The Non-Existence of certain Regular Generalized Polygons

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October 9, 2002

Abstract

We define the notion of d_i-regularity and regularity in generalized polygons, thus generalizing the notion of *'regular point'* in a generalized quadrangle or hexagon. We show that a thick generalized polygon cannot contain too many regular points unless it is a projective plane, quadrangle or hexagon. For certain polygons (thick *'odd'*-gons and 8-gons), we show that even a certain number of d_2-regular points cannot exist. As an application, we present a geometric and rather elementary proof of the non-existence of thick buildings of spherical type H_3.

1 Introduction and Main Result

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ be a connected rank 2 incidence geometry with point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation I. A *path* of *length* d is a sequence $v_0, \ldots, v_d \in \mathcal{P} \cup \mathcal{L}$ with $v_i I v_{i+1}$, $0 \leq i < d$. Define a function $\delta : (\mathcal{P} \cup \mathcal{L}) \times (\mathcal{P} \cup \mathcal{L}) \to \mathbf{N}$ by $\delta(v, v') = d$ if and only if d is the minimum of all $d' \in \mathbf{N}$ such that there exists a path of length d' joining v and v' (d always exists by connectedness).

Then $\mathcal{S}$ is a *generalized n-gon, $n \in \mathbf{N} \setminus \{0, 1, 2\}$*, or a *generalized polygon* if it satisfies the following conditions:

(GP1) There is a bijection between the sets of points incident with two arbitrary lines. There is also a bijection between the sets of lines incident with two arbitrary points.

(GP2) The image of $(\mathcal{P} \cup \mathcal{L}) \times (\mathcal{P} \cup \mathcal{L})$ under δ equals $\{0, \ldots, n\}$. For $v, v' \in \mathcal{P} \cup \mathcal{L}$ with $\delta(v, v') = d < n$ the path of length d joining v and v' is unique.

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(GP3) Each $v \in \mathcal{P} \cup \mathcal{L}$ is incident with at least 2 elements.

Generalized polygons were introduced by Tits [8]. Note that we have excluded the trivial case of generalized digons here. Usually, generalized 3-gons, 4-gons, 5-gons, 6-gons and 8-gons are called (generalized) projective planes, quadrangles, pentagons, hexagons and octagons respectively. A generalized polygons is called *thick* if every point is incident with at least three lines and dually, every line is incident with at least three points. Infinite examples of thick generalized n-gons exist for all $n \in \mathbf{N} \setminus \{0, 1, 2\}$ (via free constructions). Finite thick generalized n-gons exist only for $n \in \{3, 4, 6, 8\}$ by a result of Feit & Higman [3] and examples are known in each case. For instance, the natural geometries of the groups of Lie-type which have an irreducible (B, N)-pair of rank 2 are generalized polygons (viewed as buildings of rank 2) and they all satisfy the so-called *Moufang condition* (see Tits [10]), which is a condition on the existence of certain automorphisms (root-elations). By a theorem of Tits [12] and Weiss [17], thick generalized n-gons satisfying the Moufang condition exist only for $n \in \{3, 4, 6, 8\}$ and a classification in each of these cases is possible by work of Faulkner [2] and Tits (unpublished) for $n = 6$; Tits [13] for $n = 8$; Tits (unpublished) for $n = 4$ (the case $n = 3$ being a classical result on alternative division rings).

There exists a notion of topological generalized polygon, which was introduced in general by Grundhöfer & Van Maldeghem [1], and Knarr [5] and Kramer (unpublished) prove that thick connected compact generalized n-gons exist only for $n \in \{3, 4, 6\}$. In the disconnected case, Van Maldeghem [14] defines the notion of a generalized polygon with valuation and shows that thick generalized n-gons with valuation exist only for $n \in \{3, 4, 6\}$.

So the numbers $3, 4, 6$ and 8 play a special role in the theory of generalized polygons. In the present paper, we want to add to the list above another theorem having the same conclusion, but now assuming a purely geometric condition. Motivated by work of Payne & Thas [6] on finite generalized quadrangles and by Ronan [7] on generalized hexagons, we take as geometric condition *regularity*.

Indeed, a very important notion in the theory of (finite) generalized quadrangles is the notion of a *regular point*: a point p is called *regular* if for all points x at distance 4 from p, the set $\mathcal{S}_2(p) \cap \mathcal{S}_2(x)$ is determined by any two of its elements (where $\mathcal{S}_i(p)$ denoted the set of elements at distance i from p). This notion is generalized to *half regular* in generalized hexagons by Van Maldeghem & Bloemen [15] (for reasons made explicit in *loc. cit.*) as follows: a point p of a generalized hexagon is *half regular* if for all points x at distance 6 from p, the set $\mathcal{S}_2(p) \cap \mathcal{S}_4(x)$ is determined by any two of its elements. In the present paper, we would like to call this *distant-2-regularity*, or briefly *d_2-regularity* and say that p is d_2-regular. It is now easy to see how to define d_i-regularity in a generalized n-gon, $2 \leq i \leq n/2$. Indeed, a point p of a generalized n-gon is *d_i-regular* if for all elements v at distance n, the set $\mathcal{S}_i(p) \cap \mathcal{S}_{n-i}(v)$ is determined by any two of its elements. Clearly, a similar definition for $i = 1$ or $i > n/2$ would make every point of every generalized n-gon d_i-regular for $i = 1$ and $i > n/2$, so we do not consider that. As a result, we say that, by definition, all points (and dually all lines) of any projective plane are regular, and for $n \geq 4$, we define a *regular* point in a generalized n-gon as a point which is d_i-regular for all i, $2 \leq i \leq n/2$.

Remark that we choose the notation d_i-regular in stead of i-regular in order not to confuse with the already existing notion of 3-regularity in finite generalized quadrangles (of *order* (s, s^2), see Payne & Thas [6]).

As for generalized quadrangles and hexagons, one could define a *derivation* of a generalized polygon in a d_2-regular point and investigate assumptions under which this derivation is a projective plane.

Dually, one can define d_i-regular lines, $2 \leq i \leq n/2$, and regular lines in generalized n-gons.

Note that, a priori, there is no reason why d_i-regularity of a point or line should imply d_j-regularity of the same or any other point or line, for any i and j, $2 \leq i, j \leq n/2$, in any generalized n-gon.

If every point or every line of a generalized n-gon $\mathcal{S}$ is d_i-regular, $2 \leq i \leq n/2$, then we briefly say that $\mathcal{S}$ is *d_i-regular* (in points or lines respectively). If $\mathcal{S}$ is d_i-regular (in points or lines) for every i, $2 \leq i \leq n/2$, then we say that $\mathcal{S}$ is *regular* (in points or lines respectively). If $\mathcal{S}$ is d_i-regular in points and in lines, then we call $\mathcal{S}$ *super-d_i-regular.*

Now we observe that every point of every (thick) Moufang generalized hexagon or its dual is d_2-regular and d_3-regular. This follows immediately from Ronan [7] (in Ronan's notation, a hexagon which is d_2-regular in points is a hexagon *with ideal lines*, and a d_3-regular hexagon is a hexagon satisfying the *regulus condition*). Hence every Moufang generalized hexagon is regular. By the same reference, the converse is also true: every regular generalized hexagon is Moufang. So for $n = 6$, the geometric notion of regularity coincides with the group-theoretical notion of Moufang. Naturally one can ask if there is more similarity, and motivated by the result of Tits and Weiss mentioned above, we will prove in this paper:

MAIN RESULT. *If a thick generalized n-gon is d_i-regular, $2 \leq i < n/2$, then $i \leq (n+2)/4$. If moreover i is even* **or** *$\mathcal{S}$ is super-d_i-regular, then $n = 6$* **or** *$i \leq n/4$ and i divides n.*

As a consequence, we will show:

COROLLARY. *Thick regular generalized n-gons exist only for $n \in \{3, 4, 6\}$.*

A direct proof of this result is extremely easy. We will consider much weaker hypotheses. In fact, we believe the following statement is true:

CONJECTURE. *Thick d_2-regular generalized n-gons exist only for $n \in \{3, 4, 6\}$.*

We will prove this conjecture for n odd (this is a direct consequence of our main result) and $n = 8$ (which we consider as the most important case). Surprising in these cases is that we only need a few d_2-regular points (or lines), only depending on n, and in fact, for n odd, we do not even need the full d_2-regularity of these points (or lines). We make precise statements in the next section. Note that our main result is not the best we can do, indeed, the case $n = 8$ and $d_i = d_2$ is not covered by it. In section 4, we give a reformulation which cannot be improved.

From the corollary above, one would be tempted to say that regularity can be seen as a sort of geometric counterpart to the Moufang condition. This idea is stressed by the following fact. TITS [9] shows that, whenever a generalized polygon appears as a residue in a thick spherical building of rank ≥ 3, then it satisfies the Moufang condition. From this, it follows that no thick buildings of spherical type H_3 exist, since there are no Moufang pentagons, see TITS [11]. We will show in a rather geometric way that every residue of a thick building of type H_3 is regular. From our corollary above follows the non-existence of thick buildings of type H_3. This is the content of section 5.

As far as generalized polygons are concerned, we will use the following notation. A sub-"generalized" n-gon (of a given generalized n-gon $\mathcal{S}$) in which every point and every line is incident with exactly two elements will be called an *apartment* (of $\mathcal{S}$). A *cycle* $\mathcal{C}$ is a closed path all of which elements appear only once in $\mathcal{C}$ (except of course the first element, which appears also as the last element). The *length* of the cycle is the number of distinct elements in it and this is always at least $2n$. Two elements of a given generalized n-gon at distance n will be called *opposite*. For two opposite elements v, v', the set $\mathcal{S}_i(v) \cap \mathcal{S}_{n-i}(v')$ is called a d_i*-trace* and denoted by $v_{[i]}^{v'}$, generalizing the notation of RONAN [7]. As in the latter paper, we will occasionally delete the subscript "$[i]$" when $i = 2$. For two collinear points p_1 and p_2, we write $p_1 \sim p_2$ and denote the line joining them by p_1p_2.

Remark that the thickness assumption is really needed. If we allow *weak* polygons, in particular generalized polygons in which every point (or line) is incindent with exactly two lines (or points), but not dually (so excluding usual polygons, which are always super-regular in a trivial way), then there exist also weak regular generalized 12-gons. And every weak generalized $2m$-gon as described above is d_2-regular in both points and lines in a trivial way. See section 4 of this paper for more details.

2 Weakening of the hypotheses

Let $n \in \mathbf{N} \setminus \{0, 1, 2, 3, 4, 6\}$ and suppose $i \in \mathbf{N}$ does not divide n, $2 \leq i < n/2$. There exists a unique positive integer k such that $\frac{n}{k+1} < i < \frac{n}{k}$. Let $\mathcal{S}$ be a thick generalized n-gon. Consider a $(k+2)$-tuple $\gamma = (v_0, v_1, \ldots, v_{k+1})$ of elements (points and lines) of $\mathcal{S}$, all contained in one apartment and such that $\delta(v_0, v_\alpha) = i\alpha$ for $0 \leq \alpha \leq k$, and $\delta(v_0, v_{k+1}) = 2n - i(k+1)$. Let v be an element opposite v_α, $\alpha \in \{1, 2, \ldots, k\}$, and assume $v_{\alpha-1}, v_{\alpha+1} \in (v_\alpha)_{[i]}^v$. If for every element w opposite v_α, $|(v_\alpha)_{[i]}^w \cap (v_\alpha)_{[i]}^v| \geq 2$ implies that $(v_\alpha)_{[i]}^w = (v_\alpha)_{[i]}^v$, and if this holds for all α, $1 \leq \alpha \leq k$, then we call γ a semi-d_i-regular sequence (of length $k+1$). Clearly every generalized n-gon which is d_i-regular, i even, or super-d_i-regular, i odd, contains such sequences.

We will show:

THEOREM 2.1 *Suppose i does not divide n, $2 \leq i \leq n/2$, and $k \in \mathbf{N}$ is such that $\frac{n}{k+1} < i < \frac{n}{k}$, then there does not exist a thick generalized n-gon having a semi-d_i-regular path of length $k+1$, $m \geq 2$.*

To see the strength of this result, just put $n = 5$ and $i = 2$. Then $k = 2$ and hence already 2 collinear d_2-regular points cannot live in a thick generalized pentagon.

Now let v and w be two opposite elements of some thick generalized n-gon $\mathcal{S}$. Let $2 \leq i \leq n/2$. Then we call the pair (v, w) *d_i-regular* if $|v_{[i]}^{w} \cap v_{[i]}^{w'}| \geq 2$ implies $v_{[i]}^{w} = v_{[i]}^{w'}$, for every element w' opposite v, and $|w_{[i]}^{v} \cap w_{[i]}^{v'}| \geq 2$ implies $w_{[i]}^{v} = w_{[i]}^{v'}$, for every element v' opposite w. If n is even, then all pairs of opposite points (respectively lines) are d_i-regular if and only if $\mathcal{S}$ is d_i-regular in points (respectively lines). If n is odd, then all pairs of opposite elements are d_i-regular if and only if $\mathcal{S}$ is super-d_i-regular. These are immediate consequences of the definitions.

THEOREM 2.2 *Let $2 \leq i < n/2$. If a thick generalized n-gon has a d_i-regular pair of elements, then $i \leq (n+2)/4$. Also, if a thick generalized n-gon has two d_i-regular elements at distance j, $i \leq j < n/2$, from each other, then $i \leq (n-j)/2$.*

Putting $n = 8$ and $i = 3$ in the theorem above, one obtains the non-existence of a thick generalized octagon having a d_3-regular pair of points (or lines). We will show that a thick generalized octagon cannot contain 4 d_2-regular points in a certain position (and we leave it to the reader to deduce from the proof that position). This will immediately imply:

THEOREM 2.3 *There does not exist a thick d_2-regular generalized octagon.*

Note that by Joswig & Van Maldeghem [4] the classical Moufang Ree octagons all are super-d_4-regular, so Theorem 2.3 is the best that we can do.

As a further contribution to our conjecture (above), we will also show:

THEOREM 2.4 *There does not exist a thick super-d_2-regular generalized 10-gon.*

Finally, we remark that Theorem 2.1 for odd n is also valid for a slightly more general class of geometries, namely the thick Moore geometries (generalizing the notion of d_i-regular point in the obvious way to Moore geometries), and the same proof can be used.

3 Proofs

PROOF OF THEOREM 2.1. Let $\gamma = (v_0, v_1, v_2, \ldots, v_{k+1})$ be a semi-d_1-regular path in a generalized n-gon $\mathcal{S}$. By definition, γ lies in some apartment $\mathcal{A}$. Let w_0 be the element of $\mathcal{A}$ opposite v_1. Then $\delta(w_0, v_{k+1}) = n - ki$. Let w_1 be any element at distance i from w_0 and at distance $n-i$ from v_1. We choose $w_1 \notin \mathcal{A}$ (this is possible by the thickness assumption). Clearly v_0 and w_1 are opposite and so there are elements $(w_2, w_3, \ldots, w_{k+1})$ on a path γ_0 of length n from w_1 to v_0 with the property $\delta(w_0, w_\alpha) = i\alpha$, for all $\alpha \in \{0, 1, \ldots, k+1\}$, and v_1 and w_2 are opposite (this follows again by the thickness assumption). Now let v_1' be the unique element at distance i from v_1 and $n - 2i$ from w_1.

The d_i-trace $(v_1)^{w_0}_{[i]}$ contains v_0, v_1' and v_2. The d_i-trace $(v_1)^{w_2}_{[i]}$ contains v_0 and v_1', hence by assumption, it should also contain v_2, so $\delta(w_2, v_2) = n - i$. Let v_2' be the element on the path γ_2 connecting v_2 with w_2 at distance i from v_2. Note that γ_2 has only v_2 in common with $\mathcal{A}$ and only w_2 in common with γ_0 (otherwise there is a cycle of length j with $j < 2n$ arising from a shortcut of the way from w_0 to w_1, to v_2', to v_3, back to w_0, or of the way from v_0 to v_2', to w_3, back to v_0, a contradiction). So it follows that the i-trace $(v_2)^{w_1}_{[i]}$ contains v_1, v_2' and v_3. The i-trace $(v_2)^{w_3}_{[i]}$ contains v_1 and v_2', hence also v_3 and so $\delta(w_3, v_3) = n - i$. Define v_3' as the element at distance i from v_3 on the unique path γ_3 connecting v_3 with w_3. Considering the i-traces $(v_3)^{w_2}_{[i]}$ and $(v_3)^{w_4}_{[i]}$, we obtain similarly $\delta(w_4, v_4) = n - i$. Going on like this, we finally obtain $\delta(w_{k+1}, v_{k+1}) = n - i$ and the path γ_{k+1} connecting w_{k+1} with v_{k+1} has only one of its extremeties in common with $\mathcal{A}$ and γ_0. So there are two different paths going from w_0 to w_{k+1} and having length $(n-i) + (n-ki)$, namely one via v_0 and one via v_{k+1}. Hence $2n - ki - i \geq n$, implying $i \leq \frac{n}{k+1}$, contradicting our assumption on i and k. This proves the theorem. $\square$

PROOF OF THEOREM 2.2. First suppose that a thick generalized n-gon $\mathcal{S}$ has a d_i-regular pair (v, w), $2 \leq i < n/2$. Let $\mathcal{A}$ be an apartment containing v and w. Let v' and w' be opposite elements in $\mathcal{A}$ at some fixed distance j from v and w respectively, $i \leq j < n/2$. Let v_1 and v_2 be the two elements of $\mathcal{A}$ at distance i from v and let w_1 and w_2 be the two elements of $\mathcal{A}$ at distance i from w. By the thickness assumption, there exist paths γ and γ' joining v and w, respectively v' and w', and γ nor γ' is not contained in $\mathcal{A}$. Let w_0 and v_0 be the elements on γ at distance i from w and v respectively. Let w_0' and v_0' be the elements on γ' at distance j from w' and v' respectively. The d_i-traces $v^w_{[i]}$ and $w^v_{[i]}$ contain v_0, v_1, v_2 and w_0, w_1, w_2 respectively. But the d_i-traces $v^{w_0'}_{[i]}$ and $w^{v_0'}_{[i]}$ contain v_1, v_2 and w_1, w_2 respectively. Hence, by the d_i-regularity of (v, w), we must have $\delta(v_0, w_0') = \delta(v_0', w_0) = n - i$. So there are paths γ_v and γ_w of length $n - i$ connecting w_0' with v_0 and v_0' with w_0 respectively. The path γ_v has only w_0 in common with γ' (otherwise there is a cycle of length $\leq 2n$ through such common element and v) and it has only v_0 in common with the unique path joining v_0 and v. Similarly for γ_w. This means that the closed path γ^* starting in w_0', going to v_0, to w_0, to v_0' and back to w_0 contains a cycle $\mathcal{C}$. The length of $\mathcal{C}$ is at most $(n-i) + (n-2i) + (n-i) + (n-2j)$ and this must be at least $2n$. This implies

$$2i \leq n - j.$$

If n is odd, then we can take $j = \frac{n-1}{2}$, so in this case

$$i \leq \frac{1}{2}(n - \frac{n-1}{2}) = \frac{n+1}{4}.$$

If n is even, then we can choose $j = \frac{n-2}{2}$. In this case, we have

$$i \leq \frac{n+2}{2}.$$

This proves the first part of the theorem.

Now suppose the thich generalized n-gon $\mathcal{S}$ has two d_i-regular elements v, w at distance j from each other, with $2 \leq i \leq j < n/2$. Let $\mathcal{A}$ be any apartment containing v and w and let v' and w' be opposite v and w respectively inside $\mathcal{A}$. By the thickness assumption, there exist paths γ_v and γ_w connecting v with v' and w with w' respectively and not lying in $\mathcal{A}$. Let $v'' \in \gamma_v$ and $w'' \in \gamma_w$ be at distance j from v' and w' respectively; let v_1, v_2 and w_1, w_2 be the elements in $\mathcal{A}$ at distance i from v and w respectively, and let $v_0 \in \gamma_v$ and $w_0 \in \gamma_w$ be at distance i from v and w respectively. The d_i-traces $v_{[i]}^{v'}$ and $w_{[i]}^{w'}$ contain v_0, v_1, v_2 and w_0, w_1, w_2 respectively. The d_i-traces $v_{[i]}^{w''}$ and $w_{[i]}^{v''}$ contain v_1, v_2 and w_1, w_2 respectively. Hence $\delta(w'', v_0) = \delta(v'', w_0) = n - i$. Completely similar to the first part above, we obtain a cycle $\mathcal{C}$ in the path starting at w'', going to v_0, then to v'', to w_0, back to w'' (this is indeed a cycle if the path going from v'' to w_0 does not pass through w'', but in this case we have a cycle through v', v'', w'' and w' of length $4j$, contradicting $j < n/2$). The length of $\mathcal{C}$ is at most $(n-i) + (n-i-j) + (n-i) + (n-i-j)$. This should be at least $2n$. The result follows. $\square$

PROOF OF THEOREM 2.3. Let $(p_0, p_1, p_2, \ldots, p_7)$ be an apartment in a thick generalized octagon $\mathcal{S}$, and we assume $\mathcal{S}$ to be d_2-regular in points ($p_0, \ldots, p_7, p_0$ are consecutively collinear points of $\mathcal{S}$). Let p_0' be incident with p_0p_7 but different from both p_0 and p_7 (this is possible by the thickness assumption). Construct the path $(p_0', p_0'p_1', p_1', p_1'p_2', p_2', p_2'p_3', p_3')$ such that p_3' is incident with p_3p_4. Let $(p_1, p_1p_2'', p_2'', p_2''p_3'', p_3'', p_3''p_4'', p_4'', p_4''p_5, p_5)$ be a path of length 8 with $p_2 \neq p_2'' \neq p_0$ (again possible by the thickness assumption). The traces $p_1^{p_5}$ and $p_1^{p_2'}$ have the points p_0 and p_2 in common, so $\delta(p_2', p_2'') = 6$ and there is a path $(p_2', p_2'x_1, x_1, x_1x_2, x_2, x_2p_2'', p_2'')$. Clearly x_1 is incident with neither $p_1'p_2'$, nor $p_2'p_3'$.

Suppose first that x_2 is not incident with $p_2''p_3''$. Let x_3 be the unique point on $p_2''p_3''$ at distance 6 from p_3'. Clearly $p_2'' \neq x_3 \neq p_3''$. But the traces $(p_2'')^{p_1'}$ and $(p_2'')^{p_3'}$ share the points x_2 and p_1, hence $\delta(p_1', x_3) = 6$, so p_1' and p_3'' are opposite. The traces $p_5^{p_1}$ and $p_5^{p_1'}$ share the points p_4 and p_6, and so there is a path $(p_1', p_1'y_1, y_1, y_1y_2, y_2, y_2p_4'', p_4'')$. Clearly y_1 is incident with neither $p_0'p_1'$ nor $p_1'p_2'$. And if y_2 were incident with $p_3''p_4''$, then $\delta(p_1', p_3'') = 6$, contradicting the fact that they are opposite. So p_2'' and y_1 are opposite, but this contradicts $\{p_0', p_2', y_1\} \subseteq (p_1')^{p_5}$ and $\{p_0', p_2'\} \subseteq (p_1')^{p_2''}$. We conclude that x_2 must be incident with $p_2''p_3''$.

So suppose x_2 is incident with $p_2''p_3''$. By symmetry, y_2 (defined as in the previous paragraph), must be incident with $p_3''p_4''$. But then $(p_1', y_1, y_2, p_3'', x_2, x_1, p_2', p_1')$ forms a cycle of length 14 in $\mathcal{S}$, a contradiction. $\square$

Still the geometry of the traces in the Ree octagons has some very nice properties (see Van Maldeghem [16]), but it can never be similar to the geometry of the traces in a d_2-regular quadrangle or hexagon.

PROOF OF THEOREM 2.4. Let $\mathcal{S}$ be a thick super d_2-regular generalized 10-gon. Let $(p_0, p_0p_1, p_1, \ldots, p_8p_9, p_9, p_9p_0, p_0)$ be an apartment of $\mathcal{S}$. By the thickness assumption, there is a point p'_0 on p_0p_9, different from p_0 and p_9. There is a unique point p'_4 on p_4p_5 at distance 8 from p'_0. Let $p'_0 \sim p'_1 \sim p'_2 \sim p'_3 \sim p'_4$. Again by the thickness assumption, there are points $p''_2, p''_3, p''_4, p''_5$ not in $\mathcal{A}$ and such that $p_1 \sim p''_2 \sim p''_3 \sim p''_4 \sim p''_5 \sim p_6$. The d_2-trace $(p_1)^{p_6}_{[2]}$ contains p_0, p_2 and p''_2. But $(p_1)^{p'_3}_{[2]}$ does also contain p_0 and p_2, hence by the super d_2-regularity, $\delta(p'_3, p''_2) = 8$. Let $p''_2 \sim x_1 \sim x_2 \sim x_3 \sim p'_3$. Note that x_3 is not incident with neither $p'_2p'_3$, nor $p'_3p'_4$. By the dual argument, $\delta(p'_3p'_4, p''_2p''_3) = 8$ (look at the d_2-trace $(p_4p_5)^{p_0p_9}_{[2]}$). But if x_1 is not incident with $p''_2p''_3$, this distance is 10 (measured along the path starting in $p'_3p'_4$, going to p'_3, x_3, x_2, x_1 and ending via p''_2 in $p''_2p''_3$). Hence x_1 is incident with $p''_2p''_3$. Note however that $x_1 \neq p''_3$, otherwise $x_1, x_2, x_3, p'_3, p'_4, p_5, p_6, p''_5, p''_4$ defines a cycle of length 18. Similarly, there are points y_1, y_2, y_3 with $p''_4p''_5 \; I \; y_1 \sim y_2 \sim y_3 \sim p'_1$, $y_1 \neq p''_4$ and y_3 not on $p'_0p'_1$ nor on $p'_1p'_2$. Now look at the d_2-trace $(p'_1)^{p_6}_{[2]}$. It contains p'_0, p'_2 and y_3. Also $(p'_1)^{x_1}_{[2]}$ contains p'_0 and p'_2. Hence $\delta(x_1, y_3) = 8$, but $x_1 \sim p''_3 \sim p''_4 \sim y_1 \sim y_2 \sim y_3$ defines a path of length 10 connecting x_1 with y_3, a contradiction. The result follows. $\square$

PROOF OF THE MAIN RESULT AND THE COROLLARY. Suppose $\mathcal{S}$ is a thick d_i-regular generalized n-gon, $2 \leq i < n/2$. If n is even, then by the first part of Theorem 2.2, $i \leq \frac{n+2}{4} < \frac{n+3}{4}$. If $n \equiv 1 \bmod 4$, then we use the second part of Theorem 2.2, putting $j = (n-1)/2$, to obtain $i \leq \frac{n+1}{4} < \frac{n+2}{4}$. If $n \equiv 3 \bmod 4$, then we use the second part of Theorem 2.2, putting $j = (n-3)/2$, to obtain $i \leq \frac{n+3}{4}$. But in this case, $i \leq \frac{n+1}{4}$, because $\frac{n+3}{4}$ is not an integer. This shows the first part of the Main Result.

Suppose now moreover that i is even or $\mathcal{S}$ is super-d_i-regular. By Theorem 2.1, i must divide n. If i would be equal to $n/3$, then by the preceding paragraph, $n/3 \leq (n+2)/4$, hence $n \leq 6$, implying $n = 6$ or $n = 3$ (a trivial case). This shows the Main Result completely.

The Corollary follows immediately from the Main Result noting that for a thick d_i-regular generalized n-gon with $i = 2$, n must be even, and for one with $n \geq 8$ even, there is always an integer between $(n+2)/4$ and $n/2$. $\square$

4 Weak buildings of rank 2

A generalized polygon as defined in this paper corresponds with a weak building of rank 2 *with order*, i.e. the number of points on a line is a constant and the number of line through a point is a constant. This is due to axiom (GP1). Deleting this axiom, we obtain the definition of what we could call a *weak polygonal geometry*, a notion corresponding to a weak building of rank 2.

For example, consider a thick generalized n-gon $\mathcal{S}$. We construct for $m \geq 2$ a weak polygonal geometry $\mathcal{S}^m$ as follows. Define for every incident point-line pair (p, L) an $(m-1)$-tuple $(v_1^{(p,L)}, v_2^{(p,L)}, \ldots, v_{m-1}^{(p,L)})$. The elements of $\mathcal{S}^m$ are the points, lines and symbols $v_i^{(p,L)}$, $1 \leq i \leq m-1$, $p\ I\ L$ in $\mathcal{S}$. Denote the incidence relation in $\mathcal{S}^m$ by I^m, then we define incidence in $\mathcal{S}^m$ by

$$p\ I^m\ v_1^{(p,L)}\ I^m\ v_2^{(p,L)}\ I^m \ldots I^m\ v_{m-1}^{(p,L)}\ I^m\ L,$$

and no other incidences occur. The points of $\mathcal{S}^m$ are the points of $\mathcal{S}$ together with all symbols $v_i^{(p,L)}$, i even, and, only if m is even, also all lines of $\mathcal{S}$. The lines of $\mathcal{S}^m$ are the other elements. It is straight-forward to see that $\mathcal{S}^m$ is a weak mn-gonal geometry. In fact, Tits [10] shows that every weak polygonal geometry arises in this way (also considering generalized digons), except if it consists of the union of a number of paths of length n between two fixed opposite elements.

Remark that the weak mn-gonal geometry $\mathcal{S}^m$ of the preceding paragraph is super-d_m-regular in an almost trivial way.

Let $\mathcal{S}$ now be any weak n-gonal geometry. Then we call $\mathcal{S}$ *d_i-thick*, $1 \leq i \leq n$, if for every element v of $\mathcal{S}$ the following property holds: whenever v is incident with at least three elements, then also every element at distance i from v is incident with at least three elements. In the proof of Theorem 2.1, we only used the d_i-thickness of the generalized n-gon. Noting this, and noting the remark of the preceding paragraph, we can now reformulate our main result as follows.

MAIN RESULT – SECOND VERSION. *Let $\mathcal{S}$ be a d_i-thick, d_i-regular (for i even) or super-d_i-regular (for i odd) n-gonal geometry, $1 \leq i \leq n/2$, then i divides n. For each i dividing n, there exist d_i-thick, super-d_i-regular n-gonal geometries (which are however not thick in general).*

Note that *"d_1-thick"* is the same thing as *"thick"*.

This second version does not cover the first version entirely. Indeed, the first part of the first version has no analogue in the second version.

5 Thick buildings of type H_3

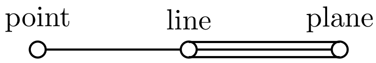
The classification of all spherical buildings of rank ≥ 3 contains a little theorem stating that no thick building of type H_3 or H_4 exists. This result (as practically the whole classification) is due to Tits [11], see also [10] and [9], addendum. The proof can be split into two major parts: the first part is the *reduction theorem*, see Tits [9],4.1.2, implying that all spherical buildings of rank ≥ 3 satisfy the so-called *Moufang condition* (which is a condition on the existence of many automorphisms of a certain type); the second part is, after noting that the residues which form generalized pentagons must also satisfy this Moufang condition, showing that there are no Moufang generalized pentagons. So the main ingredients of this proof are group-related. To the best of my knowledge, no other proof is available in the literature. It is my aim to show in this section Tits' result in a rather geometrical and combinatorial fashion, a little bit modelled on the classical proof of the fact that all projective planes of a projective spaces of rank 3 are desarguesian.

For definitions and notations not explained in this section, and related to buildings, we refer to TITS [9].

We are now ready to prove

THEOREM (TITS [11]). *There does not exist a thick spherical building of type H_3 or H_4.*

PROOF. Clearly, it is enough to show that there are no thick buildings of type H_3. Suppose by way of contradiction, that Ω is such a building. Then Ω has three types of vertices: points, lines and planes, pictured in the diagram:



Let Σ be any apartment of Ω. Note that Σ is an icosahedron. We fix a point p of Σ and the unique point q of Σ opposite p. We define the distance $d(x, y)$ of two points in Ω by the distance of x and y in the point-collinearity graph of Ω, i.e. the graph with vertex set the set of points of Ω and two points form an edge if they lie on the same line in Ω (we say that these two points are *collinear*). We have $d(x, y) = 3$ if and only if x and y are opposite (this follows immediatley from the icosahedron). Let L be any line through p. By considering an apartment through p, L and q, one sees that that exists a unique point, denoted x_L, incident with L and at distance 2 from q. Similarly, if α is a plane through p, there exists a unique line L_α incident with α and all points of which have distance 2 from q. It follows that the set p^q of points at distance 1 from p and distance 2 from q, and the set of lines having all their points in p^q, forms a generalized pentagon naturally isomorphic to the residue $R(p)$ of p in Ω. We denote that generalized pentagon by $\mathcal{S}(p^q)$. It is not so hard to see that the generalized pentagon $\mathcal{S}(q^p)$ is isomorphic to the dual of $\mathcal{S}(p^q)$. Indeed, every point x collinear with p and at distance 2 from q is collinear to all points of a line lying in a plane through q; reversing the roles of p and q, we thus obtain an anti-isomorphism $\Phi(p, q)$ from $\mathcal{S}(p^q)$ to $\mathcal{S}(q^p)$. In the future, we will view this anti-isomorphism also as one from $R(p)$ to $R(q)$. If p' is any point collinear with p and opposite q it follows that $\mathcal{S}(p^q)$ is isomorphic with $\mathcal{S}((p')^q)$. By connectedness one deduces that $\mathcal{S}(p^q)$ is self-dual. Let the points of Σ collinear with p be cyclically ordered and denoted by $p_1, p_2, \ldots, p_5$ (consecutive points are collinear) and let q_i be in Σ opposite p_i, $i \in \{1, 2, \ldots, 5\}$. Let x be any point on the line pp_2, $p \neq x \neq p_2$. Every point x_1 of the line p_1x has a unique closest point y_1 on the line qq_1. There follows that the collineation $\Phi(p, q)\Phi(q, x)\Phi(x, y_1)\Phi(y_1, p)$, $q_1 \neq y_1$, maps the line pp_1 of $R(p)$ to the line px_1. But one can check that it fixes the lines pp_2 and pp_3. Together with the fact that $R(p)$ is self-dual, this implies (using a standard argument) that the collineation group of $R(p)$ acts transitively on the set of paths of length 4 (bounded by points respectively lines). This is a first major result.

Now consider $\mathcal{S}(p_1^{q_1})$. Since this cannot be a regular generalized pentagon, we may suppose that at least two d_2-traces containing the points q_4 and p_5 do not coincide. In fact, this means that the subgeometry $\mathcal{G}$ (not necessarily *subpentagon*) of $\mathcal{S}(p_1^{q_1})$ containing the points p, p_2, q_4, q_3 and p_5, containing all lines through q_3 and through every point of $(q_3)_{[2]}^{pp_2}$, generated by the rules

1. *if $\mathcal{G}$ contains a line, then it contains all points of that line,*
2. *if $\mathcal{G}$ contains two collinear points, then it contains the joining line,*

contains at least one line meeting q_3q_4 in a point different from both q_3 and q_4. This is a second step in our proof.

Now consider in $\mathcal{S}(p_q)$ the d_2-trace $(p_1)_{[2]}^{p_3p_4}$. Clearly, this is exactly the intersection of q^p with $p_1^{q_1}$. Consider any line L of $\mathcal{S}(p^q)$ meeting p_4p_5 in a point $p_4' \neq p_5$ and at distance 3 from p_2. Let p_3' denote the point incident with L collinear with p_2. The anti-isomorphism $\Phi(p,q)$ maps the line $p_3'p_4'$ to a point q_1' and the line p_2p_3' to a point q_5'. Note that q_5' lies on q_4q_5. In fact we have an apartment Σ of Ω containing the points p, q, p_1, p_2, p_3', p_4', p_5, q_1', q_2, q_3, q_4 and q_5'. As before, the d_2-trace $(p_1)_{[2]}^L$ in $\mathcal{S}(p^q)$ is the intersection of p^q and $p_1^{q_1'}$. Let $\mathcal{G}$ be the intersection of $p_1^{q_1}$ and $p_1^{q_1'}$. Clearly $\mathcal{G}$ contains p, p_2, p_5. q_3 and q_4. Since by the anti-isomorphism $\Phi(qp_1,q_1)$, the lines of $\mathcal{S}(p_1^{q_1})$ through q_3 are in one-to-one correspondence with the points on the line qq_3, and since both q_1 and q_1' are collinear with all points of that line, $\mathcal{G}$ contains all lines through q_3. A line X through q_4 in $\mathcal{S}(p_1^{q_1})$ corresponds with a point x on qq_5 in $\mathcal{S}(q_1^{p_1})$. But it is readily verified that all points of X are collinear with the intersection x' of q_4x and qq_5' (these two lines lie in the projective plane spanned by q, q_4 and q_5). Hence, since x' is collinear with q_1', it follows that $\mathcal{G}$ contains all lines through q_4. Similarly it contains all lines through every point of $(q_3)_{[2]}^{pp_2}$. By the second step of this proof, it also contains a line through a point z on q_3q_4, $q_3 \neq z \neq q_4$. Again similarly as above, $\mathcal{G}$ contains all lines through z and hence $\mathcal{G}$ contains at least one line M through p for which $pp_2 \neq M \neq pp_5$. If u is the unique point of p^q on M, then it is clear that, since $u \in \mathcal{G}$, we have $u \in (p_1)_{[2]}^{p_3p_4} \cap (p_1)_{[2]}^L$. Combined with the first part of this proof, we have in fact shown that for *all* points p_1, p_2 and p_5 of $\mathcal{S}(p^q)$, p_1 collinear with both p_2 and p_5 but p_2 not collinear with p_5, and for all lines L^* through p_5 (with p_1 not on L^*; in the above reasoning, L^* is the line p_4p_5), there exists a point u collinear with p_1 such that $u \in (p_1)_{[2]}^L$ whenever L is a line meeting L^* and such that $(p_1)_{[2]}^L$ contains p_5 and p_2. But then a similar point u' exists for the points p_1, p_2, p_3 and the line $L^* = p_3p_4$. The line L_1 (respectively L_1') through u (respectively u') meeting p_3p_4 (respectively p_4p_5) lies at distance 3 from u' (respectively u). Hence the cycle $(u, L_1, \ldots, u', L_1', \ldots, u)$ has length 8, a contradiction. $\square$

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