Geometriae Dedicata **56:** 271–284, 1995.

Cyclic Caps in PG(3, q)

L. STORME* and H. VAN MALDEGHEM**
Department of Pure Mathematics and Computer Algebra, University of Gent, Galglaan 2, B-9000 Gent, Belgium
e-mail: (L. Storme) ls@cage.rug.ac.be; (H. Van Maldeghem) hvm@cage.rug.ac.be

(Received: 16 May 1994)

Abstract. This article investigates cyclic complete k-caps in PG(3, q). Namely, the different types of complete k-caps K in PG(3, q) stabilized by a cyclic projective group G of order k, acting regularly on the points of K, are determined. We show that in PG(3, q), q even, the elliptic quadric is the only cyclic complete k-cap. For q odd, it is shown that besides the elliptic quadric, there also exist cyclic k-caps containing $k/2$ points of two disjoint elliptic quadrics or two disjoint hyperbolic quadrics and that there exist cyclic k-caps stabilized by a transitive cyclic group G fixing precisely one point and one plane of PG(3, q). Concrete examples of such caps, found using AXIOM and CAYLEY,‡ are presented.

Mathematics Subject Classification (1991): 51E22.

1. Introduction

A *k-cap* K in PG(n, q) is a set of k points, no three of which are collinear. A point r of PG(n, q) *extends* a k-cap K to a $(k+1)$-cap if and only if $K \cup \{r\}$ is a $(k+1)$-cap. A k-cap is *complete* if it is not contained in a $(k+1)$-cap. The k-caps of PG(2, q) are also called *k-arcs* ([7, p. 285]).

In [11], we described the different types of complete k-arcs K in PG(2, q) stabilized by a cyclic projective group G, so $G \leq$ PGL(3, q), acting regularly on the points of K. The results of [11] show that either G is a subgroup of a cyclic Singer group of PG(2, q), K is a conic in PG(2, q), q odd, or K is a k-arc in an affine plane AG(2, q), $q \equiv -1 \pmod 4$, which is the union of $k/2$ points on two concentric ellipses C_1 and C_2.

We now apply the same method to cyclic complete k-caps in PG(3, q). A description of the possible types of complete k-caps K, stabilized by a cyclic projective group G, so $G \leq$ PGL(4, q), acting regularly on K, is given.

An important difference with the results of [11] is that we are able to show that in PG(3, q), q even, the elliptic quadric is the only cyclic complete k-cap. In other words, the problem is completely solved for even characteristic.

---

* Senior Research Assistant of the National Fund for Scientific Research NFWO (Belgium).
** Senior Research Associate of the National Fund for Scientific Research NFWO (Belgium).
‡ With thanks to CAGe, Computer Algebra Gent, a project supported by FGWO and FKFO, associated funds of the National Fund for Scientific Research NFWO (Belgium).

If q is odd, it is shown that, besides the elliptic quadric, there exist cyclic k-caps stabilized by a cyclic projective group G fixing one point r not on K, and that there exist cyclic k-caps having $k/2$ points in common with two disjoint elliptic quadrics or two disjoint hyperbolic quadrics. This means that results analogous to [11, Th. 4.5(ii)] are obtained. In that theorem, cyclic k-arcs in an affine plane AG$(2, q)$, $q \equiv -1 \pmod 4$, having $k/2$ points on two disjoint concentric ellipses were constructed. Now cyclic k-caps containing points on two quadrics are presented.

Throughout the article, let K be a complete k-cap of PG$(3, q)$, and let G be a cyclic projective group acting regularly on K. Put $G = \langle\alpha\rangle$ where α is induced by the linear transformation $\alpha: \bar{x} \mapsto A\bar{x}$, with A a 4×4 matrix over GF(q). By looking at the different possibilities for the eigenvalues of A, the distinct types for the cyclic complete k-caps are obtained. For each type, using AXIOM ([9]) and CAYLEY ([2]), concrete examples are presented.

2. Eigenvalue in GF(q)

2.1. MAIN THEOREM

If the matrix A has an eigenvalue in GF(q), then there is a point x of PG$(3, q)$ fixed by G. We first show that this point is unique.

LEMMA 2.1. *There is exactly one point x in* PG$(3, q)$ *fixed by α, or equivalently by G.*

Proof. Let x be a point fixed by α. Since K is complete, there is at least one bisecant to K through x. By the transitivity of G, the point x lies on $k/2$ bisecants to K. Let z and z' be two points of K collinear with x. The unique element θ of G mapping z to z' must map z' to z and hence θ is the unique involution in G. Since K is complete, $k \geq 5$ and so x is uniquely determined as the intersection of the lines zz^θ, $z \in K$. □

The unique involution $\theta = \alpha^{k/2}$ of the preceding lemma fixes more than $(\sqrt{2}\,q + 1)/2$ lines through x ([6, Th. 18.1.9]), so θ fixes at least four lines through x if $q \geq 5$. The first possibility is that θ fixes four lines through x, no three of which belong to a plane. Then θ fixes all lines through x, so θ is an involutory perspectivity. The other possibility is that all, but possibly one, fixed lines are contained in a plane π through x. This however is impossible. Indeed, if all fixed lines lie in π, then $K \subset \pi$ and K is not complete. Suppose all fixed lines through x, except the fixed line L, lie in a plane π. Then $|K \cap \pi| = k - 2$. By the transitive action of G on the bisecants to K through x, this line L must also belong to a plane π' containing $k - 2$ points of K. This is impossible. Hence θ is an involutory perspectivity if $q \geq 5$ and fixes some plane π pointwise. This plane π is also stabilized by α since it is the unique axis of the unique involution θ in $\langle\alpha\rangle$.

For the small values of q, assume $q = 2$. A complete k-cap K in PG$(3,2)$ is either an elliptic quadric or the complement of a plane in PG$(3, 2)$ ([6, p. 96]). The cyclic group stabilizing an elliptic quadric does not fix a point of PG(3, 2) ([4, p. 1170]). Suppose that there is a cyclic group G, $|G| = 8$, which has the complement of a plane π as an orbit. Since $|G| = 8$, G fixes a point x in π and G fixes a line L through x in π. This gives the situation discussed at the end of the proof of Lemma 2.3 below and this leads to a contradiction.

Suppose we have a 6-cap K in PG(3, 3). Then K consists of six points on three lines L_1, L_2, L_3 through the fixed point x. The involution $\theta = \alpha^3$ interchanges the points of $K \cap L_i$, $i = 1, 2, 3$, and fixes a point r_i on L_i. Selecting $x = (1, 0, 0, 0)$, $r_1 = (0, 1, 0, 0)$, $r_2 = (0, 0, 1, 0)$, $r_3 = (0, 0, 0, 1)$, $\theta: (x_0, x_1, x_2, x_3) \mapsto (x_0, -x_1, -x_2, -x_3)$, so θ is an involutory perspectivity.

If $q = 4$, since K is complete, $|K| > \sqrt{2}\, q + 1 > 6$ ([6, Th. 18.1.9]). As $|K|$ is even, $|K| \geq 8$, so the reasoning for $q \geq 5$ can be used.

With the notations used in these paragraphs, we have the following two results.

LEMMA 2.2. *If q is odd and G fixes a point $x \notin K$, then G fixes a plane π not passing through x, k is even and $k/2$ divides $q^2 + q + 1$.*

Proof. If q is odd, then π does not contain x since an involutory perspectivity is a homology. If we take $x = (1, 0, 0, 0)$ and $\pi: X_0 = 0$, then the matrix A has the form

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & A_i & \\ 0 & & & \end{pmatrix}.$$

Since there is only one point fixed by α, all eigenvalues of A_1 must lie in GF(q^3)\GF(q) and are conjugate. Hence A_1 is the matrix of a power of a Singer cycle in π ([11, Th. 3.1]). This means that the action of G on the points of π is semi-regular. Since $k/2$ is the smallest number of which $\alpha^{k/2}$ fixes all points of π, $k/2$ divides $q^2 + q + 1$. □

We will give an example of a complete cap, and also of some incomplete ones, of the above type at the end of this section. For q even, we now show that no complete cyclic cap K of that type exists.

LEMMA 2.3. *If q is even, then the group G does not fix any point x.*

Proof. Suppose the contrary and let x be the unique point fixed by G. Since q is even, the fixed plane π contains x. The residual geometry in x is a projective plane of which π is a line and of which the points are the lines of PG(3, q) through x. Since α stabilizes π, applying [8, p. 256] for this residual geometry, it must stabilize some line L through x. The action of α on $L \backslash \{x\}$ must be fixed-point free, hence it must be a 'translation', implying that α^2 fixes L pointwise.

If L is not contained in π, then we may assume that α does not fix any line through x in π. But since α fixes the point x in π, it must stabilize some line M in

π ([8, p. 256]). Hence A^2 has two distinct eigenvalues ω and ω^q in $\mathrm{GF}(q^2)\backslash\mathrm{GF}(q)$ with corresponding fixed points on M. This implies that $\theta = \alpha^{2(q+1)}$ fixes both L and M pointwise, so we can pick a basis such that the matrix of θ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}.$$

Since q is even, the order of θ is not equal to 2. But one easily checks that for any point y of K, the points y, y^θ and y^{θ^2} are collinear, hence θ is the identity. Now let z be a point of K and let β be the plane generated by z and M. Let z' be the intersection of β and L, let n be any integer and consider α^{2n}. Clearly, this stabilizes β and z'. If it fixed a line N through z' in β, then it would fix the intersection point of N and M and hence it would fix M pointwise as it already fixes two points of M in $\mathrm{PG}(3, q^2)$, implying, as for θ, $\alpha^{2n} = 1$. So $\langle\alpha^2\rangle$ acts semi-regularly on the lines through z' in β. Since K is contained in $\beta \cup \beta^\alpha$, it is now easily seen that z' extends K. The cap K is however complete, so we have a contradiction. This means that there is a line L in π, with $x \in L$, fixed by G.

Selecting $x = (1, 0, 0, 0)$, $L\colon X_2 = X_3 = 0$ and $\pi\colon X_3 = 0$, and taking into account that A can only have one eigenvalue in $\mathrm{GF}(q)$ as there is only one fixed point,

$$A = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It now easily follows that α^4 is the identity. Hence $k = 4$, a contradiction. □

We now have

THEOREM 2.4. *If the matrix A belonging to α has at least one eigenvalue in* $\mathrm{GF}(q)$, *then q is odd and G fixes exactly one point x and one plane π not through x.*

2.2. EXAMPLES

Suppose α fixes the point $x = (1, 0, 0, 0)$ and the plane $\pi\colon X_0 = 0$ of $\mathrm{PG}(3, q)$, q odd, then

$$\alpha\colon (x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_2, x_3)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & A & \\ 0 & & & \end{pmatrix}.$$

The matrix A to construct the transformation α is obtained in the following way. Using a primitive polynomial of degree 3, the matrix B of a Singer cycle in a projective plane is constructed ([5, p. 44]). By calculating $A = B^{(q^3-1)/k}$, a matrix of order k is obtained and this matrix then is used to define α. These examples are obtaincd using AXIOM ([9]) and CAYLEY ([2]).

(1) *A complete 134-cap in* PG*(3, 29)*
Here $X^3 - 3X^2 - 25X - 27$ is the primitive polynomial of degree 3 over GF(29) defining

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 27 & 25 & 3 \end{pmatrix} \quad \text{and} \quad A = B^{182} = \begin{pmatrix} 15 & 0 & 4 \\ 21 & 28 & 12 \\ 5 & 2 & 6 \end{pmatrix}$$

is the matrix used to define α. The orbit of (1, 1, 0, 0) is a complete 134-cap.

(2) *An incomplete 122-cap in* PG*(3, 47)*
The primitive polynomial $X^3 - 46X - 43$ over GF(47) defines

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 43 & 46 & 0 \end{pmatrix} \quad \text{and} \quad A = B^{851} = \begin{pmatrix} 18 & 39 & 31 \\ 17 & 34 & 39 \\ 32 & 25 & 34 \end{pmatrix}$$

defines α. The orbit of $(1, 1, 0, 0)$ under $\langle\alpha\rangle$ is an incomplete 122-cap K. Nine orbits in $X_0 = 0$ and 30 orbits, not contained in $X_0 = 0$, consist of points extending K.

(3) *An incomplete 86-cap in* PG*(3, 49)*
The primitive polynomial $X^2 + X + 3$ is used to define GF(49). Let ω be a root of this polynomial. Then $X^3 + \omega X + \omega^{25}$ is a primitive polynomial of degree 3 over GF(49) defining

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\omega^{25} & -\omega & 0 \end{pmatrix} \quad \text{and} \quad A = B^{1368} = \begin{pmatrix} \omega^{35} & \omega^{19} & \omega^{38} \\ \omega^{39} & \omega^{34} & \omega^{19} \\ \omega^{20} & 5 & \omega^{34} \end{pmatrix}$$

is the matrix defining α. Once again, the orbit of (1, 1, 0, 0) is an incomplete 86-cap which can be extended by the points of 23 orbits in $X_0 = 0$ and of 276 orbits, not contained in $X_0 = 0$, to a larger cap.

3. Eigenvalues in an Extension of Degree 4

3.1. MAIN RESULT

In this section, we suppose that A has four different conjugate eigenvalues ω, ω^q, ω^{q^2}, ω^{q^3} in GF$(q^4)\backslash$GF(q^2). It follows that α, and hence also G, fixes exactly four different points x_i, $i = 1, 2, 3, 4$, in PG$(3, q^4)\backslash$PG$(3, q^2)$ which correspond to the eigenvalues ω, ω^q, ω^{q^2}, ω^{q^3}, and we can choose the coordinates in such a way that $x_1^{q^3} = x_2^{q^2} = x_3^q = x_4$.

The line x_1x_3 has no point in PG(3, q), otherwise such a point x lies on both x_1x_3 and $(x_1x_3)^q = x_2x_4$. Since both x_1x_3 and x_2x_4 are stabilized by α, this would imply that α fixes their intersection x, and so A would have an eigenvalue in GF(q), a contradiction.

On the other hand, $(x_1x_3)^{q^2} = x_3x_1$ and so x_1x_3 is a line in PG(3, q^2), as similarly x_2x_4 is. Hence α stabilizes the regular spread $\mathcal{R}$ defined by the lines x_1x_3 and x_2x_4 of PG(3, q^2) which are conjugate to each other over PG(3, q) ([1, Th. 5.3]). If, for some positive integer n the element α^n fixes a point of PG(3, q^2) on x_1x_3, then α^n fixes three points on x_1x_3, so α^n fixes x_1x_3 pointwise and then also the conjugate line x_2x_4 is fixed pointwise.

In this case, all lines of the regular spread $\mathcal{R}$ are stabilized. Since any line can contain at most two points of K, this implies that α^{2n} is the identity. So we have shown that the action of G on the points of x_1x_3, or equivalently on the points of x_2x_4, has a kernel of order at most 2 and that the faithful factorgroup acts semi-regularly on the points of x_1x_3 in PG(3, q^2). So there are two possibilities. Either this kernel is trivial and then k divides $q^2 + 1$. Or the kernel has order 2 and then $k/2$ divides q^2+1 while 2 divides $q+1$ because the kernel acts fixed-point free on the points of a line of the regular spread $\mathcal{R}$. Hence in this latter case, q is odd.

Now note that the subgroup of PGL(4, q) which fixes x_i, $i = 1, 2, 3, 4$, is a cyclic Singer group S. Hence G is a subgroup of S. By Ebert ([4, p. 1170], see also [3, Th. 3.7] and [10]), the orbits of the unique subgroup of order $q^2 + 1$ of any cyclic Singer group S partition PG(3, q) into $q + 1$ disjoint elliptic quadrics.

This now enables us to present the main result of this section.

THEOREM 3.1. *If A has four different conjugate eigenvalues in* GF(q^4), *then G is a subgroup of a cyclic Singer group S and either K is an elliptic quadric, so $k = q^2 + 1$, or q is odd and K has $k/2$ points on two disjoint elliptic quadrics.*

Proof. Suppose first that the kernel mentioned above is trivial. Then k divides $q^2 + 1$ and hence α^{q^2+1} is trivial. So G is a subgroup of the unique subgroup of order q^2+1 of the cyclic Singer group S. Hence K is a subset of an elliptic quadric ([4, p. 1170], [3, Th. 3.7], [10]) and since K is complete, it coincides with that quadric.

Suppose now that the kernel mentioned above has order 2. A similar argument as above shows that G is now a subgroup of the unique subgroup C of order $2(q^2 + 1)$ of S. Using the previously mentioned result by Ebert ([4, p. 1170]), any orbit of C is the union of two disjoint elliptic quadrics. The result follows. □

3.2. EXAMPLES

The following three k-caps, found using AXIOM ([9]) and CAYLEY ([2]), consist of $k/2$ points on two disjoint elliptic quadrics.

In all examples, by using the method described in [5, p. 44], the Singer cycle β of the cyclic Singer group $S = \langle\beta\rangle$ is determined. Then by taking a suitable power of β, the generator α of the cyclic group $G = \langle\alpha\rangle$, fixing the cap, is constructed.

(1) *A complete 772-cap in* PG(3, 81)
Consider the field GF(81) generated by a root ω of the primitive polynomial $X^4 + X + 2$ over GF(3). Then the primitive polynomial $X^4 + \omega X + \omega$ over GF(81) can be used to construct the Singer cycle

$$\beta\colon (x_0,\ x_1,\ x_2\ x_3) \mapsto (x_0,\ x_1,\ x_2,\ x_3)\begin{pmatrix} 0 & 0 & 0 & -\omega \\ 1 & 0 & 0 & -\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then

$$\alpha = \beta^{697}\colon (x_0,\ x_1,\ x_2,\ x_3) \mapsto (x_0,\ x_1,\ x_2,\ x_3)\begin{pmatrix} \omega^{25} & \omega^{36} & \omega^{44} & \omega^{21} \\ \omega^{60} & \omega^{57} & \omega^{15} & \omega^{28} \\ \omega^{3} & \omega^{60} & \omega^{57} & \omega^{15} \\ \omega^{75} & \omega^{3} & \omega^{60} & \omega^{57} \end{pmatrix}$$

partitions PG(3, 81) into 697 disjoint complete 772-caps consisting of 386 points on two disjoint elliptic quadrics.

(2) *An incomplete 116-cap in* PG(3, 41)
By using the primitive polynomial $X^4 + X^3 + X^2 + X + 6$ over GF(41), the Singer cycle

$$\beta\colon (x_0,\ x_1,\ x_2,\ x_3) \mapsto (x_0,\ x_1,\ x_2,\ x_3)\begin{pmatrix} 0 & 0 & 0 & -6 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

is constructed. Then

$$\alpha = \beta^{609}\colon (x_0,\ x_1,\ x_2,\ x_3) \mapsto (x_0,\ x_1,\ x_2,\ x_3)\begin{pmatrix} 5 & 17 & 2 & 28 \\ 40 & 1 & 31 & 34 \\ 31 & 36 & 15 & 22 \\ 4 & 27 & 9 & 6 \end{pmatrix}$$

partitions PG(3, 41) into 609 incomplete 116-caps consisting of 58 points on two disjoint elliptic quadrics. Each cap can be extended by the points of 24 orbits under $\langle\alpha\rangle$ to a 117-cap.

(3) *An incomplete 164-cap in* PG(3, 73)
Here the primitive polynomial $X^4 + X^3 + X^2 + 5$ over GF(73) is used to construct the Singer cycle

$$\beta\colon (x_0,\ x_1,\ x_2,\ x_3) \mapsto (x_0,\ x_1,\ x_2,\ x_3)\begin{pmatrix} 0 & 0 & 0 & -5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Then

$$\alpha = \beta^{2405}\colon (x_0,\ x_1,\ x_2,\ x_3) \mapsto (x_0,\ x_1,\ x_2,\ x_3)\begin{pmatrix} 10 & 27 & 37 & 32 \\ 10 & 10 & 27 & 37 \\ 31 & 30 & 32 & 48 \\ 53 & 51 & 52 & 53 \end{pmatrix}$$

partitions PG(3, 73) into incomplete 164-caps consisting of 82 points on two disjoint elliptic quadrics. All caps can be extended by the points of 220 orbits under $\langle\alpha\rangle$ to a larger cap.

4. Eigenvalues in a Quadratic Extension

4.1. MAIN THEOREM

From now on, assume that A does not have eigenvalues in GF(q) and does not have 4 conjugate eigenvalues in GF(q^4). Then A has at least two conjugate eigenvalues ω and ω^q in GF(q^2). Hence α fixes two conjugate points p and p^q in PG(3, q^2)\PG(3, q). Let $L = pp^q$ be the line of PG(3, q) stabilized by G.

THEOREM 4.1. *The k-cap K contains* $2(q+1)$ *points and the group G stabilizes a second line M skew to L when q is odd.*

Proof. Consider the action of G on L. This group stabilizes the two conjugate points p and p^q of L in PG(3, q^2)\PG(3, q). So G partitions L into orbits of equal size and α^{q+1} stabilizes L pointwise.

The group G cannot fix a plane through L as the action of α on the planes is defined by the matrix $(A^T)^{-1}$ and $(A^T)^{-1}$ has no eigenvalues in GF(q). This shows that G also partitions the planes through L into orbits of equal size and so α^{q+1} stabilizes all planes $\beta_1, \ldots, B_{q+1}$ through L.

In β_i, α^{q+1} induces a perspectivity with center r_i and axis L. Only an involutory perspectivity can fix $K \cap \beta_i$, $i = 1, \ldots, q+1$, so $\alpha^{2(q+1)} = 1$ which shows that $k|2(q+1)$. Since K is complete, $k > \sqrt{2}\,q+1$ ([6, Th. 18.1.9]), so $k = 2(q+1)$.

If q is odd, then the center r_i of the involutory perspectivity α^{q+1} in β_i does not belong to the axis L ([8, p. 172]). So α^{q+1} fixes L pointwise, together with $q+1$ other points $r_1, \ldots, r_{q+1}$. This can only occur if $r_1, \ldots, r_{q+1}$ belong to a line M skew to L.

Since $G = \langle \alpha \rangle$ contains precisely one involution, M must be fixed by G. □

THEOREM 4.2. *In* PG(3, q), *q even, there does not exist a complete cyclic k-cap K stabilized by the cyclic group $G = \langle \alpha \rangle$ where $a: \bar{x} \mapsto A\bar{x}$ and where A only has eigenvalues in* GF(q^2).

Proof. Since A has eigenvalues in GF(q^2), also the matrix $(A^T)^{-1}$ which gives the action of α on the planes of PG(3, q) has two conjugate eigenvalues in GF(q^2). This means that α fixes two conjugate planes β and β^q in PG(3, q^2)\PG(3, q).

Then $L = \beta \cap \beta^q$ is a line of PG(3, q) stabilized by α. Since α does not have eigenvalues in GF(q), α fixes two conjugate points p_1 and p_1^q of L in PG(3, q^2)\PG(3, q).

Consider α as a projective transformation in PG(3, q^2) and look at the action of α in the plane β. In that plane, α fixes the two points p_1 and p_1^q. Hence, α must fix two lines in β ([8, p. 256]). One of these lines is L. Let L_1 be a second line in β fixed by α. Then L_1 is not defined over GF(q), so α fixes the conjugate line L_1^q of L_1 in β^q. These two lines L_1 and L_1^q are two conjugate skew lines of PG(3, q^2). Hence they define a regular spread $\mathcal{R}$ ([1, Th. 5.3]).

Suppose that α only fixes the point $L \cap L_1$ of L_1. Then α is a 'translation' on L_1. So α^2 fixes L_1 pointwise and then also L_1^q is stabilized pointwise by α^2. Equivalently, α^2 fixes the lines of the regular spread $\mathcal{R}$ and this means that the $(2q+2)$-cap K is the union of two lines. This is false.

It follows from the preceding reasoning that α fixes a second point p_2 on L_1. Then α also fixes the conjugate point p_2^q on L_1^q and this all implies that α fixes the skew lines $p_1p_1^q$ and $p_2p_2^q$ of PG(3, q).

Since α does not fix a point of PG(3, q), α^{q+1} stabilizes $p_1p_1^q$ and $p_2p_2^q$ pointwise. In a plane $\langle p_1p_1^q, r\rangle$, $r \in p_2p_2^q$, α^{q+1} is an involutory perspectivity with center r and axis $p_1p_1^q$ (see also the proof of 4.1). This is false since q is even, and thus α^{q+1} should be an elation in $\langle p_1p_1^q, r\rangle$ ([8, p. 172]).

We have a contradiction. The cap does not exist. □

Remark 4.3. From now on, let q be odd.

Suppose that $G = \langle \alpha \rangle$ fixes the lines $L: X_0 = X_1 = 0$ and $M: X_2 = X_3 = 0$ (Theorem 4.1). Choose the reference system in such a way that α fixes the points

$(1,\ \pm i,\ 0,\ 0)$ and $(0,\ 0,\ 1,\ \pm i)$ with $i^2 = d_1$ nonsquare in GF(q). Then, from [11, Remark 4.2],

$$\alpha\colon \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 & 0 \\ bd_1 & a & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & fd_1 & e \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

LEMMA 4.4. *Using the notations of Remark 4.3.*

The hyperbolic quadrics $\mathcal{Q}_c$: $X_0^2 - d_1^{-1}X_1^2 - cd_1X_2^2 + cX_3^2 = 0$, $c \in \mathrm{GF}(q)^* = \mathrm{GF}(q)\backslash\{0\}$, *partition* $\mathrm{PG}(3,\ q)\backslash(L \cup M)$.

All these quadrics $\mathcal{Q}_c$ *contain* $(1,\ \pm i,\ 0,\ 0)$ *and* $(0,\ 0,\ 1, \pm i)$, *the lines* L *and* M *are polar lines with respect to* $\mathcal{Q}_c$ *and* α *stabilizes* $\mathcal{Q}_c$ *if and only if* $a^2 - b^2d_1 = e^2 - f^2d_1$.

Proof. One easily verifies that all quadrics contain $(1,\ \pm i,\ 0,\ 0)$ and $(0,\ 0,\ 1,\ \pm i)$, and L and M are polar lines with respect to each quadric. Using [5, Table 5.1], it is straightforward to check that all quadrics $\mathcal{Q}_c$, $c \neq 0$, are (nonsingular) hyperbolic quadrics $\mathcal{H}_{3,q}$.

We now check that all quadrics $\mathcal{Q}_c$ partition $\mathrm{PG}(3,\ q)\backslash(L \cup M)$. Let $(z_0,\ z_1,\ z_2,\ z_3)$ be a point of $\mathcal{Q}_{c_1}$ and $\mathcal{Q}_{c_2}$, $c_1 \neq c_2$, then

$$z_0^2 - d_1^{-1}z_1^2 - c_1d_1z_2^2 + c_1z_3^2 = 0, \tag{1}$$

$$z_0^2 - d_1^{-1}z_1^2 - c_2d_1z_2^2 + c_2z_3^2 = 0. \tag{2}$$

Then (1)–(2) implies $d_1z_2^2 = z_3^2$, so $z_2 = z_3 = 0$ since d_1 is a nonsquare. Substituting $z_2 = z_3 = 0$ in (1) implies $z_0^2 = d_1^{-1}z_1^2$ and so also $z_0 = z_1 = 0$. We have a contradiction.

Since $|\mathcal{H}_{3,q}| = (q+1)^2$ ([6, p. 23]), $(q-1)(q+1)^2$ points belong to a quadric $\mathcal{Q}_c$, $c \neq 0$. Since $|\mathrm{PG}(3,\ q)\backslash(L \cup M)| = (q-1)(q+1)^2$, these quadrics partition $\mathrm{PG}(3,\ q)\backslash(L \cup M)$.

Now let $x_0' = ax_0 + bx_1$, $x_1' = bd_1x_0 + ax_1$, $x_2' = ex_2 + fx_3$ and $x_3' = fd_1x_2 + ex_3$. Then

$$\begin{aligned} x_0'^2 - d_1^{-1}x_1'^2 - cd_1x_2'^2 + cx_3'^2 = {} & (a^2 - b^2d_1)(x_0^2 - d_1^{-1}x_1^2) \\ & +(e^2 - f^2d_1)(-cd_1x_2^2 + cx_3^2). \end{aligned}$$

So $\mathcal{Q}_c$ is fixed by α if and only if $a^2 - b^2d_1 = e^2 - f^2d_1$. □

LEMMA 4.5. *Using the notations of Remark 4.3,* $a^2 - b^2d_1 = \pm(e^2 - f^2d_1)$.

Proof. Since $G = \langle\alpha\rangle$ fixes the two lines $L\colon X_0 = X_1 = 0$ and $M\colon X_2 = X_3 = 0$, but does not fix a point of PG(3, q) on $L \cup M$, α^{q+1} is the identity on L and M, so

$$\alpha^{q+1}\colon \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} \begin{pmatrix} a & b \\ bd_1 & a \end{pmatrix}^{q+1} & 0 \\ 0 & \begin{pmatrix} e & f \\ fd_1 & e \end{pmatrix}^{q+1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_1 & 0 & 0 \\ 0 & 0 & \epsilon_2 & 0 \\ 0 & 0 & 0 & \epsilon_2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

In a plane $\langle L, r\rangle$ where $r \in M$, α^{q+1} defines an involutory perspectivity (see also proof of 4.1). This implies $\epsilon_1 = -\epsilon_2$ and so $\epsilon_1^2 = (a^2 - b^2d_1)^{q+1} = (e^2 - f^2d_1)^{q+1} = \epsilon_2^2$. Equivalently, $(a^2 - b^2d_1)^{q+1} = (a^2 - b^2d_1)^2 = (e^2 - f^2d_1)^2$ which means that $a^2 - b^2d_1 = \pm(e^2 - f^2d_1)$. □

If $a^2 - b^2d_1 = e^2 - f^2d_1$, then the $(2q+2)$-cap consists of $2q+2$ points on a hyperbolic quadric $\mathcal{Q}_c$ (Lemma 4.4). If however $a^2 - b^2d_1 = -(e^2 - f^2d_1)$, then K has $q+1$ points in common with two hyperbolic quadrics $\mathcal{Q}_c$ and $\mathcal{Q}_{-c}$.

We will now show that the first possibility does not occur.

THEOREM 4.6. *Using the notations of the preceding parts, there does not exist a cyclic complete* $(2q+2)$*-cap* K *contained in a hyperbolic quadric* $\mathcal{Q}_c$.

Proof. Consider $\mathcal{Q}_c\colon X_0^2 - d_1^{-1}X_1^2 - cd_1X_2^2 + cX_3^2 = 0$, $c \neq 0$.

This quadric contains the four lines $\langle(1, \pm i, 0, 0), (0, 0, 1, \pm i)\rangle$. Choose coordinates so that the lines $M_1 = \langle(0, 0, 1, i), (1, i, 0, 0)\rangle$ and $M_1^q = \langle(0, 0, 1, -i), (1, -i, 0, 0)\rangle$ belong to the regulus $\mathcal{R}$ while $M_2 = \langle(0, 0, 1, i)(1, -i, 0, 0)\rangle$ and $M_2^q = \langle(0, 0, 1, -i), (1, i, 0, 0)\rangle$ belong to the complementary regulus $\mathcal{R}'$.

Let $G = \langle\alpha\rangle$ where α is described in Remark 4.3. Let the fixed points $(1, i, 0, 0)$ and $(1, -i, 0, 0)$ correspond to the eigenvalues α_1 and α_1^q of $\begin{pmatrix} a & b \\ bd_1 & a \end{pmatrix}$, while $(0, 0, 1, i)$ and $(0, 0, 1, -i)$ correspond to the eigenvalues α_2 and α_2^q of $\begin{pmatrix} e & f \\ fd_1 & e \end{pmatrix}$.

Then, since α^{q+1} fixes L and M pointwise,

$$\alpha^{q+1}\colon \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto A^{q+1}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha_1^{q+1} & 0 & 0 & 0 \\ 0 & \alpha_1^{q+1} & 0 & 0 \\ 0 & 0 & \alpha_2^{q+1} & 0 \\ 0 & 0 & 0 & \alpha_2^{q+1} \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

with $\alpha_1^{q+1} = -\alpha_2^{q+1}$ (see proof of Lemma 4.5).

Let $L_1, \dots, L_{q+1}$ be the lines in $\mathrm{PG}(3,q)$ of the regulus $\mathcal{R}'$ of $\mathcal{Q}_c$. Since each such line L_i can contain at most two points of K, these lines must form one orbit under G. This implies that α^{q+1} fixes all lines L_i and so α^{q+1} fixes their intersection points with M_1. Hence α^{q+1} is the identity on M_1. This is only possible if the eigenvalues α_1^{q+1} and α_2^{q+1} of A^{q+1} corresponding to $(1,i,0,0)$ and $(0,0,1,i)$ are equal to each other.

This however contradicts $\alpha_1^{q+1} = -\alpha_2^{q+1}$. So K does not exist. □

THEOREM 4.7. *Let K be a cyclic complete $(2q+2)$-cap of* $\mathrm{PG}(3,q)$, *q odd, fixed by a cyclic group $G = \langle\alpha\rangle$ which stabilizes two skew lines L and M.*

If $q \equiv -1 \pmod 4$, then G acts on at least one of the lines L and M in orbits of size strictly smaller than $q+1$.

Proof. Choose the reference system as in Remark 4.3. So $L\colon X_0 = X_1 = 0$, $M\colon X_2 = X_3 = 0$, and α of type given in that remark.

Consider

$$\alpha^{(q+1)/2}\colon \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ b_1 d_1 & a_1 & 0 & 0 \\ 0 & 0 & e_1 & f_i \\ 0 & 0 & f_1 d_1 & e_1 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Since α^{q+1} stabilizes all points of L and M and since α^{q+1} defines an involutory perspectivity in the planes $\langle L, r\rangle$ with $r \in M$ (see for instance the proof of Lemma 4.5), $\alpha^{q+1}\colon (x_0, x_1, x_2, x_3) \mapsto (\epsilon x_0, \epsilon x_1, -\epsilon x_2, -\epsilon x_3)$, so

$$\begin{cases} 2a_1 b_1 = 2e_1 f_1 = 0 \\ a_1^2 + b_1^2 d_1 = -e_1^2 - f_1^2 d_1 = \epsilon. \end{cases}$$

If $b_1 = f_1 = 0$, then $\alpha^{(q+1)/2}\colon (x_0, x_1, x_2, x_3) \mapsto (a_1 x_0, a_1 x_1, e_1 x_2, e_1 x_3)$ is the identity on L and M. So $\alpha^{(q+1)/2}$ must be the involutory perspectivity in the planes $\langle L, r\rangle$, $r \in M$, and $\alpha^{q+1} = 1$. This is false.

If $b_1 = e_1 = 0$, then $\alpha^{(q+1)/2}\colon (x_0, x_1, x_2, x_3) \mapsto (a_1 x_0, a_1 x_1, f_1 x_3, d_1 f_1 x_2)$ is the identity on M. So α acts on M in orbits of size at most $(q+1)/2$. The equations simplify to $a_1^2 = \epsilon$ and $d_1 f_1^2 = -\epsilon$, so $d_1 = -(a_1/f_1)^2$ which implies that $q \equiv$

$-1 \pmod 4$). The case $a_1 = f_1 = 0$ gives the same conclusion since L and M play an equivalent role.

Finally, if $a_1 = e_1 = 0$, then the preceding equations simplify to $b_1^2 d_1 = \epsilon$ and $d_1 f_1^2 = -\epsilon$ which implies that $-1 = (b_1/f_1)^2$ and so $q \equiv 1 \pmod 4$). □

4.2. EXAMPLES

The following examples of $(2q+2)$-caps consisting of $q+1$ points on two hyperbolic quadrics all are incomplete. No complete examples of such caps were found.

Only the elements (a, b, e, f, d_1) of the matrix

$$A = \begin{pmatrix} a & b & 0 & 0 \\ bd_1 & a & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & fd_1 & e \end{pmatrix}$$

of the transformation α: $(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_2, x_3)A$ are given.

For the given values of a, b, e, f and d_1, the orbits under $\langle\alpha\rangle$, not contained in $X_0 = X_1 = 0$ or $X_2 = X_3 = 0$, consist of $q + 1$ points on two of the hyperbolic quadrics

$$\mathcal{Q}_c\colon d_1^{-1}X_0^2 - X_1^2 - cX_2^2 + cd_1X_3^2 = 0, \quad c \in \mathrm{GF}(q)^*.$$

The equation of $\mathcal{Q}_c$ differs slightly from the one in the lemmas and theorems of this section since in the computer programs the coordinate vector (x_0, x_1, x_2, x_3) had to be calculated on the left-hand side with the matrix A.

(1) *A 20-cap in* PG(3, 9)
The field GF(9) was generated by using a root ω of the primitive polynomial X^2+X+2. By using $(a, b, e, f, d_1) = (1, 1, \omega^7, 1, \omega^5)$, the orbit of $(1, 1, 1, 1)$ is a 20-cap. This cap can only be extended by the points of eight orbits, not contained in $X_0 = X_1 = 0$ or $X_2 = X_3 = 0$, to a 21-cap.

(2) *A 24-cap in* PG(3, 11)
For $(a, b, e, f, d_1) = (1, 2, 8, 1, 2)$, the orbit of the unit point $(1, 1, 1, 1)$ is a 24-cap. This cap is extended by the points of 10 orbits, not contained in $X_0 = X_1 = 0$ and $X_2 = X_3 = 0$, and by six points on $X_2 = X_3 = 0$ to a larger cap.

(3) *A 36-cap in* PG(3, 17)
The orbit of $(1, 1, 1, 1)$ under α where α is defined by $(a, b, e, f, d_1) = (1, 10, 9, 1, 3)$ is a 36-cap. This cap is extended by 12 points on $X_2 = X_3 = 0$ and by the points of 12 orbits, not contained in $X_0 = X_1 = 0$ or $X_2 = X_3 = 0$, to a larger cap.

(4) *A 40-cap in* PG(3, 19)

For $(a, b, e, f, d_1) = (1, 1, 16, 2, 2)$, the orbit of $(1, 1, 1, 1)$ is a 40-cap K. The points of precisely 24 orbits under $\langle\alpha\rangle$, not contained in the two fixed lines $X_0 = X_1 = 0$ and $X_2 = X_3 = 0$, extend K to a 41-cap. There are also 10 points on $X_0 = X_1 = 0$ extending K to a larger cap.

(5) *A 48-cap in* PG(3, 23)
In PG(3, 23), when $(a, b, e, f, d_1) = (1, 1, 5, 8, 5)$, the orbit of $(1, 1, 1, 1)$ under $\langle\alpha\rangle$ is a 48-cap extended by the points of 41 orbits, not contained in $X_0 = X_1 = 0$ or $X_2 = X_3 = 0$, and by 20 points on $X_0 = X_1 = 0$, to a 49-cap.

5. Conclusion

From the preceding theorems follows

THEOREM 5.1. *In* PG(3, q), *q even, a cyclic complete k-cap K is an elliptic quadric. In* PG(3, q), *q odd, a cyclic complete k-cap K can only be*

(a) *an elliptic quadric,*
(b) *a k-cap K containing $k/2$ points of two disjoint elliptic quadrics,*
(c) *a $(2q+2)$-cap K containing $q+1$ points of two disjoint hyperbolic quadrics,*
(d) *a k-cap K stabilized by a cyclic group G fixing one point x not on K and one plane π with $x \notin \pi$.*

It should be emphasized that possibility (c) of this theorem may not occur. No complete cyclic k-caps of type (c) are known to the authors.

References

1. Bruck, R. H.: Construction problems of finite projective planes, *Conference on Combinatorial Mathematics and its Applications (University of North Carolina, 1967)*, University of North Carolina Press, pp. 426–514.
2. Cannon, J. and Bosma, W.: *CAYLEY Quick Reference Guide*, 1991.
3. Cossidente, A. and Storme, L.: Caps on elliptic quadrics, Preprint.
4. Ebert, G. L.: Partitioning projective geometries into caps. *Canad. J. Math.* **37**(6) (1985), 1163–1175.
5. Hirschfeld, J. W. P.: *Projective Geometries over Finite Fields*, Oxford University Press, Oxford, 1979.
6. Hirschfeld, J. W. P.: *Finite Projective Spaces of Three Dimensions*, Oxford University Press, Oxford, 1985.
7. Hirschfeld, J. W. P. and Thas, J. A.: *General Galois Geometries*, Oxford University Press, Oxford, 1991.
8. Hughes, D. R. and Piper, F. C.: *Projective Planes*, Springer-Verlag, Berlin, 1973.
9. Jenks, R. D. and Sutor, R. S.: *AXIOM: The Scientific Computation System*, Springer-Verlag, New York, 1992.
10. Maruta, T.: Optimal pseudo-cyclic codes and caps in PG(3, q), Preprint.
11. Storme, L. and Van Maldeghem, H.: Cyclic arcs in PG(2, q), *J. Algebraic Combin.* 3 (1994), 113–128.