Primitive arcs in $PG(2,q)$

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Abstract

We show that a complete arc K in the projective plane $PG(2,q)$ admitting a transitive primitive group of projective transformations is either a cyclic arc of prime order or a known arc. If the completeness assumption is dropped, then K has either an affine primitive group, or K is contained in an explicit list. As an immediate corollary, the list of complete arcs fixed by a 2-transitive projective group is obtained.

1 Introduction and main results.

A k-arc K of a projective plane $PG(2,q)$, also called a plane k-arc, is a set of k points, no 3 of which are collinear. The best known example of an arc is the point set of a conic.

A point p of $PG(2,q)$ extends a k-arc if and only if $K \cup \{p\}$ is a $(k+1)$-arc. A k-arc K of $PG(2,q)$ is called complete if and only if it is not contained in a $(k+1)$-arc of $PG(2,q)$. In $PG(2,q)$, q odd, $q>3$, a conic is complete, but in $PG(2,q)$, q even, a conic is not complete. It can be extended in a unique way to a $(q+2)$-arc by its nucleus.

In the search for other examples of arcs, various methods have been used. The bibliographies of [7, 8, 9] contain a large number of articles in which arcs are constructed.

This paper continues the work of the authors in [11, 12] where arcs fixed by a large projective group are classified. In [11], all types of complete k-arcs, fixed by a cyclic projective group of order k, were determined. This led to a new class of such arcs containing $k/2$ points of 2 concentric conics. In [12], a slight variation to [11] is treated. In this paper, all complete $(k+1)$-arcs fixed by a cyclic projective group of order k, were described. Here, no new examples were found.

Now, the classification of all complete k-arcs fixed by a transitive projective group acting primitively on the points of the arc, is presented. This is achieved by applying the

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classification of finite primitive permutation groups by O'Nan and Scott, in the version of Buekenhout [2], on the list of subgroups of $PSL_3(q)$, given by Bloom [1] for q odd, and by Suzuki [13] and Hartley [6] for q even.

In almost all cases, the completeness condition on the arc K can be dropped. The completeness of K is only assumed in Section 3 where the complete k-arcs K fixed by a transitive elementary abelian group of order k, are determined. In the following section, all classes of primitive k-arcs, $k \geq 5$, fixed by an almost simple projective group G_K, are found. They are the conic in $PG(2,q)$, the unique 5- and 6-arc in $PG(2,4)$ fixed by A_5 and A_6, and a unique 6- and 10-arc in $PG(2,q), q \equiv \pm 1 \pmod{10}$, fixed by A_5.

As an immediate corollary, all complete arcs fixed by a 2-transitive projective group, are determined.

From now on, suppose that K is an arc in $PG(2,q)$ with automorphism group Γ_K. Put $G := PGL_3(q)$ and $G_K := \Gamma_K \cap G$.

2 Preliminary lemmas.

Lemma 1 If $|K| \geq 4$, then G_K acts faithfully on K.

Proof : The group G acts regularly on the set of all ordered 4-arcs of $PG(2,q)$. ■

Lemma 2 If $|K| \geq 4$ and K is complete, then Γ_K acts faithfully on K.

Proof : If $\sigma \in \Gamma_K$ fixes every point of K, then σ must be induced by a field automorphism and it fixes a subplane π pointwise. So $K \subseteq \pi$. Let T be a line of π skew to K and let x be a point on T not in π. Then x extends K to a larger arc since every bisecant of K is a line of π. ■

Lemma 3 Suppose K is complete.

The socle S of Γ_K is either elementary abelian or simple, i.e., Γ_K is either of affine type or almost simple. Moreover, if Γ_K is almost simple, then $S \leq L_3(q)$.

Proof : Use the result of O'Nan and Scott in the version of Buekenhout [2]. According to that result, the group Γ_K is of one and only one of the following types: affine type, biregular type, cartesian type or simple type. The definition of cartesian and biregular type requires Γ_K to have a normal subgroup H isomorphic to the direct product of two or more isomorphic copies of a non-abelian simple group S [2]. Let $H \cong S_1 \times S_2 \times \cdots \times S_n$, where each S_i is isomorphic to S, $1 \leq i \leq n$. For every $i \in \{1, 2, \ldots, n\}$, the group S_i can be viewed as a subgroup of H, which is on its turn a subgroup of $P\Gamma L_3(q)$ by the previous lemmas, and either $S_i \cap L_3(q) = S_i$ or $S_i \cap L_3(q) = 1$. Suppose the latter happens, then

$$S_i \cong S_i/(S_i \cap L_3(q)) \cong S_i L_3(q)/L_3(q) \leq P\Gamma L_3(q)/L_3(q).$$

Using the **ATLAS**-notation [3], the latter is isomorphic to the group $3.h$ or h, where $q = p^h$, p prime. This is impossible since in the first case, S_i has a normal subgroup of order 3 and in the other case, S_i is cyclic and so abelian. Hence each S_i is inside $L_3(q)$ and so is H. But by inspection of the list of subgroups of $L_3(q)$, see Bloom [1, Theorem 1.1], for q odd, and Hartley [6, pp. 157-158], for q even, one sees that this is impossible for $n \geq 2$. The case $n = 1$ corresponds to $H \cong S$. So H is simple, Γ_K is almost simple [2] and the above argument shows that the socle S is a subgroup of $L_3(q)$. ■

In Section 3 we will consider the affine case and in Section 4, we will completely classify the simple case.

The following lemmas are elementary but turn out to be very useful.

Lemma 4 The group Γ_K cannot contain a subgroup H of central collineations with common center and common axis of order $r \geq 3$, when $|K| > 3$.

Proof : Every non-trivial orbit of such a group H of collineations contains r points on one line and so they cannot be points of an arc K. So K is a subset of the set of points fixed by H, but then $|K| \leq 3$. ■

Lemma 5 If a central projective transformation σ in G_K fixes at least three points of an arc K, $|K| > 3$, then it is the identity.

This holds in particular for any involution σ in G_K.

Proof : One of the three points, say x, must be the center of the central projective transformation σ. Any other point y of K is mapped onto a point y^σ with the property that x, y and y^σ are points of K on one line, but this is impossible.

This lemma is valid for the involutions of $PGL_3(q)$ since they are central [4, p. 172]. ■

Lemma 6 Any projective transformation of G_K fixing at least four points of K is the identity.

Proof : The group $PGL_3(q)$ acts regularly on the ordered quadrangles of $PG(2, q)$. ■

3 The affine case.

Assume that G_K is of affine type. This means that K bears the structure of a vector space V over some prime field GF(r) such that $G_K = H.G_0$ where H is the group of all translations of V and where G_0, the stabilizer of the origin o, is a subgroup of $GL(V)$ [2].

Using the fact that H acts regularly on K, the following proposition is obtained.

Proposition 1 Let K be a complete k-arc, $k = r^n$ with r prime, in $PG(2, q)$. Suppose $H \leq G_K$ is an elementary abelian group of order r^n, acting regularly on K. Then $n = 1$ and K is an orbit of an element of order r of a Singer group of $PGL_3(q)$, or $k = 2^2$ and K is a conic in $PG(2, 3)$ or a hyperoval in $PG(2, 2)$.

Proof : Let $r = 2$. If q is odd, then H contains $2^n - 1$ involutory homologies [4, p. 172] which commute with each other. Two homologies h_1 and h_2 commute if and only if they have common center and axis or the center of one homology h_i belongs to the axis of the other homology h_j, $\{i, j\} = \{1, 2\}$. The first possibility cannot occur since there is a unique involutory homology with given center and given axis. The second possibility clearly implies that $|H| \leq 4$. Hence, by the completeness of K, $|H| = 4$, $q = 3$ and K is a conic in $PG(2, 3)$.

If q is even, then all involutions in H are elations with either common center or common axis. If they have common center, then every non-trivial orbit of H is contained in a line through the common center contradicting the fact that K is an arc. In fact, this shows that no two elations of H have common center. Suppose all elations have common axis L. Assume that a line T is tangent to K. Then all elements of T^H are tangent to K and hence the point $T \cap L$ extends the arc K, so K is not complete, a contradiction. There are no lines tangent to K. This implies $|K| = q + 2$ and this is a power of 2 only if $q = 2$. So K is a hyperoval, the points of an affine plane, in $PG(2, 2)$.

Assume now r odd. Let O be an arbitrary orbit in $PG(2, q)$ under H. Since H is an r-group, $|O| = r^m$ for $0 \leq m \leq n$. If $m = 0$, then $O = \{x\}$ and there is at least one line T through x tangent to K since $|K|$ is odd. Applying H to T, every line through x meeting K is a tangent line, hence x extends K and K is not complete. If $0 < m < n$, then the kernel of H on O is non-trivial and so there is an element σ of order r fixing O point by point. If at least three points of O are collinear, then σ is a central projective transformation, contradicting Lemma 4. So O is an arc and hence $|O| = 3$. We can take coordinates such that $O = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. A projective transformation φ of order 3 which is not central has necessarily a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-1} \end{pmatrix}, a^3 = 1, a \neq 1.$$

Hence $r^n = 3^2$. A projective transformation ψ of order 3 permuting cyclically the points of O has, without loss of generality, matrix

$$\begin{pmatrix} 0 & 0 & d \\ 1 & 0 & 0 \\ 0 & b & 0 \end{pmatrix}, b, d \in \mathrm{GF}(q)^* = \mathrm{GF}(q) \setminus \{0\}.$$

Since $\varphi, \psi \in H$, they commute, but this implies that $a = 1$, a contradiction.

We have shown that every orbit of H must have r^n points, so r^n must divide $q^2 + q + 1$. If $r \neq 3$, then every Sylow r-subgroup of $PGL_3(q)$ must be contained in some Singer group. Indeed, r does not divide $|PGL_3(q)|/(q^2 + q + 1)$, which is shown in [11, Theorem 3.1]. This implies $n = 1$ since H must be cyclic and elementary abelian. The result follows. If $r = 3$, since $9 \not| (q^2 + q + 1)$, $k = 3$, but then K is not complete, so this case need not be considered. ∎

Every triangle, 3 non-collinear points, and every quadrilateral, 4 points no 3 of which are collinear, constitutes a primitive arc in a plane. From now on, assume $|K| \geq 5$.

4 The simple case.

In this section, assume that G_K acts primitively on K, $|K| \geq 5$, and that G_K is an almost simple group with socle S, i.e., S is a non-abelian simple group and $G_K \leq \mathrm{Aut}\, S$ [2]. By the classification of subgroups of $L_3(q)$ by Bloom [1, Theorem 1.1], see also Mitchell [10, pp. 239-242], for q odd, and Hartley [6, pp. 157-158], for q even, there are three infinite series for S, namely $L_3(q')$, $U_3(q')$ and $L_2(q')$, for suitable q' dividing q. We first deal with them and afterwards with the sporadic cases.

Set $q = p^h$, p a prime number and let K be a k-arc in $PG(2, q)$.

4.1 Infinite classes.

4.1.1 The L_3-case.

Here, $PGL_3(q') \leq PGL_3(q)$ for every prime power $q' = p^{h'}$ such that h' divides h.

Proposition 2 No arc K, $|K| \geq 5$, exists such that

$$L_3(q') \leq G_K \leq PGL_3(q') = \mathrm{Aut}(L_3(q')) \cap PGL_3(q)$$

and such that G_K acts primitively on K.

Proof : The group $L_3(q')$ contains a subgroup of elations with common center and common axis of order q', hence by Lemma 4, $q' = 2$. So there is a subplane $PG(2,2)$ in $PG(2,q)$ stabilized by G_K. Clearly $K \cap PG(2,2) = \emptyset$. If a point $x \in K$ lies on a line L of $PG(2,2)$, by applying an element of order 2 in $L_3(2)$ contained in the stabilizer of L, one sees that L contains at least two points of K, but the lines of $PG(2,2)$ partition in this way the points of K in blocks of imprimitivity, a contradiction. Now let $x \in K$ and $u \in PG(2,2)$, then xu is a line of $PG(2,q)$ not in $PG(2,2)$. The set of elations in $L_3(2)$ with center u forms a subgroup of order 4 acting semi-regularly on the points of $xu \setminus \{u\}$. So xu contains four points of K, a contradiction. ∎

4.1.2 The U_3-case.

Here, $PGU_3(q') \leq PGL_3(q)$, $q' = p^{h'}$, whenever $2h'$ divides h. This group stabilizes a Hermitian curve in a subplane $PG(2,q^2)$ of $PG(2,q)$.

Proposition 3 No arc K, $|K| \geq 5$, exists such that

$$U_3(q') \leq G_K \leq PGU_3(q') = \mathrm{Aut}(U_3(q')) \cap PGL_3(q)$$

and such that G_K acts primitively on K.

Proof : The group $U_3(q')$ acts 2-transitively on a Hermitian curve $\mathcal{H}$ in some subplane $PG(2,q'^2)$. Consider an element σ of $U_3(q')$ fixing some point x of $\mathcal{H}$ and mapping another point y to some point z on the line xy, $y, z \in \mathcal{H}$. Then σ fixes xy and its pole u w.r.t. $\mathcal{H}$. Hence σ fixes the lines xu and xy. The order of σ can be chosen to be p. So σ fixes all lines through x and it is easily seen that xu is the axis. By Lemma 4, $p = 2$. But z can be varied to obtain a group of elations with common center x and common axis xu of order q'. Hence $q' = 2$ by Lemma 4. But $U_3(2) \cong 3^2 : Q_8$ is not simple and has no non-abelian simple socle. ∎

4.1.3 The L_2-case.

Here, $PGL_2(q') \leq PGL_3(q)$, $q' = p^{h'}$, whenever h' divides h.

Proposition 4 If K is an arc in $PG(2,q)$ such that G_K, with

$$L_2(q') \leq G_K \leq PGL_2(q') = \mathrm{Aut}(L_2(q')) \cap PGL_3(q),$$

acts primitively on K, then K is a conic in some subplane $PG(2,q')$ of $PG(2,q)$.

Proof : Let C be the conic on which G_K acts naturally inside some subplane $PG(2, q')$. Note that we can assume $q' > 3$ since $PGL_2(2)$ and $PGL_2(3)$ have no non-abelian simple socle. Clearly if the arc K has a point in common with $PG(2, q')$, then it consists of either all internal points of C (p odd), all external points of C (p odd), the nucleus of C ($p = 2$), all points not on C and distinct from the nucleus of C ($p = 2$) or the conic C itself. Only the last set of points constitutes an arc. So we can assume that all points of K lie outside $PG(2, q')$. If one point of K lies on a line L of $PG(2, q')$, then all points of K do and the lines in the orbit of L under G_K define a partition of K invariant under G_K. Let $x \in K \cap L$. If L is a bisecant of C, then the cyclic subgroup of $L_2(q')$ fixing L has at least order $(q' - 1)/2$ and acts on $L \setminus C$ in orbits of at least size $(q' - 1)/4$, if L is a tangent of C in a, the cyclic subgroup of $L_2(q')$ fixing a and a second point b of C has again at least order $(q'-1)/2$ and acts semi-regularly on $L \setminus \{a\}$ and if L is skew to C in $PG(2, q')$, $L_2(q')$ contains a cyclic subgroup of order $(q'+1)/2$, fixing L, and acting semi-regularly on $L \setminus C$. Hence the partition is not trivial if $q' > 5$. The only problem occurs when $G_K = L_2(5)$ and L is a bisecant of C in $PG(2, q')$. If L contains one point of K, all bisecants of C contain one point of K, so $|K| = 15$. This is impossible since $G_K \cong L_2(5) \cong A_5$ does not act primitively on 15 points [3].

So we may assume that no point of K lies on a line of $PG(2, q')$. Let $x \in K$, $\sigma \in G_K$ and suppose that $x^\sigma = x$. If σ fixes two points a, b of C, then σ fixes four points, namely a, b, x and the pole of the line ab w.r.t. C or the nucleus of C. No three of these points are collinear, otherwise x lies on a line of $PG(2, q')$, contradicting our assumption, hence σ is the identity. Suppose now σ acts semi-regularly on C. Then σ fixes two points a, b of C in a quadratic extension of $PG(2, q')$ and as above, this leads to σ being the identity. Finally, suppose σ fixes exactly one point u of C, then it fixes the tangent line T to C through u and it fixes also the line xu. Since σ has necessarily order p, it readily follows that it fixes all lines through u. So σ is central, $p = 2$ (Lemma 4), and the axis is T. But x does not lie on T and is fixed, hence σ is the identity.
We have shown that no non-trivial element of G_K fixes a point of K. So G_K acts regularly on K and such an action can never be primitive for groups of non-prime order. ∎

This completes the investigation of the infinite classes.

4.2 The sporadic classes.

The list of these classes is given by Bloom [1, Theorem 1.1] for q odd, and by Suzuki [13, intoduction] for q even.

4.2.1 Case $L_2(7) \leq G_K \leq PGL_2(7)$.

In this case, $q^3 \equiv 1 \pmod 7$, q odd, see Bloom [1, Theorem 1.1]. By the **ATLAS** [3], $L_2(7)$ can only act primitively on either 7 or 8 elements. If $G_K \cong PGL_2(7)$ and $L_2(7)$ as a subgroup of G_K does not act transitively on K, then $|K| = 28$ or 21 [3]. If $|K| = 28$, then K can be identified with the pairs of points of $PG(1,7)$ and every involution fixes 4 pairs, contradicting lemma 6. If $|K| = 21$, then K can be identified with the pairs of conjucated points in $PG(1,49)$. The involution sending x to $-x$ fixes three such pairs, contradicting lemma 5. We now deal with $G_K \cong L_2(7)$.

Proposition 5 The group $L_2(7)$ does not act primitively on any arc in $PG(2,q)$, $q^3 \equiv 1 \pmod 7$.

Proof : Suppose $|K| = 7$. Since $L_2(7) \cong L_3(2)$, the Klein fourgroup K_4 is inside G_K, it fixes three points $x, y, z \in K$ and acts regularly on the remaining four points of K. This contradicts Lemma 5.

Suppose now $|K| = 8$. Drop the restrictions on q for the time being. It is shown that every orbit of $L_2(7)$ of length 8 which constitutes an arc in any finite projective plane must be a conic in a subplane of order 7.
We can identify the points of K with the elements of $\mathrm{GF}(7) \cup \{\infty\}$ in the natural action of $L_2(7)$. We establish this identification via the indices. So $K = \{x_0, x_1, \ldots, x_6, x_\infty\}$. We coordinatize $PG(2,q)$ and take $x_0 = (1,0,0)$, $x_\infty = (0,1,0)$ and $x_1 = (1,1,1)$. An element σ in G_K of order 3 fixing x_0 and x_∞ exists. It is multiplication by 2 or 4 in the natural action, let us assume multiplication by 2. Since $1 + q + q^2 \not\equiv 2 \bmod 3$, σ has to fix at least one other point y of $PG(2,q)$. By Lemma 4, σ cannot be central, hence $q \equiv 1 \pmod 3$ and y is not incident with the line x_0x_∞. Neither lies y on any other bisecant of K containing x_0 or x_∞. It would imply that σ has to fix that bisecant point by point and so σ would be central. Hence we can take $y = (0,0,1)$. The matrix of σ looks like

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}, a, b \in \mathrm{GF}(q)^*.$$

Clearly $a = b$ or $1 \in \{a, b\}$ implies that x_1, x_1^σ and $x_1^{\sigma^2}$ are collinear, hence $a \neq b$, $a, b \neq 1$. Since σ has order 3, both a and b are non-trivial third roots of unity, say $a = \omega$ and $b = \omega^2$, $\omega^2 + \omega + 1 = 0$. Hence $x_2 = (\omega, \omega^2, 1)$ and $x_4 = (\omega^2, \omega, 1)$. If we set $x_3 = (u, v, 1)$, then $x_6 = (\omega u, \omega^2 v, 1)$ and $x_5 = (\omega^2 u, \omega v, 1)$. Let θ be the element of $L_2(7)$ mapping x_i to x_{i+1} and fixing x_∞. Knowing the action of θ on 8 points of $PG(2,q)$, we can find its matrix, namely

$$\begin{pmatrix} 1 & 0 & b \\ 1 & a & c \\ 1 & 0 & d \end{pmatrix}, a, b, c, d \in \mathrm{GF}(q).$$

Expressing $x_1^\theta = x_2$, $x_2^\theta = x_3$ and $x_3^\theta = x_4$, the elements a, b, c, d must satisfy,

(A) $\omega + (1 + d)\omega - 1 = (\omega + d)u$,

(B) $\omega + \omega^2 a + (1+d)\omega^2 - 1 - a = (\omega + d)v$,

(C) $u + (1+d)\omega - 1 = (u+d)\omega^2$,

(D) $u + av + (1+d)\omega^2 - 1 - a = (u+d)\omega$,

(E) $b = (1+d)\omega - 1$,

(F) $c = (1+d)\omega^2 - 1 - a$.

From (A) and (C), $(u-1)(u+2) = 0$. If $u = 1$, then $d = -1$ by (A), so $a(v-1) = 0$ by (D). Clearly $a \neq 0$, so $v = 1$ and $x_1 = x_3$, a contradiction. So $u = -2$. Noting $\omega \neq -2$ $(p \neq 3)$, we deduce from (A) that $d = -3\omega - 1$ since $\omega^2 + \omega + 1 = 0$. Combining (B) and (D), gives

$$v^2(1-\omega) + v(5\omega + 4) + 14\omega + 4 = 0.$$

This implies $v = -2$ or $v = -3\omega + 1$. If $v = -2$, then $a = -3$ by (B). But $x_4^\theta = x_5$ implies

$$(2\omega + 1, -4\omega - 2, -4\omega - 2) = k.(2\omega + 2, -2\omega, 1),$$

for some $k \in \mathrm{GF}(q)^*$. This implies $-2\omega = 1$, hence $p = 3$ and $a = 0$ which is false. So $v = -3\omega + 1$. Then (B) implies $a = 3\omega + 3$ and (E) and (F) imply that θ has matrix

$$\begin{pmatrix} 1 & 0 & 3\omega + 2 \\ 1 & 3\omega + 3 & -3\omega - 7 \\ 1 & 0 & -3\omega - 1 \end{pmatrix}.$$

Expressing $x_4^\theta = x_5$, we obtain $7 = 0$ and $\omega = 4$. So $p = 7$ and all points of K satisfy $X_0 X_1 = X_2^2$, showing our assertion. ■

4.2.2 Case $A_6 \leq G_K \leq \mathrm{Aut}(A_6)$.

First, assume $G_K \cong A_6$. This can only happen for q an even power of 2 [13, intoduction] or 5 and for $q \equiv 1$ or 19 (mod 30) [1, Theorem 1.1 (8) and (9)].

Proposition 6 Under the above assumptions, if $A_6 \leq PGL_3(q)$ acts primitively on an arc K, then q is even and K is the unique hyperoval consisting of 6 points in a subplane of order 4.

Proof : By the **ATLAS** [3], there are three distinct possibilities for $|K|$. First, suppose $|K| = 6$. Select 4 points of K and give them coordinates $(1,0,0), (0,1,0), (0,0,1)$ and $(1,1,1)$. There is an element σ of order 3 fixing the first three points and acting regularly on the remaining three points of K. As in the proof of Proposition 5, σ has matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \omega \in \mathrm{GF}(q), \omega \neq 1, \omega^3 = 1.$$

The group element with matrix

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -\omega & \omega \\ 0 & 0 & \omega^2 \end{pmatrix}$$

fixes $(1,0,0)$ and $(0,1,0)$, maps $(0,0,1)$ to $(1,\omega,\omega^2)$ and $(1,1,1)$ to $(0,0,1)$. Hence, it should preserve K since A_6 acts 4-transitively on 6 points. So the image $(-1+\omega^2, -\omega^2+1, \omega)$ of $(1,\omega,\omega^2)$ must belong to K. This happens only if $p = 2$, in which case all points of K except $(1,0,0)$ lie on the conic $X_1X_2 = X_0^2$ in $PG(2,4)$. So K is the hyperoval mentioned in the statement of the proposition.

Next, suppose $|K| = 10$. Then we can think of A_6 as being $L_2(9)$ acting on the elements of $\mathrm{GF}(9) \cup \{\infty\}$. Hence, we can label the points of K as x_i, $i \in \mathrm{GF}(9) \cup \{\infty\}$, and the action of $L_2(9)$ goes via its natural action on the indices. An entirely similar argument as for the case $L_2(7)$ shows here that q must be an even power of 3 and that K is a conic in some subplane of order 9 of $PG(2,q)$. This case was treated in 4.1.3.

Finally, suppose $|K| = 15$. Here, K can be identified with the pairs of the set $\{1,2,3,4,5,6\}$ with the natural action of A_6 [3]. The involution $(1\ 2)(3\ 4) \in A_6$ fixes the pairs $\{1,2\}$, $\{3,4\}$, $\{5,6\}$ and acts semi-regularly on the remaining ones. This permutation induces an involution in $PG(2,q)$ fixing three points of K, contradicting Lemma 5. ∎

The next case deals with groups having A_6 as a socle.

Proposition 7 Under the assumptions above, if $A_6 \leq G_K \leq \mathrm{Aut}(A_6)$ acts primitively on an arc K in $PG(2,q)$, then $G_K \cong A_6$, $q = 2^{2h}$, $h \geq 1$, and K is a hyperoval in some subplane of order 4.

Proof : By the previous result, we may assume that $A_6 \ncong G_K$. By the information in the **ATLAS** [3], there are two possibilities: the action of G_K on K is equivalent to the action of $PGL_2(9)$ on pairs of points , $O_2^+(9)$'s, of $PG(1,9)$ or the action on K is equivalent to the action of $PGL_2(9)$ on pairs of conjugated points in a quadratic extension, $O_2^-(9)$'s, of $PG(1,9)$.

In the first case, any involution of $L_2(9)$ fixes five pairs of $PG(1,9)$ and acts semi-regularly on the remaining 40, contradicting Lemma 6.

In the second case, the involution $x \mapsto -x$, $x \in \mathrm{GF}(9)$, belongs to $L_2(9)$ and fixes 4 pairs of conjugated points in a quadratic extension of $\mathrm{GF}(9)$, contradicting Lemma 6 again. ∎

4.2.3 Case $A_7 \leq G_K \leq S_7$.

This occurs when $p = 5$ and h is even [1, Theorem 1.1 (8)].

Proposition 8 The group A_7 does not act primitively on any arc in $PG(2,5^{2h}), h \geq 1$.

Proof : The group A_7 has a primitive action on $7, 15, 21$ and 35 points [3]. Let $S := \{1, 2, 3, 4, 5, 6, 7\}$. The action of A_7 on 7 points is the natural one on S and is 5-transitive which is impossible by Lemma 6. The action on 21 points is the action of A_7 on the unordered pairs of S. The permutation (1 2 3) fixes 6 pairs and hence should be the identity, by Lemma 6 again. The action on 35 points is the action on the triads of S. The permutation (1 2 3) fixes five triads and hence should be the identity again.

The action on 15 points is the action of A_7 on the points of $PG(3, 2)$. Here, there is an involution fixing three points on a line of $PG(3, 2)$, contradicting Lemma 5. ∎

To conclude, we deal with $G_K \cong S_7$.

Proposition 9 The group S_7 does not act primitively on any arc in $PG(2, 5^{2h}), h \geq 1$.

Proof : By the previous proposition, we may assume that A_7, as a subgroup of S_7, does not act primitively on K. This leaves only one possibility [3]: an action of S_7 on 120 points. The group A_7 acts on these points imprimitively in blocks of size 8, the stabilizer of a block being $L_2(7)$. The 15 blocks can be identified with the points of $PG(3, 2)$. The stabilizer of a point of $PG(3, 2)$ in A_7 is $L_3(2)$. This contains an element σ of order 3 and this element σ has to fix at least 2 other points of $PG(3, 2)$. In other words, σ stabilizes 3 blocks, and in each one of them, it must fix 2 points. So σ fixes in total 6 points, contradicting Lemma 6. ∎

4.2.4 Case $A_5 \cong G_K$.

In this case, $q \equiv \pm 1 \pmod{10}$ Bloom [1, Theorem 1.1 (6)] or $q = 2^{2h}, h \geq 1$ Hartley [6, pp. 157-158]. By the **ATLAS** [3], G_K can only act primitively on 5,6 or 10 points. The action of A_5 in $PG(2, q), q \equiv \pm 1 \pmod{10}$, is uniquely determined by 2 matrices T and B [1, Lemma 6.4].

Proposition 10 Suppose A_5 fixes a 5-arc K in $PG(2, q)$. Then $q = 2^{2h}, h \geq 1$, and K is a conic in a subplane $PG(2, 4)$.

Proof : Let $K = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1), (1, x, y)\} = \{p_1, \ldots, p_5\}$ where A_5 acts naturally on the indices $i, 1 \leq i \leq 5$.

The mapping (1 2)(3 4) of A_5 is defined by the matrix

$$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and fixes $(1, x, y)$ if and only if $y = x + 1$.

The mapping (1 2 3) is defined by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and fixes $(1, x, y)$ if and only if $1 + x = \rho, 1 = \rho x$ and $x = \rho(1 + x)$ for some $\rho \neq 0$. This implies $x^2 + x - 1 = 0$ and $x^2 - x - 1 = 0$. So $2 = 0$ and $x^2 + x + 1 = 0$. This shows that $q = 2^{2h}$, $h \geq 1$, and K is a conic in a subplane $PG(2, 4)$.

Remark 1 This conic K is contained in a unique hyperoval of $PG(2, 4)$ fixed by A_6 (Proposition 6).

Proposition 11 When $q \equiv \pm 1 \pmod{10}$, then the sets $K_1 = \{(1, 1, 1), (1, 1, -1), (1, -1, 1), (1, -1, -1), (0, 4t^2, 1), (0, -4t^2, 1), (-4t^2, 1, 0), (4t^2, 1, 0), (1, 0, 4t^2), (1, 0, -4t^2)\}$ and $K_2 = \{(1, 0, 1-2t), (1, 0, 2t-1), (1, 2t, 0), (1, -2t, 0), (0, 1, 2t), (0, 1, -2t)\}$ constitute a 10-arc and a 6-arc fixed by A_5. The points of K_1 are the 10 points of $PG(2, q)$ on 3 bisecants of K_2.

Proof : This can be verified by using the matrices T and B of [1, Lemma 6.4]. ∎

Proposition 12 The 10-arc K_1 and 6-arc K_2 in $PG(2, q), q \equiv \pm 1 \pmod{10}$, are projectively unique.

Proof : (a) Suppose there is a second orbit O of size 10. Let $p \in O$, then p is fixed by a subgroup H of order 6 of A_5. The unique subgroup of order 3 in H must fix a point r_1 of K_1. Then r_1 is fixed by H. This implies that pr_1 is a tangent to K_1.

Since 10 is even, p belongs to a second tangent pr_2 to $K_1, r_2 \in K_1$. If an element of order 3 in H fixes r_2, it fixes 4 points of K_1, which is false (Lemma 6), so p belongs to at least 4 tangents $pr_i, 1 \leq i \leq 4$, to K_1. Any involution γ in H must fix two tangents through p since it fixes r_1. It cannot fix 4 tangents (Lemma 6). Assume $\gamma(r_2) = r_2$ and $\gamma(r_3) = r_4$, then $\{p, r_1, r_2, r_3, r_4\}$ is a 5-arc fixed by γ. This contradicts Lemma 5.

(b) Suppose there is a second orbit O of size 6.

If $p \in O$, then p is fixed by a subgroup H, of order 10, of A_5. Since H has a unique subgroup of order 5 and since $|K_2| = 6$, H must fix one point r_1 of K_2 and, as in (a), pr_1 is tangent to K_2. An element of order 5 in H acts transitively on $K_2 \setminus \{r_1\}$ and fixes p, so p extends K_2 to a 7-arc.

An involution γ of H fixes p, r_1 and a second point r_2 of K_2. Let $\gamma(r_3) = r_4$, $r_3, r_4 \in K_2$, then $\{p, r_1, r_2, r_3, r_4\}$ is a 5-arc fixed by γ. This again contradicts Lemma 5. ∎

Remark 2 Hexagons $\mathcal{H}$ fixed by A_5 were studied in detail by Dye [5]. These hexagons occur when $q \equiv \pm 1 \pmod{10}$, $q = 5^h$ or $q = 2^{2h}, h \geq 1$.

When $q = 2^{2h}, h \geq 1$, then $\mathcal{H}$ is a hyperoval in a subplane $PG(2,4)$ and $\mathcal{H}$ is fixed by A_6 (Propositions 6 and 10).

From now on, assume q odd. If $q = 5^h$, then $\mathcal{H}$ is a conic in a subplane $PG(2,5)$ of $PG(2,q)$, $A_5 \cong L_2(5)$, but $\mathcal{H}$ is not contained in a conic when $q \equiv \pm 1 \pmod{10}$. In both cases, this hexagon is called the Clebsch hexagon [5]. One of its particular properties is that it has exactly 10 Brianchon-points, i.e., points on exactly 3 bisecants to $\mathcal{H}$. If $q = 5^h$, these Brianchon-points are the internal points of the conic $\mathcal{H}$ in the subplane $PG(2,5)$. The 10 Brianchon-points constitute a 10-arc if $q \equiv \pm 1 \pmod{10}$ (Proposition 11).

With this hexagon correspond 5 triangles whose edges partition $\mathcal{H}$ and on which A_5 acts in a natural way. These 5 triangles are self-polar w.r.t. a unique conic C. When $q = 5^h$, $\mathcal{H} = C$. The 10 Brianchon-points belong to C if and only if $q = 3^{2h}$, $h \geq 1$, and in this case, C is a conic in a subplane $PG(2,9)$. Equivalently, when $q = 3^{2h}$, $h \geq 1$, the 10-arc K_1 (Proposition 11) is a conic in a subplane $PG(2,9)$.

This completes the proof of our main result.

5 Complete 2-transitive arcs

As an immediate consequence of the classification of primitive arcs made in Sections 3 and 4, the following list of complete 2-transitive arcs is obtained. As before, assume $|K| \geq 5$.

Proposition 13 If K is a complete k-arc of $PG(2,q)$, fixed by a 2-transitive projective group G_K, then either

(1) K is a conic in $PG(2,q)$, q odd, $q > 3$;

(2) K is the unique 6-arc in $PG(2,4)$;

(3) K is the unique 6-arc in $PG(2,9)$ fixed by A_5;

(4) K is the unique 10-arc in $PG(2,11)$ or $PG(2,19)$ fixed by A_5.

Proof : This follows from the preceding classification.

The completeness of the 6- and 10-arc fixed by A_5 in $PG(2,q), q \equiv \pm 1 \pmod{10}$, was checked by computer. The 10-arc in $PG(2,9)$ fixed by A_5 is the conic of $PG(2,9)$ (Remark 2), so this arc is included in case (1). ■

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