

The Classification of Finite Generalized Quadrangles Admitting a Group Acting Transitively on Ordered Pentagons

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Abstract

Let \mathcal{S} be a thick generalized quadrangle and let G be a group of automorphisms of \mathcal{S} . If G acts transitively on the set of non-degenerate ordered pentagons, then \mathcal{S} is one of the classical generalized quadrangles $W(q)$, $Q(4, q)$, $Q(5, q)$ or $H(3, q^2)$. The possibilities for G in each case are determined. We do not use the classification of the finite simple groups (from which this result also follows).

1 Introduction and Main Result

A finite generalized quadrangle (GQ) of order (s, t) , $s, t \in \mathbf{N} \setminus \{0\}$, is an incidence geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ in which \mathcal{P} and \mathcal{B} are disjoint non-empty sets of objects called points and lines respectively, and for which \mathcal{I} is a symmetric point-line incidence relation satisfying axioms (GQ1), (GQ2) and (GQ3).

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- (GQ1) Each point is incident with $1 + t$ lines and two distinct points are incident with at most one line.
- (GQ2) Each line is incident with $1 + s$ points and two distinct lines are incident with at most one point.
- (GQ3) For every non-incident pair $(x, L) \in \mathcal{P} \times \mathcal{B}$, there exists a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x I M I y I L$.

For terminology, notation, results, etc., concerning finite GQ, see the monograph [5], hereafter denoted by FGQ. We now introduce some further terminology. A finite GQ of order (s, t) is *thick* if $s, t \geq 2$ (the non-thick GQ are the grids (order $(s, 1)$) and the dual grids (order $(1, t)$)). A *pentagon* in a GQ is a subconfiguration consisting of five distinct points and five distinct lines such that each line (respectively point) is incident with exactly two points (respectively lines). An *ordered pentagon* is a pentagon in which the elements are ordered in such a way that two consecutive elements are incident. A *skeleton* is a subconfiguration $\Omega = (Q; L, p)$ where Q is a quadrilateral (i. e. a subquadrangle of order $(1, 1)$) and L (respectively p) is a line (respectively point) not in Q but incident with a point p_1 (respectively line L_1) of Q , where $p_1 I L_1$.

It follows easily from 9.8.3 of FGQ that the classical GQ $W(q)$, $Q(4, q)$, $Q(5, q)$ and $H(3, q^2)$ admit an automorphism group G acting transitively on the set of skeletons (the automorphism group of the GQ $H(4, q^2)$ is not transitive on ordered triples of concurrent lines). We will see in 2.1.1 that this is equivalent with G acting transitively on the set of ordered pentagons. The converse is also true. Suppose the GQ \mathcal{S} admits a group G acting transitively on the set of ordered pentagons. Then G is a group with a (B, N) -pair of type B_2 and using the classification of the finite simple groups one can show that \mathcal{S} must be classical (for an explicit proof, see [1]), but in the present case different from $H(4, q^2)$. The aim of this paper is to give a proof of this result without using the classification of the finite simple groups. We will also characterize the groups G . Therefore, we need the following notation.

Let $PGO_5(q)$ be the projective general orthogonal group in 4-dimensional projective space over the field $GF(q)$ and let q be odd. This group has a unique normal subgroup of index 2, namely the simple group $PSO_5(q)$, sometimes denoted by $O_5(q)$. Now let θ be a field automorphism of $GF(q)$ and denote by g_θ the element of $PGL_5(q)$ corresponding to the semilinear

transformation with identity matrix and field automorphism θ . Let g be any element of $PGO_5(q) \setminus PSO_5(q)$. Then we denote by $PGO_5^\theta(q)$ the group generated by $PSO_5(q)$ and $g_\theta g$. Now consider the projective general unitary group $PGU_4(q)$ (still assuming q odd). This group has a unique normal subgroup of index 2 (if $q \equiv 1 \pmod{4}$) or 4 (if $q \equiv 3 \pmod{4}$), namely the simple group $PSU_4(q)$, also denoted by $U_4(q)$. The corresponding quotient group is cyclic and hence $PGU_4(q)$ has a unique subgroup H of index 2 containing $PSU_4(q)$. Let θ be a field automorphism of $GF(q^2)$ and let $g_\theta \in PGL_4(q^2)$ be defined similarly as above. Then we denote by $PGU_4^\theta(q)$ the group generated by H and $g_\theta g$, where $g \in PGU_4(q) \setminus H$.

In this paper we will show :

MAIN RESULT. *Let \mathcal{S} be a finite thick generalized quadrangle and let G be a group of automorphisms of \mathcal{S} . Then G acts transitively on the set of ordered pentagons if and only if \mathcal{S} is one of the classical generalized quadrangles $W(q)$, $Q(4, q)$, $Q(5, q)$ or $H(3, q^2)$, and G contains one of the following groups:*

- (i) $PGO_5(q)$, if $\mathcal{S} \cong \mathcal{W}(\Pi)$ or $\mathcal{S} \cong \mathcal{Q}(\Delta, \Pi)$,
- (ii) $PGO_5^\theta(q)$, if $\mathcal{S} \cong \mathcal{W}(\Pi)$ or $\mathcal{S} \cong \mathcal{Q}(\Delta, \Pi)$ with q an odd square and θ a field automorphism of $GF(q)$ of order a power of 2,
- (iii) $PGU_4(q)$, if $\mathcal{S} \cong \mathcal{Q}(\nabla, \Pi)$ or $\mathcal{S} \cong \mathcal{H}(\exists, \Pi^\epsilon)$,
- (iv) $PGU_4^\theta(q)$, if $\mathcal{S} \cong \mathcal{Q}(\nabla, \Pi)$ or $\mathcal{S} \cong \mathcal{H}(\exists, \Pi^\epsilon)$ with q odd and θ a field automorphism of $GF(q^2)$ of order a power of 2.

The groups will be further discussed in Section 2.4.

Part of the motivation of our proof is to stimulate people to aim for weaker hypotheses (such as the (B, N) -pair hypothesis above), still giving a proof without relying on the classification of the finite simple groups.

2 Proof of the Main Result

In this section, we denote by $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ a finite thick generalized quadrangle of order (s, t) and by G a group of automorphisms of \mathcal{S} acting transitively on the set of ordered pentagons.

2.1 Some General Facts

2.1.1 Skeletons

Let $\Pi = (p_1, L_1, \dots, p_5, L_5)$, with $p_1 I L_1 I p_2 I \dots I p_5 I L_5 I p_1$, be an ordered pentagon in \mathcal{S} . Let p'_i be the point of L_{i+2} collinear with p_i (where indices are taken modulo 5). Consider the skeleton $\Omega = (Q; L_5, p'_4)$, where Q is the quadrilateral $(p_1, L_1, p_2, L_2, p_3, L_3, p'_1, p_1 p'_1)$. Then Π completely determines Ω and vice versa. Hence G acts transitively on the set of skeletons.

2.1.2 Property (H)

Fix a point u of \mathcal{S} and let (x, y, z) be a triad of points of u^\perp . Suppose $z \in cl(x, y)$. If $z \in \{x, y\}^{\perp\perp}$, then $y \in \{x, z\}^{\perp\perp}$. Now suppose $z \notin \{x, y\}^{\perp\perp}$ and let v respectively v' be a point in $\{x, y\}^\perp \setminus z^\perp$ respectively $\{x, z\}^\perp \setminus y^\perp$. This defines a unique skeleton $(Q; L_0, z')$ respectively (Q', L'_0, y') , where $uz \sim L_0 I v$ respectively $uy \sim L'_0 I v'$, where $z \sim z' I v y$ respectively $y \sim y' I v' z$, and where $Q = (v, vy, y, yu, u, ux, x, xv)$ respectively $Q' = (v', v'z, z, zu, u, ux, x, xv')$. By the transitivity of G on skeletons, we see that there is an automorphism mapping the ordered quadruple (u, x, y, z) onto the ordered quadruple (u, x, z, y) . Indeed, z is collinear with z' and incident with the unique line uy containing u and concurrent with L_0 ; z' is mapped onto y' , and the unique line containing u and concurrent with L'_0 is the line uy . As $y' \sim y I uy$, the point z is mapped onto y . Hence $y \in cl(x, z)$ and so u has property (H). By transitivity, every point of \mathcal{S} has property (H), and dually, every line has property (H). By 5.6.2 of FGQ \mathcal{S} must satisfy, up to duality, one of the following conditions:

- (i) \mathcal{S} is isomorphic to $H(4, q^2)$, $q^2 = s$.
- (ii) Every point and every line in \mathcal{S} are regular.
- (iii) Every hyperbolic line has exactly 2 points, and dually.
- (iv) Every hyperbolic line in \mathcal{S} has exactly 2 points and every line of \mathcal{S} is regular.

It is readily seen that $H(4, q^2)$ does not admit a group of automorphisms acting transitively on the set of skeletons. Also, case (ii) implies that \mathcal{S} is isomorphic to $W(2^e)$ for some positive integer e (see 5.2.1 and 3.2.1 of FGQ). So from now on, we may assume that every hyperbolic line has exactly 2 points. Using the same kind of argument to show property (H), one produces a collineation mapping (u, x, y, z) onto (u, x', y', z') , where (x, y, z) and (x', y', z') are two arbitrary triads in u^\perp (note that $z \notin \{x, y\}^{\perp\perp}$ as required for the argument). So all triads in u^\perp have a constant number of centers. By 1.7.1(i) of FGQ, this constant equals $1 + t/s$, so s divides t .

Now we consider the cases (iii) and (iv) separately.

2.2 Case (iii)

In this case, also the dual of \mathcal{S} has hyperbolic lines of length 2, hence also $1 + s/t$ is an integer. It follows that $s = t$. As each triad of points has either 0 or 2 centers, each point of \mathcal{S} is antiregular (cf. 1.3.6(iii) of FGQ). Dually, each line of \mathcal{S} is antiregular. Also $s = t$ is odd by 1.5.1(i) of FGQ.

Assume that \mathcal{S}' is a thick subquadrangle of \mathcal{S} of order (s', t') , and that \mathcal{S}' does not admit a proper thick subquadrangle. Let Ω and Ω' be skeletons contained in \mathcal{S}' . Further, let θ be an automorphism of \mathcal{S} mapping Ω to Ω' . If $\mathcal{S}'^\theta \neq \mathcal{S}'$, then $\mathcal{S}' \cap \mathcal{S}'^\theta$ is a proper thick subquadrangle of \mathcal{S}' , a contradiction. Consequently $\mathcal{S}'^\theta = \mathcal{S}'$, and so the automorphism group G' of \mathcal{S}' acts transitively on the set of skeletons of \mathcal{S}' .

In \mathcal{S}' any triad of points has at most two centers, that is, each point of \mathcal{S}' is antiregular. Hence $s' \geq t'$ by 1.3.6(i) of FGQ. Similarly, $t' \geq s'$. Consequently, $s' = t'$ and then by 1.3.6(iii) of FGQ each triad of points respectively lines has 0 or 2 centers. Also, $s' = t'$ is odd. Let us now forget the GQ \mathcal{S} ; so “ \perp ” means perpendicular in \mathcal{S}' , etc.

Since \mathcal{S}' has no proper thick subquadrangles, the identity is the only automorphism of \mathcal{S}' fixing a given skeleton of \mathcal{S}' . The group H' of automorphisms of \mathcal{S}' fixing an ordered quadrilateral $Q := (p_1, L_1, p_2, \dots, L_4)$ and a line L through p_1 , $L_4 \neq L \neq L_1$, has even order $s' - 1$. Let σ be an involution of the group H' .

By 1.3.2 of FGQ an affine plane $\pi(p_4, p_1)$ of order s' may be constructed as follows. Points of $\pi(p_4, p_1)$ are the points of p_4^\perp that are not on L_4 . Lines

are the pointsets $\{p_4, z\}^{\perp\perp} \setminus \{p_4\}$, with $p_4 \sim z \not\sim p_1$, and $\{p_4, u\}^\perp \setminus \{p_1\}$, with $p_1 \sim u \not\sim p_4$. Let us denote the pointset of a line M of \mathcal{S}' by M^* . The involution σ induces either the identity or an involution σ' in the plane $\pi(p_4, p_1)$. In the latter case, σ' fixes the point p_3 , the line L_3^* , the parallel class of lines defined by L_3^* , the line $\{p_4, p_2\}^\perp \setminus \{p_1\} = U$ which contains p_3 , the parallel class of lines defined by U (as $L_1^\sigma = L_1$), and the parallel class containing the line $\{p_4, v\}^\perp \setminus \{p_1\} = V$ with $p_1 \neq v \text{ I L}$ (as $L^\sigma = L$)

If σ' is the identity, then, by 2.4 of FGQ, σ is the identity, a contradiction; if σ' is a Baer involution, then, again by 2.4 of FGQ, the fixed elements of σ form a subquadrangle of order $(\sqrt{s'}, \sqrt{s'})$ of \mathcal{S}' , a contradiction. Hence σ' is a homothety of $\pi(p_4, p_1)$ with center p_3 . Consequently σ fixes every line through p_1 and every point of $\{p_1, p_3\}^\perp$. As \mathcal{S}' does not contain a proper thick subquadrangle, the fixed elements of σ are p_1, p_3 , the points of $\{p_1, p_3\}^\perp$, the lines through p_1 , and the lines through p_3 .

Now let x and y be distinct points of \mathcal{S}' on the line N of \mathcal{S}' , with p_1 not incident with N . By the transitivity properties of G' there is an involution γ of \mathcal{S}' fixing p_1 linewise and fixing N . Let $r^\gamma \neq r$, with r a point of \mathcal{S}' on N . Further, let δ be an element of G' fixing p_1, N , and for which $r^\delta = x$ and $r^{\gamma\delta} = y$. Then $\delta^{-1}\gamma\delta$ fixes p_1 linewise and maps x onto y . Now it is clear that the subgroup of G' fixing p_1 linewise acts transitively on $\mathcal{P}' \setminus \sqrt{\infty}^\perp$, with \mathcal{P}' the pointset of \mathcal{S}' . Then by 8.2.4 of FGQ the GQ \mathcal{S}' is an EGQ (elation generalized quadrangle) with base point p_1 and elation group \overline{G} .

Let u_1 and u_2 be distinct points of the plane $\pi(p_4, p_1)$. If ζ is the elation in \overline{G} mapping u_1 onto u_2 , then ζ induces a translation ζ' in $\pi(p_4, p_1)$ mapping u_1 onto u_2 . Consequently $\pi(p_4, p_1)$ is a translation plane. Interchanging the roles of p_1 and p_4 , we see that $\pi(p_1, p_4)$ is also a translation plane. It follows that, if ∞ is the point at infinity of $\pi(p_4, p_1)$ defined by the parallel class of lines containing L_3^* , then the projective completion $\overline{\pi(p_4, p_1)}$ of $\pi(p_4, p_1)$ is a dual translation plane with translation point ∞ .

Next, let x_1 and x_2 be points of $\pi(p_4, p_1)$ not on L_3^* and not on $\{p_2, p_4\}^\perp \setminus \{p_1\} = U$. By the transitivity of G' on the skeletons of \mathcal{S}' , there is an automorphism η in G' fixing p_1, p_2, p_3, p_4 mapping $x_1 p_4$ onto $x_2 p_4$ and mapping x_1 onto x_2 . Then η induces an automorphism η' of $\pi(p_4, p_1)$ fixing L_3^*, U and mapping x_1 onto x_2 . Hence $\pi(p_4, p_1)$ is Desarguesian [4]. Then by 5.2.7 of FGQ, the GQ \mathcal{S}' is isomorphic to $Q(4, s')$. Consequently every line of \mathcal{S}' is regular, a contradiction.

We conclude that case (iii) cannot occur.

2.3 Case (iv)

2.3.1 Some general observations

We first fix the notation.

Throughout, Ω will denote the skeleton $((p_1, L_1, p_2, L_2, p_3, L_3, p_4, L_4); L, p)$, where $Q = (p_1, \dots, L_4)$, $L I p_1$ and $p I L_1$. The group G_Ω of automorphisms in G fixing Ω has order $k \in \mathbf{N} \setminus \{0\}$.

The line L_1 is regular in \mathcal{S} . For any two lines M and M' meeting L_1 in different points, we call the collection $\{M, M'\}^{\perp\perp}$ of $s + 1$ lines the *regulus* through M and M' ; it is denoted by MM' . Every line of the regulus MM' meets L_1 . The set of all lines through a point p will be denoted by p^* .

Consider an involution σ that fixes p_1, L_1 and p_2 . Suppose that σ does not fix all points on L_1 . Let x be such a point on L_1 which is not fixed by σ , and let M be a line through x different from L_1 . The regulus MM^σ meets p_1^* respectively p_2^* in exactly one line R_1 respectively R_2 . Clearly the regulus MM^σ is fixed by σ , and since p_1 and p_2 are fixed by σ , also R_1 and R_2 are fixed. Varying M through x , one sees that the number of fixed reguli MM^σ is t . But this number also equals $t_1 t_2$, where t_i is the number of lines through p_i , different from L_1 and fixed by σ , $i = 1, 2$. If moreover σ fixes a third point u on L_1 , then the number t_u of fixed lines through u (different from L_1) satisfies $t_1 t_u = t_2 t_u = t_1 t_2 = t$ implying $t_1 = t_2 = t_u = \sqrt{t}$.

2.3.2 A useful lemma

Let $\theta (\neq 1)$ be an automorphism in G fixing all points on L_1 and the lines L_2, L_4 and L . Let $r I L_1, p_2 \neq r \neq p_1, p_1 I M, M \neq L_1, r I N, N \neq L_1$. Then the regulus MN is fixed by θ if and only if $M^\theta = M$ and $N^\theta = N$. Let $t_i + 1$ be the number of fixed lines through p_i , $i = 1, 2$, and let $t' + 1$ be the number of fixed lines through r . If w is the number of fixed reguli consisting of $s + 1$ lines concurrent with L_1 , then clearly $w = t_1 t'$. Analogously, $w = t_2 t'$ and $w = t_1 t_2$. Hence $t_1 = t_2 = t'$. So θ fixes a constant number $t' + 1$ of lines through every point of L_1 . Let \mathcal{P}' be the union of the points on all

these lines and let \mathcal{B}' be the set of lines meeting \mathcal{P}' in at least two points. Clearly, every element of \mathcal{B}' is incident with exactly $s + 1$ elements of \mathcal{P}' (by the regularity of L_1). By 2.3.1 of FGQ, $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$, with I' the restriction of I to $\mathcal{P}' \times \mathcal{B}' \cup \mathcal{B}' \times \mathcal{P}'$, is a subquadrangle of order (s, t') . If $t' < t$, then by 5.3.5 of FGQ and the transitivity properties of G , the GQ \mathcal{S} is the classical GQ $Q(5, s)$. Now let $t = t'$. As $\theta \neq 1$, we have $p_3^\theta \neq p_3$ by 2.4.1 of FGQ. For every point x on L_2 with $p_2 \neq x \neq p_3$, the conjugate θ^η of θ , where η is a collineation fixing L_1, L_2, L_4, p_3 and mapping p_3^θ to x , fixes every line concurrent with L_1 and maps p_3 to x . Therefore \mathcal{S} is half Moufang and hence Moufang by [7]. Since all lines are regular, we have $\mathcal{S} \cong \mathcal{Q}(\Delta, f)$ or $\mathcal{S} \cong \mathcal{Q}(\nabla, f)$ by 3.3.1 of FGQ.

Now we distinguish between several numerical cases.

2.3.3 k and s both odd

The subgroup H of G fixing Q and L has order $(s - 1)k$ and hence contains some involution σ . If σ would fix a third point u on L_1 , then σ would be inside G_Ω (taking without loss of generality $u = p$), contradicting the hypothesis k odd. Hence σ has no fixed points, different from p_1 and p_2 , on L_1 . From Section 2.3.1 follows immediately that $t_1 t_2 = t$, where $t_i + 1$ is the number of fixed lines (for σ) through $p_i, i = 1, 2$.

Suppose $t_i > 1$ for $i = 1, 2$. Let $M_i \neq L_1$ be any line fixed by σ and incident with p_i and let x_i be the point of M_i collinear with $p_{i+2}, i = 1, 2$. Clearly x_i is also fixed by σ and considering the point x'_2 of M_2 collinear with x_1 , one sees that σ fixes a skeleton unless $x'_2 = x_2$. Since k is odd, σ cannot fix a skeleton and hence there arises a dual $(t_1 + 1) \times (t_2 + 1)$ -grid having as points the points p_1, p_2 , the t_1 points x_1 and the t_2 points x_2 (obtained by letting vary M_1 and M_2). So there are three points with at least $t_1 + 1$ centers and three points with at least $t_2 + 1$ centers. By the observation in 2.1.2, $t_i \leq t/s$ and hence $t = t_1 t_2 \leq t^2/s^2$, implying $s^2 \leq t$, so $t = s^2$ (cf. 1.2.3 of FGQ) and $t_1 = t_2 = t/s = s$. So we have $|\{p_1, p_3, x_2\}^{\perp\perp}| \geq 1 + s$, hence the left hand side is actually equal to $1 + s$. By the transitivity on triads in x_1^\perp , we have that x_1 is 3-regular. By transitivity every point of \mathcal{S} is 3-regular, and consequently, by 5.3.2(i) of FGQ, \mathcal{S} is the classical GQ $Q(5, s)$.

Next we consider the case $t_2 = 1$. Then of course $t_1 = t$ and so σ fixes all lines through p_1 . Consider two arbitrary points y, z on L_2 with $z \neq p_2 \neq y$

and $y \neq p_3$. The subgroup H^* of G fixing p_1, p_2, y, L_4 and L has also order $(s-1)k$ and acts transitively on the points of L_2 different from p_2 and y . Let $\theta \in H^*$ map y^σ (which is different from y) to z . Then the map $\theta^{-1}\sigma\theta$ maps y to z and fixes all lines through p_1 . Varying the point p_2 on L_1 and varying afterwards the line L_1 through p_1 , we see that there exists a group K of automorphisms of \mathcal{S} fixing all lines through p_1 and acting transitively on $\mathcal{P} \setminus \sqrt{\infty}^\perp$. By 8.2.4(iv) of FGQ, there is a unique subgroup $E \trianglelefteq K$ of order s^2t acting regularly on $\mathcal{P} \setminus \sqrt{\infty}^\perp$; this normal subgroup is the set of all automorphisms of \mathcal{S} fixing all lines through p_1 and having no fixed point in $\mathcal{P} \setminus \sqrt{\infty}^\perp$. Let E_{L_2} be the subgroup of E , of order s , fixing L_2 .

Now consider the subgroup H_1 of G which fixes L_4, p_1, L_1, p_2, L_2 and L . This group acts transitively on the points of L_1 different from p_1 and p_2 . Clearly $E_{L_2} \leq H_1$. If $\eta \in E_{L_2} \setminus \{1\}$ and $\delta \in H_1$, then $\delta^{-1}\eta\delta$ fixes p_1 linewise, fixes L_2 , and has no fixed point collinear with p_1 . Hence $\delta^{-1}\eta\delta \in E_{L_2}$, and so $E_{L_2} \trianglelefteq H_1$. Hence E_{L_2} acts on the points of L_1 different from p_1 and p_2 in orbits of equal length d . So d has to divide both $s-1$ and s , hence $d=1$. It follows that every element of E_{L_2} fixes all points on L_1 . By the regularity of L_1 it is immediately clear that every element of E_{L_2} fixes each line of $\{L_4, L_2\}^{\perp\perp}$. Now let M be a line meeting L_1 , with p_1 not incident with M , p_2 not incident with M , and $M \notin \{L_4, L_2\}^{\perp\perp}$. Let N be the line through p_1 which belongs to $\{M, L_2\}^{\perp\perp}$. Each element of E_{L_2} fixes $\{N, L_2\}^{\perp\perp}$ and the common point of M and L_1 . Hence each element of E_{L_2} fixes M . It easily follows that each element of E_{L_2} fixes each line concurrent with L_1 . By transitivity of G on the lines of \mathcal{S} , \mathcal{S} is half Moufang and hence Moufang by [7]. So, by [3], \mathcal{S} is classical (cf. 9.1 of FGQ). Since all lines of \mathcal{S} are regular, we have $\mathcal{S} \cong \mathcal{Q}(\Delta, f)$ or $\mathcal{S} \cong \mathcal{Q}(\nabla, f)$.

This completes the case k and s both odd.

2.3.4 k odd and s even

We consider here the subgroup H_1 of G fixing $L_4, p_1, L_1, p_2, L_2, L$ and p . This group has order sk and hence it contains an involution σ . If σ would fix a point not incident with L_1 , then it would fix a skeleton and this contradicts the fact that k is odd. So the only fixed points for σ are on L_1 . Let x be any point on L_1 and let y be any point collinear with x but not on L_1 . If

$y^\sigma \sim y$, then clearly x is fixed by σ . If $y \not\sim y^\sigma = y'$, then $\{y, y'\}^\perp$ contains a fixed point (indeed, it contains $t + 1$ elements and since s divides t , t is even). Hence there is a point collinear with y which is fixed by σ . This must clearly be x and so we showed that σ fixes every point on L_1 . By Lemma 2.3.2, $\mathcal{S} \cong \mathcal{Q}(\Delta, f)$ or $\mathcal{S} \cong \mathcal{Q}(\nabla, f)$.

2.3.5 k and s both even

From these assumptions follows the existence of an involution σ fixing the skeleton Ω . This involution has as fixed elements the points and lines of a thick proper subquadrangle \mathcal{S}' of order (s', t') . If $s = s'$, then by the transitivity properties of G , it follows from 5.3.5 of FGQ that \mathcal{S} is the classical GQ $Q(5, s)$. Hence suppose $s \neq s'$. By Section 2.3.1, $t' = \sqrt{t}$.

Let R be a line of \mathcal{S}' and let z be a point of R not in \mathcal{S}' . Then any line through z , different from R , has no point in common with \mathcal{S}' ; so there exists a line M external to \mathcal{S}' . The line $M' = M^\sigma$ does not meet M and the $s + 1$ lines of $\{M, M'\}^{\perp\perp}$ are permuted by σ . Since $s + 1$ is odd, $\{M, M'\}^{\perp\perp}$ contains at least one fixed line N , which contains in turn exactly $s' + 1$ fixed points. A line containing any of these $s' + 1$ points and concurrent with both M and M' is fixed by σ . Hence there are exactly $s' + 1$ lines of \mathcal{S}' concurrent with a given external line. By Proposition 2.2.1 of FGQ, we have $1 + s' = 1 + t'$. But again by 2.2.1 of FGQ, we also have $s \geq s't' = t$. Hence $s = t$ and \mathcal{S} is the classical GQ $Q(4, q)$ (by the regularity of every line).

2.3.6 k even and s odd

We subdivide this case in two subcases: t odd and t even.

Subcase t even. Here we consider the subgroup H_2 of G fixing L_4, p_1, L_1, p_2, L and p . This group has order stk and hence contains a non-trivial Sylow 2-subgroup S . Suppose 2^n respectively 2^m divides t respectively k and 2^{n+1} respectively 2^{m+1} does not. Then $|S| = 2^{n+m}$. Let σ be an involution in the center of S and suppose that σ fixes a second line through p_2 , say L_2 . Since s is odd σ fixes a second point on L_4 respectively L_2 . If σ would fix all points on L_1 , then its fixed elements would be the points and lines of a thick proper subquadrangle of order (s, t') . Then, by the transitivity properties of G , it

again follows from 5.3.5 of FGQ, that $\mathcal{S} \cong \mathcal{Q}(\nabla, f)$. Hence we assume now that σ does not fix all points of L_1 . By Section 2.3.1, σ fixes \sqrt{t} lines different from L_1 through p_2 . The group S acts on these lines and the stabilizer S_M in S of any of these lines M has at most order 2^m , since S_M is a subgroup of a conjugate of the group G_Ω of order k . So the length of the orbit of M under S has at least order $|S|/2^m = 2^n$. Since M was arbitrary, this means that \sqrt{t} must be divisible by 2^n , hence 2^{2n} divides t , a contradiction. Hence σ does not fix any line through p_2 besides L_1 . But then, by 2.3.1, σ fixes every point on L_1 . On L_4 there is a second fixed point p_4 . If σ does not fix all points of L_4 , then, by 2.3.1, p_4 is on at least two fixed lines, a contradiction as every fixed line contains p_1 . Hence σ fixes every point of L_4 , and similarly it fixes every point of L . Now consider the group T generated by all conjugates of σ by elements of the subgroup H_3 of G fixing p_1, L_1, p_2 . Every element of T fixes each point of L_1 . Let M be a line different from L_1 containing p_1 , and let M', \overline{M}' be distinct lines different from L_1 containing p_2 . Then there is a $\delta \in H_3$ such that $L_4^\delta = M, N^\delta = \overline{M}', N^{\sigma\delta} = M'$, where N is any line through p_2 different from L_1 . Then $\delta^{-1}\sigma\delta \in T$ fixes each point of M and maps M' onto \overline{M}' . Now let x be a point collinear with p_2 and not incident with L_1 . Denote by O_x the orbit of x under T . Clearly, we have $|O_x| \geq t$. If $|O_x| = t$, then by the transitivity property just mentioned, all points of O_x are collinear to one single point of $M, M \neq L_1$ and M through p_1 . But M is arbitrary, so we obtain a dual $(t+1) \times (t+1)$ -grid. Consequently we have a regular pair of points and by 1.3.6(i) of FGQ, $s \geq t$. But all lines are regular, so $t \geq s$ implying $s = t$, which means that \mathcal{S} is the classical GQ $Q(4, s)$. Hence we may assume $|O_x| > t$. This means that there exists a collineation γ in T preserving L_2 and not inducing the identity map on the set of points of L_2 . By interchanging roles of p_1 and p_2 , we see that there exists an element $\xi \in G$ fixing all points of L_1 , all points of L_2 and mapping L_4^γ onto L_4 . Then $\gamma\xi = \zeta$ fixes L_2, L_4 , all points of L_1 , but not every point of L_2 . Let $x^\zeta \neq x, x \notin L_2$. Choose on L_2 distinct points y, z , not on L_1 . By the transitivity properties of G , there is an element $\delta \in H_3$ which fixes L_4 , maps x onto y , and x^ζ onto z . Then $\delta^{-1}\zeta\delta$ fixes all points of L_1 , fixes L_4 and L_2 , and maps y onto z . Consequently the group T_0 generated by all conjugates of ζ by elements of H_3 fixes all points of L_1 , fixes L_4 and L_2 , and acts transitively on the s points of L_2 not on L_1 . Hence s divides $|T_0|$. Assume that no element of $T_0 \setminus \{1\}$ fixes a third line through p_1 . Then T_0 acts faithfully and semi-regularly on the lines through p_1 , different from L_1 and L_4 . Hence $|T_0|$ divides $t-1$, so s divides $t-1$. As s divides t , we have a contradiction.

Hence there is an element $\theta \in T_0$ fixing all points of L_1 , fixing L_2 and L_4 , and fixing a third line, say L , containing p_1 . By Lemma 2.3.2, $\mathcal{S} \cong \mathcal{Q}(\Delta, f)$ or $\mathcal{S} \cong \mathcal{Q}(\nabla, f)$, so s and t have the same parity, a contradiction.

Subcase t odd. Here we consider the subgroup H_4 of G fixing Q and p , which has order $(t-1)k$. We consider a Sylow 2-subgroup S in H_4 . Suppose 2^n respectively 2^m divides $t-1$ respectively k and 2^{n+1} respectively 2^{m+1} does not. Then $|S| = 2^{n+m}$. Let σ be an involution in the center of S . If σ does not fix every point on L_1 , then it fixes exactly $\sqrt{t}-1$ lines different from L_1 and L_4 through p_1 . Let M be one of these lines. The group S acts on these $\sqrt{t}-1$ fixed lines and similarly as in the previous case, we obtain that the size of the orbit of M is divisible by 2^n . Hence 2^n divides $\sqrt{t}-1$, so 2^{n+1} divides $(\sqrt{t}-1)(\sqrt{t}+1) = t-1$, a contradiction. Hence σ fixes all points on L_1 . Assume there is no element of G fixing all points of L_1 , the lines L_2 , L_4 , and at least three lines containing p_1 or p_2 . Let T' be the group generated by the conjugates of σ by the elements of the subgroup H_5 of G fixing L_4 , p_1 , L_1 , p_2 and L_2 . Then the elements of T' fix all points of L_1 , fix L_2 and L_4 . Suppose first that σ does not fix all points of L_2 . Then T' acts transitively on the set A of points of L_2 different from p_2 . By an argument similar to the one in the case t even, we obtain a collineation of \mathcal{S} fixing all points of L_1 , the line L_2 and at least three lines containing p_1 . Hence σ fixes all points of L_2 . Then the fixed elements of σ are all elements of the $(s+1) \times (s+1)$ -grid containing the lines L_2 and L_4 . Let N be a line through p_2 different from L_1 and L_2 . Then $N^\sigma \neq N$. Let H_3 be the subgroup of G fixing p_1 , L_1 , p_2 , let M be a line different from L_1 containing p_1 , and let M' , \overline{M}' be distinct lines different from L_1 containing p_2 . Then there is a $\delta \in H_3$ such that $L_4^\delta = M$, $N^\delta = \overline{M}'$, $N^{\sigma\delta} = M'$. So $\delta^{-1}\sigma\delta$ fixes each point of L_1 , fixes each point of M and maps M' onto \overline{M}' . Let T'' be the subgroup of G generated by the conjugates of σ by the elements of the group H_3 . Then the elements of T'' fix all points of L_1 . Using an argument analogous to the one in the case t even (replacing T by T''), we find that necessarily $\mathcal{S} \cong \mathcal{Q}(\Delta, f)$. So we may assume that \mathcal{S} admits a collineation fixing all points of L_1 , the lines L_2 , L_4 , and at least three lines containing p_1 or p_2 . Then by Lemma 2.3.2, the GQ \mathcal{S} is isomorphic to either $Q(4, s)$ or $Q(5, s)$.

2.4 The possible groups

Let \mathcal{S} be one of the generalized quadrangles $W(q)$, $Q(4, q)$, $Q(5, q)$ or $H(3, q^2)$ and let G be a group of automorphisms acting transitively on the set of ordered pentagons. By duality, we may restrict ourselves to $Q(4, q)$ and $H(3, q^2)$. First let $\mathcal{S} \neq \mathcal{Q}(\Delta, \epsilon)$. By a theorem of Seitz [6], G has to contain the simple group $O_5(q)$ ($q \neq 2$), respectively $U_4(q)$. If q is even, then $O_5(q)$ respectively $U_4(q)$ coincides with $PGO_5(q)$ respectively $PGU(4, q)$ and each of these groups contains all generalized homologies of the corresponding GQ; hence these groups have the desired transitivity properties. If $\mathcal{S} = \mathcal{Q}(\Delta, \epsilon)$, then $O_5(2)$ is the full automorphism group of \mathcal{S} and $|O_5(2)|$ is the number of ordered pentagons in \mathcal{S} . As $O_5(2)$ acts semiregularly on the set of ordered pentagons, and consequently also regularly, it follows that G must coincide with $O_5(2)$ in this case. Suppose now that q is odd.

First let $\mathcal{S} \cong \mathcal{Q}(\Delta, \Pi)$. The full group of automorphisms of $Q(4, q)$ is $PGO_5(q).h = P\Gamma O_5(q)$, where $q = p^h$ with p a prime number. If h is odd, then the group $O_5(q).h$ does not act transitively on the ordered pentagons since its order is not divisible by the number of ordered pentagons. This number of ordered pentagons equals $2|O_5(q)|$, so $O_5(q)$ must have even index in G . Since the outer automorphism group of $O_5(q)$ has a unique involution (this is the diagonal automorphism, see [2]), every extension of $O_5(q)$ in $P\Gamma O_5(q)$ in which $O_5(q)$ has even index must contain $PGO_5(q)$. Suppose now that h is even. Consider a skeleton $(Q; L, x)$ in \mathcal{S} . The stabilizer in $O_5(q)$ of Q acts transitively on the points not in Q and incident with the line of Q containing x ; but the stabilizer in $O_5(q)$ of Q and x acts on the set V of lines not in Q but incident with the point of Q on L as the permutation group on $GF(q) \setminus \{0\}$ consisting of the elements $\gamma_a : x \mapsto a^2x$, $a, x \in GF(q) \setminus \{0\}$. If G does not contain $PGO_5(q)$, then it contains an element involving a field automorphism θ , $\theta \neq 1$. So G contains the group $PGO_5^\theta(q)$ generated by $O_5(q)$ and $g_\theta g$, where $g \in PGO_5(q)$ and where g_θ is the element of $P\Gamma L_5(q)$ corresponding to the semilinear transformation with identity matrix and field automorphism θ . Clearly $PGO_5^\theta(q)$ contains an element that fixes Q and x , and acts on V as the permutation $\gamma : x \mapsto bx^\theta$, with b a non-square in $GF(q)$, that is, we may assume that $g \in PGO_5(q) \setminus O_5(q)$. Also, θ has even order n (otherwise $(g_\theta g)^n \in PGO_5(q) \setminus O_5(q)$ and then G contains $PGO_5(q)$). Conversely, each such $PGO_5^\theta(q)$ acts transitively on the set of skeletons of \mathcal{S} . Let $n = l.2^e$ with l odd and put $\theta' = \theta^l$. Then θ' has

order 2^e , $PGO_5^{\theta'}(q) \leq PGO_5^{\theta}(q)$, and $PGO_5^{\theta'}(q)$ also acts transitively on the set of skeletons, and hence on the set of ordered pentagons. This shows (i) and (ii) of the main result.

Now suppose $\mathcal{S} \cong \mathcal{H}_{\supset}(\mathbb{II}^{\epsilon})$, q odd. First suppose that $q \equiv 3 \pmod{4}$. Then the Sylow 2-subgroup P of the outer automorphism group of $U_4(q)$ is a semi-direct product of a cyclic group C_4 (of diagonal automorphisms) with a group of order 2 (the unique involutory field automorphism). Now the number of ordered pentagons equals $4|U_4(q)|$ and so G must contain $U_4(q)$ as a subgroup of index divisible by 4. Hence $G/U_4(q)$ contains a group isomorphic to a subgroup of P of order at least 4. But every such subgroup clearly contains the unique involutory diagonal automorphism (the unique involution of C_4). So G contains the group H (with the notation of the introduction). This is trivially true if $q \equiv 1 \pmod{4}$. But now, a similar reasoning as in the case $\mathcal{S} \cong \mathcal{Q}(\Delta, \mathbb{II})$, q odd and h even, of the preceding paragraph shows that only cases (iii) and (iv) of the main result are possible.

This completes the proof of our main result.

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