{PG}(n, \mathbb{L})$ is a duality of $\mathsf{PG}(n, \mathbb{L})$.*

The cases of $\Gamma^n_{a;\geq a-1}(\mathbb{L}) \equiv \Gamma^n_{a;a-1}(\mathbb{L})$ and $\Gamma^n_{a;-1}(\mathbb{L})$ of the previous theorem were already handled by Chow [2] and Lim [5], respectively.

Our second important tool is the following 'observation'. We call a graph trivial if it is one of the exceptions in the main results.

Proposition 3.3. (*i*) *Let $0 \leq i \leq j \leq k \leq n-1$, $k \leq \ell \leq n$ with $n \geq 2$, and suppose that $\Gamma_{i,j,k;\leq \ell}$ is not trivial. Then every automorphism of the tripartite 3-graph $\Gamma_{i,j,k;\leq \ell}$ permutes the classes of vertices if we do not have $j = k = l$. Also, if it fixes all these classes and fixes at least one of the classes vertexwise, it is the identity.*

(*ii*) *Let $0 \leq i, k \leq \ell \leq n$, and suppose that $\Gamma_{i,\{i,k\};\leq \ell}$ is not trivial. Then every automorphism of the bipartite 3-graph $\Gamma_{i,\{i,k\};\leq \ell}$ preserves the classes. Also, if it fixes at least one of the classes vertexwise, it is the identity.*

(*iii*) *Let* $0 \leq i \leq j \leq k \leq n-1$, $k \leq \ell \leq n$ *with* $n \geq 2$, *and suppose that* $\Gamma_{i,j,k;\ell}$ *is not trivial. Then every automorphism of the tripartite* 3*-graph* $\Gamma_{i,j,k;\ell}$ *permutes the classes of vertices if we do not have* $j = k = l$. *Also, if it fixes all these classes and fixes at least one of the classes vertexwise, it is the identity.*

(*iv*) *Let* $0 \leq i, k \leq n-1$, $i, k \leq \ell \leq n$ *with* $n \geq 2$, *and suppose that* $\Gamma_{i,\{i,k\};\ell}$ *is not trivial. Then every automorphism of the bipartite* 3*-graph* $\Gamma_{i,\{i,k\};\ell}$ *preserves both classes. Also, if it that fixes at least one of the classes vertexwise, it is the identity.*

Proof

(*i*) Note that $\Gamma_{i,j,k;\leq\ell}$ being non-trivial implies $\ell \leq n-1$. Let θ be an automorphism of $\Gamma_{i,j,k;\leq\ell}$. We will deduce the classes of vertices in terms of $\mathcal{T}$ and hence these classes are permuted by θ. Consider any $\{I, J, K\} \in \mathcal{T}$. We will show that a vertex I' is contained in the same class as I if and only if there exist vertices $I_1, I_2, \ldots, I_{m-1}, J_1, \ldots, J_m, K_1, \ldots, K_m$, with $m \in \mathbb{N}$, such that

$$\{I_{a-1}, J_a, K_a\}, \{I_a, J_a, K_a\} \in \mathcal{T}$$

for all $a \in \{1, \ldots, m\}$, where we set $I_0 = I$ and $I_m = I'$. If $I \in V_i$, then clearly $I' \in V_i$ if it satisfies these conditions. Conversely, suppose $I' \in V_i$. Assume $\dim(I \cap I') = i-1$. Since our graph is non-trivial we have $\ell \leq i+j+k+2$, so we can find $J \in V_j$ and $K \in V_k$ such that $\langle I, J, K\rangle$ is an ℓ-space coinciding with $\langle I', J, K\rangle$, unless $i = j = k = l$ but then our graph is also trivial. Since $\Gamma^n_{i,i-1}$ is connected we obtain the above claim. Similarly if $I \in V_j$ or $I \in V_k$, unless $\ell = k$. Since we excluded the case $j = k = l$, we get $j < \ell$. Then $K \in V_k$ if and only if for all vertices I, J, K', $\{I, J, K\}, \{I, J, K'\} \in \mathcal{T}$ implies $K = K'$. Hence V_k is mapped onto itself, and the previous arguments show that V_i and V_j are (possibly trivially) permuted.

Now suppose θ fixes V_i and V_j and fixes V_k vertexwise. Let $\mathcal{T}$ be the set of triangles of $\Gamma_{i,j,k;\leq\ell}$. We claim that, if $I_1, I_2 \in V_i$, $J_1, J_2 \in V_j$, with $\ell - k \leq \dim\langle I_1, J_1\rangle \leq \dim\langle I_2, J_2\rangle \leq \ell$, then the sets

$$\{K \in V_k : \{I_1, J_1, K\} \in \mathcal{T}\} \text{ and } \{K \in V_k : \{I_2, J_2, K\} \in \mathcal{T}\}$$

coincide if and only if $\langle I_1, J_1\rangle = \langle I_2, J_2\rangle$. The "if"-part is clear. We now show the "only if"-part. Set $\langle I_1, J_1\rangle = U_1 \neq U_2 = \langle I_2, J_2\rangle$. Since $\ell \leq n-1$, we can select an ℓ-space L containing U_1 but not containing U_2. Since $\dim U_1 \geq \ell - k$, we can select $K \in V_k$ such that $\langle U_1, K\rangle = L$, and $K \cap U_1$ is not empty. There are two possibilities.

- If $U_1 \subseteq U_2$, then $\dim U_2 > \dim U_1$ and

$$\ell = \dim\langle U_1, K\rangle = \dim L < \dim\langle L, U_2\rangle = \dim\langle U_1, U_2, K\rangle = \dim\langle U_2, K\rangle,$$

  which shows that $\{I_1, J_1, K\}$ is a triangle of $\Gamma_{i,j,k;\leq\ell}$, but $\{I_2, J_2, K\}$ is not.
- If $U_1 \setminus U_2 \neq \emptyset$, then it is convenient to again distinguish two cases.
  - Suppose $\dim U_1 > \dim(L \cap U_2)$. Then we may re-choose K such that $\dim(U_1 \cap K) > \dim(U_2 \cap K)$. It then follows that

$$\begin{aligned} \ell &= \dim\langle U_1, K\rangle \\ &= \dim U_1 + \dim K - \dim(U_1 \cap K) \\ &< \dim U_2 + \dim K - \dim(U_2 \cap K) \\ &= \dim\langle U_2, K\rangle, \end{aligned}$$

    which shows that $\{I_1, J_1, K\}$ is a triangle of $\Gamma_{i,j,k;\leq\ell}$, but $\{I_2, J_2, K\}$ is not.

– Suppose $\dim U_1 \le \dim(L \cap U_2)$. Then we can re-choose K such that $\langle K, L \cap U_2\rangle = L$. It then follows that

$$\ell = \dim\langle U_1, K\rangle < \dim\langle L, U_2\rangle = \dim\langle U_2, K\rangle,$$

which again shows that $\{I_1, J_1, K\}$ is a triangle of $\Gamma_{i,j,k;\le\ell}$, but $\{I_2, J_2, K\}$ is not.

Our claim follows. Now we claim that, if $I_1, I_2 \in V_i$, $J_1, J_2 \in V_j$, with $\ell - k \le \dim\langle I_1, J_1\rangle \le \ell$, then, regardless of $\dim\langle I_2, J_2\rangle$, the sets

$$\{K \in V_k : \{I_1, J_1, K\} \in \mathcal{T}\} \text{ and } \{K \in V_k : \{I_2, J_2, K\} \in \mathcal{T}\}$$

are distinct, unless $\langle I_1, J_1\rangle = \langle I_2, J_2\rangle$. Indeed, if $\ell - k \le \dim\langle I_2, J_2\rangle \le \ell$, this follows from the previous claim; if $\dim\langle I_2, J_2\rangle < \ell - k$, then $\{K \in V_k : \{I_1, J_1, K\} \in \mathcal{T}\} \ne V_k$ whereas $\{K \in V_k : \{I_2, J_2, K\} \in \mathcal{T}\} = V_k$; if $\ell < \dim\langle I_2, J_2\rangle$, then $\{K \in V_k : \{I_1, J_1, K\} \in \mathcal{T}\} \ne \emptyset$ whereas $\{K \in V_k : \{I_2, J_2, K\} \in \mathcal{T}\} = \emptyset$.

We have $j \le \ell$ and $i+j+k+1 \ge \ell$ (since $\Gamma_{i,j,k;\le\ell}$ is non-trivial). Hence, we can select an arbitrary natural number r, with $\max\{j, \ell-k\} \le r \le \min\{\ell, i+j+1\}$. Consider any pair $(I, J) \in V_i \times V_j$ with $\dim(I \cap J) = i + j - r =: d$, and note that by the choice of r, we have $-1 \le i + j - r \le i$. By our second claim, θ maps the pair $\{I, J\}$ onto a pair $\{I', J'\}$ with $\langle I, J\rangle = \langle I', J'\rangle$. In particular, $\dim(I \cap J) = \dim(I' \cap J')$. Hence θ is an automorphism of the graph $\Gamma^n_{i,j;d}(\mathbb{L})$.

Suppose $i = j$ and the only possibility for r is i. Then $i = \min\{\ell, i+j+1\}$ and so, since $i \ge 0$, we have $i = j = k = \ell$ and $\Gamma_{i,j,k;\le\ell}$ is a 3-matching. Hence we can avoid $i = j = d$. Likewise, if $i = j = n - 1$, then $k = n - 1$ and either we have a 3-matching ($\ell = n - 1$) or a complete tripartite graph ($\ell = n$). Finally, $n + d = n - r + i + j \ge i + j$. Hence, according to Theorem 3.1(2), there are two possibilities.

(a) The graph $\Gamma^n_{i,j;d}(\mathbb{L})$ is non-trivial. Then θ comes from a collineation or duality of $\mathsf{PG}(n, \mathbb{L})$. Since θ fixes every r-space (since every r-space is generated by some pair $(I, J) \in V_i \times V_j$), it must be the identity and the assertion is proved.

(b) $i = j = 0$ and $d = -1$. Then θ maps any pair of points $(p, q) \in V_0 \times V_0$ to a pair (p', q') with $\langle p, q\rangle = \langle p', q'\rangle$. If $p \ne p'$, then we choose a point $x \notin \langle p, p'\rangle$. The line $\langle p', x^\theta\rangle$ is certainly distinct from $\langle p, x\rangle$, a contradiction. Hence $p = p'$ and θ is the identity. The assertion again follows.

A very similar proof, which we shall not repeat here, holds if θ fixes each vertex of V_j, or each vertex of V_i.

(ii) This is completely similar to (i).

(iii) The proof of (i) can be copied, except for the cases $\ell = n$, and $i+j+k+2 = \ell$, since these cases did not occur in (i). The proof of the first part of (i) still holds in these cases. Now we adapt the proof of the second part.

First suppose that $i + j + k + 2 = \ell$ and that θ is an automorphism of $\Gamma_{i,j,k;\ell}$ which fixes every vertex of V_k and stabilises V_i and V_j. The only pairs $(I, J) \in V_i \times V_j$ contained in a triangle are those for which $I \cap J = \emptyset$. Hence, θ maps the pair (I, J) onto a pair $(I', J') \in V_i \times V_j$ with $\langle I, J\rangle = \langle I', J'\rangle$, as distinct $(i + j + 1)$-spaces do not intersect the same k-spaces non-trivially. Similarly as in (i), we conclude that θ is the identity.

Now suppose that $\ell = n$. It is more instructive to assume now that an automorphism of $\Gamma_{i,j,k;\ell}$ fixes every vertex of V_i and stabilises V_j and V_k. Then, exactly as above, for every pair $(J_a, K_a) \in V_j \times V_k$, with $n - i - 1 \le \dim\langle J_a, K_a\rangle < n$, $a = 1, 2$, we find an i-space I such that $\dim\langle I, J_1, K_1\rangle =$

$n-1$ and $\dim\langle I, J_2, K_2\rangle = n$ if and only if $\langle J_1, K_1\rangle \neq \langle J_2, K_2\rangle$. Then we can finish the proof exactly as before, unless $j = k = n-1$, as in this case the graph $\Gamma^n_{j,k;d}(\mathbb{L})$ we want to use can not be chosen distinct from the matching $\Gamma^n_{n-1,n-1;n-1}$ (in all other cases there are choices that give non-trivial graphs). But in this case, for every $J \in V_j$, the $I \in V_i$ such that for at least one $K \in V_k$, the triple $\{I, J, K\}$ is *not* a triangle, are precisely the $I \in V_i$ such that $I \subseteq J$. Since this set is different for different J (noting that $i < n-1$), the permutation θ must fix every vertex of V_j and likewise of V_k.

(*iv*) This is again similar to the proof of (*iii*).

The proof of the proposition is complete. □

The next lemma shows that, if our graph is non-trivial, the tripartite parts V_i, V_j and V_k are often stabilised.

Lemma 3.4. (*i*) *Let $0 \leq i \leq j \leq k \leq n-1$, $k \leq \ell < n$ with $n \geq 2$, and suppose that $\Gamma_{i,j,k;\leq\ell}$ is not trivial. Then every automorphism of the tripartite 3-graph $\Gamma_{i,j,k;\leq\ell}$ fixes every class V_a for which (a, a, a) and (i, j, k) differ in exactly two positions.*

(*ii*) *Let $0 \leq i \leq j \leq k \leq n-1$, $k \leq \ell \leq n$ with $n \geq 2$, and suppose that $\Gamma_{i,j,k;\ell}$ is not trivial. Then every automorphism of the tripartite 3-graph $\Gamma_{i,j,k;\ell}$ fixes every class V_a for which (a, a, a) and (i, j, k) differ in exactly two positions.*

Proof

(*i*) Suppose first $\ell \leq j + k$. Consider $I \in V_i$ and $J \in V_j$. By $\ell - k \leq j \leq \ell$ and the claim proven in the second paragraph of the proof of Proposition 3.3 we get that $I \subseteq J$ if and only if $\{K \in V_k \mid \{I, J, K\} \in \mathcal{T}\} = \{K \in V_k \mid \{I', J', K\} \in \mathcal{T}\}$ implies $J = J'$, for any $I' \in V_i$ and $J' \in V_j$. Similarly one can describe when $I \subseteq K$, with $K \in V_k$, purely in terms of $\mathcal{T}$. If $i < j$ we get that we can describe, using $\mathcal{T}$, that an element of $I \in V_i$ is strictly contained in elements of the two other classes and hence V_i is mapped onto itself. So if $a = i$, then by assumption $i < j$ and we are done. Assume $i = j$. Then by assumption $a = k > j$. We can describe when $I \subseteq K$ and $J \subseteq K$, for $I \in V_i$, $J \in V_j$ and $K \in V_k$. So vertices properly contained in vertices of another class are mapped onto each other, and hence V_k is stabilised. So we may assume $i < j < k$ and $a = j$ or $a = k$. The j-spaces J and J' intersect in a $(j-1)$-space if and only if there exist $I, I' \in V_i$ such that $I, I' \subseteq J$, $I, I' \not\subseteq J'$, and for any such I and I' we have $\langle I, J'\rangle = \langle I', J'\rangle$. Again, by the claim proven in the second paragraph of the proof of Proposition 3.3 we can describe all these properties in terms of $\mathcal{T}$, as long as $j + 1 \leq \ell$; but this holds since $j < k \leq \ell$. Similarly we can describe when two distinct k-spaces intersect maximally, as long as $k < l$. So, if $k < l$, then $J_0 \in V_j$ if and only if there exist $J_1, \ldots, J_{j-i}$ in the same component as J_0, with J_m and J_{m+1} distinct and maximally intersecting, such that there is a unique $I \in V_i$ contained in all these subspaces. Here we used $j < k$. We obtain that V_j and V_k are also stabilised. If $\ell = k > j$, then as shown in the first paragraph of Proposition 3.3 we get that V_k (and hence V_j) is stabilised.

Suppose now $\ell = j+k+d$, with $d \geq 1$. Consider $I \in V_i$, $J \in V_j$ and $K \in V_k$ arbitrary. Note that the nontriviallity of $\Gamma_{i,j,k;\leq\ell}$ implies $i - d + 1 \geq 0$. Then $\dim(I \cap J) \geq i - d + 1$ if and only if $\dim(\langle I, J\rangle) \leq j + d - 1$ if and only if $\{I, J, K\} \in \mathcal{T}$ for all $K \in V_k$. Similarly, one can deduce when $\dim(I \cap K) \geq i - d + 1$ and $\dim(J \cap K) \geq i - d + 1$ from $\mathcal{T}$. Now clearly $I \subseteq J$ if and only if for all $K \in V_k$: $\dim(I \cap K) \geq i - d + 1$ implies $\dim(J \cap K) \geq i - d + 1$.

Similarly we can describe $J \subseteq K$ and $I \subseteq K$ in terms of $\mathcal{T}$. Hence V_a is stabilised.

(ii) The above proof can be easily adopted to cover this case as well, except if $\ell = j + k + 1$ or $\ell = n$. We first handle the former case. For $I \in V_i$ and $J \in V_j$ we have $I \subseteq J$ if and only if $\{K \in V_k \mid \{I, J, K\} \in \mathcal{T}\} = \{K \in V_k \mid \{I', J', K\} \in \mathcal{T}\}$ implies $J = J'$. Indeed, note that if $I \not\subseteq J$, we can easily find $J' \in V_j$ with $J' \neq J$ and $\langle I, J\rangle = \langle I, J'\rangle$, and note that a j-space is uniquely defined by all k-spaces not intersecting it. Similarly $I \subseteq K$ can be described in terms of $\mathcal{T}$, with $K \in V_k$. Now the argument to show that V_a is stabilised is the same as in the first part. For $\ell = n$, note that the previous arguments still apply if $\ell \geq j + k + 1$. If $\ell \leq j + k$, one combines the arguments in the first part of this proof with these of part (iii) of Proposition 3.3.

□

Strategy—We proceed as follows to prove the main results. We generally assume that θ is an automorphism of one of the 3-graphs $\Gamma_{i,j,k;\ell}$, $\Gamma_{i,j,k;\leq\ell}$, $\Gamma_{i,\{i,k\};\leq\ell}$, $\Gamma_{k,\{i,k\};\leq\ell}$, $\Gamma_{i,\{i,k\};\ell}$, $\Gamma_{k,\{i,k\};\ell}$, $\Gamma_{i;\leq\ell}$ or $\Gamma_{i;\ell}$. In the tripartite case, we initially and additionally assume that θ preserves the tripartition classes. Then we show that, under the appropriate conditions, θ acts on two classes (in the tripartite or bipartite case) or a single class (in the ordinary case, or the class containing two vertices of each triangle in the bipartite case) as the restriction of an automorphism of $\mathsf{PG}(n, \mathbb{K})$, using Theorems 3.1 and 3.2 above. Below, we will show that this automorphism can never be a duality. Then we apply a known automorphism of the 3-graphs to fix one of the classes, and apply Proposition 3.3. We gradually reduce the tripartite case to the bipartite one, and then to the ordinary one. After that, using Lemma 3.4, we treat the automorphisms that permute the classes non-trivially.

3.2. **Dualities.** We first rule out dualities.

Lemma 3.5. *The permutation of the vertices of* $\Gamma_{i,j,k;\ell}$, $\Gamma_{i,j,k;\leq\ell}$, $\Gamma_{i,\{i,k\};\leq\ell}$, $\Gamma_{i,\{i,k\};\ell}$, $\Gamma_{i;\leq\ell}$ *or* $\Gamma_{i;\ell}$ *induced by a duality of* $\mathsf{PG}(n, \mathbb{L})$ *is never an automorphism of the respective graph.*

Proof If no confusion is possible, we denote for short by $\Gamma = (\mathcal{V}, \mathcal{T})$ the appropriate 3-graph under consideration.

Suppose σ is a duality of $\mathsf{PG}(n, \mathbb{L})$ and induces an automorphism of $\Gamma_{i,j,k;\ell}$ or $\Gamma_{i,j,k;\leq\ell}$. Then $i + k = 2j = n - 1$. Suppose first that $i < j$ and $\ell < n$. Let J be an arbitrary j-space. Let K be an arbitrary k-space containing J, and let I be an arbitrary i-space intersecting K in an $(n - 1 - \ell)$-space and such that $I \cap J$ has dimension $\max\{i + j - \ell, -1\}$ (the latter means that $I \cap J$ is as small as possible). Then $\{I, J, K\} \in \mathcal{T}$. However, $\langle I^\sigma, J^\sigma, K^\sigma\rangle = \langle J^\sigma, I^\sigma\rangle$. The latter has dimension $\min\{n, k - j + \ell\} > \ell$ and so $\{I^\sigma, J^\sigma, K^\sigma\} \notin \mathcal{T}$, a contradiction.

If $i < j$ and $\ell = n$, then we may assume $\Gamma = \Gamma_{i,j,k;n}$. We choose an i-space I contained in a k-space K. Then we select a j-space J intersecting K in a $(j+k-n)$-space and intersecting I in a space of dimension $\min\{j - i - 1, i\}$ (an as large as possible intersection). Then $\{I, J, K\} \in \mathcal{T}$ whereas $\{I^\sigma, J^\sigma, K^\sigma\} \notin \mathcal{T}$ since $\langle I^\sigma, J^\sigma, K^\sigma\rangle = \langle J^\sigma, I^\sigma\rangle$, and the latter has dimension $\max\{n - i - 1, n - j + i\} < n = \ell$.

Now assume $i = j = k$. Then $\ell \geq i + 1$. The following argument also applies to the graphs $\Gamma_{i,\{i,i\};\leq\ell}$, $\Gamma_{i,\{i,i\};\ell}$, $\Gamma_{i;\leq\ell}$ and $\Gamma_{i;\ell}$. First let $\ell < n$. Then we can select two i-spaces I, I' generating an ℓ-space L. We may then select an i-space $I'' \subseteq L$

with $I \cap I'$ not contained in I''. Then $\{I, I', I''\} \in \mathcal{T}$, but I''^{σ} is not contained in $\langle I^{\sigma}, I'^{\sigma}\rangle = (I \cap I')^{\sigma}$, yielding $\{I^{\sigma}, I'^{\sigma}, I''^{\sigma}\} \notin \mathcal{T}$. If $\ell = n$, then the previous argument for $\ell = n-1$ yields $\{I, I', I''\} \notin \mathcal{T}$ and $\{I^{\sigma}, I'^{\sigma}, I''^{\sigma}\} \in \mathcal{T}$, a contradiction (note that $\ell = n$ only applies to the graphs $\Gamma_{i,\{i,i\};n}$ and $\Gamma_{i;n}$).

This completes the proof of the lemma. □

In fact, we need a slightly stronger result for the graphs $\Gamma_{i,j,k;\ell}$, $\Gamma_{i,j,k;\leq\ell}$ (with $|\{i, j, k\}| \leq 2$), $\Gamma_{i,\{i,k\};\leq\ell}$ and $\Gamma_{i,\{i,k\};\ell}$.

Lemma 3.6. (*i*) *No automorphism θ of the nontrivial 3-graphs $\Gamma_{i,j,k;\ell}$ or $\Gamma_{i,j,k;\leq\ell}$ has the same action on two (stabilised) tripartition classes as some common duality σ of $\mathsf{PG}(n, \mathbb{L})$.*

(*ii*) *No automorphism θ of the nontrivial 3-graphs $\Gamma_{i,\{i,k\};\leq\ell}$ or $\Gamma_{i,\{i,k\};\ell}$ has the same action on the (stabilised) class V_i as some duality of $\mathsf{PG}(n, \mathbb{L})$.*

Proof As in the previous proof, we let $\Gamma = (\mathcal{V}, \mathcal{T})$ be the 3-graph under consideration.

(*i*) We first consider the case $\ell < n$. The assumptions imply that two of i, j, k equal $\frac{n-1}{2}$, say $i = j = \frac{n-1}{2}$, and θ restricted to $V_i \sqcup V_j$ is induced by a duality of $\mathsf{PG}(n, \mathbb{L})$ (preserving the two classes separately). We abandon for once the assumption $j \leq k$ and also allow $k < j$. Consider four i-spaces $I, I' \in V_i$ and $J, J' \in V_j$ such that $\langle I, J\rangle = \langle I', J'\rangle$ is an ℓ-space and $I \cap J \neq I' \cap J'$. Then the set of k-spaces K such that $\{I, J, K\} \in \mathcal{T}$ coincides with the set of k-spaces K' such that $\{I', J', K'\} \in \mathcal{T}$ (namely, all K and K' in $\langle I, J\rangle$). But the set of k-spaces K such that $\{I^{\sigma}, J^{\sigma}, K\} \in \mathcal{T}$ is the set of k-spaces contained in $(I \cap J)^{\sigma}$ and differs from the set of k-spaces contained in $(I' \cap J')^{\sigma}$ (by the choice of I, J, I', J'), which equals the set of k-spaces K' such that $\{I'^{\sigma}, J'^{\sigma}, K'\} \in \mathcal{T}$. This contradiction shows the non-existence of θ.

Now suppose $\ell = n$. We select I, I', J, J' exactly as in the previous paragraph for $\ell = n-1$. Then the set of k-spaces K such that $\{I, J, K\} \in \mathcal{T}$ coincides with the set of k-spaces not contained in the hyperplane $\langle I, J\rangle = \langle I', J'\rangle$. But the set of k-spaces K such that $\{I^{\sigma}, J^{\sigma}, K\} \in \mathcal{T}$ coincides with the set of k-spaces not contained in the hyperplane $(I \cap J)^{\sigma} \neq (I' \cap J')^{\sigma}$. The non-existence of θ follows again.

(*ii*) This is proved in completely the same way, now picking the i-spaces in V_i.

□

3.3. **The tripartite case.** We start with the most general cases $\Gamma := \Gamma_{i,j,k;\ell} = (\mathcal{V}, \mathcal{T}_{i,j,k;\ell})$ and $\Gamma_{\leq} := \Gamma_{i,j,k;\leq\ell} = (\mathcal{V}, \mathcal{T}_{i,j,k;\leq\ell})$ and gradually reduce to special cases.

So let θ be an automorphism of Γ or of $\Gamma_{\leq}$. We have to show that, under the assumptions stated in the main results, θ is induced by a collineation of $\mathsf{PG}(n, \mathbb{L})$, and otherwise there exist automorphisms of Γ or $\Gamma_{\leq}$ that do not come from collineations of $\mathsf{PG}(n, \mathbb{L})$.

It is clear that, if $\ell \geq i + j + k + 3$, then $\Gamma_{i,j,k;\ell}$ is the empty 3-graph and $\Gamma_{i,j,k;\leq\ell}$ is a complete tripartite 3-graph. In these cases the result follows easily.

Now let $\ell \leq i+k$. Take $J \in V_j$ and $K \in V_k$. Suppose that $\dim(J\cap K) \leq j+k-\ell-1$. Then $\dim\langle J, K\rangle \geq \ell + 1$. So $\{J, K\}$ is not part of any triangle of neither Γ nor $\Gamma_{\leq}$. Suppose now that $\dim(J \cap K) > j + k - \ell - 1$. Then $\dim\langle J, K\rangle \leq \ell$. Since $\dim\langle J, K\rangle \geq k$, it follows that $\{J, K\}$ is part of a triangle of both Γ and $\Gamma_{\leq}$. Hence the bipartite graph with vertex set $V_j \sqcup V_k$ and $J \in V_j$ adjacent with $K \in V_k$ if $\{J, K\}$

is part of a triangle of Γ (or $\Gamma_{\leq}$) is isomorphic to the graph $\Gamma^n_{j,k;\geq j+k-\ell}(\mathbb{L})$ and is non-trivial (not a matching, not empty, not complete bipartite, not the bipartite complement of a matching), except if $\ell = n$ or $j = k = \ell$. Indeed, we check when the conditions of Theorem 3.1(1) are satisfied. First, $j + k \geq n + (j + k - \ell)$ is equivalent with $\ell \geq n$, hence $\ell = n$. Also, $j+k-\ell = -1$ contradicts the assumption $\ell \leq i + k \leq j + k$. Finally, $j = k = j + k - \ell$ implies $j = k = \ell$.

Suppose first that we do not have $\ell = n$ or $j = k = \ell$. By Theorem 3.1(3), the restriction of θ to $V_j \sqcup V_k$ is then induced by a collineation (and not a duality by Lemma 3.6) θ' of $\mathsf{PG}(n, \mathbb{L})$. Let α be the automorphism of Γ or $\Gamma_{\leq}$ induced by θ'. By Proposition 3.3(i),(iii), $\theta\alpha^{-1}$ is the identity. Hence, $\theta = \alpha$ and θ is induced by a collineation of $\mathsf{PG}(n, \mathbb{L})$. In the subsequent cases, we will not repeat the latter argument and only content ourselves with proving that θ acts on at least two tripartition classes as the restriction of a collineation or duality. Note that, if the parameters imply that a duality is impossible (for instance, if $i \neq j$ for the bipartite graph $\Gamma^n_{i,j;\leq\ell}(\mathbb{L})$ since we only consider automorphisms preserving the classes), we will not mention the need for Lemma 3.5 or 3.6.

Suppose now that $j = k = \ell$, still assuming $\ell \leq i + k$. In that case, θ restricted to $V_j \sqcup V_k$ is an automorphism of $\Gamma^n_{\ell,\ell;\geq\ell}$. Hence, we can recognise when $V \in V_j$ and $W \in V_k$ represent the same ℓ-space (namely, when they are contained in a common triangle), and θ induces an automorphism of $\Gamma_{\ell,\{i,\ell\};\leq\ell}$ and we reduced the problem to the bipartite case. Note that if $i = \ell$, we clearly have (as claimed in Main Result 2.1 and 2.2) that Γ and $\Gamma_{\leq}$ are 3-matchings.

Finally suppose $\ell = n$, still assuming $\ell \leq i + k$. Then $\Gamma_{\leq}$ is complete tripartite (this also holds if we have $\ell > i + k$). Also, for $J \in V_j$ and $K \in V_k$ we have $\dim(J \cap K) = j + k - \ell$ if and only if $\{J, K\}$ is contained in a triangle of Γ together with every $I \in V_i$. This yields the graph $\Gamma^n_{j,k;j+k-\ell}(\mathbb{L})$ and, if that graph is non-trivial, we are likewise done, now using Theorem 3.1(2),(3) and Proposition 3.3(iii) (the parameters imply that no duality can appear). Note that $\Gamma^n_{j,k;j+k-\ell}(\mathbb{L})$ is trivial if $j = k = n - 1$ and $n - 2 = j + k - n$, or $j = k = j + k - n$ (and thus $j = k = \ell = n$, a trivial case). In the former case, just as before, we can recognise when $V \in V_j$ and $W \in V_k$ represent the same space, and as in the previous case, we get that θ induces an automorphism of $\Gamma_{j,\{i,j\};\ell}$. So we reduced the problem to the bipartite case.

The argument also works for $n = \ell = i + k + 1$, since then the only case where $\Gamma_{j,k;j+k-\ell}$ is trivial (excluding the cases previously discussed) is $j = k = i = 0$, $\ell = n = 1$. But as claimed in Main Result 2.2, Γ is then the tripartite complement of a 3-matching.

If $n > \ell = i + k + 1$ and $i < j$, then the argument in the previous paragraphs yields the graph $\Gamma^n_{j,k;\geq j-i-1}(\mathbb{L})$, which is non-trivial by Theorem 3.1(1) (since $j + k < j - i - 1 + n$ by assumption, and $j - i - 1 = -1$ is impossible). By Theorem 3.1(3) and Proposition 3.3, θ is induced by a collineation (and, according to Lemma 3.6, not a duality) of $\mathsf{PG}(n, \mathbb{L})$.

Suppose now $i = j$ and $\ell = i + k + 1$. In view of the paragraph preceding the previous one, we may assume that $\ell < n$. Recall that we here consider V_i and V_j as distinct sets (copies of the same set however). Let $I \in V_i$ and $J \in V_j$ and suppose that $\dim\langle I, J\rangle > i$ (i.e., as i-spaces of $\mathsf{PG}(n, \mathbb{L})$, I and J are distinct). Let $J' \in V_j$ be such that $\langle I, J'\rangle = \langle I, J\rangle$ and $J \neq J'$. Then clearly $\{K \in V_k : \{I, J, K\} \in \mathcal{T}_{i,j,k;\ell}\} = \{K \in V_k : \{I, J', K\} \in \mathcal{T}_{i,j,k;\ell}\}$ and $\{K \in V_k : \{I, J, K\} \in \mathcal{T}_{i,j,k;\leq\ell}\} = \{K \in V_k : \{I, J', K\} \in \mathcal{T}_{i,j,k;\leq\ell}\}$. Also, if I and J coincide as i-spaces, and $J' \in V_j$ is arbitrary but distinct from J, then, since $\ell < n$, we can select an ℓ-space L such that $I \subseteq L$

but $J' \not\subseteq L$. Since $\ell = i + k + 1$, there exists a k-space K such that $\langle I, K\rangle = L$. It follows that $\{I, J, K\} \in \mathcal{T}_{i,j,k;\ell} \cap \mathcal{T}_{i,j,k;\leq\ell}$ but $\{I, J', K\} \notin \mathcal{T}_{i,j,k;\ell} \cup \mathcal{T}_{i,j,k;\leq\ell}$. Hence we can recognise the elements of V_i and V_j that coincide in $\mathsf{PG}(n, \mathbb{L})$ and we can deduce the bipartite 3-graphs $\Gamma_{i,\{i,k\};\ell}$ and $\Gamma_{i,\{i,k\};\leq\ell}$ from the 3-graphs $\Gamma_{i,i,k;\ell}$ and $\Gamma_{i,i,k;\leq\ell}$, respectively. So we reduced this case to the bipartite case.

Now suppose $\ell = i + k + 2$. Let $J \in V_j$ and $K \in V_k$ be arbitrary. Then $J \subseteq K$ in $\mathsf{PG}(n, \mathbb{L})$ if, and only if, they are not contained in a member of $\mathcal{T}_{i,j,k;i+k+2}$. Hence, if $j = k$, we can reduce to the bipartite case $\Gamma_{j,\{i,j\};i+j+2}$. If $j < k$, then this determines the non-trivial bipartite graph $\Gamma^n_{j,k;j}(\mathbb{L}) \cong \Gamma^n_{j,k;\geq j}(\mathbb{L})$, and Theorem 3.1(3) and Proposition 3.3(iii) settle this case. Also, in general, $\dim(J \cap K) \geq j-1$ if and only if $\{I, J, K\}$ is a triangle of $\mathcal{T}_{i,j,k;\leq i+k+2}$ for every $I \in V_i$. This determines the bipartite graph $\Gamma^n_{j,k;\geq j-1}(\mathbb{L})$. Theorem 3.1(3), Proposition 3.3(i) and Lemma 3.6 settle this case (note that $j - 1 = -1$ implies $i + j + k + 2 = \ell$, and this case is covered in the next paragraph; also $k < n - 1$ as otherwise ℓ exceeds n by assumption).

Now let $i+k+3 \leq \ell \leq i+j+k+2$. Then $\ell-i-2 \geq k+1$ and so there exist j-spaces J and k-spaces K with $\dim\langle J, K\rangle \leq \ell - i - 2$, or $\dim\langle J, K\rangle \leq \ell - i - 1$. In this case, however, no i-space I exists such that $\dim\langle I, J, K\rangle \geq \ell$, or $\dim\langle I, J, K\rangle > \ell$, respectively. Hence $\{J, K\}$ is not contained in any triangle of Γ, or $\{J, K\}$ is contained in a triangle of $\Gamma_{\leq}$ with every element of V_i, respectively. On the other hand, if for a given $J \in V_j$ and $K \in V_k$ we have $\dim\langle J, K\rangle > \ell - i - 2$, then, since $\ell - i - 2 \leq j + k$, there always exists an i-space I such that $\dim\langle I, J, K\rangle = \ell$. Hence the bipartite graph with vertex set $V_j \sqcup V_k$ and $J \in V_j$ adjacent with $K \in V_k$ if $\{J, K\}$ is not part of a triangle of Γ is isomorphic to the graph $\Gamma^n_{j,k;\geq i+j+k+2-\ell}(\mathbb{L})$ and is non-trivial by Theorem 3.1(1) ($j = k = i + j + k + 2 - \ell$ is impossible since $\ell > i + j + 2$, and $i + j + k + 2 - \ell > -1$ by assumption). Also, if for a given $J \in V_j$ and $K \in V_k$ we have $\dim\langle J, K\rangle > \ell - i - 1$, then, since $\ell < n$, there always exists an i-space I such that $\dim\langle I, J, K\rangle > \ell$. Hence the bipartite graph with vertex set $V_j \sqcup V_k$ and $J \in V_j$ adjacent with $K \in V_k$ if $\{J, K\}$ is part of a triangle of $\Gamma_{\leq}$ with any element of V_i is isomorphic to the graph $\Gamma^n_{j,k;\geq i+j+k+1-\ell}(\mathbb{L})$ and is non-trivial, like above, except if $\ell = i + j + k + 2$, in which case clearly $\Gamma_{\leq}$ is also trivial (complete tripartite graph). By Theorem 3.1(3), Proposition 3.3(i),(iii) and Lemma 3.6, θ is induced by a collineation of $\mathsf{PG}(n, \mathbb{L})$.

If $i < j < k$, then we exhausted all cases with the previous arguments, and we reduced the other cases to bipartite cases (or solved them).

3.4. **The bipartite case.** In this paragraph we consider $\Gamma := \Gamma_{i,\{i,j\};\ell} = (\mathcal{V}, \mathcal{T}_{i,\{i,j\};\ell})$ and $\Gamma_{\leq} := \Gamma_{i,\{i,j\};\leq\ell} = (\mathcal{V}, \mathcal{T}_{i,\{i,j\};\leq\ell})$ and we start with the case $i \leq j$. The reduction uses similar arguments as above, so we will leave out some details. We will also always assume that $\ell < n$ when we consider $\Gamma_{\leq}$.

THE CASE $i \leq j$.

First of all, when $\ell \geq 2i+j+3$, then we either have an empty 3-graph or a complete bipartite one.

When $\ell \leq i + j$, then for $I \in V_i$ and $J \in V_j$ we have $\dim(I \cap J) \leq i + j - \ell - 1$ if and only if $\{I, J\}$ is not contained in any triangle of Γ or of $\Gamma_{\leq}$. We derive $\Gamma^n_{i,j;\geq i+j-\ell}(\mathbb{L})$ from this, and Theorem 3.1 together with Lemma 3.5 concludes this case, if we do not have $\ell = n$ or $i = j = \ell$. In the latter case, Γ and $\Gamma_{\leq}$ have no triangles. If $n = \ell$, then we derive $\Gamma^n_{i,j;i+j-\ell}(\mathbb{L})$ from the fact that for $I \in V_i$ and $J \in V_j$ we have $\dim(I \cap J) = i + j - \ell$ if and only if $\{I, J\}$ is contained in a triangle of Γ together with every $I' \in V_i$, $I' \neq I$. By Theorem 3.1(2), $\Gamma^n_{i,j;i+j-\ell}(\mathbb{L})$ is trivial if $i = j = 0$, $\ell = n = 1$ (which contradicts $\ell \leq i + j$) or $i = j = n - 1$. In the latter case, Γ is a complete bipartite 3-graph, so we're done. So if we do not have $i = j = n - 1$, we can apply Theorem 3.1(3) and Lemma 3.5 to conclude this case.

Suppose now $i + j + 2 \leq \ell \leq 2i + j + 2$ and $i < j$. Let $I \in V_i$ and $J \in V_j$ be arbitrary. Then $\dim\langle I, J\rangle \leq \ell - i - 2$ if and only if $\dim(I \cap J) \geq 2i + j + 2 - \ell$ if and only if $\{I, J\}$ is not contained in any triangle of Γ. Also, $\dim\langle I, J\rangle \leq \ell - i - 1$ if and only if $\dim(I \cap J) \geq 2i + j + 1 - \ell$ if and only if $\{I, J\}$ is contained in a triangle of $\Gamma_{\leq}$ together with any $I' \in V_i$, $I' \neq I$. Again we can conclude using Theorem 3.1(3) and Lemma 3.5, except if $\ell = 2i + j + 2$ for $\Gamma_{\leq}$. But in this case, the latter is trivial (complete bipartite, i.e., every pair in V_i forms a triangle with every vertex in V_j).

Next consider $\Gamma_{\leq}$ and suppose $\ell = i + j + 1$ and $i < j$. Let $I \in V_i$ and $J \in V_j$ be arbitrary. Then $I \subseteq J$ as subspaces of $\mathsf{PG}(n, \mathbb{L})$ if and only if $\{I, J\}$ is contained in a triangle of $\Gamma_{\leq}$ together with any $I' \in V_i$, $I' \neq I$. This yields the graph $\Gamma^n_{i,j;i}(\mathbb{L})$ and Theorem 3.1(3) concludes this case.

Now suppose $i = j$ (but remember we keep writing i and j to distinguish the biparts). The case $2i + 3 \leq \ell \leq 3i + 2$ is similarly as above: For $I \in V_i$ and $J \in V_j$, we have $\dim(I \cap J) \geq 3i + 2 - \ell$ if and only if $\{I, J\}$ is not contained in any triangle of Γ; $\dim(I \cap J) \geq 3i + 1 - \ell$ if and only if $\{I, J\}$ is contained in a triangle of $\Gamma_{\leq}$ together with every $I' \in V_i$, $I' \neq I$ (and the case $\ell = 3i + 2$ yields a complete bipartite 3-graph); we can now again apply Theorem 3.1(3).

Still for $i = j$, consider the case $\ell = 2i+2$. The vertices $I \in V_i$ and $J \in V_j$ represent the same subspace of $\mathsf{PG}(n, \mathbb{L})$ if and only if $\{I, J\}$ is not contained in any triangle of Γ. This allows reduction to the 3-graph $\Gamma_{i,\ell}$. Also, the vertices $I \in V_i$ and $J \in V_j$ represent i-subspaces of $\mathsf{PG}(n, \mathbb{L})$ that intersect in at least an $(i - 1)$-space if and only if $\{I, J\}$ is contained in a triangle of $\Gamma_{\leq}$ together with every $I' \in V_i$, $I' \neq I$ (recall $\ell < n$). Theorem 3.1(1) implies that $\Gamma^n_{i,j;\geq i-1}(\mathbb{L})$ is trivial if, and only if, $i = j = 0$, but then $\ell = 2 \geq 3i + 2$, and thus $\Gamma_{\leq}$ is a complete bipartite 3-graph. In all other cases, Theorem 3.1(3) concludes this case.

Still for $i = j$, consider now the case $\ell = 2i + 1$. The vertices $I \in V_i$ and $J \in V_j$ represent the same subspace of $\mathsf{PG}(n, \mathbb{L})$ if and only if $\{I, J\}$ is contained in a triangle of $\Gamma_{\leq}$ together with every $I' \in V_i$, $I' \neq I$. Also, the vertices $I \in V_i$ and $J \in V_j$ represent the same subspace of $\mathsf{PG}(n, \mathbb{L})$ if and only if there does not exist $I' \in V_i$, $I' \neq I$, such that $\{I^* \in V_i : \{I, I^*, J\} \in \mathcal{T}_{i,\{i,i\};2i+1}\} = \{I^* \in V_i : \{I', I^*, J\} \in \mathcal{T}_{i,\{i,i\};2i+1}\}$. This again allows reduction to the 3-graphs $\Gamma_{i;\leq\ell}$ and $\Gamma_{i;\ell}$.

So, in conclusion, we have either solved the case, or reduced to $\Gamma_{i;\leq\ell}$, or to $\Gamma_{i;\ell}$, or we have the unique case $\Gamma_{i,\{i,j\};i+j+1}$, with $i < j$. In this case we can also assume that $n > i + j + 1$, since otherwise $I \in V_i$ and $J \in V_j$ are disjoint if and only if for all $I' \in V_i$, the triple $\{I, I', J\}$ is a triangle of Γ, and Theorem 3.1(3) concludes this case. If $n > i + j + 1$, then we claim that $I \in V_i$ and $J \in V_j$ are disjoint if and only if there exists $I' \in V_i$ such that $\{I, I', J\}$ is a triangle of Γ, and $\Omega(I, J) := \{I^* \in V_i : \{I, I^*, J\} \in \mathcal{T}_{i,\{i,j\};i+j+1}, I^* \neq I'\} = \{I^* \in V_i : \{I', I^*, J\} \in \mathcal{T}_{i,\{i,j\};i+j+1}, I^* \neq I\} =: \Omega(I', J)$. Indeed, if I and J are disjoint, then clearly $\Omega(I, J)$ consists of all i-spaces in $\langle I, J\rangle$ except for I and I'. We can then take I' such that $I \neq I'$ and $\langle I, J\rangle = \langle I', J\rangle$, which is easy to do. If I and J are not disjoint, and $I' \in V_i$ is arbitrary but such that $\{I, I', J\}$ is a triangle in Γ, then we can choose $I^* \in V_i$ such that $\langle I, J\rangle = \langle I^*, J\rangle$. It follows that $\langle I', I, J\rangle = \langle I', I^*, J\rangle$, hence $\{I', I^*, J\}$ is a triangle of Γ, but $\{I, I^*, J\}$ is clearly not a triangle of Γ, proving our claim. We can apply Theorem 3.1(3) to conclude this case.

THE CASE $i > j$.

We again put $\Gamma := \Gamma_{i,\{i,j\};\ell} = (\mathcal{V}, \mathcal{T}_{i,\{i,j\};\ell})$ and $\Gamma_{\leq} := \Gamma_{i,\{i,j\};\leq\ell} = (\mathcal{V}, \mathcal{T}_{i,\{i,j\};\leq\ell})$, with $i > j$.

For $\ell \geq 2i + j + 3$, the 3-graph Γ is empty, and for $\ell \geq 2i + j + 2$, the 3-graph $\Gamma_{\leq}$ is complete bipartite.

If $\ell \leq i + j + 2$ and $\ell < n$, then, as before, we derive $\Gamma^n_{i;\geq 2i-\ell}(\mathbb{L})$ from Γ and from $\Gamma_{\leq}$. This graph is trivial either if $\ell = i$ (in which case both Γ and $\Gamma_{\leq}$ are also trivial, namely, empty), or if both $\ell = i + j + 2$ and $i = j + 1$. If $\ell = n$, then $\Gamma_{\leq}$ is complete bipartite, and we derive $\Gamma^n_{i;2i-\ell}(\mathbb{L})$ from Γ. This graph is trivial (using Theorem 3.2(2)) if, and only if, $i = n - 1 = l - 1$. In that case Γ is a complete bipartite 3-graph. If $\Gamma^n_{i;2i-\ell}(\mathbb{L})$ is non-trivial, we conclude these cases with Theorem 3.2(3), Proposition 3.3(*iv*) and Lemma 3.6.

Also the cases $i + j + 3 \leq \ell \leq 2i + j + 2$ are similarly as before. So the only case remaining is when $\ell = i + j + 2$ and $i = j + 1$, implying $\ell = 2i + 1$. But then a j-space J is contained in an i-space I if and only if $\{I, J\}$ is contained in a triangle of $\Gamma_{\leq}$ together with every $I' \in V_i$, $I \neq I$, and we again reduced the situation to Theorem 3.1. So we are only left with the unique case $\Gamma_{i,\{i,i-1\};2i+1}$. Put $j = i - 1$ for ease of notation. Also, for $I, I' \in V_i$, put $\Omega(I, I') = \{J \in V_j : \{I, I', J\} \in \mathcal{T}_{i,\{i,j\};2i+1}\}$. We claim that, in this case, $I, I' \in V_i$ span an $(i+1)$-space if, and only if, for every $I^* \in V_i$ such that $\Omega(I, I') = \Omega(I, I^*)$, we also have $\Omega(I, I') = \Omega(I', I^*)$. Indeed, suppose first that $\langle I, I'\rangle$ is $(i+1)$-dimensional and $I^* \in V_i$ is not contained in $\langle I, I'\rangle$. If $I' \subseteq \langle I, I^*\rangle$, then $\dim\langle I, I^*\rangle \geq i+2$ and we can choose a j-space J not disjoint from I' such that $\langle I, I^*, J\rangle$ is $(2i + 1)$-dimensional. Otherwise we can find a j-space J containing a point of $I' \setminus \langle I, I^*\rangle$ and spanning a $(2i+1)$-space together with $\langle I, I^*\rangle$, except if I and I^* are disjoint. In the latter case we select the j-space J in $\langle I, I^*\rangle$ not disjoint from I. In each of these three cases J belongs to $\Omega(I, I^*)$ but not to $\Omega(I, I')$. We conclude that, in order to have $\Omega(I, I') = \Omega(I, I^*)$, we must have $I^* \subseteq \langle I, I'\rangle$. But then also $\Omega(I, I') = \Omega(I', I^*)$. Now suppose that I and I' span a space of dimension at least $i + 2$. We select an i-space I^* in $\langle I, I'\rangle$ such that $\langle I, I^*\rangle = \langle I, I'\rangle$ and $\dim\langle I', I^*\rangle = i + 1$ (this is easily done). Clearly $\Omega(I, I') = \Omega(I, I'')$, but by the foregoing, we know $\Omega(I, I') \neq \Omega(I', I^*)$. Our claim is proved. Hence we can construct the graph $\Gamma^n_{i;i-1}(\mathbb{L})$, which is non-trivial (note that $i = 0$ is impossible since $j = i - 1 \geq 0$). Theorem 3.2(3), Proposition 3.3(*iv*) and Lemma 3.6 conclude this case.

3.5. **The ordinary case.** Let Γ be the 3-graph $\Gamma_{i;\ell}$ and $\Gamma_{\leq}$ the 3-graph $\Gamma_{i;\leq\ell}$. If $\ell \geq 3i + 3$, then Γ has no triangles, and $\Gamma_{\leq}$ is complete. If $\ell \leq 2i$, then $I_1, I_2 \in V_i$ intersect in a space of dimension at most $2i - 1 - \ell$ if and only if $\{I_1, I_2\}$ is not contained in any triangle of Γ or $\Gamma_{\leq}$. By considering the complement, we get $\Gamma^n_{i;\geq 2i-l}(\mathbb{L})$, which is, by Theorem 3.2(1), trivial if, and only if, $\ell = n$ or $i = \ell$. In the latter case, Γ and $\Gamma_{\leq}$ have no triangles. Now consider $\ell = n$, then $I_1, I_2 \in V_i$ intersect in a space of dimension $2i - n$ if and only if $\{I_1, I_2, I_3\}$ is a triangle of Γ for all $I_3 \in V_i, I_1 \neq I_3 \neq I_2$. By Theorem 3.2(2), $\Gamma^n_{i;2i-n}(\mathbb{L})$ is trivial if, and only if, $0 < i = n - 1 = l - 1$ ($i = 0, l = 1$ contradicts $\ell \leq 2i$). But then, Γ is a complete 3-graph and we are done. If $n \neq l \neq i$, Theorem 3.2(3) and Lemma 3.5 conclude this case.

Now suppose $2i + 3 \leq \ell \leq 3i + 2$. Let $I_1, I_2 \in V_i$. Then $\dim(I_1 \cap I_2) \geq 3i + 2 - \ell$ if and only if $\{I_1, I_2\}$ is not contained in any triangle of Γ, and $\dim(I_1 \cap I_2) \geq 3i + 1 - \ell$ if and only if $\{I_1, I_2\}$ is contained in a triangle of $\Gamma_{\leq}$ together with every $I \in V_i$, $I \neq I_1$. Noting that $\Gamma_{\leq}$ is complete if $\ell = 3i + 2$, we can conclude these cases with Theorem 3.2(3). This also holds for $\Gamma_{\leq}$ if $\ell = 2i + 2$.

So only the cases $\Gamma_{i;\leq 2i+1}$, $\Gamma_{i;2i+1}$ and $\Gamma_{i;2i+2}$ remain. Moreover, in these cases we may assume that $n > 2i+1$. Indeed, this is trivial for $\Gamma_{i;2i+2}$. Concerning $\Gamma_{i;\leq 2i+1}$, we obtain a complete 3-graph for $n = 2i+1$. Finally, if $n = 2i+1$ for $\Gamma_{i;2i+1}$, then $I, I' \in V_i$ are disjoint if and only if for all $I'' \in V_i$, $I \neq I'' \neq I'$, the triple $\{I, I', I''\}$ is a triangle of Γ, and Theorem 3.2(3) and Lemma 3.5 conclude this case, except if $i = 0$. But in that case, we use the Fundamental Theorem of Projective Geometry (see [1]).

First let $\Gamma_{\leq} = (V_i, \mathcal{T})$ be the 3-graph $\Gamma_{i;\leq 2i+1}$, with $n > 2i+1$. We are going to characterise the disjoint pairs of i-spaces.

Lemma 3.7. *Two i-spaces I_1 and I_2 of $\mathsf{PG}(n,\mathbb{L})$ are disjoint if, and only if, for all $I_3, I_4 \in V_i$, the fact that $\{I_1, I_2, I_3\}$ and $\{I_1, I_2, I_4\}$ are triangles of $\Gamma_{\leq}$ implies that $\{I_1, I_3, I_4\}$ and $\{I_2, I_3.I_4\}$ are triangles of $\Gamma_{\leq}$.*

Proof Suppose first that I_1 and I_2 are two disjoint i-spaces. Let $I_3, I_4 \in V_i$ be such that $\{I_1, I_2, I_3\}$ and $\{I_1, I_2, I_4\}$ are triangles of $\Gamma_{\leq}$. Then $I_3, I_4 \subseteq \langle I_1, I_2\rangle$ and hence $\langle I_1, I_3, I_4\rangle \subseteq \langle I_1, I_2\rangle$, implying $\{I_1, I_3, I_4\} \in \mathcal{T}$. Likewise $\{I_2, I_3, I_4\} \in \mathcal{T}$.

Now suppose I_1 and I_2 are two i-spaces intersecting in a subspace of dimension $\ell \geq 0$. Then $\dim\langle I_1, I_2\rangle = 2i - \ell$ and hence we can find a subspace U of $\mathsf{PG}(n,\mathbb{L})$ of dimension $\ell+1$ disjoint from $\langle I_1, I_2\rangle$. We can choose two distinct subspaces U_3 and U_4 of U of dimension ℓ and a subspace W of $\langle I_1, I_2\rangle$ of dimension $i - \ell - 1$ disjoint from I_1. Then the subspaces $I_3 = \langle U_3, W\rangle$ and $I_4 = \langle U_4, W\rangle$ are i-dimensional. Moreover $\dim\langle I_1, I_2, I_3\rangle = (2i - \ell) + \ell + 1 = 2i + 1$, so that $\{I_1, I_2, I_3\} \in \mathcal{T}$. Similarly, $\{I_1, I_2, I_4\} \in \mathcal{T}$. But, since $\langle I_1, I_3, I_4\rangle = \langle I_1, W, U_3, U_4\rangle = \langle I_1, I_2, U\rangle$, we have $\dim\langle I_1, I_3, I_4\rangle = 2i - \ell + (\ell + 1) + 1 = 2i + 2$ and so $\{I_1, I_3, I_4\} \notin \mathcal{T}$. □

As before, recognising disjoint i-spaces concludes with Theorem 3.2(3) the proof of this case, except if $i = 0$. We then, again, use the Fundamental Theorem of Projective Geometry (see [1]).

Now consider $\Gamma_{i;2i+1}$ and $\Gamma_{i;2i+2}$. Let Γ be either one of these two graphs. Put $\ell = 2i+1$ or $2i+2$, respectively.

Lemma 3.8. *Let $I_1, I_2, I_3 \in V_i$ be three distinct i-spaces of $\mathsf{PG}(n,\mathbb{L})$, where $n > 2i+1$. If $\dim\langle I_1, I_2\rangle = i+1$ and I_3 is not contained in $\langle I_1, I_2\rangle$, then there exists an i-space $I \in V_i$ such that $\dim\langle I_1, I_2, I\rangle = 2i+1$ and $\dim\langle I_1, I_3, I\rangle = 2i+2$.*

Proof Let $x \in I_2 \setminus I_1$. Select a subspace S of $\langle I_1, I_2, I_3\rangle$ disjoint from $\langle I_1, I_2\rangle$ and such that $\dim\langle I_1, I_2, S\rangle = -1 + \dim\langle I_1, I_2, I_3\rangle$ (possibly S is the empty subspace). Note that $\dim S \leq i - 1$. Finally, select a subspace S' disjoint from $\langle I_1, I_2, I_3\rangle$ such that $\dim S + \dim S' = i - 2$ (possibly S' is empty; this happens if and only if $\dim S = i - 1$ if and only if I_3 is disjoint from $\langle I_1, I_2\rangle$). Then $I = \langle x, S, S'\rangle$ is i-dimensional and we have $\dim\langle I_1, I_2, I\rangle = \dim\langle I_1, I_2\rangle + \dim I = 2i+1$. Also, $\dim\langle I_1, I_3, I\rangle = \dim\langle I_1, I_2, I_3\rangle + \dim S' + 1 = \dim\langle I_1, I_2\rangle + \dim S + 2 + \dim S' + 1 = 2i + 2$. □

Let, for distinct i-spaces $I, I' \in V_i$, the set $\Omega(I, I')$ be the set of all i-spaces I^* such that $\{I, I', I^*\}$ is a triangle of our 3-graph. It is clear, for $I_1, I_2, I_3 \in V_i$, that $\Omega(I_1, I_2) = \Omega(I_1, I_3)$ as soon as $\langle I_1, I_2\rangle = \langle I_1, I_3\rangle$ (and $\dim\langle I_1, I_2\rangle \neq \ell$). The previous lemma also implies that $\Omega(I_1, I_2) \neq \Omega(I_1, I_3)$ if $\dim\langle I_1, I_2\rangle = i+1$ and $\langle I_1, I_2\rangle \neq \langle I_1, I_3\rangle$. Hence, if $i \neq 0$, I_1 and I_2 intersect in a space of dimension $i-1$ if and only if, for all $I \in V_i$, as soon as $\Omega(I_1, I_2) = \Omega(I_1, I)$, then $\Omega(I_1, I_2) = \Omega(I_2, I)$ and $\{I_1, I_2, I\}$ is not a triangle of Γ. So we constructed $\Gamma^n_{i;i-1}(\mathbb{L})$, which is trivial if, and only if, $i = 0$ or $i = n-1$. In the former case, the claim of Main Result 2.2 follows from the Fundamental Theorem of Projective Geometry. In the latter case,

$n \geq l \geq 2i+1 = 2n-1$ yields $n = 1 = l$ and thus $\Gamma = \Gamma_{0;1}$ is a complete 3-graph, as claimed in Main Result 2.2. In all other cases, the result follows from Theorem 3.2(3).

3.6. **Automorphisms that do not preserve all classes.** By Proposition 3.3, this only applies to the tripartite cases $\Gamma_{i,j,k;\ell}$ and $\Gamma_{i,j,k;\leq\ell}$. So let θ be an automorphism of $\Gamma_{i,j,k;\ell}$ or $\Gamma_{i,j,k;\leq\ell}$ (which we shall refer to as Γ below) not stabilising each tripartition class.

First suppose that θ does not permute the classes. By Proposition 3.3(i) we then have $i < j = k = \ell$ and it is easy to see that V_i is preserved. The ordinary graph induced on the classes V_j and V_k is a matching, and clearly interchanging any two adjacent vertices and fixing all others induces an automorphism of Γ. Hence composing with an appropriate number of such automorphisms brings us back to the classes preserving case.

Hence from now on we may assume that θ permutes the classes nontrivially. According to Lemma 3.4 this only happens when $i = j$ or $j = k$ (or both). If $i = j < k$, then we can compose θ with the automorphism of Γ that maps every vertex of $V_i \sqcup V_j$ to its copy, and we are reduced to the classes preserving case. Similarly if $i < j = k$, and likewise for the case $i = j = k < \ell$.

This completes the proof of our Main Results.

4. The thin case

The analogue of the previous problems for sets is the following (we only consider the cases where we do not have a tripartite or bipartite graph, since similar arguments as before can reduce these cases to the given cases): Given a set S, possibly infinite, and given two natural numbers i, ℓ, with $i < \ell$, define the 3-graphs $\Gamma_{i;\ell}$ and $\Gamma_{i,\leq\ell}$ as follows. The vertices are the i-subsets of S and the triangles are the triples of i-subsets whose union is an ℓ-subset, or whose union is contained in an ℓ-subset, respectively. It is easy to see that $\Gamma_{i;\ell}$ is non-trivial (meaning there is at least one triangle) if and only if $i+1 \leq \ell \leq 3i$ and $\ell \leq |S|$. Also, $\Gamma_{i;\leq\ell}$ is non-trivial (meaning there is at least one triangle and not every triple is a triangle) if and only if $i+1 \leq \ell \leq 3i-1$ and $\ell < |S|$.

We now show the following theorem.

Theorem 4.1. *Whenever $\Gamma_{i;\ell}$ or $\Gamma_{i,\leq\ell}$ is non-trivial, every automorphism of it is induced by a permutation of S.*

Proof By Theorem 4.8 and (the remarks after) Theorem 4.2 of [3], it suffices to recognise pairs of i-subsets intersecting in either at least n elements, for some n, such that $1 \leq n \leq i-1$ and $2i - n < |S|$, or in exactly n elements, for some n, such that $1 \leq n \leq i-1$ and $2i - n \leq |S|$, excluding the case $(|S|, i, n) = (4n^*-1, 2n^*-1, n^*-1), n^* \geq 2$.

Suppose first that $\ell \leq 2i-1$. Then it is clear that two i-subsets are contained in a triangle if and only if their union is at most an ℓ-subset, and that happens if and only if their intersection contains at least $2i-\ell$ elements. Note that $1 \leq 2i-\ell \leq i-1$ and $2i-(2i-\ell) = \ell \leq |S|$. So, if $\ell < |S|$, we are done. Suppose now $\ell = |S|$. Then we are dealing with the 3-graph $\Gamma_{i;\ell}$, and it is easy to see that two i-subsets intersect in exactly $2i-\ell$ elements if, and only if, they are contained in a triangle with every other i-subset.

Next suppose that $\ell \geq 2i+1$ and first consider $\Gamma_{i;\leq \ell}$. Then it is clear that two i-subsets intersect in at least $3i-\ell$ elements if and only if they are contained in a triangle with every other i-subset (because it is equivalent with the two i-subsets contained in an $(\ell-i)$-subset). Note $1 \leq 3i-\ell \leq i-1$, $2i-(3i-\ell) = \ell-i < |S|$ and $(|S|, i, 3i-l) = (4n^*-1, 2n^*-1, n^*-1)$ does not hold for any $n^* \geq 2$, since otherwise $\ell < |S| = 2i+1$.

Now suppose $\ell \geq 2i+2$ and consider $\Gamma_{i;\ell}$. Here it is clear that two i-subsets are not contained in any triangle if and only if their union has at most $\ell-i-1$ elements, and this happens if and only if they intersect in at least $3i-\ell+1$ elements. We have $1 \leq 3i-\ell+1 \leq i-1$, $2i-(3i-\ell+1) = \ell-i-1 < |S|$ and $(|S|, i, 3i-l+1) = (4n^*-1, 2n^*-1, n^*-1)$ does not hold for any $n^* \geq 2$, since otherwise $\ell \leq |S| = 2i+1$

So only the three graphs $\Gamma_{i;\leq 2i}$, $\Gamma_{i;2i}$ and $\Gamma_{i;2i+1}$ remain (note that this fact is completely similar to the "thick" case).

First assume $n = 2i$, then we just have to consider $\Gamma_{i;2i}$ since the other graphs are trivial. Clearly two i-subsets are disjoint if and only if all other i-subsets form a triangle with these two subsets. So we can characterise when two i-subsets intersect in at least one element, note that $2i-1 < |S|$. We may now assume $n > 2i$. Consider two i-subsets I_1, I_2 intersecting in an $(i-1)$-subset J. Let I_3 be an i-subset (different from I_1 and I_2) with the property that the set $\Omega(I_1, I_2)$ of i-subsets I such that $\{I_1, I_2, I\}$ is a triangle coincides with the set $\Omega(I_1, I_3)$ of i-subsets I such that $\{I_1, I_3, I\}$ is a triangle. Suppose I_3 is not contained in $I_1 \cup I_2$ and let $x \in I_3 \setminus (I_1 \cup I_2)$. Let I be an i-subset disjoint from I_1, containing $((I_2 \setminus I_1) \cup (I_3 \setminus I_1)) \setminus \{x\}$ and not containing x, we can find such set because $n > 2i$. Then clearly $|I_1 \cup I_2 \cup I| = i+1+i-1 = 2i$, whereas $|I_1 \cup I_3 \cup I| = 1+i+i-1+1 = 2i+1$. This contradicts our assumption on I_3. On the other hand, if $I_3 \subseteq I_1 \cup I_2$ and $I_1 \neq I_3 \neq I_2$, then clearly, for all i-subsets I we have $I_1 \cup I_2 \cup I = I_1 \cup I_3 \cup I = I_2 \cup I_3 \cup I$. We conclude that, if two i-subsets I_1, I_2 intersect in an $(i-1)$-subset, then for each i-subset I_3 holds that, if $\Omega(I_1, I_2) = \Omega(I_1, I_3)$, then $\Omega(I_1, I_2) = \Omega(I_2, I_3)$.

Now suppose the i-subsets I_1, I_2 are such that $|I_1 \cup I_2| \geq i+2$. Suppose $I_1 \cap I_2 = \emptyset$, and suppose that we are considering $\Gamma_{i,\leq 2i}$ or $\Gamma_{i,2i}$. Then $\Omega(I_1, I_2)$ consists of all i-subsets in $I_1 \cup I_2$ distinct from I_1 and I_2. Hence $\Omega(I_1, I_3) = \Omega(I_1, I_2)$ for a certain i-subset I_3 implies $I_3 = I_2$. For the graph $\Gamma_{i,2i+1}$ one obtains the same result in a similar fashion. Now suppose that I_1 and I_2 intersect non-trivially. Then we can easily choose an i-subset I_3 such that $|I_2 \cup I_3| = i+1$ and $I_1 \cup I_2 = I_1 \cup I_3$. It follows from the previous paragraph that $\Omega(I_2, I_3) \neq \Omega(I_1, I_2)$, whereas clearly $\Omega(I_1, I_2) = \Omega(I_1, I_3)$. Hence, we conclude that two i-subsets I_1, I_2 intersect in an $(i-1)$-subset if, only if, there exist an i-subset I_3 (distinct from I_1 and I_2) such that $\Omega(I_1, I_2) = \Omega(I_1, I_3)$, and for any such subset $\Omega(I_1, I_2) = \Omega(I_2, I_3)$.

Hence we can recognise pairs of i-subsets intersecting in exactly $i-1$ elements (note that $(|S|, i, i-1) = (4n^*-1, 2n^*-1, n^*-1)$ for some $n^* \geq 2$ is impossible). This concludes the proof of the theorem. □

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Ghent University, Department of Mathematics: Algebra and Geometry, Krijgslaan 281- S25, B-9000 Ghent

Email address: `Jeroen.Meulewaeter@hotmail.com`

Email address: `Hendrik.VanMaldeghem@UGent.be`