The geometry of traces in Ree Octagons

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Abstract

In this paper, we prove some geometric properties of traces of perfect Ree octagons. It is shown, for instance, that a derived geometry can be defined and that it is isomorphic to the generalized quadrangle $T_3(\mathcal{O})$ of Tits-type, where $\mathcal{O}$ is a Suzuki-Tits ovoid.

1 Introduction

Generalized polygons were introduced by Tits [6] and have since then been studied by several authors. The main examples arise from groups of Lie type or their twisted analogues. A great deal of research concerning polygons is devoted to characterizing these Lie polygons in a geometric fashion. One of the most beautiful and complete results in this direction is Ronan's [4] characterization of all Moufang hexagons by ideal lines. Also various classes of "classical" generalized quadrangles, mainly finite ones, are characterized by the same idea of looking at intersections of traces. No such characterization of the Moufang octagons is known. From a geometric point of view however, it is already an interesting question to ask what kind of properties traces have in the Moufang octagons. We will answer this question in the present paper for a large subclass of Moufang octagons, thus establishing the geometric foundation necessary to reconstruct the ambient metasymplectic space for these geometries, which should eventually lead to a geometric characterization of all such octagons. We will briefly sketch at the end of the paper how to do this.

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2 Notation

We will assume that the reader is familiar with the definition of a generalized polygon, in particular a generalized octagon. Also, we will use some common building terminology such as *opposite elements* for 2 points or 2 lines at maximal distance; the projection of an element x onto a non-opposite element y for the unique element incident with y closest to x (see e.g. TITS [8]). Also, for any point x, we will denote by $x^\perp$ the set of points collinear to x.

One important class of thick generalized octagons arises from the Ree group of type ${}^2F_4$, see TITS [10], and we call the members of this class the *Ree octagons*. For every field K of characteristic 2 and every endomorphism σ in K whose square is the Frobenius endomorphism, there exists such a Ree octagon, which we will denote by $O_R(K,\sigma)$. In the infinite case, we also have some other examples arising from some 'free' constructions, see TITS [9].

It is known that the Ree octagon $O_R(K,\sigma)$ can be viewed as the set of absolute points and lines of a certain polarity in the metasymplectic space over K, see e.g. SARLI [5]. In the present paper, we want to clear the way for reconstructing this metasymplectic space entirely in terms of the geometry of the Ree octagon. This will establish the foundation for a geometric characterization of these octagons, which will be done elsewhere. However, in order not to drown in notation and technicalities, we restrict ourselves to the perfect case, i. e. the case where σ is an automorphism. The non-perfect case is – geometrically – very much more complicated and so, in this paper, we do not want to spend twice as much space for objects only half as important (as a figure of speech).

3 The Ree octagon $O_R(K,\sigma)$.

The following description of $O_R(K,\sigma)$, K and σ as in the previous section, is due to JOSWIG & VAN MALDEGHEM [2].

Let $K_\sigma^{(2)}$ be the group on the set of all pairs $(k_0,k_1)\in K\times K$ with operation law $(k_0,k_1)\oplus(l_0,l_1)=(k_0+l_0,k_1+l_1+l_0k_0^\sigma)$. For $k=(k_0,k_1)$, set $tr(k)=k_0^{\sigma+1}+k_1$ (the *trace* of k) and set $N(k)=k_0^{\sigma+2}+k_0k_1+k_1^\sigma$ (the *norm* of k). Define a multiplication $a\otimes k=a\otimes(k_0,k_1)=(ak_0,a^{\sigma+1}k_1)$. Also write $(k_0,k_1)^\sigma$ for (k_0^σ,k_1^σ). Then the points of $O_R(K,\sigma)$ are the elements of $\{(\infty)\}\cup K\cup K_\sigma^{(2)}\times K\cup\ldots\cup K_\sigma^{(2)}\times K\times K_\sigma^{(2)}\times K\times K_\sigma^{(2)}\times K\cup K\times K_\sigma^{(2)}\times K\times K_\sigma^{(2)}\times K\times K_\sigma^{(2)}\times K$ (and these are all denoted by round parantheses); the lines of $O_R(K,\sigma)$ are the elements of $\{[\infty]\}\cup K_\sigma^{(2)}\cup K\times K_\sigma^{(2)}\cup\ldots\cup K\times K_\sigma^{(2)}\times K\times K_\sigma^{(2)}\times K\times K_\sigma^{(2)}\cup$

$K_\sigma^{(2)} \times K \times K_\sigma^{(2)} \times K \times K_\sigma^{(2)} \times K \times K_\sigma^{(2)}$ (and denoted by square brackets); incidence is given by the sequence

$$
\begin{gathered}
(a,l,a',l',a'',l'',a''')\ I\ [a,l,a',l',a'',l'']\ I\ (a,l,a',l',a'')\ I\ \ldots \\
\ldots(a)\ I\ [\infty]\ I\ (\infty)\ I\ [k]\ I\ (k,b)\ I\ \ldots \\
\ldots[k,b,k',b',k'']\ I\ (k,b,k',b',k'',b'')\ I\ [k,b,k',b',k'',b'',k'''],
\end{gathered}
$$

and the rule : (a,l,a',l',a'',l'',a''') is incident with $[k,b,k',b',k'',b'',k''']$ if and only if the following six equations hold:

$$
\begin{aligned}
(k_0''', k_1''') &= (l_0,l_1)\ \oplus\ a\otimes(k_0,k_1)\ \oplus\ (0, al_0' + a^\sigma l_0'') && (I1)\\
b'' &= a' + a^{\sigma+1}N(k) + k_0(al_0' + a^\sigma l_0'' + tr(l)) \\
&\quad + a^\sigma(a''' + l_0k_1) + al_0''^\sigma + l_0l_0' && (I2)\\
(k_0'', k_1'') &= a^\sigma \otimes (k_1, tr(k)N(k))\ \oplus\ k_0\otimes(l_0,l_1)^\sigma \\
&\quad \oplus\ (0, tr(k)N(l) + a^{\sigma+1}l_0N(k)^\sigma + tr(k)(aa' + a^\sigma l_0 l_0'' + a^{\sigma+1}a''') \\
&\quad + tr(l)(k_1^\sigma a + a''') + k_1^\sigma a^{\sigma+1}l_0'' + k_0^{\sigma+1}a^2 l_0''^\sigma \\
&\quad + k_0(a' + al_0''^\sigma + k_1a^\sigma l_0 + a^\sigma a''')^\sigma \\
&\quad + k_0^\sigma l_0(a' + al_0''^\sigma + k_1a^\sigma l_0 + a^\sigma a''') \\
&\quad + a(l_1'' + a'''^\sigma l_0 + a'''l_0') + l_0''(a' + a^\sigma a''') + a''l_0 + l_0l_0'l_0'') \\
&\quad \oplus\ (l_0', l_1') && (I3)\\
b' &= a'' + a^{\sigma+1}N(k)^\sigma + a(k_0l_0'' + l_0k_1 + a''')^\sigma + tr(k)(l_1 + a^\sigma l_0'') \\
&\quad + k_0^\sigma(a' + a^\sigma a''') + l_0'l_0'' + l_0^\sigma a''' && (I4)\\
(k_0', k_1') &= (l_0'', l_1'')\ \oplus\ a\otimes(tr(k), k_0N(k)^\sigma)\ \oplus\ l_0\otimes(k_0,k_1)^\sigma \\
&\quad \oplus\ (0, N(k)(a^\sigma l_0'' + l_1) + k_0(a'' + l_0'l_0'' + aa'''^\sigma + l_0^\sigma a''') \\
&\quad + k_1(k_1l_0a^\sigma + a' + al_0''^\sigma + a^\sigma a''') + k_0k_1^\sigma al_0^\sigma + a'''^\sigma l_0 + a'''l_0') && (I5)\\
b &= a''' + aN(k) + l_0k_1 + k_0l_0'' && (I6)
\end{aligned}
$$

where $a,a',a'',a''',b,b',b'' \in K$ and $k,k',k'',k''',l,l',l'' \in K_\sigma^{(2)}$ and $k=(k_0,k_1)$, etc. These elements are called *the coordinates*.

This description is also valid in the non-perfect case. From now on however, we assume that K is perfect. This includes every finite field $GF(2^{2e+1})$. In the latter case, the corresponding generalized octagon is denoted by $O_R(2^{2e+1})$.

Note that every Ree octagon has a lot of automorphisms (it has the *Moufang property* and it is characterized in this way, see TITS [10]). In particular, the twisted Chevalley group ${}^2F_4(K,\sigma)$ is an automorphism group of $O_R(K,\sigma)$. It acts transitively on the set of opposite pairs of points, and also dually, on the set of pairs of opposite lines. The stabilizer of a pair of opposite points acts on the set of lines incident with either one of these points as a doubly transitive automorphism group of a Suzuki-Tits ovoid (see next section) and the stabilizer of a pair of opposite lines acts on the set of points of either one of these lines as $PGL_2(K)$. We will use these properties in order to choose certain arbitrary elements in a suitable way "without loss of generality".

4 Geometric properties of $O_R(K,\sigma)$

4.1 Properties of the Suzuki-Tits ovoids

Let $W(K)$ be the symplectic generalized quadrangle over K, i. e. the generalized quadrangle arising from a symplectic polarity τ in $PG(3,K)$. Let ρ be a polarity in $W(K)$, then it is known that the set of absolute points (resp. lines) of $W(K)$ (i. e. the points (resp. lines) incident with their image under ρ) forms an ovoid $\mathcal{O}_{\mathcal{ST}}$ (resp. spread $\mathcal{S}_{\mathcal{L}}$) in $W(K)$, called the *Suzuki-Tits ovoid* (resp. *Lüneburg spread*), see TITS [7]. Let π be a plane of $PG(3,K)$. It is easily seen that the intersection of π with $\mathcal{O}_{\mathcal{ST}}$ is exactly the set of points of $\mathcal{O}_{\mathcal{ST}}$ collinear in $W(K)$ to the point π^τ. We call such a set a *circle*. Now, every three points of $\mathcal{O}_{\mathcal{ST}}$ determine a unique plane, and hence a unique circle. So we obtain an inversive plane. But the Lüneburg spread puts an extra structure on this inversive plane, indeed, given a circle C lying in the plane π, the point π^τ is incident with a unique element M of $\mathcal{S}_{\mathcal{L}}$. And M is incident with a unique point x of $\mathcal{O}_{\mathcal{ST}}$, which belongs to C. Hence every circle C contains a special element x which we will call the *corner* of the circle and denote by ∂C. We list now some immediate properties.

LEMMA 4.1 *Let $\mathcal{O}_{\mathcal{ST}}$ be a Suzuki-Tits ovoid and let $x \in \mathcal{O}_{\mathcal{ST}}$. Let $\mathcal{C}_\S$ be the set of circles C with $\partial C = x$. Then the $C \setminus \{x\}$ partitions $\mathcal{O}_{\mathcal{ST}} \setminus \{\S\}$. In other words, any circle is uniquely determined by its corner and a second point, and, conversely, every pair of points (x,y) in $\mathcal{O}_{\mathcal{ST}}$ defines a unique circle C such that $y \in C$ and $\partial C = x$.*

LEMMA 4.2 *Let $\mathcal{O}_{\mathcal{ST}}$ be a Suzuki-Tits ovoid and let $x \in \mathcal{O}_{\mathcal{ST}}$ be the corner of a circle C. Let $\mathcal{D}_\mathcal{C}$ be the set of circles C' with $\partial C = y \in C \setminus \{x\}$ and $x \in C'$.*

Then the $C' \setminus \{x\}$ *partitions* $\mathcal{O}_{\mathcal{ST}} \setminus \{\S\}$.

PROOFS. Every circle is determined by a point u in $W(K)$, $u \notin \mathcal{O}_{\mathcal{ST}}$. For the first lemma, let u vary along the line x^ρ of $\mathcal{S}_{\mathcal{L}}$; for the second lemma, let u vary along the line y^ρ, where y is the point of $W(K)$ defining C. ⌺

REMARK. Lemmas 4.1 and 4.2 allow one to reconstruct $W(K)$ in a more axiomatized setting.

Following TITS [7], we can describe $\mathcal{O}_{\mathcal{ST}}$ by the set $K_\sigma^{(2)} \cup \{\infty\}$. Using, e.g., the coordinates in HANSSENS & VAN MALDEGHEM [1], one calculates that the circle containing ∞, $(0,0)$ and (k_0,k_1), $k_0,k_1 \in K$, contains, besides ∞, all points (tk_0,tk_1), $t \in K$. The circle containing $(0,0)$ with corner ∞ contains, besides ∞, the points $(0,k_1)$, $k_1 \in K$. By the action of the Suzuki group, one obtains the other circles. But we will need no explicite description of them.

Now let L_1 and L_2 be two elements of the Lüneburg spread $\mathcal{S}_{\mathcal{L}}$ and let $x_i \; I \; L_i$, $i = 1,2$, be the corresponding points of the Suzuki-Tits ovoid $\mathcal{O}_{\mathcal{ST}}$. The set of lines in the quadrangle $W(K)$ meeting all lines which meet both L_1 and L_2 is a *regulus* $\mathcal{R}$ and each element of $\mathcal{R}$ is incident with one point of $\mathcal{O}_{\mathcal{ST}}$. The set $\mathcal{T}$ of these points will be called a *transversal* with extremeties x_1 and x_2. It is completely determined by x_1 and x_2. Now consider the set $\{x_1,x_2\}^{\perp\perp}$ of points colinear to all points collinear to both x_1 and x_2 (this is the *span* of x_1 and x_2, see PAYNE & THAS [3],p.2) and let $y \in \{x_1,x_2\}^{\perp\perp} \setminus \{x_1,x_2\}$. The circle C defined by y (via intersection with y^τ) has as corner a point incident with y^ρ; but $y^\rho \in \mathcal{R}$, hence the corner of C lies in $\mathcal{T}$. Now note that the plane y^τ contains all points of the hyperbolic line H consisting of all points collinear to both x_1 and x_2 and H does not meet the ovoid. Hence the set of circles defined by the elements of $\{x_1,x_2\}^{\perp\perp} \setminus \{x_1,x_2\}$ partitions $O_{ST} \setminus \{x_1,x_2\}$ and the set of corners of the circles is the transversal $\mathcal{T}$. We logically call this partition the *transversal partition* with extremities x_1 and x_2.

Following HANSSENS & VAN MALDEGHEM [1], we can take for the symplectic polarity ρ the bilinear form

$$x_0y_1 + x_1y_0 + x_2y_3 + x_3y_2.$$

We choose $x_1 = (1,0,0,0)$ and $x_2 = (0,1,0,0)$. The line H is determined by $(0,0,1,0)$ and $(0,0,0,1)$. The Suzuki-Tits ovoid can be chosen to contain the points (see [1],5.6)

$$\{(N(k),1,k_1,k_2) | k = (k_0,k_1) \in K_\sigma^{(2)}\} \cup \{(1,0,0,0)\}.$$

It is then clear that a circle of the transversal partition described above contains the points

$$\{(N(k), 1, k_1, k_0) | N(k) = \text{Constant}\}.$$

Since every line of $PG(3, K)$ which is not a line of $W(K)$ meets $\mathcal{O}_{\mathcal{ST}}$ in either two or zero points, the following lemma is readily verified.

LEMMA 4.3 *Let x be a point of the Suzuki-Tits ovoid $\mathcal{O}_{\mathcal{ST}}$ and let C be a circle of $\mathcal{O}_{\mathcal{ST}}$ not containing x. Then there exists a unique transversal partition containing C and having x as one of its extremities.*

A Suzuki-Tits ovoid with the additional structure of the inversive plane, corners for all circles and transversal partitions for each pair of points will be called a *Suzuki-Tits inversive plane*, or briefly, an *STi-plane*.

4.2 Properties of $O_R(K, \sigma)$

Let $O_R(K, \sigma)$ be the *perfect* Ree Octagon described in section 3 (i.e. K is a perfect field). The lines through the point (∞) are parametrized by the set $K_\sigma^{(2)} \cup \{\infty\}$. By the preceding paragraph, we can give this set the structure of an STi-plane $\mathcal{P}_{(\infty)}$ (in an algebraic fashion). We will now reconstruct $\mathcal{P}_{(\infty)}$ geometrically. Note that, by transitivity, all points p define an STi-plane $\mathcal{P}_{\surd}$.

Let p be a point of $O_R(K, \sigma)$ opposite (∞). Clearly p has seven coordinates. The *trace* of p, denoted by $(\infty)^p$, (with respect to (∞)) is the set of projections of p onto the lines incident with (∞). Let o be the point with coordinates $(0,0,0,0,0,0,0)$ and suppose p has coordinates $(a, l, a', l', a'', l'', a''')$. The set of lines incident with (∞) and with one of the points of the intersection $(\infty)^o \cap (\infty)^p$ will be called the *support* of that intersection and, by (I6), it consists, besides possibly $[\infty]$, of all lines $[k]$, $k \in K_\sigma^{(2)}$, such that

$$f(k) =: a''' + aN(k) + l_0 k_1 + k_0 l_0'' = 0. \qquad (*)$$

If $a = 0$, then $[\infty]$ belongs to the support. So assume $a \neq 0$. Put $L = (L_0, L_1) = (\frac{l_0}{a}, \frac{l_0''}{a})$, then we have $f(k \oplus L) = aN(k) + f(L)$. We deduce from this that $k = (L_0, L_1 + \left(\frac{f(L)}{a}\right)^{\sigma^{-1}})$ is a solution of $(*)$. Hence we have shown that every two traces meet in at least one point. Without loss of generality we can take this point to be (0), i. e. $p = (0, l, a', l', a'', l'', a''')$. For any further points in the intersection, the equation $(*)$ reduces to

$$a''' + l_0k_1 + l_0''k_0 = 0. \qquad (**)$$

If $l_0 = l_0'' = a''' = 0$, then the two traces coincide; if $l_0 = l_0'' = 0$ and $a''' \neq 0$, then the two traces meet only in (0); if $(l_0, l_0'') \neq (0,0)$, then clearly, the set of lines through (∞) incident with a point of $(\infty)^o \cap (\infty)^p$ is a circle C in $\mathcal{P}_{(\infty)}$ and by the transitivity of the stabilizer of (∞), all circles of $\mathcal{P}_{(\infty)}$ arise in this way. So without loss of generality, let us consider the circle $C = \{[\infty]\} \cup \{[(0,k_1)] | k_1 \in K\}$. The set of points p such that the support of $(\infty)^p \cap (\infty)^o$ contains C is, by (I6), equal to

$$\{(0,(0,l_1),a',l',a'',l'',0) | l_1, a', a'' \in K, l', l'' \in K_\sigma^{(2)}\}.$$

The projection of the point $q = (0,(0,l_1),a',l',a'',l'',0)$ onto the line $[0]$ is the line $[0,0,l'']$, hence every line through $(0,0)$ (except for $[0]$ of course) arises in this way. This remains true for all points $((0,k_1),0)$, by transitivity. However, the projection of q onto $[\infty]$ is the line $[0,(0,l_1)]$, and here, not all lines arise. In fact, only lines of a circle in $\mathcal{P}_{(\prime)}$ with corner $[\infty]$ arise. This characterizes the corner of C in a geometric fashion. It also follows that the corner, defined in this geometric way, is independent of the chosen intersection of traces with C as support. We will call the intersection of two traces *trivial* if it contains only one element, or if the two traces are equal. Let X be a non-trivial intersection of two traces. We shall refer to the set of lines M through $x \in X$ such that M is the projection of a point p whose trace contains X *the gate set with respect to X through x*. We say such a gate set is trivial if it contains all lines through x except its support. Then we can summarize the above results as follows:

LEMMA 4.4 *Let x be any point of a perfect Ree octagon. Consider traces with respect to x. Then two traces always meet. If two traces X and Y meet non-trivially, then there exists a unique point u in $X \cap Y$ such that the gate set of u with respect to $X \cap Y$ is non-trivial. The line xu thus obtained from the support C of $X \cap Y$ is independent of the choice of $X \cap Y$. If we define ux to be the corner of C, then the set of all such supports, together with their corners define an STi-plane over the set of lines through x (the transversal partitions will follow from Lemma 4.7).*

Now consider again traces with respect to (∞). Suppose we are given 4 pairwise non-collinear points collinear to (∞) and such that the 4 respective projections onto (∞) do not lie on one circle of $\mathcal{P}_{(\infty)}$ (this defines a *general position* for 4 points collinear to one fixed point). We can take without loss of generality the points (0), $(0,0)$, (k,b) and (k^*,b^*), $k,k^* \in K_\sigma^{(2)}$, $b,b^* \in K$, with $k = (k_0,k_1)$ not proportional to $k^* = (k_0^*,k_1^*)$. If $p = (a,l,a',l',a'',l'',a''')$ defines a trace containing these 4 points, then its coordinates must satisfy $a = a''' = 0$, $b = l_0k_1 + l_0''k_0$ and $b^* = l_0k_1^* + l_0''k_0^*$. It is clear that under the given assumptions these equations define uniquely a pair (l_0, l_0''), and hence we have shown:

LEMMA 4.5 *Let x be any point of a perfect Ree octagon, then there is a unique trace containing 4 points in general position collinear to x.*

Suppose now two traces (with respect to (∞)) meet in exactly one point, say (0). Using (I6), one can check that, if one trace is defined by o, then the other must be defined by a point p with coordinates $(0,(0,l_1),a',l',a'',(0,l_1''),a''')$ with $a''' \neq 0$ and consequently it consists of the points (k,a'''), $k \in K_\sigma^{(2)}$. So the set of traces meeting $(\infty)^o$ in only (0) is a set of traces with trivial intersection. We call such a set a *pencil* of traces based at (0). We have shown:

LEMMA 4.6 *Let x be any point of a perfect Ree octagon and let X be a trace with respect to x. For every point $y \in X$, there exists a unique pencil of traces based at y and containing X.*

Next, we look at intersections of pencils. Let $\mathcal{E}_\infty$ and $\mathcal{E}_\prime$ be two pencils of traces based at respectively (0) and $(0,0)$, both containing the trace $(\infty)^o$. Put $u = (0,0,0,0,0,0,a''')$ and $v = (a,0,0,0,0,0,0)$. Then $(\infty)^u$ and $(\infty)^v$ are arbitrary elements of $\mathcal{E}_\infty$ and $\mathcal{E}_\prime$. They meet on the line $[k]$, $k \in K_\sigma^{(2)}$, if and only if $a''' = aN(k)$ (by (I6) again). Hence their intersection is non-trivial and the support is a circle defined by $N(k) = a'''/a =$constant. This shows that the set of supports of the intersections is the transversal partition with extremities the lines $[\infty]$ and $[0]$. Hence the lemma:

LEMMA 4.7 *Let x be any point of a perfect Ree octagon and consider, with respect to x, two pencils (based at resp. y_1 and y_2) sharing a common trace Y. Then the set of supports of all intersections of elements of one pencil with elements of the other pencil (excluding Y) is the transversal partition in $\mathcal{P}_\S$ with extremities xy_1 and xy_2.*

As a consequence we have:

COROLLARY 4.8 *There do not exist three distinct traces (with respect to a fixed point) in $O_R(K,\sigma)$ with pairwise trivial intersection and not contained in a pencil.*

Now we can define the following geometry $O_R(K,\sigma)_x$ for any point x of $O_R(K,\sigma)$. The points are of two types :

(i) the traces with respect to x,

(ii) the points collinear to x in $O_R(K,\sigma)$, including x itself.

The lines are also of two types :

(a) the pencils of traces (with respect to x),

(b) the lines of $O_R(K,\sigma)$ through x.

The incidence between points of type (i) (resp. type (ii)) and lines of type (a) (resp. type (b)) is containment (resp. the incidence in $O_R(K,\sigma)$). No point of type (i) is incident with a line of type (b). A point of type (ii) is incident with a line of type (a) if the pencil in question is based at the point in question. Incidence in $O_R(K,\sigma)_x$ will be denoted by I_x.

PROPOSITION 4.9 *The geometry $O_R(K,\sigma)_x$ as defined above is the generalized quadrangle of Tits-type $T_3(\mathcal{O}_{\mathcal{ST}})$.*

PROOF. We give a geometric proof using the lemmas above. A group-theoretic proof or another algebraic one is also possible. For instance, one can coordinatize $O_R(K,\sigma)_x$ to identify it. Or one could determine the automorphism group of this geometry inside the stabilizer of a point of the automorphism group of $O_R(K,\sigma)$. One would find an affine group with a Suzuki group acting.

We first show that $O_R(K,\sigma)_x$ is a generalized quadrangle. Let Π and Λ be a point and a line of $O_R(K,\sigma)_x$ which are not incident. We have to show that there is a unique point $\Pi'\ I_x\ \Lambda$, and a unique line $\Lambda'\ I_x\ \Pi$ such that $\Pi'\ I_x\ \Lambda'$. Suppose first that Π is of type (ii) and Λ is of type (b). Then $\Pi\ I_x\ M\ I_x\ x\ I_x\ \Lambda$, where M is the line in $O_R(K,\sigma)$ joining Π and x, and this path is unique. Suppose now Π is of type (i) and Λ is of type (b). Then there is a unique point p in $O_R(K,\sigma)$ incident with the line Λ and contained in the trace Π. There is also a unique pencil Λ' of traces based at p and containing Π, by Lemma 4.6. Again we have $\Pi\ I_x\ \Lambda'\ I_x\ p\ I_x\ \Lambda$ and no other such path exists. Next, let Π be of type (ii) and Λ of type (a). Let Λ be based at p. If $\Pi = x$, then $\Pi\ I_x\ xp\ I_x\ p\ I_x\ \Lambda$. Suppose now $\Pi = y$ is distinct from x. If y is collinear to p in $O_R(K,\sigma)$, then clearly $\Pi\ I_x\ xy\ I_x\ p\ I_x\ \Lambda$; if not, then $\Pi\ I_x\ \Lambda^*\ I_x\ X\ I_x\ \Lambda$, where X is the member of the pencil Λ containing y, and Λ^* is the pencil of traces based at y and containing X. Finally, suppose Π is of type (i) and Λ is of type (a). Let Λ be based at y. If $y \in \Pi$, then $\Pi \ I_x \ \Lambda^* \ I_x \ y \ I_x \ \Lambda$, where Λ^* is the pencil of traces based at y and containing Π. If $y \notin \Pi$, then by Corollary 4.8, there exists at most one member of the pencil Λ meeting the trace Π trivially. We now show that there exists at least one such element of Λ. Let X be any member of Λ. Suppose that X meets Π non-trivially (otherwise we are done) and let C be the support of the intersection $\overline{C}$. So C is a circle in $\mathcal{P}_\S$. By Lemma 4.3, there exists a unique transversal partition of $\mathcal{P}_\S$ containing C and having xy as one of its extremities. Let M (a line in $O_R(K,\sigma)$ through x) be the other extremity and let p be the unique point incident with M and lying on the trace Π. Let Λ^* be the pencil of traces containing Π and based at p, and Y be the unique member of Λ^* containing y. let Λ^{**} be the pencil of traces containing Y and based at y. Then by Lemma 4.7, there exists a trace $Z \in \Lambda^{**}$ meeting Π in $\overline{C}$, hence $\Lambda^{**} = \Lambda$, $Y \in \Lambda$ and $Y \cap \Pi = \{p\}$. So $\Pi \ I_x \ \lambda^* \ I_x \ Y \ I_x \ \Lambda$. We leave it to the reader to check that no other such paths exist and hence $O_R(K,\sigma)$ is a generalized quadrangle.

We still have to show that $O_R(K,\sigma)_x$ is isomorphic to $T_3(\mathcal{O}_{\mathcal{ST}})$. If K is a finite field of order 2^{2e+1}, it follows from Payne & Thas [3],5.3.1 that $O_R(2^{2e+1})_x$ is isomorphic to some $T_3(\mathcal{O})$, $\mathcal{O}$ an ovoid in $PG(3,q)$ since the point x of $O_R(2^{2e+1})_x$ is a 3-regular point (this can be verified easily). From the first part of the proof of 5.3.1 of *loc.cit.*, it follows that $\mathcal{O}$ is the Suzuki-Tits ovoid $\mathcal{O}_{\mathcal{ST}}$. In the infinite case, one has essentially the same proof, replacing some counting arguments in 5.3.1 of *loc.cit.* by arguments using the lemma's of this section. Alternatively, an algebraic proof goes as follows. The map

$$x^\perp \to x^\perp : (k,b) \mapsto (k, b + A''' + AN(k) + L_0k_1 + L_0''k_0)$$

defines an automorphism of $\mathcal{P}_{(\infty)}$ (this follows from (I6)). The group Ω of all such maps is isomorphic to $K \times K \times K \times K, +$, where the above element corresponds to (A''', A, L_0, L_0''). But Ω acts regularly on the set of traces (this is immediately verified), hence we can put the structure of the affine space $AG(4,K)$ on the set of traces in $(\infty)^\perp$ (induced by Ω). One can easily check that the set of lines of $O_R(K,\sigma)_x$ of type (a) are all lines of that affine space of certain parallel classes, and these parallel classes determine exactly a Suzuki-Tits ovoid at infinity. ⌬

REMARK. In the finite case, the above proof simplifies. Indeed, a counting argument replaces the use of Lemma 4.3, Lemma 4.7 and Corollary 4.8.

5 The metasymplectic space $\mathcal{M}(\mathcal{K})$

Dual traces.
One can also ask what the geometry of the *dual* traces look like (defined dually). From the relation (I1), we can deduce the following. Let L be a line of $O_R(K,\sigma)$, x_1 and x_2 two different points on L and L_1, L_2 two lines incident with x_1 resp. x_2, not equal to L. Let C_i, $i = 1,2$, be the circle in $\mathcal{P}_{\S\rangle}$ with corner L and containing L_i. If M is a line, varying over the set of all lines whose (dual) trace with respect to L contains an element of C_i for $i = 1,2$, then the set of lines contained in the trace of M and incident with any point x of L varies over a circle C in $\mathcal{P}_\S$ with corner L. We call the set of lines of traces with respect to L of all such lines M a *Suzuki regulus*. These sets will play an important role in the reconstruction of the ambient metasymplectic space for $O_R(K,\sigma)$.

One further property of traces.
Let x be any point in $O_R(K,\sigma)$ and let p be any point opposite x. Let $\mathcal{G}$ be the set of points y opposite x such that $x^y = x^p$. We define a graph on $\mathcal{G}$ by the rule: two points of G are adjacent if they are not opposite each other. One can show that two such points have distance 6 in the octagon and the unique middle element of the shortest path joining them has distance 5 to x. The graph $\mathcal{G}$ is not connected, in fact, it has exactly $|K|$ connected components. A connected component of $\mathcal{G}$ will be called a *trace direction*.

The reconstruction.
We briefly sketch how one can now reconstruct the ambient metasymplectic space $\mathcal{M}(\mathcal{K})$. The points of $\mathcal{M}(\mathcal{K})$ are of three types. Type (I) consist of the points of the octagon $O_R(K,\sigma)$ itself. The points of type (II) are the Suzuki reguli. A point of type (III) is a trace direction. The lines and planes must then be defined using the properties of traces listed in this paper. The hyperlines then follow rather easily, as well as the polarity. This will be proved in detail elsewhere.

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