A Geometric Construction of a Class of Twisted Field Planes by Ree-Tits Unitals

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Dedicated to H. Reiner Salzmann for his sixtieth birthday

Abstract

We investigate the intersection pattern of Ree-Tits unitals in the split Cayley Hexagon $H(q)$ associated to Dickson's group $G_2(q)$. Using these patterns, we are able to define an incidence geometry Γ which turns out to be a twisted field plane of order 3^{2h+1}, non-desarguesian if $h \neq 0$. We also show that a general point of the underlying generalized hexagon defines an oval in Γ.

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1 Introduction

Reiner Salzmann's name will be attached for ever to the theory of *topological planes*. However also results in the general theory of projective planes are due to him, mainly in the late 50's and 60's. About that time, or a little bit earlier, another eminent mathematician, A. A. Albert [1, 2, 3] constructed what is now known under the name *Albert twisted fields*, giving rise to a huge class of non-desarguesian finite projective planes. Remarkable about that construction is the fact that it is one of the (relatively) few classes containing ternary fields of non-square order. To the best of our knowledge,

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the corresponding projective planes have never been constructed in a purely geometric way. We will show that a subclass of Albert twisted field planes of order $q = 3^{2h+1}$, $h \in \mathbb{N}$, can be constructed geometrically with Ree-Tits unitals in the generalized hexagon $H(q)$ associated to the finite simple group $G_2(q)$. It turns out however, that the twisted field we obtain is not one of the twisted fields which have been shown to be non-associative in general, see P. Dembowski [5]. So we will have to convince ourselves that they are indeed non-associative, provided $h > 1$. As an application we construct in three different ways an oval inside every twisted field plane that we have found.

Our paper runs as follows. In section 2 we introduce the notation and the coordinates we need to describe the Ree-Tits unitals in $H(q)$. In section 3 we investigate the intersection pattern of Ree-Tits unitals in $H(q)$. In section 4 we define the twisted field planes Γ. In section 5 we show that they are non-desarguesian whenever their order is at least 27. In section 6 we close with the construction of an oval in Γ.

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2 A description of the Ree-Tits unitals in $H(q)$

V. De Smet and H. Van Maldeghem [7] give the following description of the generalized hexagon $H(q)$ (due to J. Tits [9]) arising from Dickson's simple group $G_2(q)$ (we only give the description in the case q is a power of 3, since that is the case in which we will be interested in the present paper). The points are 0-, 1-, 2-, 3-, 4- and 5-tuples over $GF(q)$ denoted with round parentheses; the lines are similarly but denoted with square brackets (a 0-tuple is here denoted by (∞) for a point and by $[\infty]$ for a line); incidence (denoted by I; in fact, I will serve as an abbreviation of *"is incident with"* in whichever geometry we consider, of course we will explicitly mention which one when confusion is possible) is defined as follows. For all $a, a', a'', b, b', k, k', k'', l, l' \in GF(q)$, we have

$$(a, l, a', l', a'')\,\mathbf{I}\,[a, l, a', l']\,\mathbf{I}\,(a, l, a')\,\mathbf{I}\,[a, l]\,\mathbf{I}\,(a)\,\mathbf{I}\,[\infty]\,\mathbf{I}$$

$$(\infty)\,\mathbf{I}\,[k]\,\mathbf{I}\,(k, b)\,\mathbf{I}\,[k, b, k']\,\mathbf{I}\,(k, b, k', b')\,\mathbf{I}\,[k, b, k', b', k''],$$

no other incidences occur except that

$$(a, l, a', l', a'')\,\mathbf{I}\,[k, b, k', b', k'']$$

$$\Updownarrow$$

$$\begin{cases} a'' = ak + b, \\ a' = a^2k + b' + 2ab, \\ k'' = ka^3 + l, \\ k' = k^2a^3 + l' - kl. \end{cases}$$

It was shown by J. Tits [10] that $H(q)$ admits a polarity if and only if q is an odd power of 3. In this case, a flag (a *flag* is an incident point-line pair) mapped onto itself by the polarity is called an *absolute flag*. The points and the lines of the absolute flags are called the *absolute points* and *absolute lines* respectively. It is also known that the set of absolute points forms an *ovoid* in the corresponding generalized hexagon, i.e. a set of $q^3 + 1$ such that every other point is collinear with exactly one of them. Now, all polarities of $H(q)$ are equivalent (see J. Tits [10], also proved by V. De Smet and H. Van Maldeghem [7]) and the corresponding ovoid is called the *Ree-Tits unital*. The automorphism group of the Ree-Tits unital inside $H(q)$ is the simple Ree group ${}^2G_2(q)$. Every involution in ${}^2G_2(q)$ fixes exactly $q + 1$ points in the Ree-Tits unital and these are defined to constitute the blocks of the unital. In this paper, we will regard these unitals as sets of absolute flags, absolute points and absolute lines. This will unable us to talk about the points, lines and flags of the unitals and also about the points, lines or flags of blocks of the unitals. In V. De Smet and H. Van Maldeghem [7], it is shown that the following set of points is the set of points of some fixed Ree-Tits unital $\mathcal{U}_{RT}(q)$ (putting $s = 3^{h+1}$ and $3^{2h+1} = q$)

$$\{(a, a''^s - a^{3+s}, a', a^{3+2.s} + a'^s + a^s a''^s, a'')\|a, a', a'' \in GF(3^{2h+1})\} \cup \{(\infty)\}.$$

The corresponding polarity interchanges the point (a, l^s, a', l'^s, a'') with the line $[a^s, l, a'^s, l', a''^s]$ (the action on the other points and lines is obtained by restricting these coordinates) and one can write down the set of lines of $\mathcal{U}_{RT}(q)$ easily. In particular, $[\infty]$ is an absolute line and all other absolute lines have five coordinates, namely

$$[a^s, a'' - a^{1+s}, a'^s, a^{2+s} + a' + aa'', a''^s].$$

If we denote by $P(a, a', a'')$ the point $(a, a''^s - a^{3+s}, a', a^{3+2.s} + a'^s + a^s a''^s, a'')$ of $\mathcal{U}_{RT}(q)$, then it is shown by V. De Smet and H. Van Maldeghem [8] that the points of an arbitrary block $B(a, a')$, $a, a' \in GF(q)$, through (∞) of $\mathcal{U}_{RT}(q)$ are

$$\{P(a, a', A'') \| A'' \in GF(q)\} \cup \{(\infty)\}.$$

It is easily checked that the following map $\Psi_y(K, K', K'')$ defines a collineation of $H(q)$:

$$\begin{array}{llcl}
\Psi_y(K, K', K'') & : \qquad H(q) & \rightarrow & H(q) \\
& : \ (a, l, a', l', a'') & \mapsto & (a, yl - Ka^3 + K'', ya' + Ka^2, \\
& & & y^2 l' + K^2 a^3 + yKl + K', ya'' + Ka) \\
& : \ [k, b, k', b', k''] & \mapsto & [yk + K, yb, y^2 k' - K'' yk + K' - KK'', yb', yk'' + K''],
\end{array}$$

where $y, K, K', K'' \in GF(q)$, $y \neq 0$ and the action on the other elements of $H(q)$ is obtained by restricting the above action to the appropriate coordinates. Also, all these collineations form a group Ψ of order $q^3(q-1)$. It now takes an elementary computation to see that $\Psi_y(K, K', K'')$ stabilizes $\mathcal{U}_{RT}(q)$ if and only if $K = K' = K'' = 0$ and $y = 1$. Hence the orbit of $\mathcal{U}_{RT}(q)$ under Ψ has length $q^3(q-1)$.

The stabilizer in $G_2(q)$ of the Ree-Tits unital $\mathcal{U}_{RT}(q)$ is, as already mentioned, the group ${}^2G_2(q)$, which has order $q^3(q^3+1)(q-1)$. Noting that the order of $G_2(q)$ is $q^6(q^6-1)(q^2-1)$ (see for instance the $\mathbb{ATLAS}$ [4]), we derive from this the total number of Ree-Tits unitals in $H(q)$, namely

$$\frac{|G_2(q)|}{|R(q)|} = \frac{q^6 \cdot (q^6-1) \cdot (q^2-1)}{(q^3+1) \cdot q^3 \cdot (q-1)} = q^3 \cdot (q^3-1) \cdot (q+1).$$

The number n of Ree-Tits unitals in $H(q)$ which have a fixed incident point-line pair as a flag, is obtained by counting all triples $(x, L, \mathcal{U})$ with $x \in H(q)$, L a line of $H(q)$, $\mathcal{U}$ any Ree-Tits unital of $H(q)$, with x on L and (x, L) a flag of $\mathcal{U}$. We get

$$(q^3-1)q^3(q+1) \cdot (q^3+1) = (1+q)(1+q^2+q^4) \cdot (q+1) \cdot n \ ,$$

hence $n = q^3(q-1)$. This implies that an arbitrary Ree-Tits unital containing the flag $((\infty), [\infty])$ is of the form $\mathcal{U}_{RT}(q)^{\Psi_y(K,K',K'')}$.

3 Flag-intersection of Ree-Tits unitals in $H(q)$

We now investigate the intersection pattern on the flags of two distinct Ree-Tits unitals of $H(q)$.

Theorem 3.1
If $q = 3$, then two distinct Ree-Tits unitals of $H(q)$ have 0, 1 or $q+1$ flags in common.
If $q = 3^{2h+1}$, $h \geq 1$, then two distinct Ree-Tits unitals of $H(q)$ have 0, 1, 2 or $q+1$ flags in common.

PROOF. We fix one Ree-Tits unital, namely $\mathcal{U}_{RT}(q)$. We investigate the flag-intersection of $\mathcal{U}_{RT}(q)$ with all other Ree-Tits unitals of $H(q)$. This will be done in two steps. First we suppose they have at least one flag in common, say $((\infty), [\infty])$. In the second step, we will investigate if there exist Ree-Tits unitals that have no flag in common with $\mathcal{U}_{RT}(q)$.

$\boxed{\textbf{First step}}$

By the arguments in section 2, a general Ree-Tits unital $\mathcal{U}_{RT}(q)^{\Psi_y(K,K',K'')}$ of $H(q)$ which has $((\infty), [\infty])$ as a flag, has besides this fixed flag the set of the following flags:

$$\{(\Big(a, ya''^s - ya^{3+s} + K'' - Ka^3, ya' + Ka^2,$$

$$y^2a^{3+2s} + y^2a'^s + y^2a^sa''^s + K' + K^2a^3 + Kya''^s - Kya^{s+3}, ya'' + Ka\Big),$$

$$\Big[ya^s + K, ya'' - yaa^s, y^2a'^s + K' - K''ya^s - KK'', ya^sa^2 + ya' + yaa'', ya''^s + K''\Big])$$

$$\| \; a, a', a'' \in GF(q)\}$$

with $K, K', K'' \in GF(q)$ and $y \in GF(q) \setminus \{0\}$.

Recall that the set of flags of $\mathcal{U}_{RT}(q)$ is given by

$$\{((\infty), [\infty]\} \cup \{(\Big(b, b''^s - b^{3+s}, b', b^{3+2s} + b'^s + b^sb''^s, b''\Big),$$

$$\left[b^s, b'' - b^s b, b'^s, b^s b^2 + b' + bb'', b''^s\right]) \mid\mid b, b', b'' \in GF(q)\} \ .$$

So the flags lying in both $\mathcal{U}_{RT}(q)$ and $\mathcal{U}_{RT}(q)^{\Psi_y(K,K',K'')}$ are found by solving the following system of equations (of course, the flag $((\infty), [\infty])$ is automatically in the intersection):

$$(A) \quad \begin{cases} b &= a & (1) \\ b' &= ya' + Ka^2 & (2) \\ b'' &= ya'' + Ka & (3) \\ b^s &= ya^s + K & (1)' \\ b'^s &= y^2 a'^s + K' - K'' y a^s - KK'' & (2)' \\ b''^s &= ya''^s + K'' & (3)' \end{cases}$$

and

$$(B) \quad \begin{cases} b''^s - b^{3+s} &= ya''^s - ya^{3+s} + K'' - Ka^3 \\ b^{3+2s} + b'^s + b^s b''^s &= y^2 a^{3+2s} + y^2 a'^s + y^2 a^s a''^s + K' + \\ & \quad K^2 a^3 + Kya''^s - Kya^{3+s} \\ b'' - b^s b &= ya'' - yaa^s \\ b^s b^2 + b' + bb'' &= ya^s a^2 + ya' + yaa'' \ . \end{cases}$$

We solve this system of equations as follows (assuming there is a solution).

- Substituting (1) in $(1)'$, we get two possibilities :

  case I : $\begin{cases} y &= 1 \\ K &= 0 \end{cases}$ **case II :** $\begin{cases} y &\neq 1 \\ a^s &= -\frac{K}{y-1} \end{cases}$.

- Substituting (2) in $(2)'$, we get the following conditions :

  - **case I :**
    $a'^s + K' - K'' a^s = a'^s$
    If $K'' = 0$, then $K' = 0$ and we have the unital $\mathcal{U}_{RT}(q)^{\Psi_1(0,0,0)} = \mathcal{U}_{RT}(q)$.
    If $K'' \neq 0$, we have that $a^s = \frac{K'}{K''}$.
  - **case II :**

$$y^2 a'^s + K' + K'' K \frac{y}{y-1} - KK'' = y^s a'^s + K^s a^{2s}$$

or

$$a'^s = (\frac{K^2K^s}{(y-1)^2} - \frac{KK''}{y-1} - K') \cdot (y^2 - y^s)^{-1} \ .$$

- If we substitute (3) in (3)$'$, we get :

$$ya''^s + K'' = y^s a''^s + K^s a^s \ .$$

  - **case I :**
    We have $K = 0, y = 1$ and therefore $K'' = 0$. Hence $\mathcal{U}_{RT}(q)^{\Psi_y(K,K',K'')} = \mathcal{U}_{RT}(q)$. So **case I** is trivial.
  - **case II :**
    case II(i). If $y \neq y^s$, then

$$a''^s = (K'' + \frac{KK^s}{y-1}) \cdot (y^s - y)^{-1} \ .$$

    case II(ii). If $y = -1$, then

$$\begin{cases} K'' = -KK^s \\ a'' \in GF(q) \ . \end{cases}$$

- The set of equations (B) is now satisfied in both **cases II(i) and II(ii)**.

Conclusion

- **case II(ii) :**
  The q^2 Ree-Tits unitals $\mathcal{U}_{RT}(q)^{\Psi_y(K,K',K'')}$ with

$$\begin{cases} y = -1 \\ K'' = -KK^{3^{h+1}} \\ K, K' \in GF(q) \end{cases}$$

  have $q+1$ flags in common with $\mathcal{U}_{RT}(q)$ namely

$$\text{the flag} \ \ ((\infty), [\infty]) \ \ \text{and the set of flags}$$

$$\{(\Big(-K^{3^h}, -a''^{3^{h+1}} + KK^{3^{h+1}}, -K'^{3^h}, -K' + Ka''^{3^{h+1}}, -a'' - KK^{3^h}\Big),$$

$$\Big[-K, -a''+KK^{3^h}, -K', -K'^{3^h}+a''K^{3^h}, -a''^{3^{h+1}}-KK^{3^{h+1}}\Big]) \mid\mid a'' \in GF(q)\} \ .$$

- **case II(i) :**
The $q^3 \cdot (q-3)$ Ree-Tits unitals $\mathcal{U}_{RT}(q)^{\Psi_y(K,K',K'')}$ with

$$\begin{cases} y \notin GF(3) \\ K, K'', K' \in GF(q) \end{cases}$$

have 2 flags in common with $\mathcal{U}_{RT}(q)$ namely

$$((\infty), [\infty]) \quad \text{and}$$

$$\Bigg(\bigg(-\frac{K^{3^h}}{(y-1)^{3^h}}, \frac{y^{3^{h+1}}K''}{y^{3^{h+1}}-y} + \frac{y^{3^{h+1}}KK^{3^{h+1}}}{(y-1)^{3^{h+1}} \cdot (y^{3^{h+1}}-y)}, -\frac{yK^{3^h}K''^{3^h}}{(y^{2\cdot 3^h}-y)\cdot(y-1)^{3^h}}$$
$$-\frac{yK'^{3^h}}{y^{2\cdot 3^h}-y} + \frac{y^{2\cdot 3^h}KK^{2\cdot 3^h}}{(y^{2\cdot 3^h}-y)(y-1)^{2\cdot 3^h}}, \frac{y^{3^{h+1}}(y^{3^{h+1}}+y)K^2K^{3^{h+1}}}{(y-1)^{3^{h+1}}\cdot(y^2-y^{3^{h+1}})\cdot(y^{3^{h+1}}-y)}$$
$$-\frac{y^{1+s}KK''}{(y^{3^{h+1}}-y)\cdot(y^2-y^{3^{h+1}})} - \frac{y^{3^{h+1}}K'}{y^2-y^{3^{h+1}}}, \frac{yK''^{3^h}}{y-y^{3^h}} + \frac{y^{3^h}KK^{3^h}}{(y-1)^{3^h}\cdot(y-y^{3^h})}\bigg),$$
$$\bigg[-\frac{K}{y-1}, \frac{yK''^{3^h}}{y-y^{3^h}} + \frac{yKK^{3^h}}{(y-1)\cdot(y-y^{3^h})}, -\frac{y^{3^{h+1}}KK''}{(y^2-y^{3^{h+1}})\cdot(y-1)} - \frac{y^{3^{h+1}}K'}{y^2-y^{3^{h+1}}}$$
$$+\frac{y^2K^{3^{h+1}}K^2}{(y^2-y^{3^{h+1}})(y-1)^2}, \frac{y(y+y^{3^h})K^{2\cdot 3^h}K}{(y-1)\cdot(y^{2\cdot 3^h}-y)\cdot(y-y^{3^h})} - \frac{y^{3^h+1}K^{3^h}K''^{3^h}}{(y-y^{3^h})\cdot(y^{2\cdot 3^h}-y)}$$
$$-\frac{yK'^{3^h}}{y^{2\cdot 3^h}-y}, \frac{y^{3^{h+1}}K''}{y^{3^{h+1}}-y} + \frac{yKK^{3^{h+1}}}{(y-1)\cdot(y^{3^{h+1}}-y)}\bigg]\Bigg) .$$

- The q^3-1 Ree-Tits unitals $\mathcal{U}_{RT}(q)^{\Psi_y(K,K',K'')}$ with

$$\begin{cases} y = 1 \\ (K, K'', K') \in GF(q)^3 \setminus (0,0,0) \end{cases}$$

have only the flag $((\infty), [\infty])$ in common with $\mathcal{U}_{RT}(q)$.

- The $q^2 \cdot (q-1)$ Ree-Tits unitals $\mathcal{U}_{RT}(q)^{\Psi_y(K,K',K'')}$ with

$$\begin{cases} y = -1 \\ K, K' \in GF(q) \\ K'' \neq -K \cdot K^{3^{h+1}} \end{cases}$$

have only the flag $((\infty), [\infty])$ in common with $\mathcal{U}_{RT}(q)$.

$\boxed{\textbf{Second step}}$

Now we will investigate the existence of Ree-Tits unitals which have no flags in common with $\mathcal{U}_{RT}(q)$.
Recall that the total number of Ree-Tits unitals in $H(q)$ is $q^3 \cdot (q^3-1) \cdot (q+1)$.

In order to count the number r_i of Ree-Tits unitals which have $i = 1, 2$ or $q+1$ flags in common with $\mathcal{U}_{RT}(q)$, we count in two different ways the couples $(f,\mathcal{U})$ with f a flag of $\mathcal{U}_{RT}(q)$, $\mathcal{U}$ a Ree-Tits unital of $H(q)$ which has 1, respectively 2 or $q+1$, flags in common with $\mathcal{U}_{RT}(q)$ and f a flag of $\mathcal{U}$. We obtain :

(1) $(q^3+1) \cdot [q^3 - 1 + q^2 \cdot (q-1)] = r_1$.

Hence the number of Ree-Tits unitals of $H(q)$ which have 1 flag in common with $\mathcal{U}_{RT}(q)$ is

$$2q^6 - q^5 + q^3 - q^2 - 1.$$

(2) $(q^3+1) \cdot [q^3 \cdot (q-3)] = r_2 \cdot 2$.

Hence the number of Ree-Tits unitals of $H(q)$ which have 2 flags in common with $\mathcal{U}_{RT}(q)$ is

$$\frac{1}{2} \cdot (q^7 - 3q^6 + q^4 - 3q^3).$$

Remark that if $q = 3$, there are no Ree-Tits unitals which have 2 flags in common with $\mathcal{U}_{RT}(q)$.

(3) $(q^3+1) \cdot q^2 = r_{q+1} \cdot (q+1)$.

Hence the number of Ree-Tits unitals of $H(q)$ which have $q+1$ flags in common with $\mathcal{U}_{RT}(q)$ is

$$q^4 - q^3 + q^2.$$

So we have $\frac{1}{2} \cdot (q^7 + q^6 - 2q^5 + 3q^4 - 3q^3 - 2)$ Ree-Tits unitals which have at least one flag in common with $\mathcal{U}_{RT}(q)$. But this number is less than the total number of Ree-Tits unitals in $H(q)$. In fact, there are exactly $\frac{1}{2}(q^7 + q^6 + 2q^5 - 5q^4 + q^3)$ unitals missing.
This proves the theorem. $\square$

Remark. From the shape of the intersection, it is clear that whenever two Ree-Tits unitals meet in $q+1$ flags, the points of these flags constitute a block in both unitals. Also, since there are exactly q^2 blocks through one point in any Ree-Tits unital, there are exactly two Ree-Tits unitals $\mathcal{U}_1$ and $\mathcal{U}_2$ containing the flags corresponding to any block B of any Ree-Tits unital $\mathcal{U}_1$. Group-theoretically, this means that every maximal subgroup $2 \times L(q)$ inside a copy ${}^2G_2(q)$ in $G_2(q)$ is in exactly two copies of ${}^2G_2(q)$ in $G_2(q)$ and these copies are interchanged by an involution σ. It is easy to see that the group generated by σ and $2 \times L(q)$ is isomorphic to $2 \times PGL_2(q)$. Since the stabilizer of B either fixes $\mathcal{U}_1$ and $\mathcal{U}_2$ or interchanges them, the normalizer of the intersection of the corresponding Ree groups is isomorphic to $2 \times PGL_2(q)$.

4 Geometric construction of a twisted field projective plane

In this section, by a Ree-Tits unital we always mean a Ree-Tits unital of $H(q), q = 3^{2h+1}$, which contains the flag $((\infty), [\infty])$. We will show that, using a certain class of such Ree-Tits unitals, we can construct a non-desarguesian plane of order q, if $q > 3$.

We first note that an elementary calculation proves

$$(\mathcal{U}_{RT}(q)^{\Psi_{y_1}(K_1,K_1',K_1'')})^{\Psi_{y_2}(K_2,K_2',K_2'')} = U_{RT}(q)^{\Psi_{y_1y_2}(y_2K_1+K_2,y_2^2K_1'+y_2K_1''K_2+K_2',y_2K_1''+K_2'')}.$$

Lemma 4.1 *Consider all Ree-Tits unitals* $\mathcal{U}_{RT}(q)^{\Psi_y(K,K'',K')}$ *in* $H(q)$ *(through the flag* $((\infty), [\infty])$*). On this set of unitals, the relation* R *defined as follows :*

$$\mathcal{U}_{RT}(q)^{\Psi_{y_1}(K_1,K_1',K_1'')} \ R \ \mathcal{U}_{RT}(q)^{\Psi_{y_2}(K_2,K_2',K_2'')}$$

$$\Updownarrow$$

these unitals intersect in either 1 *or* $q+1$ *flags,*

is an equivalence relation with $\frac{q-1}{2}$ *equivalence classes of size* $2q^3$*.*

PROOF. For $q = 3$, two such distinct Ree-Tits unitals have either 1 or $q+1$ flags in common (see Theorem 3.1), so there is only one equivalence class.

Let $q > 3$. We have to prove

$$\mathcal{U}_1 \; R \, \mathcal{U}_2 \; \text{ and } \; \mathcal{U}_2 \; R \, \mathcal{U}_3 \; \Rightarrow \; \mathcal{U}_1 \; R \, \mathcal{U}_3.$$

By transitivity of $G_2(q)$ on the Ree-Tits unitals of $H(q)$ which contain $((\infty), [\infty])$, we can take for $\mathcal{U}_1$ the unital $\mathcal{U}_{RT}(q)$. In the previous section we showed that the unitals which have a flag-intersection of size 1 or $q+1$ with $\mathcal{U}_1$, are the unitals $\mathcal{U}_{RT}(q)^{\Psi_y(K,K',K'')}$, with $y \in GF(3) \setminus \{0\}, K, K', K'' \in GF(q)$. So $\mathcal{U}_2 = \mathcal{U}_{RT}(q)^{\Psi_{y_2}(K_2,K_2',K_2'')}$ must be such a unital. The unitals $\mathcal{U}_{RT}(q)^{\Psi_y(K,K',K'')}$ which have a flag-intersection of size 1 or $q+1$ with $\mathcal{U}_2$, are the images under $\Psi_{y_2}(K_2, K_2', K_2'')$ of the unitals which have 1 or $q+1$ flags in common with $\mathcal{U}_1$. We find

$$\mathcal{U}_3 = \mathcal{U}_1^{\Psi_{y_3}(K_3,K_3',K_3'')} \in \{\mathcal{U}_1^{\Psi_{y_2 \cdot y}(K,K',K'')} \,||\, y = \pm 1, \; K, K'', K' \in GF(q)\}, \text{ with } y_2 = \pm 1.$$

It follows that $y_3 = \pm 1$, so $\mathcal{U}_3$ has 1 or $q+1$ flags in common with $\mathcal{U}_1$. Hence $\mathcal{U}_1 \; R \, \mathcal{U}_3$.

By the above, we have

$$\forall K, K', K'', M, M', M'' \in GF(q) \; : \; \mathcal{U}_{RT}(q)^{\Psi_1(K,K',K'')} \, R \, \mathcal{U}_{RT}(q)^{\Psi_z(M,M',M'')} \; \Leftrightarrow \; z = \pm 1 \, .$$

Hence if we apply the collineation $\Psi_y(0, 0, 0)$, we get

$$\forall K, K', K'', M, M', M'' \in GF(q) \; : \mathcal{U}_{RT}(q)^{\Psi_y(K,K',K'')} \; R \; \mathcal{U}_{RT}(q)^{\Psi_z(M,M',M'')} \; \Leftrightarrow \; y = \pm z.$$

So there are $\frac{q-1}{2}$ equivalence classes. $\square$

Lemma 4.2 *Let $\mathcal{U}$ be any Ree-Tits unital and let $\mathcal{R}$ be the set of the q^2 Ree-Tits unitals which have $q+1$ flags in common with $\mathcal{U}$. Then there are exactly $q-1$ other Ree-Tits unitals which all have a flag-intersection of size $q+1$ with every member of $\mathcal{R}$.*

PROOF. We can assume again that $\mathcal{U} = \mathcal{U}_{RT}(q)$. We know from the proof of theorem 3.1 that

$$\mathcal{R} = \{\mathcal{U}^{\Psi_2(K,K',-K^{1+s})} \| K, K' \in GF(q)\}.$$

Let $\mathcal{U}^{\Psi_z(M,M',M'')}$ be any Ree-Tits unital. Applying $\Psi_z(M, M', M'')$, we find that the unital $\mathcal{U}^{\Psi_z(M,M',M'')}$ has a flag-intersection of size $q+1$ with the unitals

$$\mathcal{U}^{\Psi_{-z}(zN+M, z^2N'-zN^{1+s}M+M', -zN^{1+s}+M'')}, \;\; \forall N, N' \in GF(q) \; . \qquad (1)$$

Hence $\mathcal{U}^{\Psi_z(M,M',M'')}$ has a flag-intersection of size $q+1$ with every member of $\mathcal{R}$ if and only if every unital (1) coincides with a member of $\mathcal{R}$. This is equivalent with the conditions

$$\begin{cases} -z = -1 \\ -zN^{1+s} + M'' = -(zN+M)^{1+s}, \ \forall N \in GF(q) \ . \end{cases}$$

Hence $z = 1$ and $M = M'' = 0$. It follows that the q Ree-Tits unitals $\mathcal{U}^{\Psi_1(0,M',0)}$, with $M' \in GF(q)$, have a flag-intersection of size $q+1$ with the q^2 Ree-Tits unitals of $\mathcal{R}$ and these are the only ones. $\square$

We call a set of q Ree-Tits unitals (still through the flag $((\infty),[\infty])$) having a flag-intersection of size $q+1$ with q^2 other Ree-Tits unitals a *sleeper*. Given a Ree-Tits unital $\mathcal{U}$, there exists, by the preceding lemma, a unique sleeper containing $\mathcal{U}$. Also, this sleeper is a subset of the equivalence class under the relation R containing $\mathcal{U}$. Hence every equivalence class modulo R contains exactly $2 \cdot q^2$ sleepers. We now show that all these can be seen as the points and lines of a helicopter plane (i.e. a net of order q and degree $q+1$, or in other terms, an affine plane with one parallel class eliminated). Note that a helicopter plane can be extended uniquely to a projective plane. The order of that projective plane is by definition the order of the original helicopter plane.

First we need a lemma.

Lemma 4.3 *The equation*

$$a \cdot x + a^{s/3} \cdot x^s = b$$

has exactly 1 solution in x for every $a, b \in GF(q)$, $a \neq 0$ (and $q = 3^{2h+1}$, $s = 3^{h+1}$).

PROOF. By substituting $x = a^{s/3} \cdot y$ and $b = a^{1+s/3}c$ we obtain the equivalent equation

$$y + y^s = c.$$

Since $y \mapsto y + y^s$ is an additive map, it suffices to show that $y + y^s = 0$ implies $y = 0$. But if $y^s = -y$, then $y^3 = y^{s^2} = -y^s$ (since s is odd), hence $y = y^3$, implying either $y = 0$, or $y = \pm 1$. The latter contradicts $y^s = -y$. $\square$

Theorem 4.4 *The graph $\Delta(q)$ with as set of vertices the set of sleepers in one fixed equivalence class modulo R and with edges the pairs of sleepers the members of which have $q+1$ flags in common with every member of the other sleeper, is the incidence graph of a helicopter plane of order q and its isomorphism class is independent of the choice of the equivalence class modulo R.*

PROOF. By the transitivity, we may assume that the equivalence class modulo R consists of all Ree-Tits unitals of the form $\mathcal{U}_{RT}(q)^{\Psi_{\pm 1}(K,K',K'')}$.

By the proof of lemma 4.2 the set $\{\mathcal{U}_{RT}(q)^{\Psi_1(0,K',0)} \| K' \in GF(q)\}$ is a sleeper. Applying $\Psi_{\pm 1}(K, 0, K'')$ we see that a general sleeper looks like

$$S^{\pm}_{K,L} = \{\mathcal{U}^{\Psi_{\pm 1}(K,x,L)} \| x \in GF(q)\}.$$

Now we construct a projective plane as follows. Define the geometry $\Gamma(q) = (\mathcal{P}, \mathcal{L}, I)$ where

$\mathcal{P}$ **:** (1) the sleepers $S^{+}_{K,L}$,
(2) elements P_x, with $x \in GF(q)$
(3) a point P_∞,

$\mathcal{L}$ **:** (1) the sleepers $S^{-}_{K,L}$,
(2) elements L_y, with $y \in GF(q)$,
(3) a line L_∞,

I **:** $P_x \; I \; S^{-}_{x,L}$, $\quad \forall L \in GF(q)$, $\quad P_x \; I \; L_\infty$,
$P_\infty \; I \; L_k$, $\quad \forall k \in GF(q)$, $\quad P_\infty \; I \; L_\infty$,
$S^{+}_{K,L} \; I \; L_K$, $\quad \forall L \in GF(q)$,
$S^{+}_{K,L} \; I \; S^{-}_{K',L'}$ if and only if they form an edge in $\Delta(q)$.

It is clear that the theorem will be proved if we show that $\Gamma(q)$ is a projective plane.

There are q^2+q+1 points and q^2+q+1 lines and the group $\{\Psi_{\pm 1}(K, 0, K'') \| K, K'' \in GF(q)\}$ acts transitively on the set of points and lines of type (1). Hence it suffices to show that there is a unique line incident with every two points.

Take two different points of type (1), say $S^{+}_{K_1,L_1}$ and $S^{+}_{K_2,L_2}$. The lines incident with $S^{+}_{K_1,L_1}$ are L_{K_1} and all lines $S^{-}_{K+K_1,-K\cdot K^s+L_1}$, for $K \in GF(q)$.

If $K_2 = K_1$, then the line L_{K_1} is also incident with $S^+_{K_1,L_2}$. Since in this case $L_2 \neq L_1$, there is no other line incident with these two points.
Suppose $K_2 \neq K_1$. Then L_{K_1} is not incident with $S^+_{K_2,L_2}$.
Consider the lines $S^-_{K+K_1,-K\cdot K^s+L_1}$, with $K \in GF(q)$. Such a line is incident with all points $S^+_{-K'+K+K_1,K'\cdot K'^s-K\cdot K^s+L_1}$, $K' \in GF(q)$. So $S^-_{K+K_1,-K\cdot K^s+L_1}$ is incident with the point $S^+_{K_2,L_2}$ if and only there exists K' for which :

$$\begin{cases} K_2 &= -K' + K + K_1 \\ L_2 &= K' \cdot K'^s - K \cdot K^s + L_1 \ . \end{cases}$$

Sustituting $K = K_2 + K' - K_1$ in the second equation, we get the condition

$$\begin{aligned} L_2 - L_1 &= K' \cdot K'^s - K_2 \cdot K_2^s - K_2 \cdot K'^s + K_2 \cdot K_1^s - K' \cdot K_2^s - \\ &\quad K' \cdot K'^s + K' \cdot K_1^s + K_1 \cdot K_2^s + K_1 \cdot K'^s - K_1 \cdot K_1^s \\ &= (K_2 - K_1) \cdot (K_1 - K')^s + (K_2 - K_1)^s \cdot (K_1 - K') + \\ &\quad K_1 \cdot K_1^s - K_2 \cdot K_2^s \ . \end{aligned}$$

This equation can be written as

$$a \cdot x + a^{3^h} \cdot x^{3^{h+1}} = b$$

with $a = (K_2 - K_1)^s \neq 0$ and $x = K_1 - K'$. We know by the previous lemma that this equation has exactly one solution in x. So we have exactly one solution for K' and therefore one solution for K. So there is exactly one line of type (1) incident with $S^+_{K_1,L_1}$ and $S^+_{K_2,L_2}$.

The unique line incident with a point $S^+_{K,L}$ of type (1) and a point P_x of type (2), is the line $S^-_{x,-(x-K_1)\cdot(x-K_1)^s+L_1}$.

The line L_K is the unique line incident with $S^+_{K,L}$ and P_∞.

There is only one line incident with a point of type (2) and the point P_∞, namely L_∞.

So we have indeed a projective plane of order q. $\square$

In the next section, we show that $\Gamma(q)$ is non-desarguesian whenever $q > 3$.

5 $\Gamma(q)$ is non-desarguesian for $q > 3$

In order to show that the plane $\Gamma(q)$ is non-desarguesian for $q > 3$, we use the concept of coordinatization as described by D. R. Hughes and F. C. Piper

[6]. So we coordinatize $\Gamma(q)$ by a ternary field (R, T). We use the same notation as in [6]. In particular, we use R for the ring (this cannot cause confusion with the relation R of the previous sections since we do not need that relation anymore) and we use the symbol I for a certain point (and we write "is incident with" always in words).

The set R of q symbols will be $GF(q)$. We set $X = P_0$, $Y = P_\infty$ and $O = S^+_{0,0}$. The point $S^+_{1,2}$ is not incident with any side of the chosen triangle and so we can put $I = S^+_{1,2}$. One can calculate easily that the point $A = XI \cap OY$ is the sleeper $S^+_{0,1}$. Similarly, $B = YI \cap OX$ is $S^+_{1,1}$ and $J = AB \cap XY$ is P_2. We assign to the point $S^+_{0,c}$ the coordinates $(0, c)$. It is now a long but elementary job to calculate the coordinates of all elements of $\Gamma(q)$. One obtains

elements of $\Gamma(q)$	coordinates
P_∞	(∞)
P_m	$(m + m^s)$
$S^+_{x,y}$	$(-x - x^s, -x^{1+s} + y)$
L_∞	$[\infty]$
L_x	$[-x - x^s]$
$S^-_{m,k}$	$[m + m^s, m^{1+s} + k]$

We define a function ϕ as follows :

$$\forall x \in GF(q) \ : \ \phi(x) = y \ \Leftrightarrow \ y + y^s = x.$$

From lemma 4.3 we know that ϕ is a bijection. This will enable us to calculate the ternary operation T, which is, following [6], defined as follows.

$T(a, b, c) = k$ if and only if (b, c) is incident with $[a, k]$, $a, b, c, k \in GF(q)$.

Thus $(0, k)$ is the intersection of OY with the line joining (a) to (b, c) so that, given a, b, c, the value of k is uniquely determined.

After some more computations, one finds

$$\boxed{T(a, b, c) = k = -\phi(a) \cdot \phi(b)^s - \phi(b) \cdot \phi(a)^s + c.}$$

This defines a twisted field in the sense of A. A. Albert [1], see also P. Dembowski [5].

It is convenient to define a (new) addition and multiplication in $R = GF(q)$ as follows.

$$\forall a, b \in GF(q) \ : \ \begin{cases} a \oplus b &= T(1, a, b) \\ a \otimes b &= T(a, b, 0). \end{cases}$$

With the definition of the ternary operation we obtain

$$a \oplus b = \phi(a)^s + \phi(a) + b = a + b$$

$$a \otimes b = -\phi(a) \cdot \phi(b)^s - \phi(b) \cdot \phi(a)^s.$$

$\boxed{\textbf{CASE } q = 3}$

Here

$$\forall x \ \in GF(3) \ : \ \phi(x) = -x.$$

It follows that $a \otimes b = -\phi(a) \cdot \phi(b) - \phi(b) \cdot \phi(a) = a \cdot b$.

So we have the ordinary addition and multiplication. Hence $\Gamma(3)$ is desarguesian.

$\boxed{\textbf{CASE } q > 3}$

It is again an elementary exercise to show that $R, \oplus$ is an abelian group (this is trivial), $R, \otimes$ is a commutative loop with identity and both the left and right distributive laws are satisfied.

We now show that the multiplicative associative law is never satisfied.

First we will prove that

$$\boxed{\phi(x) = -x - x^3 - \cdots - x^{3^h} + x^{3^{h+1}} + \cdots + x^{3^{2h}}}$$

Recall that the definition of ϕ is the following: $\phi(x) = y \Leftrightarrow y + y^s = x$. Put

$$y = -x - x^3 - \cdots - x^{3^h} + x^{3^{h+1}} + \cdots + x^{3^{2h}} \ ;$$

then

$$y^s = -x + x^3 + \cdots + x^{3^h} - x^{3^{h+1}} - \cdots - x^{3^{2h}} \ .$$

It follows that $y + y^s = x$.

Next, we show that

$$\boxed{a \otimes b = -\phi(a) \cdot \phi(b) - b \cdot \phi(a) - a \cdot \phi(b)}$$

Indeed, $a \otimes b$

$= -\phi(a) \cdot \phi(b)^s - \phi(b) \cdot \phi(a)^s$

$$= \left[a + a^3 + \cdots + a^{3^h} - a^{3^{h+1}} - \cdots - a^{3^{2h}}\right] \cdot \left[-b - b^3 - \cdots - b^{3^h} + b^{3^{h+1}} + \cdots + b^{3^{2h}}\right]^{3^{h+1}}$$
$$+ \left[b + b^3 + \cdots + b^{3^h} - b^{3^{h+1}} - \cdots - b^{3^{2h}}\right] \cdot \left[-a - a^3 - \cdots - a^{3^h} + a^{3^{h+1}} + \cdots + a^{3^{2h}}\right]^{3^{h+1}}$$

$$= \left[a + a^3 + \cdots + a^{3^h} - a^{3^{h+1}} - \cdots - a^{3^{2h}}\right] \cdot \left[-b + b^3 + \cdots + b^{3^h} - b^{3^{h+1}} + \cdots - b^{3^{2h}}\right]$$
$$+ \left[b + b^3 + \cdots + b^{3^h} - b^{3^{h+1}} - \cdots - b^{3^{2h}}\right] \cdot \left[-a + a^3 + \cdots + a^{3^h} - a^{3^{h+1}} - \cdots - a^{3^{2h}}\right]$$

$= \phi(a) \cdot \phi(b) + \phi(a) \cdot \phi(b) - b \cdot \phi(a) - a \cdot \phi(b)$

$= -\phi(a) \cdot \phi(b) - b \cdot \phi(a) - a \cdot \phi(a)$.

Finally, we prove that

$$\boxed{\phi(a^{3^h}) = -\phi(a) + a^{3^h}}$$

This is true since

$$\begin{aligned} \phi(a) &= -a - a^3 - \cdots - a^{3^h} + a^{3^{h+1}} + \cdots + a^{3^{2h}} \\ \phi(a^{3^h}) &= -a^{3^h} - a^{3^{h+1}} - \cdots - a^{3^{2h}} + a + a^3 + \cdots + a^{3^{h-1}}. \end{aligned}$$

To prove the non-associativity, take $b = a$ and $c = a^{3^h}$.

$$\begin{aligned} (a \otimes a) \otimes a^{3^h} &= -\phi\left[-\phi(a)^2 + a \cdot \phi(a)\right] \cdot \left(-\phi(a) + a^{3^h}\right) \\ &\quad -a^{3^h} \cdot \phi\left[-\phi(a)^2 + a \cdot \phi(a)\right] \\ &\quad -\left[-\phi(a)^2 + a \cdot \phi(a)\right] \cdot \left(-\phi(a) + a^{3^h}\right) \\ &= -\phi(a) \cdot \phi\left(\phi(a)^2\right) + a^{3^h} \phi\left(\phi(a)^2\right) + \phi\left(a \cdot \phi(a)\right) \cdot \phi(a) \\ &\quad -\phi\left(a \cdot \phi(a)\right) \cdot a^{3^h} + a^{3^h} \cdot \phi\left(\phi(a)^2\right) - a^{3^h} \cdot \phi\left(a \cdot \phi(a)\right) \\ &\quad -\phi(a)^3 + \phi(a)^2 \cdot a^{3^h} + a \cdot \phi(a)^2 - a \cdot \phi(a) \cdot a^{3^h} \end{aligned}$$

and

$$\begin{aligned} a \otimes (a \otimes a^{3^h}) &= -\phi(a) \cdot \phi\left[\phi(a)^2 - a^{3^h} \cdot \phi(a) - \phi(a) \cdot a^{3^h} + a \cdot \phi(a) - a \cdot a^{3^h}\right] \\ &\quad - \left[\phi(a)^2 - a^{3^h} \cdot \phi(a) - \phi(a) \cdot a^{3^h} + a \cdot \phi(a) - a \cdot a^{3^h}\right] \cdot \phi(a) \\ &\quad -a \cdot \phi\left[\phi(a)^2 - a^{3^h} \cdot \phi(a) - \phi(a) \cdot a^{3^h} + a \cdot \phi(a) - a \cdot a^{3^h}\right] . \end{aligned}$$

If $\otimes$ is associative then the following must hold for all a in $GF(q)$:

$$-\phi\left(\phi(a)^2\right) \cdot [a - a^{3^h}] + \phi\left(\phi(a) \cdot a^{3^h}\right) \cdot [-a - \phi(a)]$$

$$+\phi\left(a \cdot \phi(a)\right) \cdot [-a + \phi(a) - a^{3^h}] + \phi(a \cdot a^{3^h}) \cdot [a + \phi(a)]$$

$$+\phi(a) \cdot a^{3^h} \cdot [\phi(a) - a] + a \cdot \phi(a)^2 = 0.$$

Now, let $\phi(a) = x$, so $x + x^{3^{h+1}} = a$. We obtain

$$-\phi(x^2)\cdot[x^{3^{h+1}} - x^{3^h}] + \phi(x^2 + x^{3^h+1})\cdot[x - x^{3^{h+1}}] + \phi(x^2 + x^{3^{h+1}+1})\cdot[-x - x^{3^{h+1}} - x^{3^h}]$$
$$+\phi(x^{3^h+1} + x^2 + x^{3^h+3^{h+1}} + x^{3^{h+1}+1})\cdot[-x + x^{3^{h+1}}] + (x^2 + x^{3^h+1})\cdot(-x^{3^{h+1}}) + (x + x^{3^{h+1}})\cdot x^2$$

or

$$(-x + x^{3^{h+1}}) \cdot \phi(x^2 + x^{3^h+3^{h+1}}) + (x - x^{3^h}) \cdot \phi(x^{1+3^{h+1}}) + x^3 - x^{1+3^h+3^{h+1}} = 0.$$

If this must hold for all x in $GF(q)$, then replacing $x^{3^{2h+1}}$ by x, we must have a zero-identity.

Remark that if $h = 0$ and hence $q = 3$, then we get indeed a zero-identity.

If $h = 1$ and hence $q = 27$, then this equation is

$$(-x + x^9) \cdot \phi(x^2 + x^{3+9}) + (x - x^3) \cdot (x^{1+9}) + x^3 - x^{1+3+9} = 0 ,$$

or

$(-x + x^9) \cdot (-x^2 - x^{3+9} - x^{2\cdot 3} - x^{9+1} + x^{2\cdot 9} + x^{1+3}) + \quad (x - x^3) \cdot (-x^{1+9} - x^{3+1} + x^{9+3}) + x^3 - x^{1+3+9} = 0$.

It follows that this equation is not identical zero.

If $h > 1$, then we obtain a polynomial in x of degree $2 \cdot 3^{2h} + 3^{h+1}$ of which the coefficient of $x^{2\cdot 3^{2h}+3^{h+1}}$ is equal to 1.

We conclude that the $\otimes$ is not associative if $q > 3$. Hence $\Gamma(q)$ is desarguesian if and only if $q = 3$.

6 Polarities and ovals in $\Gamma(q)$

In this section, we write down a special property of $\Gamma(q)$ concerning ovals and polarities without detailed proof.

First notice that the maps $\Psi_1(0,0,K'')$ induce translations in $\Gamma(q)$ with center P_∞ and axis L_∞. The other translations are not induced by collineations of $H(q)$ (this can be checked by looking at the Sylow 3-subgroup of $G_2(q)_{((\infty),[\infty])}$).

Theorem 6.1 *The set of points* $\mathcal{A} = \{S^+_{K,0} \parallel K \in GF(q)\} \cup \{P_\infty\}$ *is an oval in the projective plane* $\Gamma(q)$. *It can be obtained in three different ways.*

(i) The elements of $\mathcal{A}$ *are precisely the absolute points of the polarity* $S^+_{K,L} \mapsto S^-_{K,-L}$, $S^-_{K,L} \mapsto S^+_{K,-L}$, $L_K \mapsto P_K$, $P_K \mapsto L_K$, $P_\infty \mapsto L_\infty$ *and* $L_\infty \mapsto P_\infty$.

(ii) The elements of $\mathcal{A}$ *distinct from* P_∞ *form the orbit of* $S^+_{0,0}$ *under the action of the group* $\{\Psi_1(x,0,0) \| x \in GF(q)\}$.

(iii) The elements of $\mathcal{A}$ *distinct from* P_∞ *are precisely those sleepers which contain a member containing a flag through the point* $(0,0,0,0,0)$ *of* $H(q)$.

PROOF. First we show that $\mathcal{A}$ is an oval. We have to prove that every line through a point of $\mathcal{A}$ meets the set $\mathcal{A}$ in at most one other point. Indeed, the line L_∞ only contains the point P_∞ of $\mathcal{A}$ and the lines L_k meet $\mathcal{A}$ in one other point, namely $S^+_{k,0}$. Clearly, the line L_{K_i} is a bisecant. One can now easily check that the other lines through $S^+_{K_i,0}$ are the lines $S^-_{K+K_i,-K\cdot K^s}$. Similarly, the points on such a line are the points $S^+_{-K'+K+K_i,K'\cdot K'^s-K\cdot K^s}$. This is a point of $\mathcal{A}$ if and only if $K' = \pm K$. So $S^-_{K_i,0}$ is a tangent line of $\mathcal{A}$ and the other lines are bisecants.

The properties *(i), (ii)* and *(iii)* now follow by a simple calculation (using the fact that $S^+_{K,L}$ is incident with $S^-_{M,N}$ if and only if $S^+_{0,0}$ is incident with $S^-_{M-K,N-L}$ (use $\Psi_1(-K,-L)$)). $\square$

Remark that the mapping $\Psi_2(0,0,0)$ also induces a polarity in $\Gamma(q)$. The set of absolute points is

$$\{S^+_{K,-K^{1+s}} \| K \in GF(q)\} \cup \{P_\infty\}.$$

This again constitutes an oval in $\Gamma(q)$ which is probably not equivalent with $\mathcal{A}$ above. But we were not able to prove that.

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