Remarks on Finite Generalized Hexagons and Octagons with a Point-Transitive Automorphisme Group

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Abstract

We show that the only point-transitive representations of the groups displayed in the ATLAS [2] on a finite generalized hexagon or octagon are the natural ones.

1. Introduction.

Let Γ be a thick, finite generalized hexagon (resp. octagon) of order (s,t) and G a group of automorphisms of Γ acting transitively on the points. Assume furthermore that G is almost simple, so there is a nonabelian simple group S with

$$S \trianglelefteq G \leq \mathrm{Aut}S.$$

We want to show that "small" G are ruled out, in particular that S cannot be a sporadic group. As a matter of fact, we consider all groups displayed in the main section of the ATLAS [2]. Let us call these groups "ATLAS-groups", then we can formulate our main results as follows:

Theorem 1.1 *If an* ATLAS*-group acts transitively on the points of a generalized hexagon, then it is has socle $G_2(q)$ $(q = 2,3,4,5)$ or ${}^3D_4(2)$ and it acts in the natural way on a 'classical' generalized hexagon or its dual.*

Theorem 1.2 *If an* ATLAS*-group acts transitively on the points of a generalized octagon, then it is has socle ${}^2F_4(2)'$ and it acts in the natural way on the 'classical' generalized octagon of order $(2,4)$ or its dual.*

Theorem 1.1 will be proved in section 3 and theorem 1.2 will be proved in section 4.

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2. Some known facts.

2.1. Generalized Hexagons.

Let Γ be a generalized hexagon of v points and order (s,t). Then $v=(s+1)(1+st+s^2t^2)$, st is a perfect square (Feit & Higman [3]) and $s \le t^3 \le s^9$ (Haemers & Roos [4]). Also, the rational number

$$\frac{st(s+1)(t+1)(1+st+s^2t^2)}{2[s^2t+t^2s-st+s+t\pm(s-1)(t-1)\sqrt{st}]} \tag{1}$$

is an integer (Higman [5]).

2.2. Generalized Octagons.

Let Γ be a generalized octagon of v points and order (s,t). Then $v=(s+1)(1+st)(1+s^2t^2)$, $2st$ is a perfect square (Feit & Higman [3]) and $s \le t^2 \le s^4$ (Higman [5]). Also, the rational number

$$\frac{st(s+1)(t+1)(1+st)(1+s^2t^2)}{4[s^2t+t^2s-2st+s+t\pm(s-1)(t-1)\sqrt{2st}]} \tag{2}$$

is an integer (Higman [5]).

3. Generalized hexagons.

In this section, we prove theorem 1.1.

We use the notation above and put $u=\sqrt{st}$ and $w=s+t$. Rewriting condition (1) we have that

$$\frac{u^2(1+w+u^2)(1\pm u+u^2)}{2(w-u)} \tag{3}$$

must be an integer for both choices of signs.

Suppose that G acts transitively on the v points of a thick generalized hexagon Γ of order (s,t) and G is a one of the simple groups listed in the Atlas [2], see also tables 2 and 3 below. Since $v=(1+s)(1+st+s^2t^2)$, the latter expression divides $|G|$. Let p be a prime dividing $1+st+s^2t^2$. Then $1+st+s^2t^2 \equiv 0 \pmod p$, hence $s^3t^3 \equiv 1 \pmod p$. If $st \equiv 1 \pmod p$, then clearly $1+st+s^2t^2 \equiv 3 \pmod p$ and so $p=3$. In the other case, 1 must have three distinct third roots in $GF(p)$, so $p-1$ is divisible by 3 or in other words,

$p \equiv 1 \pmod 3$. Note that for any integer n, $1+n+n^2$ is never divisible by 9. Put $a(G)$, or simply a, for the largest integer divisible by 3, but not by 9, all of whose other prime divisors are congruent to 1 (mod 3) and such that $a(G)$ divides $|G|$. We now distinguish between "small" groups and "larger" ones, the larger ones being $E_7(2)$, M and $E_8(2)$.

3.1. Small Groups.

Given G, it turns out that $a(G)$ only depends on its "socle" S except for $S \cong Sz(8)$ in which case we consider $a(\mathrm{Aut}Sz(8))$. Obviously, un upper bound U for u is given by the fourth root of $a(G)$. We can then look at table 1; it contains all values for $(1+st+s^2t^2)$ for given u, $2 \leq u \leq 136$. We consider the largest number $U^* \leq U$ such that $1+(U^*)^2+(U^*)^4$ divides $|G|$. This is clearly a new upper bound for u. Hence $st \leq (U^*)^2$ and since $s \leq t^3$, this implies

$$s \leq \sqrt[4]{(st)^3} \leq \sqrt{(U^*)^3}.$$

So

$$v \leq \lfloor(\sqrt{(U^*)^3}+1)\rfloor(1+(U^*)^2+(U^*)^4)$$

and we denote the latter by $h(G)$. If $a(G) > 3$ (that means, if $U > 1$), then we list the values for $a(G)$, U, U^* and $h(G)$ (if $U^* \geq 2$) in table 2, in which we also include the number $P(G)$ defined as the smallest permutation degree of S. The value for $P(G)$ follows from LIEBECK & SAXL [6] for ${}^2E_6(2)$ and $E_6(2)$; from MAZUROV [7] for the sporadic groups and from the ATLAS [2] for the other groups. The "ATLAS-groups" with $a(G) \leq 3$ are A_5, A_6, $L_2(11)$, $L_2(17)$, $L_2(16)$, $L_2(23)$, M_{11}, $U_4(2)$, M_{12}, $S_4(4)$ and $U_5(2)$.

In a lot of cases, we have $h(G) < P(G)$ which is a contradiction. If $U^* = 2$, then $s = t = 2$ and by TITS [8], Γ is the unique classical generalized hexagon $H(2)$ arising from the classical group $U_3(3) \cong G_2(2)'$. Only one simple group is a proper subgroup of $U_3(3)$, namely $L_3(2)$. But this group does not act transitively on the 63 points of $H(2)$ because 63 does not divide $|L_3(2)| = 168$. Of course if $S \cong U_3(3)$, then G acts transitively on exactly two generalized hexagons, namely $H(2)$ and its dual. The only remaining sporadic group is Suz. The largest possible value for s or t is 8 (when u=4; in general the largest value for s or t is $\sqrt{u^3}$, see above). Now Suz contains an element θ of order 11. Since $11 \equiv 2 \pmod 3$ and $11 > s+1, t+1$, θ fixes at least one point x, all lines through x, all points on all lines through x, etc. So θ fixes everything, a contradiction. In the sequel, we shall refer to this argument by the expression: *a group element of order* 11 *cannot live in* Γ. We consider the other groups in turn. Note that $u > 2$ (by the argument above), so $(s,t) \neq (2,2)$. A similar argument kills $(s,t) = (2,8)$ and $(s,t) = (8,2)$.

u	$1+u^2+u^4$	u	$1+u^2+u^4$	u	$1+u^2+u^4$	u	$1+u^2+u^4$
2	3.7	36	13.31.43.97	70	3.1657.4831	104	3.67.163.3571
3	7.13	37	3.7.31.43.67	71	3.1657.5113	105	67.163.11131
4	3.7.13	38	3.7.67.1483	72	7.751.5113	106	3.19.199.11131
5	3.7.31	39	7.223.1483	73	3.7.751.1801	107	3.7.13.19.127.199
6	31.43	40	3.7.223.547	74	3.7.13.61.1801	108	7.13.61.127.193
7	3.19.43	41	3.547.1723	75	7.13.61.5701	109	3.7.61.193.571
8	3.19.73	42	13.139.1723	76	3.1951.5701	110	3.7.571.12211
9	7.13.73	43	3.13.139.631	77	3.1951.6007	111	12211.12433
10	3.7.13.37	44	3.7.283.631	78	6007.6163	112	3.4219.12433
11	3.7.19.37	45	7.19.109.283	79	$3.7^2.43.6163$	113	3.13.991.4219
12	7.19.157	46	3.7.19.103.109	80	$3.7^2.43.6481$	114	7.13.991.1873
13	3.61.157	47	3.7.37.61.103	81	7.13.73.6481	115	3.7.1873.4447
14	3.61.211	48	13.37.61.181	82	3.7.13.73.2269	116	$3.7^2.277.4447$
15	211.241	49	3.13.19.43.181	83	3.19.367.2269	117	$7^2.277.13807$
16	3.7.13.241	50	3.19.43.2551	84	19.37.193.367	118	3.31.151.13807
17	3.7.13.307	51	7.379.2551	85	3.37.193.2437	119	3.31.151.14281
18	$7^3.307$	52	3.7.379.919	86	3.7.1069.2437	120	13.1117.14281
19	$3.7^3.127$	53	3.7.409.919	87	7.13.19.31.1069	121	3.7.13.19.37.1117
20	3.127.421	54	7.409.2971	88	3.7.13.19.31.373	122	3.7.19.37.43.349
21	421.463	55	3.13.79.2971	89	3.7.373.8011	123	7.43.349.2179
22	$3.13^2.463$	56	3.13.31.79.103	90	8011.8191	124	3.7.2179.5167
23	$3.7.13^2.79$	57	31.103.3307	91	3.2791.8191	125	3.19.829.5167
24	7.79.601	58	3.7.163.3307	92	3.43.199.2791	126	13.19.829.1231
25	3.7.31.601	59	3.7.163.3541	93	7.43.199.1249	127	3.13.1231.5419
26	3.7.19.31.37	60	7.523.3541	94	3.7.13.229.1249	128	$3.7^2.337.5419$
27	19.37.757	61	3.7.13.97.523	95	3.7.13.229.1303	129	$7^2.31.337.541$
28	3.271.757	62	3.13.97.3907	96	7.67.139.1303	130	3.7.31.541.811
29	3.13.67.271	63	37.109.3907	97	3.67.139.3169	131	3.7.811.17293
30	$7^2.13.19.67$	64	3.19.37.73.109	98	3.31.313.3169	132	97.181.17293
31	$3.7^2.19.331$	65	3.7.19.73.613	99	31.313.9901	133	3.13.97.181.457
32	3.7.151.331	66	7.613.4423	100	3.7.13.37.9901	134	3.13.79.229.457
33	7.151.1123	67	$3.7^2.31.4423$	101	3.7.13.37.10303	135	7.43.61.79.229
34	3.397.1123	68	$3.7^2.13.19^2.31$	102	7.19.79.10303	136	3.7.43.61.6211
35	3.13.97.397	69	$13.19^2.4831$	103	3.7.19.79.3571	137	3.7.37.73.6211

Table 1.

S	a	U	U^*	$h(S)$	$P(G)$	S	a	U	U^*	$h(G)$	$P(G)$
$L_3(2)$	3.7	2	2	63	7	HS	3.7	2	2	63	100
$L_2(8)$	3.7	2	2	63	9	J_3	3.19	2	1		6156
$L_2(13)$	3.7.13	4	4	2457	14	$U_3(11)$	3.37	3	1		1332
A_7	3.7	2	2	63	7	$O_8^+(2)$	3.7	2	2	63	120
$L_2(19)$	3.19	2	1		20	$O_8^+(2)$	3.7	2	2	63	119
$L_3(3)$	3.13	2	1		13	${}^3D_4(2)$	$3.7^2.13$	6	4	2457	819
$U_3(3)$	3.7	2	2	63	28	$L_3(11)$	3.7.19	4	2	63	133
$L_2(25)$	3.13	2	1		26	A_{12}	3.7	2	2	63	12
$L_2(27)$	3.7.13	4	4	2457	28	M_{24}	3.7	2	2	63	24
$L_2(29)$	3.7	2	2	63	30	$G_2(4)$	3.7.13	4	4	2457	416
$L_2(31)$	3.31	3	1		32	McL	3.7	2	2	63	275
A_8	3.7	2	2	63	8	A_{13}	3.7.13	4	4	2457	13
$L_3(4)$	3.7	2	2	63	21	He	3.7^2	3	2	63	2058
$Sz(8)$	3.7.13	4	4	2457	65	$O_7(3)$	3.7.13	4	4	2457	351
$L_2(32)$	3.31	3	1		33	$S_6(2)$	3.7.13	4	4	2457	364
$U_3(4)$	3.13	2	1		65	$G_2(5)$	3.7.31	5	5	7929	3906
$U_3(5)$	3.7	2	2	63	50	$U_6(2)$	3.7	2	2	63	672
J_1	3.7.19	4	2	63	266	$R(27)$	3.7.13.19.37	20	11	54.10^4	2.10^4
A_9	3.7	2	2	63	9	$S_8(2)$	3.7	2	2	63	120
$L_3(5)$	3.31	3	1		31	Ru	3.7	2	2	63	4060
M_{22}	3.7	2	2	63	22	Suz	3.7.13	4	4	2457	1782
J_2	3.7	2	2	63	100	$O'N$	$3.7^3.19.31$	27	5	7929	122760
$S_6(2)$	3.7	2	2	63	28	Co_3	3.7	2	2	63	276
A_{10}	3.7	2	2	63	10	$O_8^+(3)$	3.7.13	4	4	2457	1080
$L_3(7)$	$3.7^3.19$	11	2	63	57	$O_8^-(3)$	3.7.13	4	4	2457	1066
$U_4(3)$	3.7	2	2	63	112	$O_{10}^+(2)$	3.7.31	5	5	7929	496
$G_2(3)$	3.7.13	4	4	2457	351	$O_{10}^-(2)$	3.7	2	2	63	495
$S_4(5)$	3.13	2	1		156	Co_2	3.7	2	2	63	2300
$U_3(8)$	3.7.19	4	2	63	513	Fi_{22}	3.7.13	4	4	2457	3510
$U_3(7)$	$3.7^3.43$	14	2	63	344	HN	3.7.19	4	2	63	1140000
$L_4(3)$	3.13	3	1		40	$F_4(2)$	$3.7^3.13$	6	4	2457	69615
$L_5(2)$	3.7.31	5	5	7929	31	Ly	3.7.31.37.67	136	5	7929	8835156
M_{23}	3.7	2	2	63	23	Th	$3.7^2.13.19.31$	32	5	7929	$> 10^8$
$L_3(8)$	$3.7^2.73$	10	2	63	73	Fi_{23}	3.7.13	4	4	2457	31671
${}^2F_4(2)'$	3.13	2	1		1600	Co_1	$3.7^2.13$	6	4	2357	98280
A_{11}	3.7	2	2	63	11	J_4	3.7.31.37.43	31	6	19995	$> 10^8$
$Sz(32)$	31	2	0		1025	${}^2E_6(2)$	$3.7^2.13.19$	13	4	2457	3968055
$L_3(9)$	3.7.13	4	4	2457	91	$E_6(2)$	$3.7^3.13.31.73$	74	9	186004	139503
$U_3(9)$	3.73	3	1		730	Fi'_{24}	$3.7^3.13$	10	4	2457	306936
						B	$3.7^2.13.19.31$	32	5	7929	$> 10^{10}$

Table 2.

Indeed, the generalized hexagons with these orders are unique by COHEN & TITS [1] and the related simple group is ${}^3D_4(2)$. Its proper simple subgroups are $L_3(2)$, $L_2(8)$ and $U_3(3)$ (ATLAS [2]). None of these groups has a divisible by 13, which shows our assertion.

$\boxed{L_2(13)}$ Here $u \leq 4$ and so $(s,t) = (3,3)$ or $(s,t) = (4,4)$. In the latter case, $v = 1365$ and this does not divide $|\mathrm{Aut}(L_2(13)| = 2184$. In the former case, the stabilizer G_x of a point x contains an element θ of order 3. In G, there are exactly 91 Sylow 3-subgroups. Hence θ must fix exactly 4 points of Γ. These 4 points form a set of imprimitivity. Since there are 4 lines through x, θ must fix one of these lines, say l. If y is another fixed point for θ, then θ also fixes the point on l nearest to y. Consequently, θ fixes l pointwise. The 91 lines thus obtained form a partition of the point set and G acts transitively on the set $\mathcal{L}$ of such lines. Hence G acts primitively on that set and since the stabilizer G_l of l normalizes θ, G_l is isomorphic to D_{12} or D_{24} (see the ATLAS [2]). So we can identify the 91 lines in $\mathcal{L}$ with the 91 pairs of points of the projective line over $GF(13)$.

Suppose first $G \cong L_2(13)$. Then $|G_x| = 3$, so no involution can fix a point in Γ. Every involution fixes exactly 7 lines of $\mathcal{L}$ (that is the number of pairs it stabilizes on the projective line $PG(1,13)$). These seven lines are mutually *opposite* (on maximal distance) since otherwise a point is fixed. Identify an arbitrary line $l \in \mathcal{L}$ with the pair $\{(0),(\infty)\}$. All pairs $\{(r),(s)\}$, $r,s \in GF(13)^\dagger$, with r/s a square in $GF(13)$ can be stabilized under a certain involution also stabilizing $\{(0),(\infty)\}$, and the others cannot. There are 30 such pairs and by the preceding argument they are all opposite l. Left are five orbits of length 12 under the stabilizer of $\{(0),(\infty)\}$. Two of these orbits contain all pairs of the form $\{(0),(r)\}$ and $\{(\infty),(r)\}$ with r a square, resp. a non-square, denote them by $O_\square$, resp. $O_{\not\square}$. The other three orbits contain pairs $\{(s),(2s)\}$, resp. $\{(s),(5s)\}$, $\{(s),(6s)\}$ and we denote them by O_i, $i = 2,5,6$ respectively. Since there are 36 elements of $\mathcal{L}$ at distance 4 from l, at least one of the sets O_i, $i = 2,5,6$, must have all its elements at distance 4 from l. Suppose $\{(r),(s)\} \in O_2 \cup O_5$ is a line at distance 4 from l. Let $y_{r,s}$ (resp. $l'_{r,s}$) be the unique point (resp. line) of Γ at distance 1 (resp. 2) from l and at distance 3 (resp. 2) from $\{(r),(s)\}$. Applying θ, we see that also $\{(3r),(3s)\}$ is at distance 4 from l. Define $y_{3r,3s}$ and $l'_{3r,3s}$ as above. Obviously $y_{r,s} = y_{3r,3s}$. One can verify that $\{(r),(s)\}$ and $\{(3r),(3s)\}$ lie at distance 6 from each other, so $l'_{r,s} \neq l'_{3r,3s}$. Hence θ acts transitively on the lines distinct from l through each point of l. This implies that G acts transitively on the pairs of lines (m,m'), with $m \in \mathcal{L}$ and $m' \notin \mathcal{L}$. So G acts transitively on the lines not in $\mathcal{L}$, but this action is imprimitive with sets of imprimitivity of order 3. The stabilizer of such a set is A_4 (see ATLAS [2]). So A_4 acts regularly on the 12 points of the three lines of a set of imprimitivity. This implies the existence of an involution swapping any two points on a line not in $\mathcal{L}$, or in other words, swapping two lines of $\mathcal{L}$ at distance 4. But the pairs $\{(0),(\infty)\}$ and $\{(r),(s)\}$ are swapped by an element of $PGL_2(13) \setminus L_2(13)$. Hence the lines of $\mathcal{L}$ at distance 4 from l are precisely all elements of O_6, $O_{\square}$ and $O_{\not\square}$ and θ fixes all lines meeting l. Therefore, the sets $L = \{\{(\infty),(r)\}||r \in \{0,1,3,9\}\}$ and $L' = \{\{(\infty),(r)\}||r \in \{0,2,5,6\}\}$ consist of 4 lines meeting a common line. But the automorphism determined by adding 12 to each coordinate maps $\{(\infty),(1)\}$ to l and $\{(\infty),(3)\}$ to $\{(\infty),(2)\}$, hence L should be mapped to L', but it is not, as one can verify immediately.

Next suppose $G \cong PGL_2(13)$. Then $|G_x| = 6$ and so there is an involution θ fixing x. Also, θ fixes exactly 7 lines of $\mathcal{L}$, among them the unique line l of $\mathcal{L}$ incident with x. If all other fixed lines of $\mathcal{L}$ have distance 4 from l, then θ fixes at least two points on each of these lines and so θ fixes points at distance 5 from l and from other fixed lines, hence θ fixes an ordinary hexagon. In the other case, this is trivially true. Hence θ fixes a subhexagon of order $(1,3)$ or $(3,1)$ (since θ fixes at least 7 lines from $\mathcal{L}$, any other possible configuration of fixed structure contains at most 5 elements of $\mathcal{L}$). So *theta* fixes either 26 or 52 points. Since there are no involutions in G_x other than θ, this implies that θ has exactly 14, resp. 7 conjugates in G, which means that θ is normalized by a group of order at least 2.3.13, contradicting the information on $L_2(13)$ given in the ATLAS [2].

$\boxed{L_2(27)}$ As in the previous case, $(s,t) = (3,3)$ or $(s,t) = (4,4)$. Also $(s,t) = (4,4)$ is eliminated the same way. If $(s,t) = (3,3)$, then the stabilizer of a point x has order $3^3.[G : L_2(27)]$ and it follows from the ATLAS [2] that it normalizes an elementary abelian subgroup of order 3^3. There are 28 such groups (two by two disjoint) and therefore any element of such group fixes exactly 13 points of Γ. As before, these 13 points form a set Ω of imprimitivity containing the points of some line l through x (considering an element of order 3 in G_x). Any other point of Ω defines a point nearest to l which must also be in Ω. Hence we can assume that there is another line l' through x all of whose points are in Ω. By transitivity, every point of Ω is incident with two lines all of whose points are in Ω. This is impossible in view of $|\Omega| = 13$.

$\boxed{Sz(8)}$ Again $(s,t) = (3,3)$ or $(s,t) = (4,4)$. In the former case, a group element of order 5 cannot live in Γ. In the latter case, we deduce as above that every of the 65 Sylow 2-groups fixes 21 points, which form a set of imprimitivity. As before, such a set cannot exist.

$\boxed{G_2(3)}$ Again $(s,t) = (3,3)$ or $(s,t) = (4,4)$. In the latter case, v does not divide $|G|$. In the former case, Γ must be the classical generalized hexagon since G essentially has only one transitive representation on 364 points.

$\boxed{L_5(2)}$ In view of $a = 3.7.31$, only $(s,t) = 5$ is possible. Consider a point x. The stabilizer G_x contains a group H of order 2^9. A line through x is in an orbit of length 1,2 or 4 under H. If it is in an orbit of length 4, then some other line through x is in an orbit of length 2. Hence a group H^* of order 2^8 fixes a line l through x. Since $s = t = 5$, this group must fix another line through x and another point on l, etc. So at least an ordinary hexagon is fixed by H^*. A point y on one of the sides has an orbit of size at most 4, the stabilizer of y fixes also another point on the same side and there are only 2 points left on that side. Hence a group of order at least 2^5 fixes a hexagon and all points on one of its sides. Similarly, a group of order at least 2^2 fixes a hexagon, all points on one of its sides and all lines through one of its vertices. But then all elements of Γ are fixed (since this generates a subhexagon of order $(5,5)$), a contradiction.

$\boxed{L_3(9)}$ Here $(s,t) = (3,3)$ or $(s,t) = (4,4)$. In the former case, a group element of order 5 cannot live in G. In the latter case, the stabilizer G_x of a point x contains a group of order 3^5. As above, this group fixes a hexagon and a subgroup of order 3^3 fixes everything, a contradiction.

$\boxed{{}^3D_4(2)}$ We have to rule out $(s,t) = (3,3)$ and $(s,t) = (4,4)$. In both cases, the stabilizer G_x of a point x contains an element of order 7 and such an element cannot live in Γ.

$\boxed{G_2(4)}$ Order $(3,3)$ is ruled out by the presence of an element of order 5 in G. The representation on 1365 points is essentially unique, hence the "classical" generalized hexagon of order $(4,4)$ and its dual arise.

$\boxed{A_{13}}$ Every order is ruled out by the presence of an element of order 13, which cannot live in Γ.

$\boxed{O_7(3)}$ If $(s,t) = (3,3)$, then a group element of order 5 cannot live in G. If $(s,t) = (4,4)$, then, as in the case of $L_3(9) \trianglelefteq G$, the presence of a group of order 3^8 in the stabilizer of a point leads to a contradiction.

$\boxed{S_6(3)}$ See $O_7(3) \trianglelefteq G$; both groups have the same order.

$\boxed{G_2(5)}$ Here, only $(s,t) = (5,5)$ is possible, it occurs and it is unique up to duality (by the information in the Atlas [2]).

$\boxed{R(27)}$ Here, $u = 11, 10, 4, 3, 2$. But $u \leq 4$ is impossible in view of $P(G)$. If $u = 11$, then $s = t = 11$ and a group element of order 13 cannot live in Γ. If $u = 10$, then $s = 4, 5, 10, 20$ or 25. But $s = 4, 10, 20$ or 25 implies v does not divide $|G|$ ($s = 4$ and $s = 25$ are also ruled out by the fact that (3) is not an integer in these cases). Hence $(s, t) = (5, 20)$. The order of the stabilizer of a point is $2^2.3^b.19$, with $b = 8$ or 9 depending on $G \cong R(27)$ or $G \cong R(27) : 3$. So G_x properly contains the normalizer of a Sylow 19-subgroup, which is a maximal subgroup, a contradiction.

$\boxed{O_8^+(3)}$ In view of $P(S) = 1080$, the only possibility here is $s = t = 4$. The order of the stabilizer of a point of Γ is divisible by $2^{12}.3^{11}.5$. The presence of a Sylow 3-subgroup of G_x of order at least 3^{11} leads to a contradiction as in the case $S \cong L_3(9)$ or $S \cong O_7(3)$.

$\boxed{O_8^-(3)}$ Whatever the order of Γ, a group element of order 41 cannot live in G.

$\boxed{O_{10}^+(2)}$ Whatever the order of Γ, a group element of order 17 cannot live in Γ.

$\boxed{E_6(2)}$ In view of table 1, $u = 9$ or $u \leq 5$. In the latter case, v would be smaller than $P(G)$; in the former case, a group element of order 31 cannot live in Γ (because $s, t \leq 27$).

This completes the case of small groups.

3.2. Larger Groups.

In this case, we can still compute U as above, but it is too large to use table 1 to find U^*. But we can use U to compute $h(G)$ as in the previous paragraph. For $G \cong M$, we obtain

$$\begin{array}{rcl} a(G) & = & 3.7^6.13^3.19.31 \\ h(G) & \approx & 107.10^{14} \\ P(G) & \approx & 927.10^{17} \end{array}$$

This rules out $G \cong M$. In general, there is obviously a minimal value U_* for u such that the derived number $h(G)$ is larger then $P(G)$. So $U_* \leq u \leq U$. We now develop a method to reduce the bound U until it is below U_* without having to calculate all values for $1 + u^2 + u^4$. Consider a divisor d of $a(G)$, preferably larger than or just a little bit smaller than U. The number of u such that d divides $1 + u^2 + u^4$ is limited and usually none of these values for u (except maybe very small ones which are in conflict with U_* anyway) give a $1 + u^2 + u^4$ dividing $a(G)$. Hence we can recalculate U starting from $a(G)/p$, where P is the smallest prime dividing d. We refer to this procedure as "*reduction modulo d*". We usually take for d a prime or a prime power, so that we can do this reduction a few times and end up with no value of u left. Let us illustrate this in the case of $G \cong E_7(2)$. We have here

$$\begin{aligned} a(G) &= 3.7^3.13.19.31.43.73.127 \\ P(G) &= 277347807 \\ U &= 1331 \\ U_* &= 35 \end{aligned}$$

It is easy to calculate by hand that only $u = 1250$ gives rise to $1 + u^2 + u^4$ divisible by 73.127. Indeed, every u giving $1 + u^2 + u^4$ divisible by 73 is congruent to 8,9,64 or 65 modulo 73, similarly, every suitable u must also be congruent to 19,20, 107 or 108 modulo 127. Only 1250 satisfies these conditions and is smaller 1331. But in the same way, we see that 43 does not divide $1 + u^2 + u^4$ in that case, so $d = 43.73.127$ gives us no solutions and the new upper bound is $U' = 519$. We now see that $1250 > 519$, so the new upper bound becomes 455. Putting $d = 127$, the possible values for u are 19, 107, 146, 234, 273, 361 and 400. This gives us:

u	$1 + u^2 + u^4$	u	$1 + u^2 + u^4$
19	$3.7^3.127$	20	3.127.421
107	3.7.13.19.127.199	108	7.13.61.127.193
146	$3.13^2.127.7057$	147	$13^2.127.21757$
234	7.127.433.7789	235	3.7.19.127.139.433
273	19.31.127.74257	274	3.19.31.127.25117
361	$3.7^3.13^2.127.769$	362	$3.7^3.127.331.397$

Hence only $u = 19$ would do, but it is smaller than U_*. The new upper bound now becomes 396. A reduction modulo 43.73 (with no possible u smaller than 396) gives the new upper bound 154. Reduction modulo 7^3 yields $u = 19$ or 20 (too small) and the new upper bound 95. Table 1 now shows that $u < 35$, ruling out $G \cong E_7(2)$.

The group $E_8(2)$ is much harder to handle because it is much larger. We have:

$$\begin{aligned} a(G) &= 3.7^4.13^2.19.31^2.43.73.127.151.241.331 \\ P(G) &\approx 293.10^{15} \\ U &= 571575 \\ U_* &= 1500 \end{aligned}$$

Using reductions again, we have ruled out $E_8(2)$ by computer using CAYLEY.

We will apply this method of reduction again in the next section. We will not have to use the computer again.

This completes the proof of theorem 1.1.

4. Generalized octagons.

In this section, we prove theorem 1.2.

We use the notation of subsection 2.2. Put $u = \sqrt{\frac{st}{2}}$ and $w = s + t$. We rewrite the rational number (2) of section 2.2 as

$$\frac{u^2(1+w+2u^2)(1+2u^2)(1 \pm 2u + 2u^2)}{2(w \pm 2u)}. \tag{4}$$

This must be an integer for both choices of signs.

Suppose that G acts transitively on the v points of a thick generalized octagon Γ of order (s,t) and G is again one of the simple groups listed in the Atlas [2]. Obviously $v = (1+s)(1+st)(1+s^2t^2)$ divides $|G|$. Let p be a prime dividing $1+s^2t^2$. Then $1+s^2t^2 \equiv 0 \pmod p$, hence $s^2t^2 \equiv -1 \pmod p$ and -1 is a square in $GF(p)$ which implies $p=2$ or $p \equiv 1 \pmod 4$. Since st is even, $p \neq 2$. So we put $a(G)$, or simply a, for the largest integer all of whose other prime divisors are congruent to 1 (mod 4) and such that $a(G)$ divides $|G|$. We now again distinguish between "small" groups and "larger" ones, this time, the larger ones being only M and $E_8(2)$.

4.1. Small Groups.

As before, $a(G)$ only depends on the socle S of G except in the cases $S \cong Sz(32)$ and $S \cong L_2(32)$ in which case we consider the respective automorphism groups. We can copy the arguments of subsection 3.1 almost word by word. An upper bound U for u is given by the fourth root of $a(G)/2$. We can then look at table 4; it contains all values for $(1+s^2t^2)$ for given u, $2 \leq u \leq 31$. We consider the largest number $U^* \leq U$ such that $1+4(U^*)^4$ divides $|G|$. This is clearly a new upper bound for u. By inspection of the orders of the small Atlas-groups, it turns out that only for 24 among them $U^* > 1$. We list them is table 5 together with their order, the order d of their outer automorphism group, a, U and U^*.

Note that, if $u = 3$, then $1 + st = 19$, hence $|G|$ must be divisible by 19. For the groups in table 5 with $U^* \geq 3$, this is only true for Th, ${}^2E_6(2)$, B and $E_7(2)$. In this case however, $\{s,t\} = \{3,6\}$ and no group element of prime 31 nor 17 can live in Γ contradicting the fact that one of these primes divides the order of the four groups mentioned. So we may assume $u \neq 3$. There is only one case where $U^* > 3$ and that is if $S \cong Sz(32)$. Here $u = 4$ and $\{s,t\} = \{4,8\}$, so a group element of order 31 cannot live in Γ. Hence, for the rest of the proof, we have $u = 2$ and hence $(s,t) = (2,4)$ or $(s,t) = (4,2)$. So

u	$1+4u^4$	u	$1+4u^4$	u	$1+4u^4$
2	5.13	12	5.53.313	22	5^2.37.1013
3	5^2.13	13	5.73.313	23	5.13.17.1013
4	5^2.41	14	5.73.421	24	5.13.17.1201
5	41.61	15	13.37.421	25	1201.1301
6	5.17.61	16	5.13.37.109	26	5.281.1301
7	5.17.113	17	5.109.613	27	5.17.89.281
8	5.29.113	18	5.137.613	28	5^3.13.17.89
9	5.29.181	19	5.137.761	29	5^3.13.1741
10	13.17.181	20	29^2.761	30	1741.1861
11	5.13.17.53	21	$5^2.29^2$.37	31	5.397.1861

Table 3.

S	$\|S\|$	d	a	U	U^*
$L_2(25)$	$2^3.3.5^2.13$	2^2	$5^2.13$	3	3
$U_3(4)$	$2^6.3.5^2.13$	2^2	$5^2.13$	3	3
$S_4(5)$	$2^6.3^2.5^4.13$	2	$5^4.13$	6	3
$L_4(3)$	$2^7.3^6.5.13$	2^2	5.13	2	2
${}^2F_4(2)'$	$2^1 1.3^3.5^2.13$	2	$5^2.13$	3	3
$Sz(32)$	$2^{10}.5^2.31.41$	5	$5^3.41$	5	4
$L_3(9)$	$2^7.3^6.5.7.13$	2^2	5.13	2	2
$G_2(4)$	$2^{12}.3^3.5^2.7.13$	2	$5^2.13$	3	3
A_{13}	$2^9.3^5.5^2.7.11.13$	2	$5^2.13$	3	3
$O_7(3)$	$2^9.3^9.5.7.13$	2	5.13	2	2
$S_6(3)$	$2^9.3^9.5.7.13$	2	5.13	2	2
Ru	$2^{14}.3^3.5^3.7.13.29$	1	$5^3.13.29$	10	3
Suz	$2^{13}.3^7.5^2.7.11.13$	2	$5^2.13$	3	3
$O_8^+(3)$	$2^{12}.3^{12}.5^2.7.13$	$2^3.3$	$5^2.13$	3	3
$O_8^-(3)$	$2^{10}.3^{12}.5.7.13.41$	2^2	$5.13.41$	5	2
Fi_{22}	$2^{17}.3^9.5^2.7.11.13$	2	$5^2.13$	3	3
$F_4(2)$	$2^{24}.3^6.5^2.7^2.13.17$	2	$5^2.13.17$	6	3
Th	$2^{15}.3^{10}.5^3.7^2.13.19.31$	1	$5^3.13$	4	3
Fi_{23}	$2^{18}.3^{13}.5^2.7.11.13.17.23$	1	$5^2.13.17$	6	3
Co_1	$2^{21}.3^9.5^4.7^2.11.13.23$	1	$5^4.13$	6	3
${}^2E_6(2)$	$2^{36}.3^9.5^2.7^2.11.13.17.19$	2.3	$5^2.13.17$	6	3
$E_6(2)$	$2^{36}.3^6.5^2.7^3.13.17.31.73$	2	$5^2.13.17.73$	17	3
Fi'_{24}	$2^{21}.3^{16}.5^2.7^3.11.13.17.23.29$	2	$5^2.13.17.29$	14	3
B	$2^{41}.3^{13}.5^6.7^2.11.13.17.19.23.31.47$	1	$5^6.13.17$	30	3
$E_7(2)$	$2^{63}.3^{11}.5^2.7^3.11.13.17.19.31.43.73.127$	1	$5^2.13.17.73$	17	3

Table 4.

if $|G|$ contains a prime p distinct from 13 and greater then 6, then we obtain a contradiction since a group element of order p could not live in Γ. Only for the first five groups of table 5 we have that $p = 7$ does not divide $|G|$. Moreover, the orders of the first two groups are not divisible by $3^2 = 1 + st$, a contradiction. We consider the other groups in turn.

$\boxed{S_4(5)}$ Note that necessarily $s = 4$ because otherwise $3^3 = (1+s)(1+st)$ and this does not divide $|G|$. The order of the stabilizer G_x of a point x of Γ is divisible by 5, so consider an element θ of order 5 in G_x. It has to fix all three lines through x and all points other than x on these lines, etc. So θ is the identity, a contradiction.

$\boxed{L_4(3)}$ Here, $s = 2$ since 5^2 does not divide $|G|$. The stabilizer G_x of a point x contains a Sylow 3-subgroup H of order 3^3. The latter fixes at least two lines through x, all points on these lines, at least one other line through those points, etc. This is enough to conclude that H fixes a suboctagon Γ' of order $(2, 1)$ and H has all orbits of length 27 on the points and lines not in Γ' (otherwise a group element fixes the "geometric closure" of Γ' and a fixed element not in Γ', which is Γ itself). But the number of points outside Γ' in Γ is 1710 and this is not divisible by 27.

$\boxed{{}^2F_4(2)'}$ Every transitive action on 1755 or 2304 points is primitive by the information in the ATLAS [2], hence Γ is the usual generalized octagon or its dual.

This completes the case of small groups.

4.2. Larger Groups.

We first deal with $G \cong M$. We have

$$\begin{array}{rcl} a(G) & = & 5^9.13^3.17.29.41 \\ P(G) & \approx & 927.10^{17} \\ U & = & 2158 \\ U_* & = & 373 \end{array}$$

The lower bound U_* is achieved as in subsection 3.2. We do a reduction modulo 5^5 (cp. subsection 3.2). Suppose $5^5 | 1 + s^2t^2$. This means that $s^2t^2 = 4u^4 \equiv -1 \pmod{5^5}$ or $u^4 \equiv 781 \pmod{5^5}$. This has only four solutions not larger than 2158, namely 1028, 1029, 2096 and 2097. But for none of these values $(1 + st)(1 + s^2t^2)$ is a divisor of $|M|$. Hence

$$1 + s^2t^2 | 5^4.13^3.17.29.41 \approx 2775.10^7,$$

giving us the new upper bound $U' = 298 < 375 = U^*$.

Similarly, we deal with $G \cong E_8(2)$. Here

$$\begin{array}{rcl} a(G) & = & 5^5.13^2.17^2.41.73.241 \\ P(G) & \approx & 293.10^{15} \\ U & = & 2290 \\ U_* & = & 170 \end{array}$$

Reduction modulo 5^5 gives the new bound 1531. Reduction modulo 5^4 gives the new bound (possibilities for u are 221, 222, 403, 404, 846, 847, 1471, 1472) gives the new bound 1024. Reduction modulo 73.241 ($u = 570$) gives the new bound 350. Reduction modulo 241 ($u = 88$, 89, 152, 153, 329, 330) gives the new bound 259. Reduction modulo 17^2 ($u = 125$, 126, 163, 164) gives finally the upper bound 128, contradicting $U_* = 170$.

This completes the proof of theorem 1.2.

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