Primitive arcs in $PG(2, q)$

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Abstract

We show that a complete arc $K$ in the projective plane $PG(2, q)$ admitting a transitive primitive group of projective transformations is either a cyclic arc of prime order or a known arc. If the completeness assumption is dropped, then $K$ has either an affine primitive group, or $K$ is contained in an explicit list. As an immediate corollary, the list of complete arcs fixed by a 2-transitive projective group is obtained.

1 Introduction and main results.

A $k$-arc $K$ of a projective plane $PG(2, q)$, also called a plane $k$-arc, is a set of $k$ points, no 3 of which are collinear. The best known example of an arc is the point set of a conic.

A point $p$ of $PG(2, q)$ extends a $k$-arc if and only if $K \cup \{p\}$ is a $(k + 1)$-arc. A $k$-arc $K$ of $PG(2, q)$ is called complete if and only if it is not contained in a $(k + 1)$-arc of $PG(2, q)$. In $PG(2, q)$, $q$ odd, $q > 3$, a conic is complete, but in $PG(2, q)$, $q$ even, a conic is not complete. It can be extended in a unique way to a $(q + 2)$-arc by its nucleus.

In the search for other examples of arcs, various methods have been used. The bibliographies of [7, 8, 9] contain a large number of articles in which arcs are constructed.

This paper continues the work of the authors in [11, 12] where arcs fixed by a large projective group are classified. In [11], all types of complete $k$-arcs, fixed by a cyclic projective group of order $k$, were determined. This led to a new class of such arcs containing $k/2$ points of 2 concentric conics. In [12], a slight variation to [11] is treated. In this paper, all complete $(k + 1)$-arcs fixed by a cyclic projective group of order $k$, were described. Here, no new examples were found.

Now, the classification of all complete $k$-arcs fixed by a transitive projective group acting primitively on the points of the arc, is presented. This is achieved by applying the

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classification of finite primitive permutation groups by O’Nan and Scott, in the version of
Bukenhout [2], on the list of subgroups of $PSL_3(q)$, given by Bloom [1] for $q$ odd, and by

In almost all cases, the completeness condition on the arc $K$ can be dropped. The complete-
ness of $K$ is only assumed in Section 3 where the complete $k$-arcs $K$ fixed by a transitive
elementary abelian group of order $k$, are determined. In the following section, all classes of
primitive $k$-arcs, $k \geq 5$, fixed by an almost simple projective group $G_K$, are found. They
are the conic in $PG(2, q)$, the unique 5- and 6-arc in $PG(2, 4)$ fixed by $A_5$ and $A_6$, and a
unique 6- and 10-arc in $PG(2, q), q \equiv \pm 1 \pmod{10}$, fixed by $A_5$.

As an immediate corollary, all complete arcs fixed by a 2-transitive projective group, are
determined.

From now on, suppose that $K$ is an arc in $PG(2, q)$ with automorphism group $\Gamma_K$. Put
$G := PGL_3(q)$ and $G_K := \Gamma_K \cap G$.

2 Preliminary lemmas.

Lemma 1 If $|K| \geq 4$, then $G_K$ acts faithfully on $K$.

Proof : The group $G$ acts regularly on the set of all ordered 4-arcs of $PG(2, q)$.

Lemma 2 If $|K| \geq 4$ and $K$ is complete, then $\Gamma_K$ acts faithfully on $K$.

Proof : If $\sigma \in \Gamma_K$ fixes every point of $K$, then $\sigma$ must be induced by a field automorphism
and it fixes a subplane $\pi$ pointwise. So $K \subseteq \pi$. Let $T$ be a line of $\pi$ skew to $K$ and let $x$
be a point on $T$ not in $\pi$. Then $x$ extends $K$ to a larger arc since every bisecant of $K$ is a
line of $\pi$.

Lemma 3 Suppose $K$ is complete.

The socle $S$ of $\Gamma_K$ is either elementary abelian or simple, i.e., $\Gamma_K$ is either of affine type
or almost simple. Moreover, if $\Gamma_K$ is almost simple, then $S \leq L_3(q)$.

Proof : Use the result of O’Nan and Scott in the version of Bukenhout [2]. According
to that result, the group $\Gamma_K$ is of one and only one of the following types: affine type,
bi-regular type, cartesian type or simple type. The definition of cartesian and bi-regular
type requires $\Gamma_K$ to have a normal subgroup $H$ isomorphic to the direct product of two or
more isomorphic copies of a non-abelian simple group $S$ [2]. Let $H \cong S_1 \times S_2 \times \cdots \times S_n$,
where each $S_i$ is isomorphic to $S$, $1 \leq i \leq n$. For every $i \in \{1, 2, \ldots, n\}$, the group $S_i$ can
be viewed as a subgroup of $H$, which is on its turn a subgroup of $PGL_3(q)$ by the previous
lemmas, and either $S_i \cap L_3(q) = S_i$ or $S_i \cap L_3(q) = 1$. Suppose the latter happens, then

$$S_i \cong S_i/(S_i \cap L_3(q)) \cong S_iL_3(q)/L_3(q) \leq PGL_3(q)/L_3(q).$$

Using the ATLAS-notation [3], the latter is isomorphic to the group $3.h$ or $h$, where $q = p^h$, $p$ prime. This is impossible since in the first case, $S_i$ has a normal subgroup of order 3 and in the other case, $S_i$ is cyclic and so abelian. Hence each $S_i$ is inside $L_3(q)$ and so is $H$. But by inspection of the list of subgroups of $L_3(q)$, see Bloom [1, Theorem 1.1], for $q$ odd, and Hartley [6, pp. 157-158], for $q$ even, one sees that this is impossible for $n \geq 2$. The case $n = 1$ corresponds to $H \cong S$. So $H$ is simple, $\Gamma_K$ is almost simple [2] and the above argument shows that the socle $S$ is a subgroup of $L_3(q)$.

In Section 3 we will consider the affine case and in Section 4, we will completely classify the simple case.

The following lemmas are elementary but turn out to be very useful.

Lemma 4 The group $\Gamma_K$ cannot contain a subgroup $H$ of central collineations with common center and common axis of order $r \geq 3$, when $|K| > 3$.

Proof : Every non-trivial orbit of such a group $H$ of collineations contains $r$ points on one line and so they cannot be points of an arc $K$. So $K$ is a subset of the set of points fixed by $H$, but then $|K| \leq 3$.

Lemma 5 If a central projective transformation $\sigma$ in $G_K$ fixes at least three points of an arc $K$, $|K| > 3$, then it is the identity.

This holds in particular for any involution $\sigma$ in $G_K$.

Proof : One of the three points, say $x$, must be the center of the central projective transformation $\sigma$. Any other point $y$ of $K$ is mapped onto a point $y^\sigma$ with the property that $x, y$ and $y^\sigma$ are points of $K$ on one line, but this is impossible.

This lemma is valid for the involutions of $PGL_3(q)$ since they are central [4, p. 172].

Lemma 6 Any projective transformation of $G_K$ fixing at least four points of $K$ is the identity.

Proof : The group $PGL_3(q)$ acts regularly on the ordered quadrangles of $PG(2, q)$. 

4
3 The affine case.

Assume that $G_K$ is of affine type. This means that $K$ bears the structure of a vector space $V$ over some prime field $GF(r)$ such that $G_K = H.G_0$ where $H$ is the group of all translations of $V$ and where $G_0$, the stabilizer of the origin $o$, is a subgroup of $GL(V)$ [2]. Using the fact that $H$ acts regularly on $K$, the following proposition is obtained.

Proposition 1 Let $K$ be a complete $k$-arc, $k = r^n$ with $r$ prime, in $PG(2, q)$. Suppose $H \leq G_K$ is an elementary abelian group of order $r^n$, acting regularly on $K$. Then $n = 1$ and $K$ is an orbit of an element of order $r$ of a Singer group of $PGL_3(q)$, or $k = 2^2$ and $K$ is a conic in $PG(2, 3)$ or a hyperoval in $PG(2, 2)$.

Proof : Let $r = 2$. If $q$ is odd, then $H$ contains $2^n - 1$ involutory homologies [4, p. 172] which commute with each other. Two homologies $h_1$ and $h_2$ commute if and only if they have common center and axis or the center of one homology $h_i$ belongs to the axis of the other homology $h_j$, $\{i, j\} = \{1, 2\}$. The first possibility cannot occur since there is a unique involutory homology with given center and given axis. The second possibility clearly implies that $|H| \leq 4$. Hence, by the completeness of $K$, $|H| = 4$, $q = 3$ and $K$ is a conic in $PG(2, 3)$.

If $q$ is even, then all involutions in $H$ are elations with either common center or common axis. If they have common center, then every non-trivial orbit of $H$ is contained in a line through the common center contradicting the fact that $K$ is an arc. In fact, this shows that no two elations of $H$ have common center. Suppose all elations have common axis $L$. Assume that a line $T$ is tangent to $K$. Then all elements of $T^H$ are tangent to $K$ and hence the point $T \cap L$ extends the arc $K$, so $K$ is not complete, a contradiction. There are no lines tangent to $K$. This implies $|K| = q + 2$ and this is a power of 2 only if $q = 2$. So $K$ is a hyperoval, the points of an affine plane, in $PG(2, 2)$.

Assume now $r$ odd. Let $O$ be an arbitrary orbit in $PG(2, q)$ under $H$. Since $H$ is an $r$-group, $|O| = r^m$ for $0 \leq m \leq n$. If $m = 0$, then $O = \{x\}$ and there is at least one line $T$ through $x$ tangent to $K$ since $|K|$ is odd. Applying $H$ to $T$, every line through $x$ meeting $K$ is a tangent line, hence $x$ extends $K$ and $K$ is not complete. If $0 < m < n$, then the kernel of $H$ on $O$ is non-trivial and so there is an element $\sigma$ of order $r$ fixing $O$ point by point. If at least three points of $O$ are collinear, then $\sigma$ is a central projective transformation, contradicting Lemma 4. So $O$ is an arc and hence $|O| = 3$. We can take coordinates such that $O = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. A projective transformation $\varphi$ of order 3 which is not central has necessarily a matrix of the form

$$\begin{pmatrix}
1 & 0 & 0 \\
0 & a & 0 \\
0 & 0 & a^{-1}
\end{pmatrix}, a^3 = 1, a \neq 1.$$
Hence $r^n = 3^2$. A projective transformation $\psi$ of order 3 permuting cyclically the points of $O$ has, without loss of generality, matrix

$$
\begin{pmatrix}
0 & 0 & d \\
1 & 0 & 0 \\
0 & b & 0
\end{pmatrix}, \quad b, d \in GF(q)^* = GF(q) \setminus \{0\}.
$$

Since $\varphi, \psi \in H$, they commute, but this implies that $a = 1$, a contradiction.

We have shown that every orbit of $H$ must have $r^n$ points, so $r^n$ must divide $q^2 + q + 1$. If $r \neq 3$, then every Sylow $r$-subgroup of $PGL_3(q)$ must be contained in some Singer group. Indeed, $r$ does not divide $|PGL_3(q)|/(q^2 + q + 1)$, which is shown in [11, Theorem 3.1]. This implies $n = 1$ since $H$ must be cyclic and elementary abelian. The result follows. If $r = 3$, since $9 \nmid (q^2 + q + 1)$, $k = 3$, but then $K$ is not complete, so this case need not be considered.

Every triangle, 3 non-collinear points, and every quadrilateral, 4 points no 3 of which are collinear, constitutes a primitive arc in a plane. From now on, assume $|K| \geq 5$.

4 The simple case.

In this section, assume that $G_K$ acts primitively on $K$, $|K| \geq 5$, and that $G_K$ is an almost simple group with socle $S$, i.e., $S$ is a non-abelian simple group and $G_K \leq \text{Aut} S$ [2]. By the classification of subgroups of $L_3(q)$ by Bloom [1, Theorem 1.1], see also Mitchell [10, pp. 239-242], for $q$ odd, and Hartley [6, pp. 157-158], for $q$ even, there are three infinite series for $S$, namely $L_3(q')$, $U_3(q')$ and $L_2(q')$, for suitable $q'$ dividing $q$. We first deal with them and afterwards with the sporadic cases.

Set $q = p^h$, $p$ a prime number and let $K$ be a $k$-arc in $PG(2, q)$.

4.1 Infinite classes.

4.1.1 The $L_3$-case.

Here, $PGL_3(q') \leq PGL_3(q)$ for every prime power $q' = p^h'$ such that $h'$ divides $h$.

Proposition 2 No arc $K$, $|K| \geq 5$, exists such that

$$L_3(q') \leq G_K \leq PGL_3(q') = \text{Aut}(L_3(q')) \cap PGL_3(q)$$

and such that $G_K$ acts primitively on $K$. 

6
Proof: The group $L_3(q')$ contains a subgroup of elations with common center and common axis of order $q'$, hence by Lemma 4, $q' = 2$. So there is a subplane $PG(2,2)$ in $PG(2,q)$ stabilized by $G_K$. Clearly $K \cap PG(2,2) = \emptyset$. If a point $x \in K$ lies on a line $L$ of $PG(2,2)$, by applying an element of order 2 in $L_3(2)$ contained in the stabilizer of $L$, one sees that $L$ contains at least two points of $K$, but the lines of $PG(2,2)$ partition in this way the points of $K$ in blocks of imprimitivity, a contradiction. Now let $x \in K$ and $u \in PG(2,2)$, then $xu$ is a line of $PG(2,q)$ not in $PG(2,2)$. The set of elations in $L_3(2)$ with center $u$ forms a subgroup of order 4 acting semi-regularly on the points of $xu \setminus \{u\}$. So $xu$ contains four points of $K$, a contradiction.

4.1.2 The $U_3$-case.

Here, $PGU_3(q') \leq PGL_3(q)$, $q' = p^{h'}$, whenever $2h'$ divides $h$. This group stabilizes a Hermitian curve in a subplane $PG(2,q^2)$ of $PG(2,q)$.

Proposition 3 No arc $K$, $|K| \geq 5$, exists such that

$$U_3(q') \leq G_K \leq PGU_3(q') = Aut(U_3(q')) \cap PGL_3(q)$$

and such that $G_K$ acts primitively on $K$.

Proof: The group $U_3(q')$ acts 2-transitively on a Hermitian curve $\mathcal{H}$ in some subplane $PG(2,q^2)$. Consider an element $\sigma$ of $U_3(q')$ fixing some point $x$ of $\mathcal{H}$ and mapping another point $y$ to some point $z$ on the line $xy$, $y, z \in \mathcal{H}$. Then $\sigma$ fixes $xy$ and its pole $u$ w.r.t. $\mathcal{H}$. Hence $\sigma$ fixes the lines $xu$ and $xy$. The order of $\sigma$ can be chosen to be $p$. So $\sigma$ fixes all lines through $x$ and it is easily seen that $xu$ is the axis. By Lemma 4, $p = 2$. But $z$ can be varied to obtain a group of elations with common center $x$ and common axis $xu$ of order $q'$. Hence $q' = 2$ by Lemma 4. But $U_3(2) \cong 3^2 : Q_8$ is not simple and has no non-abelian simple socle.

4.1.3 The $L_2$-case.

Here, $PGL_2(q') \leq PGL_3(q)$, $q' = p^{h'}$, whenever $h'$ divides $h$.

Proposition 4 If $K$ is an arc in $PG(2,q)$ such that $G_K$, with

$$L_2(q') \leq G_K \leq PGL_2(q') = Aut(L_2(q')) \cap PGL_3(q),$$

acts primitively on $K$, then $K$ is a conic in some subplane $PG(2,q')$ of $PG(2,q)$.
Proof: Let \( C \) be the conic on which \( G_K \) acts naturally inside some subplane \( PG(2, q') \). Note that we can assume \( q' > 3 \) since \( PGL_2(2) \) and \( PGL_2(3) \) have no non-abelian simple socle. Clearly if the arc \( K \) has a point in common with \( PG(2, q') \), then it consists of either all internal points of \( C \) \((p \text{ odd})\), all external points of \( C \) \((p \text{ odd})\), the nucleus of \( C \) \((p = 2)\), all points not on \( C \) and distinct from the nucleus of \( C \) \((p = 2)\) or the conic \( C \) itself. Only the last set of points constitutes an arc. So we can assume that all points of \( K \) lie outside \( PG(2, q') \). If one point of \( K \) lies on a line \( L \) of \( PG(2, q') \), then all points of \( K \) do and the lines in the orbit of \( L \) under \( G_K \) define a partition of \( K \) invariant under \( G_K \). Let \( x \in K \cap L \). If \( L \) is a bisecant of \( C \), then the cyclic subgroup of \( L_2(q') \) fixing \( L \) has at least order \((q' - 1)/2\) and acts on \( L \setminus C \) in orbits of at least size \((q' - 1)/4\), if \( L \) is a tangent of \( C \) in \( a \), the cyclic subgroup of \( L_2(q') \) fixing \( a \) and a second point \( b \) of \( C \) has again at least order \((q' - 1)/2\) and acts semi-regularly on \( L \setminus \{a\} \) and if \( L \) is skew to \( C \) in \( PG(2, q') \), \( L_2(q') \) contains a cyclic subgroup of order \((q' + 1)/2\), fixing \( L \), and acting semi-regularly on \( L \setminus C \). Hence the partition is not trivial if \( q' > 5 \). The only problem occurs when \( G_K = L_2(5) \) and \( L \) is a bisecant of \( C \) in \( PG(2, q') \). If \( L \) contains one point of \( K \), all bisecants of \( C \) contain one point of \( K \), so \(|K| = 15\). This is impossible since \( G_K \cong L_2(5) \cong A_5 \) does not act primitively on 15 points [3].

So we may assume that no point of \( K \) lies on a line of \( PG(2, q') \). Let \( x \in K \), \( \sigma \in G_K \) and suppose that \( x^\sigma = x \). If \( \sigma \) fixes two points \( a, b \) of \( C \), then \( \sigma \) fixes four points, namely \( a, b, x \) and the pole of the line \( ab \) w.r.t. \( C \) or the nucleus of \( C \). No three of these points are collinear, otherwise \( x \) lies on a line of \( PG(2, q') \), contradicting our assumption, hence \( \sigma \) is the identity. Suppose now \( \sigma \) acts semi-regularly on \( C \). Then \( \sigma \) fixes two points \( a, b \) of \( C \) in a quadratic extension of \( PG(2, q') \) and as above, this leads to \( \sigma \) being the identity. Finally, suppose \( \sigma \) fixes exactly one point \( u \) of \( C \), then it fixes the tangent line \( T \) to \( C \) through \( u \) and it fixes also the line \( xu \). Since \( \sigma \) has necessarily order \( p \), it readily follows that it fixes all lines through \( u \). So \( \sigma \) is central, \( p = 2 \) (Lemma 4), and the axis is \( T \). But \( x \) does not lie on \( T \) and is fixed, hence \( \sigma \) is the identity.

We have shown that no non-trivial element of \( G_K \) fixes a point of \( K \). So \( G_K \) acts regularly on \( K \) and such an action can never be primitive for groups of non-prime order.

This completes the investigation of the infinite classes.

4.2 The sporadic classes.

The list of these classes is given by Bloom [1, Theorem 1.1] for \( q \text{ odd} \), and by Suzuki [13, introduction] for \( q \text{ even} \).

4.2.1 Case \( L_2(7) \leq G_K \leq PGL_2(7) \).

In this case, \( q^3 \equiv 1 \pmod{7} \), \( q \text{ odd} \), see Bloom [1, Theorem 1.1]. By the **ATLAS** [3], \( L_2(7) \) can only act primitively on either 7 or 8 elements. If \( G_K \cong PGL_2(7) \) and \( L_2(7) \) as a
subgroup of $G_K$ does not act transitively on $K$, then $|K| = 28$ or $21$ [3]. If $|K| = 28$, then $K$ can be identified with the pairs of points of $PG(1, 7)$ and every involution fixes 4 pairs, contradicting lemma 6. If $|K| = 21$, then $K$ can be identified with the pairs of conjugated points in $PG(1, 49)$. The involution sending $x$ to $-x$ fixes three such pairs, contradicting lemma 5. We now deal with $G_K \cong L_2(7)$.

Proposition 5 The group $L_2(7)$ does not act primitively on any arc in $PG(2, q)$, $q^2 \equiv 1 \pmod{7}$.

Proof : Suppose $|K| = 7$. Since $L_2(7) \cong L_3(2)$, the Klein fourgroup $K_4$ is inside $G_K$, it fixes three points $x, y, z \in K$ and acts regularly on the remaining four points of $K$. This contradicts Lemma 5.

Suppose now $|K| = 8$. Drop the restrictions on $q$ for the time being. It is shown that every orbit of $L_2(7)$ of length 8 which constitutes an arc in any finite projective plane must be a conic in a subplane of order 7.

We can identify the points of $K$ with the elements of $GF(7) \cup \{\infty\}$ in the natural action of $L_2(7)$. We establish this identification via the indices. So $K = \{x_0, x_1, \ldots, x_6, x_\infty\}$. We coordinatize $PG(2, q)$ and take $x_0 = (1, 0, 0), x_\infty = (0, 1, 0)$ and $x_1 = (1, 1, 1)$. An element $\sigma$ in $G_K$ of order 3 fixing $x_0$ and $x_\infty$ exists. It is multiplication by 2 or 4 in the natural action, let us assume multiplication by 2. Since $1 + q + q^2 \not\equiv 2 \pmod{3}$, $\sigma$ has to fix at least one other point $y$ of $PG(2, q)$. By Lemma 4, $\sigma$ cannot be central, hence $q \equiv 1 \pmod{3}$ and $y$ is not incident with the line $x_0x_\infty$. Neither lies $y$ on any other bisecant of $K$ containing $x_0$ or $x_\infty$. It would imply that $\sigma$ has to fix that bisecant point by point and so $\sigma$ would be central. Hence we can take $y = (0, 0, 1)$. The matrix of $\sigma$ looks like

$$
\begin{pmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & 1
\end{pmatrix}, a, b \in GF(q)^*.
$$

Clearly $a = b$ or $1 \in \{a, b\}$ implies that $x_1, x_1^a$ and $x_1^b$ are collinear, hence $a \neq b, a, b \neq 1$. Since $\sigma$ has order 3, both $a$ and $b$ are non-trivial third roots of unity, say $a = \omega$ and $b = \omega^2$, $\omega^2 + \omega + 1 = 0$. Hence $x_2 = (\omega, \omega^2, 1)$ and $x_4 = (\omega^2, \omega, 1)$. If we set $x_3 = (u, v, 1)$, then $x_6 = (\omega u, \omega^2 v, 1)$ and $x_5 = (\omega^2 u, \omega v, 1)$. Let $\theta$ be the element of $L_2(7)$ mapping $x_i$ to $x_{i+1}$ and fixing $x_\infty$. Knowing the action of $\theta$ on 8 points of $PG(2, q)$, we can find its matrix, namely

$$
\begin{pmatrix}
1 & 0 & b \\
1 & a & c \\
1 & 0 & d
\end{pmatrix}, a, b, c, d \in GF(q).
$$

Expressing $x_1^\theta = x_2$, $x_2^\theta = x_3$ and $x_3^\theta = x_4$, the elements $a, b, c, d$ must satisfy,

(A) $\omega + (1 + d)\omega - 1 = (\omega + d)u$,
(B) $\omega + \omega^2 a + (1 + d)\omega^2 - 1 - a = (\omega + d)v$, 
(C) $u + (1 + d)\omega - 1 = (u + d){\omega^2}$, 
(D) $u + av + (1 + d)\omega^2 - 1 - a = (u + d)\omega$, 
(E) $b = (1 + d)\omega - 1$, 
(F) $c = (1 + d)\omega^2 - 1 - a$.

From (A) and (C), $(u - 1)(u + 2) = 0$. If $u = 1$, then $d = -1$ by (A), so $a(v - 1) = 0$ by (D). Clearly $a \neq 0$, so $v = 1$ and $x_1 = x_3$, a contradiction. So $u = -2$. Noting $\omega \neq -2 (p \neq 3)$, we deduce from (A) that $d = -3\omega - 1$ since $\omega^2 + \omega + 1 = 0$. Combining (B) and (D), gives 

$$v^2(1 - \omega) + v(5\omega + 4) + 14\omega + 4 = 0.$$ 

This implies $v = -2$ or $v = -3\omega + 1$. If $v = -2$, then $a = -3$ by (B). But $x_4^0 = x_5$ implies 

$$(2\omega + 1, -4\omega - 2, -4\omega - 2) = k(2\omega + 2, -2\omega, 1),$$

for some $k \in \text{GF}(q)^*$. This implies $-2\omega = 1$, hence $p = 3$ and $a = 0$ which is false. So $v = -3\omega + 1$. Then (B) implies $a = 3\omega + 3$ and (E) and (F) imply that $\theta$ has matrix 

$$
\begin{pmatrix}
1 & 0 & 3\omega + 2 \\
1 & 3\omega + 3 & -3\omega - 7 \\
1 & 0 & -3\omega - 1
\end{pmatrix}.
$$

Expressing $x_4^0 = x_5$, we obtain $7 = 0$ and $\omega = 4$. So $p = 7$ and all points of $K$ satisfy $X_0X_1 = X_2^2$, showing our assertion.

4.2.2 Case $A_6 \leq G_K \leq \text{Aut}(A_6)$.

First, assume $G_K \cong A_6$. This can only happen for $q$ an even power of 2 [13, introduction] or 5 and for $q \equiv 1$ or 19 (mod 30) [1, Theorem 1.1 (8) and (9)].

Proposition 6 Under the above assumptions, if $A_6 \leq PGL_3(q)$ acts primitively on an arc $K$, then $q$ is even and $K$ is the unique hyperoval consisting of 6 points in a subplane of order 4.

Proof: By the ATLAS [3], there are three distinct possibilities for $|K|$. First, suppose $|K| = 6$. Select 4 points of $K$ and give them coordinates $(1,0,0),(0,1,0),(0,0,1)$ and $(1,1,1)$. There is an element $\sigma$ of order 3 fixing the first three points and acting regularly on the remaining three points of $K$. As in the proof of Proposition 5, $\sigma$ has matrix 

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^2
\end{pmatrix}, \omega \in \text{GF}(q), \omega \neq 1, \omega^3 = 1.
$$
The group element with matrix
\[
\begin{pmatrix}
-1 & 0 & 1 \\
0 & -\omega & \omega \\
0 & 0 & \omega^2 \\
\end{pmatrix}
\]
fixes \((1, 0, 0)\) and \((0, 1, 0)\), maps \((0, 0, 1)\) to \((1, \omega, \omega^2)\) and \((1, 1, 1)\) to \((0, 0, 1)\). Hence, it should preserve \(K\) since \(A_6\) acts 4-transitively on 6 points. So the image \((-1 + \omega^2, -\omega^2 + 1, \omega)\) of \((1, \omega, \omega^2)\) must belong to \(K\). This happens only if \(p = 2\), in which case all points of \(K\) except \((1, 0, 0)\) lie on the conic \(X_1X_2 = X_3^2\) in \(PG(2, 4)\). So \(K\) is the hyperoval mentioned in the statement of the proposition.

Next, suppose \(|K| = 10\). Then we can think of \(A_6\) as being \(L_2(9)\) acting on the elements of \(GF(9) \cup \{\infty\}\). Hence, we can label the points of \(K\) as \(x_i, i \in GF(9) \cup \{\infty\}\), and the action of \(L_2(9)\) goes via its natural action on the indices. An entirely similar argument as for the case \(L_2(7)\) shows here that \(q\) must be an even power of 3 and that \(K\) is a conic in some subplane of order 9 of \(PG(2, q)\). This case was treated in 4.1.3.

Finally, suppose \(|K| = 15\). Here, \(K\) can be identified with the pairs of the set \(\{1, 2, 3, 4, 5, 6\}\) with the natural action of \(A_6\) [3]. The involution \((1 2)(3 4) \in A_6\) fixes the pairs \(\{1, 2\}, \{3, 4\}, \{5, 6\}\) and acts semi-regularly on the remaining ones. This permutation induces an involution in \(PG(2, q)\) fixing three points of \(K\), contradicting Lemma 5. 

The next case deals with groups having \(A_6\) as a socle.

Proposition 7 Under the assumptions above, if \(A_6 \leq G_K \leq \text{Aut}(A_6)\) acts primitively on an arc \(K\) in \(PG(2, q)\), then \(G_K \cong A_6, q = 2^{2h}, h \geq 1, \) and \(K\) is a hyperoval in some subplane of order 4.

Proof : By the previous result, we may assume that \(A_6 \not\cong G_K\). By the information in the ATLAS [3], there are two possibilities: the action of \(G_K\) on \(K\) is equivalent to the action of \(PGL_2(9)\) on pairs of points, \(O_2^+(9)\)'s, of \(PG(1, 9)\) or the action on \(K\) is equivalent to the action of \(PGL_2(9)\) on pairs of conjugated points in a quadratic extension, \(O_2^-(9)\)'s, of \(PG(1, 9)\).

In the first case, any involution of \(L_2(9)\) fixes five pairs of \(PG(1, 9)\) and acts semi-regularly on the remaining 40, contradicting Lemma 6.

In the second case, the involution \(x \mapsto -x, x \in GF(9)\), belongs to \(L_2(9)\) and fixes 4 pairs of conjugated points in a quadratic extension of \(GF(9)\), contradicting Lemma 6 again.

4.2.3 Case \(A_7 \leq G_K \leq S_7\).

This occurs when \(p = 5\) and \(h\) is even [1, Theorem 1.1 (8)].

Proposition 8 The group \(A_7\) does not act primitively on any arc in \(PG(2, 5^{2h}), h \geq 1,\).
Proof: The group $A_7$ has a primitive action on 7, 15, 21 and 35 points [3]. Let $S := \{1, 2, 3, 4, 5, 6, 7\}$. The action of $A_7$ on 7 points is the natural one on $S$ and is 5-transitive which is impossible by Lemma 6. The action on 21 points is the action of $A_7$ on the unordered pairs of $S$. The permutation $(1\ 2\ 3)$ fixes 6 pairs and hence should be the identity, by Lemma 6 again. The action on 35 points is the action on the triads of $S$. The permutation $(1\ 2\ 3)$ fixes five triads and hence should be the identity again.

The action on 15 points is the action of $A_7$ on the points of $PG(3, 2)$. Here, there is an involution fixing three points on a line of $PG(3, 2)$, contradicting Lemma 5.

To conclude, we deal with $G_K \cong S_7$.

Proposition 9 The group $S_7$ does not act primitively on any arc in $PG(2, 5^{2h}), h \geq 1$.

Proof: By the previous proposition, we may assume that $A_7$, as a subgroup of $S_7$, does not act primitively on $K$. This leaves only one possibility [3]: an action of $S_7$ on 120 points. The group $A_7$ acts on these points imprimitively in blocks of size 8, the stabilizer of a block being $L_2(7)$. The 15 blocks can be identified with the points of $PG(3, 2)$. The stabilizer of a point of $PG(3, 2)$ in $A_7$ is $L_3(2)$. This contains an element $\sigma$ of order 3 and this element $\sigma$ has to fix at least 2 other points of $PG(3, 2)$. In other words, $\sigma$ stabilizes 3 blocks, and in each one of them, it must fix 2 points. So $\sigma$ fixes in total 6 points, contradicting Lemma 6.

4.2.4 Case $A_5 \cong G_K$.

In this case, $q \equiv \pm 1 \pmod{10}$ Bloom [1, Theorem 1.1 (6)] or $q = 2^{2h}, h \geq 1$ Hartley [6, pp. 157-158]. By the ATLAS [3], $G_K$ can only act primitively on 5, 6 or 10 points. The action of $A_5$ in $PG(2, q), q \equiv \pm 1 \pmod{10}$, is uniquely determined by 2 matrices $T$ and $B$ [1, Lemma 6.4].

Proposition 10 Suppose $A_5$ fixes a 5-arc $K$ in $PG(2, q)$. Then $q = 2^{2h}, h \geq 1$, and $K$ is a conic in a subplane $PG(2, 4)$.

Proof: Let $K = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1), (1, x, y)\} = \{p_1, \ldots, p_5\}$ where $A_5$ acts naturally on the indices $i, 1 \leq i \leq 5$.

The mapping $(1\ 2)(3\ 4)$ of $A_5$ is defined by the matrix

$$
\begin{pmatrix}
0 & -1 & 1 \\
-1 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}
$$

and fixes $(1, x, y)$ if and only if $y = x + 1$. 

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The mapping \((1 2 3)\) is defined by the matrix
\[
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]
and fixes \((1, x, y)\) if and only if \(1 + x = \rho, 1 = \rho x\) and \(x = \rho(1 + x)\) for some \(\rho \neq 0\). This implies \(x^2 + x - 1 = 0\) and \(x^2 - x - 1 = 0\). So \(2 = 0\) and \(x^2 + x + 1 = 0\). This shows that \(q = 2^{2h}, h \geq 1\), and \(K\) is a conic in a subplane \(PG(2, 4)\).

Remark 1 This conic \(K\) is contained in a unique hyperoval of \(PG(2, 4)\) fixed by \(A_6\) (Proposition 6).

Proposition 11 When \(q \equiv \pm 1 \pmod{10}\), then the sets \(K_1 = \{(1, 1, 1), (1, 1, -1), (1, -1, 1), (1, -1, -1), (0, 4t^2, 1), (0, -4t^2, 1), (-4t^2, 1, 0), (4t^2, 1, 0), (1, 0, 4t^2), (1, 0, -4t^2)\}\) and \(K_2 = \{(1, 0, 1 - 2t), (1, 0, 2t - 1), (1, 2t, 0), (1, -2t, 0), (0, 1, 2t), (0, 1, -2t)\}\) constitute a 10-arc and a 6-arc fixed by \(A_5\). The points of \(K_1\) are the 10 points of \(PG(2, q)\) on 3 bisecants of \(K_2\).

Proof: This can be verified by using the matrices \(T\) and \(B\) of [1, Lemma 6.4].

Proposition 12 The 10-arc \(K_1\) and 6-arc \(K_2\) in \(PG(2, q)\), \(q \equiv \pm 1 \pmod{10}\), are projectively unique.

Proof: (a) Suppose there is a second orbit \(O\) of size 10. Let \(p \in O\), then \(p\) is fixed by a subgroup \(H\) of order 6 of \(A_5\). The unique subgroup of order 3 in \(H\) must fix a point \(r_1\) of \(K_1\). Then \(r_1\) is fixed by \(H\). This implies that \(pr_1\) is a tangent to \(K_1\).

Since 10 is even, \(p\) belongs to a second tangent \(pr_2\) to \(K_1, r_2 \in K_1\). If an element of order 3 in \(H\) fixes \(r_2\), it fixes 4 points of \(K_1\), which is false (Lemma 6), so \(p\) belongs to at least 4 tangents \(pr_i, 1 \leq i \leq 4\), to \(K_1\). Any involution \(\gamma\) in \(H\) must fix two tangents through \(p\) since it fixes \(r_1\). It cannot fix 4 tangents (Lemma 6). Assume \(\gamma(r_2) = r_2\) and \(\gamma(r_3) = r_4\), then \(\{p, r_1, r_2, r_3, r_4\}\) is a 5-arc fixed by \(\gamma\). This contradicts Lemma 5.

(b) Suppose there is a second orbit \(O\) of size 6.

If \(p \in O\), then \(p\) is fixed by a subgroup \(H\), of order 10, of \(A_5\). Since \(H\) has a unique subgroup of order 5 and since \(|K_2| = 6\), \(H\) must fix one point \(r_1\) of \(K_2\) and, as in (a), \(pr_1\) is tangent to \(K_2\). An element of order 5 in \(H\) acts transitively on \(K_2 \setminus \{r_1\}\) and fixes \(p\), so \(p\) extends \(K_2\) to a 7-arc.

An involution \(\gamma\) of \(H\) fixes \(p, r_1\) and a second point \(r_2\) of \(K_2\). Let \(\gamma(r_3) = r_4, r_3, r_4 \in K_2\), then \(\{p, r_1, r_2, r_3, r_4\}\) is a 5-arc fixed by \(\gamma\). This again contradicts Lemma 5.
Remark 2 Hexagons $\mathcal{H}$ fixed by $A_5$ were studied in detail by Dye [5]. These hexagons occur when $q \equiv \pm 1 \pmod{10}$, $q = 5^h$ or $q = 2^{2h}, h \geq 1$.

When $q = 2^{2h}, h \geq 1$, then $\mathcal{H}$ is a hyperoval in a subplane $PG(2,4)$ and $\mathcal{H}$ is fixed by $A_6$ (Propositions 6 and 10).

From now on, assume $q$ odd. If $q = 5^h$, then $\mathcal{H}$ is a conic in a subplane $PG(2,5)$ of $PG(2,q)$, $A_5 \cong L_2(5)$, but $\mathcal{H}$ is not contained in a conic when $q \equiv \pm 1 \pmod{10}$. In both cases, this hexagon is called the Clebsch hexagon [5]. One of its particular properties is that it has exactly 10 Brianchon-points, i.e., points on exactly 3 bisecants to $\mathcal{H}$. If $q = 5^h$, these Brianchon-points are the internal points of the conic $\mathcal{H}$ in the subplane $PG(2,5)$. The 10 Brianchon-points constitute a 10-arc if $q \equiv \pm 1 \pmod{10}$ (Proposition 11).

With this hexagon correspond 5 triangles whose edges partition $\mathcal{H}$ and on which $A_5$ acts in a natural way. These 5 triangles are self-polar w.r.t. a unique conic $C$. When $q = 5^h$, $\mathcal{H} = C$. The 10 Brianchon-points belong to $C$ if and only if $q = 3^{2h}, h \geq 1$, and in this case, $C$ is a conic in a subplane $PG(2,9)$. Equivalently, when $q = 3^{2h}, h \geq 1$, the 10-arc $K_1$ (Proposition 11) is a conic in a subplane $PG(2,9)$.

This completes the proof of our main result.

5 Complete 2-transitive arcs

As an immediate consequence of the classification of primitive arcs made in Sections 3 and 4, the following list of complete 2-transitive arcs is obtained. As before, assume $|K| \geq 5$.

Proposition 13 If $K$ is a complete $k$-arc of $PG(2,q)$, fixed by a 2-transitive projective group $G_K$, then either

1. $K$ is a conic in $PG(2,q)$, $q$ odd, $q > 3$;
2. $K$ is the unique 6-arc in $PG(2,4)$;
3. $K$ is the unique 6-arc in $PG(2,9)$ fixed by $A_5$;
4. $K$ is the unique 10-arc in $PG(2,11)$ or $PG(2,19)$ fixed by $A_5$.

Proof: This follows from the preceding classification.

The completeness of the 6- and 10-arc fixed by $A_5$ in $PG(2,q), q \equiv \pm 1 \pmod{10}$, was checked by computer. The 10-arc in $PG(2,9)$ fixed by $A_5$ is the conic of $PG(2,9)$ (Remark 2), so this arc is included in case (1).
References


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