Hyperbolic lines in generalized polygons

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Abstract

In this paper we develop the theory of hyperbolic lines in generalized polygons. In particular, we investigate the extremal situation where hyperbolic lines are long. We give restrictions on the existence of long hyperbolic lines and characterize the symplectic generalized quadrangles $W(k)$ and generalized hexagons $H(k)$ related to the groups $G_2(k)$ by the existence of certain classes of long hyperbolic lines.

1. Introduction

A *generalized n-gon* Γ is an incidence structure $(\mathcal{P}, \mathcal{L})$ consisting of a set $\mathcal{P}$ of *points* and a set $\mathcal{L}$ of *lines* with symmetric incidence relation whose incidence graph satisfies

1. any two vertices are connected by a path of length at most n;
2. any smallest circuit has length $2n$;
3. for any vertex x there is a vertex y at distance n from x.

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Γ is called *thick* if any element is incident with at least 3 others.

Generalized polygons were introduced by Tits in his paper [15]. However, even before this definition one can find characterizations of the symplectic generalized quadrangles (i.e. generalized 4-gon) by the existence of large hyperbolic lines (where for finite generalized quadrangles of order (s,t) large means that they contain at least $s+1$ points), see [6] and [11]. In this paper we extend the study of hyperbolic lines in quadrangles to the study of hyperbolic lines in arbitrary generalized polygons.

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a generalized n-gon, $n \geq 4$, with point set $\mathcal{P}$ and set of lines $\mathcal{L}$ and distance function δ on the incidence graph of Γ whose vertex set is $\mathcal{P} \cup \mathcal{L}$. Two elements are called *opposite* if their distance equals n. For each element (point or line) x of Γ we denote by $x^\perp$ the set of elements not opposite x, i.e. $x^\perp = \{y \in \mathcal{P} \cup \mathcal{L} \mid \delta(x,y) < n\}$. If X is a subset of $\mathcal{P} \cup \mathcal{L}$, then we denote by $X^\perp$ the intersection of the sets $x^\perp$ for all $x \in X$. Suppose x and y are two noncollinear points of Γ, i.e., their distance is at least 4. The set $H(x,y) = \{x,y\}^{\perp\perp}$ is called the *hyperbolic line* on x and y. If the distance between x and y equals $2j$, then $H(x,y)$ is called a *distance-j hyperbolic line* of Γ. In the next section it will be shown that a distance-j hyperbolic line only contains points that are at mutual distance $2j$.

If H is a hyperbolic line and $X \in \mathcal{P} \cup \mathcal{L}$ an element of Γ at distance $n-1$ from at least two of the points on H, then it is at distance at most $n-1$ from all the points of H. In particular, if X is at distance $n-1$ from all the points of H, then the projection $h \in H \mapsto x$, where x is the unique element of Γ which is adjacent to X and in $h^\perp$, is either constant or injective. We will call the hyperbolic line H *long* if and only if any such injective projection is also surjective (see Corollary 2.3 below).

Our first result puts some restrictions on the existence of long hyperbolic lines.

Theorem 1.1 *If every distance-j hyperbolic line of a thick generalized n-gon is long, $2 \leq j < n/2$, $n \geq 6$, then $j \leq (n+2)/4$. If moreover the same is true in Γ^D, the dual of Γ, or j is even, then j divides n and $j \leq n/4$.*

The first case to consider according to the above theorem is the case where Γ is a generalized hexagon (that is, a 6-gon) containing long distance-2 hyperbolic lines.

The generalized hexagon $H(k)$ associated to the group $G_2(k)$, where k is some field, is constructed by Tits [15] as the set of lines fixed by some triality automorphism of a polar space of type D_4. The points of $H(k)$ are the points of a geometric hyperplane of this polar space isomorphic to the polar space on the isotropic points of a 6-dimensional orthogonal space. Moreover, two points are at distance at most 4, if and only if they are collinear in the polar space. This clearly implies that the distance-2 hyperbolic lines of $H(k)$ coincide with some of the lines of the polar space. We immediately find that these hyperbolic lines are long. The following result characterizes the hexagon $H(k)$ by this property:

Theorem 1.2 *If Γ is a thick generalized hexagon all of whose distance-2 hyperbolic lines are long, then it is isomorphic to the generalized hexagon $H(k)$ for some field k.*

The above theorem is a consequence of Ronan's characterization [12] of the Moufang hexagons by the existence of *ideal lines*, see also Yanuska [19] for the finite case. An ideal line l is a distance-2 hyperbolic line with the property that if x is a point at distance 2 from two points of l, then each line on x meets l nontrivially. Ronan's work depends on some deep results of Tits on rank 3 polar spaces and Moufang hexagons. In Section 4 of this paper however, we give two elementary proofs of the above theorem not relying on Ronan's work.

The second interesting case is where Γ is a thick generalized octagon. If all distance-j hyperbolic lines are long then we cannot have $j = 3$. The following result shows that we cannot have $j = 2$ either.

Theorem 1.3 *There does not exist a thick generalized octagon, all of whose distance-*2 *hyperbolic lines are long.*

In our first theorem we did not consider the case of long distance-$n/2$ hyperbolic lines, where n is even. This case will be handled completely by the following result.

Theorem 1.4 *Suppose* Γ *is a thick generalized* n*-gon, with* $n > 2$ *and even. If all distance-*$n/2$ *hyperbolic lines are long, then either* $n = 4$ *and* Γ *is isomorphic to the symplectic quadrangle* $W(k)$*, for some field* k*, or* $n = 6$ *and* Γ *is isomorphic to the generalized hexagon* $H(k')$*, where* k' *is some field of even characteristic.*

Indeed, the symplectic quadrangle has long hyperbolic lines, and so does the generalized hexagon $H(k)$ for fields k of even characteristic. For, the isomorphism of the polar space of isotropic points in a 6-dimensional orthogonal space with a symplectic polar space in 5 dimensions in even characteristic yields an embedding of the hexagon $H(k)$ into a symplectic space. The distance-3 hyperbolic lines of the hexagon then coincide with the hyperbolic lines of the symplectic space and are obviously long.

We conclude with a corollary of the above result for finite or compact connected topological polygons with order (s, t).

Corollary 1.5 *Let Γ be a thick finite or compact topological generalized n-gon, $n \geq 4$, of order (s,t), $t \geq s$. If all points of Γ are distance-2 regular, then Γ is isomorphic to the finite generalized quadrangle $W(s)$ or hexagon $H(s)$, or the topological quadrangle $W(k)$ or hexagon $H(k)$, where $k = \mathbb{R}$ or $\mathbb{C}$.*

In the next section we will give some preliminary results on hyperbolic lines in generalized n-gons. Moreover, various restrictions on the existence of thick, ideal or long hyperbolic lines will be given. Section 3, 4 and 5 are devoted to the proofs of Theorem 1.2 and Theorem 1.3, Theorem 1.4, respectively Corollary 1.5.

2. Hyperbolic lines

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a generalized n-gon, $n > 2$, and δ the distance function on Γ.

Let v be any element of Γ and let v' be an element opposite v. For any $i \in \{1, 2, \ldots, n-1\}$, the set $v_{[i]}^{v'} = \{w | \delta(v,w) = i = n - \delta(v', w)\}$ is called a *distance-i trace* in $v^\perp$.

If x and y are two elements at distance j, with $j \leq n-1$, then the *projection* of y on x is the unique element z at distance 1 from x and $j-1$ from y.

An *apartment* of Γ is a non-trivial closed circuit of length $2n$ of distinct consecutively incident elements of Γ.

Lemma 2.1 *Let x and y be two points in Γ at distance $2j \leq n$. Let v be an element of Γ at distance j from both x and y. Then $H(x,y)$ is contained in every distance-j trace in $v^\perp$ containing x and y.*

Proof. Suppose $z \in H(x,y)$ and let $i = \delta(v,z)$. Clearly, z is not opposite v (because $v \in \{x,y\}^\perp$ and $z \in H(x,y)$). Up to permutation of x and y we can assume that the projection of z onto v is distinct from the projection of y onto v.

Let Σ be any apartment containing x, y and v. Let w be in Σ opposite v.

If z' is the element on the path from v through y to w at distance $n-i$ from v, then $z' \in y^{\perp}$, but z' is opposite z. Hence, as $z \in H(x,y)$ the element z' is opposite x, and $n-i+j=n$. Thus $i=j$.

Now assume that the projection of z onto v is also distinct from the projection of x onto v. Let γ be a path from w to v containing the projection of z onto v. Let γ' be the path from z to v and let k be the length of $\gamma' \cap \gamma$, $1 \leq k \leq j$. If $2k \geq j$, then let z'' be the unique element of Σ at distance $n-2k$ from y and $n-2k+j$ from v. Then z'' is opposite z and hence it must be opposite x. But $\delta(z'',x) = n-2j+2k$. This implies $k=j$ and z belongs to $v^w_{[j]}$. If $2k < j$, then let z'' be the point on γ which is at distance $j-2k$ from w. The point z'' is opposite z, but at distance $n-j+j-2k < n$ from both x and y. This contradicts z to be in $H(x,y)$. Hence $z \in v^w_{[j]}$, and since w was arbitrary, the result follows.

So we may assume that the unique path γ' defined above (connecting z and v) meets the unique path from x to v in a path of length $k \geq 1$. Now we consider the point u in Σ at distance $n-2j+2k$ from x and distance $n-2k$ from y. By the definition of k, u is opposite z, but it is not opposite y (since $k>0$), hence it should be opposite x, implying $k=j$ and $z=x$.

This completes the proof of the lemma. $\square$

Lemma 2.2 *Let x and y be two points at distance $2j > 2$. If x' and y' are distinct points in $H(x,y)$, then $H(x',y') = H(x,y)$.*

Proof. Suppose x and y are at distance $2j$, and v is an element at distance j from both x and y. Let z be a point in $H(x,y)$ but distinct from x and

y. Then by the above lemma z is also at distance j from v. By definition of $H(x, y)$ we have that $x^\perp \cap y^\perp \subset z^\perp$. We will show that $x^\perp \cap y^\perp = y^\perp \cap z^\perp$.

By way of contradiction, assume that there is an element in $y^\perp \cap z^\perp$ but opposite x. Let a be such an element with $\delta(y, a) = k$ minimal. Let $\gamma = (y_0 = a, y_1, \ldots, y_k = y)$ be a minimal path from a to y. If a is the only point on γ opposite x, then the minimal path from z to a contains v and $\delta(x, a) = \delta(x, v) + \delta(v, a) = \delta(z, v) + \delta(v, a) < n$. This is a contradiction.

Thus there is an element on γ distinct from a but opposite x. Let $l > 0$ be minimal with y_{2l} opposite x. Then y_l is at distance $n - l$ from x. Let $(x_0 = y_l, x_1, \ldots, x_{n-l} = x)$ be the minimal path from y_l to x. Then $x_1 \neq y_{l+1}$. All points y_m with $1 \leq m \leq 2l - 1$ and $x_{m'}$ with $0 \leq m' \leq l$ are in $x^\perp \cap y^\perp$ and hence also in $z^\perp$. This implies, together with $a = y_0 \in y^\perp \cap z^\perp$, that the point y_l is at distance $\leq n - 1 - l$ from z. However, then y_{2l} is at distance at most $n - 1$ from z, and hence in $y^\perp \cap z^\perp$, which contradicts the choice of a. Hence $x^\perp \cap y^\perp = y^\perp \cap z^\perp$. But then for all distinct $x', y' \in H(x, y)$ we have $x'^\perp \cap y'^\perp = x^\perp \cap y^\perp$, and thus $H(x', y') = H(x, y)$. $\square$

The above lemmas justify the name distance-j hyperbolic line.

Corollary 2.3 *Let H be a distance-j hyperbolic line, and v an element of Γ at distance $n - 1$ from two elements in H. Then either v is at distance $\leq n-2$ from a unique point on H or all points of H have distance $n-1$ to v. In the last situation the projection of H on v is either injective or constant.*

Proof. Suppose v is at distance $n - 1$ from distinct x and y in H. By definition of H and the above lemma, $v \in a^\perp$ for all $a \in H$. If a point u is adjacent to v and opposite to x, then by 2.2 it is opposite to at least all but one of the points of H. So, there is at most one point in H at distance

$\leq n-2$ from v. For a point z at distance at most $n-1$ from v we will denote the projection of z on v with v_z.

Suppose v is at distance $n-1$ from all the points in H. If $v_x = v_y$ for some distinct x and y in H, then by the above lemma and the definition of H, all $v_x \in z^\perp$ for all points $z \in H$. In particular, $v_x = v_z$, and the projection of H on v is constant. □

It is straightforward to see that that in a thick generalized n-gon each of the above mentioned situations do occur.

Let $2 \leq j \leq n/2$, and x and y be two points at distance $2j$. The distance-j hyperbolic line $H = H(x, y)$ will be called *thick* if it contains more than 2 points, and *ideal* if it consists of all the points of any j-trace $v^w_{[j]}$, where v is at distance j from both x and y and w is opposite v. The hyperbolic line H is called *long* if the projection of H onto any element of Γ at distance $n-1$ of at least two and thus, by 2.3, all elements of H, is bijective whenever it is injective.

Lemma 2.4 *Let $2 \leq j \leq n/2$. Every long distance-j hyperbolic line is ideal.*

Proof. Let x and y be two arbitrary distinct points of a long distance-j hyperbolic line $H = H(x, y)$. Let v be an element at distance j from both x and y. Let w be opposite v such that $x, y \in v^w_{[j]}$. By lemma 2.1, H is contained in $v^w_{[j]}$. Suppose there is a point $z \in v^w_{[j]}$ not belonging to H. Let v' be the projection of x onto v and let w' be the projection of z onto w. Clearly v' and w' are opposite, so by the thickness assumption, there exists a path $\gamma = (v' = v_0, v_1, \ldots, v_{n-1}, v_n = w')$ in Γ with v_1 and v_{n-1} not on the apartment containing v, x, w and z. Consider the element v_{n-j}. We have $\delta(v_{n-j}, w') = j$, $\delta(z, w') = n-j-1$ and $\delta(x', w') = n-j+1$, for all $x' \in H$. Hence $\delta(x', v_{n-j+1}) = n$ and x' is opposite v_{n-j+1}, for all $x' \in H$.

Moreover, $\delta(x', v_{n-j}) = n - 1$ for all $x' \in H$. Similarly, one shows easily that $\delta(x, v_{n-j-1}) = n - 2$ and, with the help of 2.1, $\delta(x', v_{n-j-1}) = n$, for all $x' \in H$ different from x. Hence the projection of H onto v_{n-j} is not a constant mapping, so it is injective, but since H is long, it should be bijective. This contradicts the fact that v_{n-j+1} is opposite every element of H. Hence the result. $\square$

An element v of Γ is called *distance-j regular*, if any two of its distance-j traces of the form $v^w_{[j]}$, where w is a point opposite v, are either equal or meet in at most one element. The above implies.

Corollary 2.5 *Let v be an element of a generalized n-gon Γ, $n > 2$. If every distance-j hyperbolic line defined by two points at distance j from v, $2 \leq j \leq n/2$, j even if v is a point, j odd if v is a line, is long, then v is distance-j regular.* $\square$

Combining this with the main result of [18] we obtain strong restrictions on the existence of ideal or long hyperbolic lines:

Theorem 2.6 *If every distance-j hyperbolic line of a generalized n-gon is ideal or long, $2 \leq j < n/2$, $n \geq 6$, then $j \leq (n+2)/4$. If moreover the same is true in Γ^D, the dual of Γ, or j is even, then j divides n and $j \leq n/4$.*

Theorem 1.1 follows from this result and Theorem 1.3 now follows from Theorem 2.3 of the above mentioned paper.

However, using almost the same proofs as in [18] we can obtain results similar to the above just on thick hyperbolic lines:

Theorem 2.7 *Suppose $2 \leq j < n/2$, and n even. If all distance-j hyperbolic lines are thick, then either $j = n/2 - 1$ or $j \leq n/4 + 1$.*

Proof. Suppose $j \neq n/2 - 1$, then there exists an maximal even number k, with $j \leq k < n/2$. Let v and w be two points, if j is even, or two lines, for odd j, at distance k, and fix an apartment Σ on v and w. Denote by v', respectively, w' the element opposite v, respectively, w in Σ. Let v_1 and v_2 be the two points at distance j from v, and w_1 and w_2 the two points at distance j from w inside Σ. Let v_3, respectively, w_3 be a third point on the hyperbolic line $H(v_1, v_2)$, respectively, $H(w_1, w_2)$. There are paths γ_v and γ_w from v to v', respectively, w to w' of length n that contain v_3, respectively, w_3. Let v'' be the unique element on γ_v at distance k from v', and w'' the element on γ_w at distance k from w'. Then v'' is opposite w and at distance $n - j$ from both w_1 and w_2. Hence, by Lemma 2.1 v'' is also at distance $n - j$ from w_3. Similarly we find w'' to be at distance $n - j$ from v_3. ¿From $2j + 2k \leq 4k < 2n$ it follows that $\delta(v_3, w_3) > k$. Now we have found a circuit through v_3, v'', w_3, w'' of length at most $(n-j-k)+(n-j)+(n-j-k)+(n-j)$. Hence $4n - 4j - 2k \geq 2n$, which implies $j \leq (n - k)/2$. This proves the theorem. □

A point x of a generalized polygon is called *projective* if it is distance-2 regular and any two distance-2 traces of the form $x^w_{[2]}$ with w opposite to x meet nontrivially. If x is a projective point, then these distance-2 traces and the lines through x define a projective plane on the set of all the points at distance at most 2 from x (called the *derivation at x*).

Proposition 2.8 *Let $n > 3$ and suppose that H is a distance-2 hyperbolic line in the generalized n-gon Γ. Let x be a point at distance 2 from all points of H and let v be opposite x. If H is long, then $H \cap x^v_{[2]}$ is non-empty. In particular, all hyperbolic lines whose points all lie at distance 2 from x are long, if and only if x is projective.*

Proof. Let x, H and v be as stated. Let $y \in H$ and let w be the projection of the line xy onto v and let u be the projection of xy onto w.

We may assume that the projection y' of v onto xy is different from y. Hence it is clear that $\delta(x,u) = n-2$, $\delta(x,w) = n-1$ and $\delta(y,u) = n-2$. Let $z \in H\backslash\{y\}$. Then $\delta(xz,w) = n$, so $\delta(z,w) = n-1$ and $\delta(z,u) = n$. Consequently w is at distance $n-1$ from all points of H and the projection of H onto w is bijective. Hence there is a point $a \in H$ at distance $n-2$ from v, implying $a \in H \cap x^v_{[2]}$. This implies the first part of the proposition. Moreover, it also shows that the point x is projective if all the hyperbolic lines whose points are at distance 2 from x are long.

Finally suppose that x is projective and let y and z be two points at distance 2 from x but with $\delta(y,z) = 4$. Denote by H the hyperbolic line $H(z,y)$. Let v be opposite to x but in $z^\perp \cap y^\perp$. Then $H \subset x^v_{[2]}$. Suppose w is in $z^\perp \cap y^\perp$. If w is opposite x then $x^v_{[2]} = x^w_{[2]}$. If $\delta(w,x) < n-2$ or equal to $n-1$, then clearly $x^v_{[2]} \subset w^\perp$. Since both y and z are in $w^\perp$ $\delta(x,w)$ can not be $n-2$. For then both the projection of y and of z on x are at distance $n-3$ from w, which is impossible. Thus $x^v_{[2]} = H$ and any element incident to x is incident to some point of H.

Now assume u is at distance $n-1$ from all points of H, and suppose that the projection of H on u is injective, but not surjective. Hence there is an element r, at distance 1 from u, but at distance n from all the points of H. Since x is projective, r is not opposite x. Thus r is at distance $n-2$ from x. The projection of r on x is at distance $n-3$ from r and adjacent to some point in H. This is a contradiction. Hence the projection of H onto v is surjective and H is long. □

Theorem 2.6 does not cover the case of long distance-$n/2$ hyperbolic lines. We take care of that in the next theorem. Note that n is automatically even.

Theorem 2.9 *If all distance-$n/2$ hyperbolic lines of a generalized n-gon Γ are long, $n > 2$ and even, then all points are distance-$\frac{n-2}{2}$ regular. In particular, $n = 4$ or $n = 6$. If Γ is finite of order (s, t), then the existence in Γ of one long distance-$n/2$ hyperbolic line implies $s = t$.*

Proof. For $n = 4$, there is nothing to prove, so assume $n \geq 6$.

Let y and z be two opposite points and let $H = H(y, z)$ be a long distance-$n/2$ hyperbolic line. Let x be collinear with y but opposite z. Let v be the element of Γ at distance $n/2$ from z and $n/2-1$ from the line xy. By Lemma 2.4, $\delta(v, a) = n/2$ for all $a \in H$. Now let $\gamma = (x = v_0, v_1, \ldots, v_{n-1}, v_n = z)$ be any path of length n joining x with z but not containing xy. All the elements $v_1, v_2, v_3, \ldots, v_{n-3}$ are neither opposite z nor opposite y, hence they are not opposite a, for any $a \in H$. So let $a \in H$, $z \neq a \neq y$. Since x is opposite z, but not opposite y, we find that a is opposite x and $\delta(v_2, a) = n - 2$. Suppose the path γ' joining v_2 and a contains v_j but not v_{j+1}, for some integer j with $2 \leq j \leq (n-4)/2$. Then clearly, $\delta(a, v_{2j}) = \delta(a, v_j) + \delta(v_j, v_{2j}) = n$, a contradiction. Hence γ' contains $v_{n/2-1}$. Since $\delta(x, v_{n/2-1}) = n/2 - 1$ and $\delta(x, v) = n/2$, we conclude $x^z_{[n/2-1]} = x^a_{[n/2-1]}$, for all $a \in H$, $a \neq y$.

Since v_{n-2} is opposite y it is also opposite all $a \in H$ distinct from z. In particular, this implies that $v_{n/2}$ is not on γ'. Moreover, we have $\delta(a, v_{n/2-2}) = \delta(a, v_{n/2}) = n/2 + 2$ and $\delta(a, v_{n/2-1}) = n/2 + 1$ for all $a \in H$ distinct from y and z. Suppose that a_1 and a_2 be two points in H, both different from y and z. If the projection w of a_1 onto $v_{n/2-1}$ would coincide with the projection of a_2 onto $v_{n/2-1}$, then we can let a_2 play the role of z and consequently w can play the role of $v_{n/2}$. As above we find that a_1 is at distance $n/2 + 2$ from w. However, we have $\delta(a_1, w) = n/2$, a contradiction. This implies that we have the projection of H onto $v_{n/2-1}$ is injective.

Suppose by way of contradiction that the above projection is not bijective and let u be an element not in the image. So u is incident with $v_{n/2-1}$ and at distance $n/2+2$ from all points of H. Let v' be the projection of z onto v. Then v' and $v_{n/2-1}$ are opposite and there is a unique element L in Γ at distance $n/2$ from v' and distance $n/2-1$ from u. We have $\delta(L,z) = \delta(L,v') + \delta(v',z) = n-1$. Let x_0 be the projection of u onto L, then $\delta(a,u) + \delta(u,x_0) = n/2+2+n/2-2 = n$, hence x_0 is opposite every element a of H. This implies that $\delta(a,L) = n-1$ for all points a of H, and since the projection of H on L is clearly not bijective (the image misses x_0), it must be constant. So $\delta(y,x_1) = n-2$, where x_1 is the projection of z onto L. But x_1 is also the projection of v' onto L; indeed, $\delta(z,x_1) = \delta(z,v')+n/2-1$. Since $\delta(y,v') = n/2+1$, we find y is opposite x_1, a contradiction. This shows our claim.

This proves for finite Γ that $s=t$. Indeed, by the Feit -Higman Theorem, see [1], we only have to consider the cases $n=6$ and $n=8$. If $n=6$, then H contains $s+1$ points and must be in bijective correspondence with the set of all the $t+1$ lines through a point. If $n=8$, then H must have $t+1$ points and be in bijective correspondence with all the $s+1$ points on a line.

It is in fact from now on that we assume that *all* distance-$n/2$ hyperbolic lines are long (until now we dealt with only *one* long distance-$n/2$ hyperbolic line). Suppose now z' is any other point opposite x different from z, with $\{v'', v_{n/2-1}\} \in x^z_{[n/2-1]} \cap x^{z'}_{[n/2-1]}$, where v'' is the projection of x onto v. Suppose first that $\delta(v',z') = n/2-1$. There is a unique element u' incident with $v_{n/2-1}$ at distance $n/2$ from z'. By the above, there exists a unique $a \in H$ such that $\delta(a,u') = n/2$. Hence $\delta(z',v) = \delta(a,v) = \delta(z',u') = \delta(a,u') = n/2$ which implies that $z' \in H(a,x)$. Hence clearly $x^a_{[i]} = x^{z'}_{[i]}$, for all $i \in \{2,\ldots,n/2\}$. Therefore $x^z_{[n/2-1]} = x^{z'}_{[n/2-1]}$ by the conclusion of the first paragraph above.

So we have shown that, if two distance-$(n/2-1)$ traces $x^{z}_{[n/2-1]}$ and $x^{z'}_{[n/2-1]}$ meet in at least two elements, and if z and z' either lie in the same hyperbolic line with x, or there exists an element v' at distance $n/2+1$ from x and distance $n/2-1$ from both z and z', then these two traces coincide.

So now let z'' be some point such that $x^{z''}_{[n/2-1]}$ meets $x^{z}_{[n/2-1]}$ in at least two elements, say b_1 and b_2. Let u_1, u_2, u''_1 and u''_2 be the projections of z and z'' onto b_1 and b_2 (where indices and primes are self-explanatory). Let w_1 be the unique element of Γ at distance 2 from b_1 and $n/2-1$ from z. Then w_1 is opposite b_2. Let z^* be the unique point at distance $n/2$ from u''_2 and $n/2-1$ from w_1. By the previous paragraph, $x^{z}_{[n/2-1]} = x^{z^*}_{[n/2-1]}$. But now there exists a unique point $z^{**} \in H(x, z^*)$ such that the projection of z^{**} onto u''_2 coincides with the projection of z'' onto u''_2. From the previous paragraph follows again $x^{z^*}_{[n/2-1]} = x^{z^{**}}_{[n/2-1]} = x^{z''}_{[n/2-1]}$. This shows that all points are $n/2-1$ regular. By [18] we obtain the theorem. □

The last result of this section is concerned with finite thick polygons. In particular, by the Feit-Higman Theorem, see [1], we are only concerned with n-gons of order (s,t), where $n = 4, 6$ or 8. We already have seen that the presence of one long distance-$n/2$ hyperbolic line forces $s = t$. A long distance-2 hyperbolic line on points is ideal and hence contains $t+1$ points. On the other hand, the projection of such a long distance-2 hyperbolic line on a line at diistance $n-1$ from at least two of its points is a bijection. Hence, such a long distance-2 hyperbolic line contains $s+1$ points and $s = t$. We will now show the same for distance-3 hyperbolic lines in thick finite generalized octagons.

Proposition 2.10 *If a finite thick generalized n-gon ($n = 4, 6, 8$) of order (s,t) contains a long distance-i hyperbolic line, where $2 \le i \le n/2$, then*

$s = t$.

In particular, a finite thick generalized octagon does not contain a long distance-j hyperbolic line for any $j > 1$.

Proof. As remarked above, it suffices to consider $n = 8$ and $i = 3$. So let L be a line of a thick generalized octagon Γ of order (s,t), with $s \neq t$, and let x, y be two points at distance 3 from L with $\delta(x,y) = 6$. Suppose $H(x,y)$ is long. Then by Lemma 2.4, $H(x,y)$ coincides with every distance-3 trace $L^M_{[3]}$, where M is opposite L and at distance 5 from at least two elements of $H(x,y)$. Let M be such a line. Let z be a point on the line M' defined by: M' meets M and $\delta(x,M') = 3$. We can assume that z does not lie on M and at distance 4 from x. Let M_z be any line through z different from M' and let z' be the projection of y onto M_z. So z' is not opposite x nor y. Hence z' is not opposite a, for all $a \in H(x,y)$. Notice that $\delta(a,M) = 5$ for all $a \in H(x,y)$ and $\delta(M,z') = 5$, hence $\delta(z',a) = 6$. Let $a_1, a_2 \in H(x,y)$ be different and such that the projection N of a_1 onto z' coincides with the projection of a_2 onto z'. Then N is opposite to L and at distance 5 from both a_1 and a_2 and so it should be at distance 5 from all points of $H(x,y)$. But clearly, it is at distance 7 from either x or y, a contradiction. So the projection of $H(x,y)$ onto z' is injective. This already implies $t > s$. Hence this projection is not bijective, and so there exists a line L' though z' at distance 7 from every point of $H(x,y)$. Note that $\delta(L,L') = 8$. Indeed, assume that $\delta(L,L') = 6$, then let x' be the projection of L' onto L. There is a unique point $x'' \in H(x,y)$ collinear with x'. Unless $\delta(x'',L') = 5$, we have $\delta(x'',z') = 8$, a contradiction, so L' is the projection of x'' onto z'. Hence L is opposite L'. Note that every point of L' except for z', is opposite every point of $H(x,y)$. Let y' be the projection of y onto L and let y'' be the projection of y' onto L'. Let L'' be the projection of y' onto y'' and y_0 the projection of y' onto L''. Then $\delta(y, y_0) = 6$ and all points of L'' except for y_0 are opposite y. Also, all points of $H(x, y)$, except for y, are opposite y_0. Hence $\delta(L'', a) = 7$ for all $a \in H(x, y)$ and the projection of $H(x, y)$ onto L'' is injective. It follows that it must also be bijective, but this contradicts the fact that y'' is opposite all points of $H(x, y)$ as y'' is on L' and different from z'. This completes the proof of the first part of the proposition.

The second part is a consequence of the first part and the fact that in a finite thick generalized octagon of order (s, t) the parameters s and t have to be different, see [1].$\square$

3. Long hyperbolic lines in generalized quadrangles

In this short section we present the classification of generalized quadrangles with long hyperbolic lines. The following result is probably folklore, see [13].

Theorem 3.1 *Suppose* $\Gamma = (\mathcal{P}, \mathcal{L})$ *is a thick generalized quadrangle and all hyperbolic lines of* Γ *are long. Then* Γ *is isomorphic to the symplectic quadrangle* $W(k)$ *for some field* k.

Proof. Let $\mathcal{H}$ be the set of hyperbolic lines of Γ, and consider the linear space $\Pi = (\mathcal{P}, \mathcal{L} \cup \mathcal{H})$. Suppose p, q, r are three points of this space not on a line. By Proposition 2.8 there is a point x on the line pq that is in $r^\perp$. Consider the line xr of Γ. On this line we find a point y perpendicular to all points of pq and thus perpendicular to p, q and r. But since this point y is projective, cf. Proposition 2.8, p, q and r generate a projective plane in Π. Thus Π is a projective space and $\perp$ clearly induces a symplectic polarity on Π. But then Γ is isomorphic to the generalized quadrangle defined by this polarity, thus to $W(k)$ where k is the underlying field of Π. $\square$

4. Long distance-2 hyperbolic lines in generalized hexagons

Generalized hexagons whose distance-2 hyperbolic lines are all long do have ideal lines as follows by Lemma 2.4. Thus by Ronan's results [12], they are Moufang and known by Tits' classification of Moufang hexagons, see [17]. However, we do not need the full power of these results and could refer to some of the intermediate results of Ronan, especially Theorem 8.19 of [12]. The proofs of these results are quite long and complicated and rely on the classification of the rank 3 polar spaces. Here we present two new and elementary proofs of the following result.

Theorem 4.1 *Let Γ be a thick generalized hexagon whose distance-2 hyperbolic lines are all long. Then Γ is isomorphic to the $G_2(k)$ hexagon $H(k)$ for some field k.*

In both proofs we use only elementary facts on polar spaces. The deepest result we use is the classification of polar spaces of type D_3, respectively, D_4. See [17].

In the first proof we will show uniqueness of the hexagon Γ by identifying all its points and lines with some subconfigurations of a D_3 polar space defined over some field k. The idea of using this D_3 geometry also appears in [12], as well as [3], but the first to describe the G_2 hexagons using this D_3 geometry seems to be S. Payne, [10].

In the second proof we construct a D_4 geometry in which we can recognize the lines of Γ as those lines that are being fixed by some triality automorphism of the D_4 geometry. This is the original way in which J. Tits has introduced the generalized hexagon $H(k)$ for the groups $G_2(k)$, see [15].

At various points in the proof we will identify subspaces of incidence structures or geometries under consideration with the set of points they contain.

Suppose $\Gamma = (\mathcal{P}, \mathcal{L})$ is a hexagon as in the hypothesis. Set $\mathcal{H}$ to be the set of hyperbolic lines in Γ, and denote by $\mathcal{S}$ the space $(\mathcal{P}, \mathcal{L} \cup \mathcal{H})$. The initial step of both proofs is the following observation:

Proposition 4.2 *$\mathcal{S}$ is a nondegenerate rank 3 polar space.*

Proof. By Proposition 2.8, we find that the space $\mathcal{S}$ satisfies the Buekenhout-Shult axiom for a polar space, [2]. Since every point has an opposite, the space $\mathcal{S}$ is nondegenerate.

If x is a point, then the set of points at distance at most 2 from x forms the point set of a projective plane, see 2.8, which is a maximal singular subspace of $\mathcal{S}$. Thus $\mathcal{S}$ has polar rank 3. □

Let x be a point of Γ and consider all the points collinear with x. As already noticed in the above proof this set of points forms a projective plane in $\mathcal{S}$, which we will call the *focal plane* on x. The point x is called the *focal point* or *focus* of the plane. In fact, every plane of $\mathcal{S}$ containing a line of Γ consists of a point x and all the lines of Γ through x, and thus has a focus.

Now we start with the first proof.

Fix an apartment in Γ, and in the apartment two triples of points $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$ at mutual distance 4, such that a_1 is opposite to a_2, $a = x, y$ or z. The points x_1, y_1 and z_1, respectively, x_2, y_2 and z_2 are contained in a singular subspace π_1, respectively, π_2 of $\mathcal{S}$ which are projective planes, see [2]. These planes consist entirely of hyperbolic lines.

Let H be a line in π_1, and suppose f is the *focus*, or *focal point* of that line (i.e. the unique point collinear in Γ with all the points of h). Since H meets the three lines $a_2^\perp \cap \pi_1$ for $a = x, y$ and z, we find that f is in $x_2^\perp \cap y_2^\perp \cap z_2^\perp$ and thus in π_2. Vice versa we find that any focal point of a hyperbolic line of π_2 is a point of π_1. Let Δ be the union of all the points of lines meeting both π_1 and π_2.

Lemma 4.3 *Δ is a geometric hyperplane of $\mathcal{S}$.*

Proof. Let $L \in \mathcal{L}$ be a line meeting Δ in at least 2 points, say a and b. Let a_1 and b_1 be two points of π_1 collinear with a, respectively, b. Since $\delta(a_1, b_1) = 4$, we find that L is either at distance 1 from a_1 or from b_1 and thus contained in Δ.

Now suppose $H \in \mathcal{H}$ meets Δ in 2 points a and b say. Let c be the focal point of H. Let a_1 and b_1 be the points in π_1 collinear with a, respectively, b. If $a_1 = b_1$, then H is contained in Δ, as it constists of points on lines through a_1. Thus assume $a_1 \neq b_1$. Let d be the focal point of the hyperbolic line on a_1 and b_1. If $a = a_1$ or $b = b_1$ then $H \subset \pi_1$, since its points are on the lines through the focal point $d = c \in \pi_2$. Thus we may assume that the points a, b, a_1, b_1, c and d are all distinct. The points a, b, c, d, a_1 and b_1 are then the 6 points on an apartment. As before, we can consider the two planes π_1' and π_2' of $\mathcal{S}$ generated by a, b and d, respectively, a_1, b_1 and c. The points of π_1' are the focal points of any line in π_2'. In particular, each point of H is collinear with some point of a_1b_1 and thus $H \subseteq \Delta$.

The above shows that Δ is a subspace of $\mathcal{S}$. It remains to show that every line in $\mathcal{L} \cup \mathcal{H}$ meets Δ.

Suppose $L \in \mathcal{L}$ is not in Δ. Then in the polar space $L^\perp \cap \pi_1$ is a point x_1 say. But then x_1 is collinear to some point on L which is then a point in Δ.

Let $H \in \mathcal{H}$ is a hyperbolic line not in Δ. Let f be the focal point of H. If $f \in \Delta$, then there is at least one line on f inside Δ and H meets Δ nontrivially. Thus assume $f \notin \Delta$. Then all lines on f contain a unique point in Δ, and the projective plane on the points at distance at most 2 from f

contains a line in Δ as well as H. In particular, the intersection point of these two lines is a point of H in Δ. $\square$

Fix a point x outside Δ, then every line of $\mathcal{S}$ on x meets Δ in a unique point. The union of all these points form a nondegenerate hyperplane $\Delta_x = x^\perp \cap \Delta$ of Δ. This implies the following.

Lemma 4.4 *Δ is isomorphic to the polar space $D_3(k)$ for some field k.*

Proof. By the above we find that each line of $\Gamma \cap \Delta$ is in exactly two planes of Δ, being the two focal planes with focus in π_1 or π_2. Thus by Proposition 7.13 of [16] we find that Δ is isomorphic to $D_3(k)$ for some skewfield k. The subspace Δ_x is a nondegenerate hyperplane of Δ, and Proposition 5.1 of [4] implies that the underlying skewfield of the D_3 geometry Δ is commutative. $\square$

We will proceed by identifying the points and lines of Γ with subconfigurations in Δ.

The focal plane on x meets Δ in a hyperbolic line denoted by L_x. Each point of L_x is on a unique line from Γ contained in Δ. These lines of Γ inside Δ and meeting L_x generate a grid in Δ denoted by G_x. If y is another point outside Δ distinct from x, then clearly L_y is also distinct from L_x. Moreover, for each hyperbolic line H inside Δ not meeting the two planes π_1 and π_2, there is a point z, the focus of that line, with $L_z = H$. If x and y are two points outside Δ such that the two lines L_x and L_y meet at some point z of Δ, then x and y are at distance 2 or 4. Moreover, if L_x and L_y meet in the point z, then the line L_y is contained in Δ_x if and only if x and y are collinear. In that case Δ_y contains G_y and $z^\perp \cap \Delta_x$, and hence is generated by G_y and $z^\perp \cap \Delta_x$. In particular, Δ_y is uniquely determined by L_y and (L_x, Δ_x).

Lemma 4.5 $\Gamma \setminus \Delta$ *is connected.*

Proof. Suppose x and y are two points of $\Gamma \setminus \Delta$ at distance 6 in Γ.

If $x^y_{[2]} \neq L_x$ and $y^x_{[2]} \neq L_y$, then we can find collinear points $x' \in x^y_{[2]} \setminus L_x$ and $y' \in y^x_{[2]} \setminus L_y$, and there exists a path from x to y outside Δ.

If $x^y_{[2]} = L_x$ or $y^x_{[2]} = L_y$, then there is a line of Γ meeting both L_x and L_y in points outside π_1 and π_2. In particular, lines of Γ contain at least 4 points. So, if $x^y_{[2]} = L_x$, but $y^x_{[2]} \neq L_y$ then we can replace y by some point y' collinear with y and at distance 6 from x but not in Δ. Clearly we have $x^{y'}_{[2]} \neq L_x$ and $y'^x_{[2]} \neq L_{y'}$, and by the above there is a path from x to y' and hence also y outside Δ. Similarly we can handle the case where $x^y_{[2]} \neq L_x$, but $y^x_{[2]} = L_y$.

Finally, if $x^y_{[2]} = L_x$ and $y^x_{[2]} = L_y$, then by replacing x by some point x' collinear with x and not in Δ we are in one of the above situations. Thus in all cases we can find a path outside Δ connecting x and y.

Now suppose x and y are at distance 4. Then y is collinear with some point $y' \notin \Delta$ that is at distance 6 from x. By the above there is a path outside Δ connecting x to y', which clearly implies that there is a path from x to y outside Δ. This proves the lemma. □

We can identify each point x outside Δ with the hyperbolic line L_x meeting neither π_1 nor π_2. Moreover, by the preceding and the connectedness of $\Gamma \setminus \Delta$, we find that, after having fixed the subspace Δ_x on L_x for some point x outside Δ, all other subspaces Δ_y with y not in Δ are uniquely determined by L_y. Two points in Δ are collinear in Γ if and only if they are on a line of Δ meeting both π_1 and π_2. A point x of Δ is collinear in Γ with a line L_y of Δ not meeting π_1 nor π_2 if and only if $x \in L_y$. And finally two lines L_x and L_y of Δ disjoint from π_1 and π_2 are collinear if and only if they intersect in a point z which is in the radical of the subspace $\Delta_x \cap \Delta_y$. Thus the isomorphism type of Γ only depends on the isomorphism type of Δ, the choice of the planes π_1 and π_2 and finally the pair (L_x, Δ_x) for some fixed point x.

But the automorphism group of Δ contains the orthogonal group $PO_6^+(k)$. This group is transitive on the nondegenerate hyperplanes Δ_x of Δ. The stabilizer of such a hyperplane Δ_x contains the group $2 \times PO_5(k)$ and is transitive on the planes of Δ, and then also on the pairs of nonintersecting planes. If we fix such a pair of planes, then their intersection with Δ_x consists of two disjoint singular lines L_1 and L_2 generating a grid. The stabilizer in $PO_5(k)$ of such a grid contains the group $PSL_2(k) \times PSL_2(k)$, and we see that the stablizer of $\{L_1, L_2\}$ is still transitive on the lines of the grid not meeting L_1 and L_2. Thus up to isomorphism, there is only one choice for the planes π_1, π_2 and the pair (L_x, Δ_x). In particular, the isomorphism type of Γ only depends on k.

Since the classical $G_2(k)$ hexagon $H(k)$ satisfies the hypothesis of the Theorem, the above implies that Γ is isomorphic to this $G_2(k)$ hexagon, which finishes the proof of the Theorem.

Remark. The above not only shows uniqueness of the hexagon Γ, it also shows a way to construct the classical generalized $G_2(k)$ hexagon from a $D_3(k)$ polar space. For finite hexagons this construction has been discussed in the appendix of [3], and by Payne in [10].

Now to the second proof. Consider the dual polar space Θ of $\mathcal{S}$. We can embed Γ into this dual polar space by identifying each point p of Γ with the unique focal plane p^* of which p is the focus. Each line of Γ is mapped to itself. Let $\mathcal{P}^*$ be the image of $\mathcal{P}$ under this embedding, and denote by Γ^* the image of Γ.

Lemma 4.6 *Γ^* is a geometric hyperplane of Θ isomorphic to Γ. Moreover, the distances in Γ^* coincide with those in Θ.*

Proof. Let L be a line of Θ. Then L is also a line of $\mathcal{S}$. Thus it is either singular or hyperbolic. All planes on a singular line are focal planes and hence in $\mathcal{P}^*$, and a hyperbolic line is contained in a unique focal plane. Thus $\mathcal{P}^*$ is indeed a hyperplane of Θ. Moreover, all lines of the hyperplane are singular lines and thus in $\mathcal{L}$. The last statement of the Lemma readily follows. □

Now consider the point-line space of Θ. Distance-2 hyperbolic lines can be defined on Θ in the same way as they are defined for a generalized hexagon. If we fix two points x and y at distance 4 in Θ, then they are contained in a unique quad, Q say. Since Θ is classical, every point z is either in the quad Q or is collinear with a unique point z' of the quad. Moreover, in the last case we have that $\delta(z,u) = \delta(z',u) + 2$ for all points $u \in Q$. (Here δ denotes the distance in the incidence graph of Θ.) We find that the hyperbolic line on x and y is just the hyperbolic line on x and y inside the quad Q. By $\mathcal{H}^*$ we denote the set of distance-2 hyperbolic lines of this near hexagon only containing points of $\mathcal{P}^*$ and by $\mathcal{H}^{**}$ the remaining distance-2 hyperbolic lines. Let Σ^* be the space whose points are the points of Θ and whose lines are those of Θ, i.e. in $\mathcal{L} \cup \mathcal{H}$ together with the lines from $\mathcal{H}^* \cup \mathcal{H}^{**}$.

Proposition 4.7 *All quads of Θ are isomorphic to $W(k)$ for some k. All distance-2 hyperbolic lines of Θ are long, and Σ^* is a polar space of type D_4.*

Proof. Fix a quad Q of Θ. This quad meets Γ^* in a point q (called the *center* of Q) and all the lines in Q (and in Γ^*) through that point. If $p \in Q$ is at distance 4 from q, then there is a point $r \in P^*$ collinear with p but at distance 6 from q inside Γ^*. The unique quad R of Θ with center r is disjoint from Q. Furthermore, all common neighbors of p and q inside Q are just the points in the trace q_r of Γ^*. As q is projective in Γ^*, it is also projective in Q. Similarly r is projective in R. But then p is also projective in Q, since the projection of R onto Q, where each point $x \in R$ is mapped to its unique neighbor in Q, is an isomorphism between R and Q mapping r to p. Thus by varying p we find all points of Q not collinear to q to be projective, and similarly all points in R not collinear to r are projective. The projection of R to Q reveals that all points in Q not collinear to p and q are projective, and as we can vary p, that all points of Q are projective.

By Theorem 3.1 and Proposition 2.8 Q is isomorphic to $W(k)$ for some field k.

Now consider Σ^*. Since Θ is a classical near hexagon, we find that the distance-2 hyperbolic lines of Θ coincide with the distance-2 hyperbolic lines inside the quads of Θ. Let p be a point and L be a line of Σ^*. If L is not in $\mathcal{H}^* \cup \mathcal{H}^{**}$, then clearly p is collinear with one or all points of L. If $L \in \mathcal{H}^* \cup \mathcal{H}^{**}$, then there is a quad Q of Θ containing L. If $p \in Q$, then p is collinear to all points of L. Thus assume that p is not in Q. Then inside Θ there is a unique point q of Q collinear to p, and $p^\perp \cap Q$ consists of all the points of Q collinear to q, and hence contains one or all points of L. This shows that Σ^* is a polar space, and that all distance-2 hyperbolic lines are long. The singular subspace induced on a quad is clearly maximal and is isomorphic to $PG(3,k)$. Moreover, it is easy to check that the maximal singular subspaces of Σ^* containing a line from Θ, i.e. from $\mathcal{L} \cup \mathcal{H}$, are the quads of Θ, and the set of points of Θ collinear (inside Θ) with some fixed point p. In particular, if we fix a point p and two lines of Θ on p, then there are just two maximal singular subspaces containing p and these two lines. This implies that all singular planes of Σ^* are in just two maximal singular subspaces. Hence Σ^* is of type D_4. □

Lemma 4.8 *The points of $\mathcal{P}^*$ form a geometric hyperplane of Σ^*.*

Proof. Each line L of Σ^* is contained in a quad of Θ. This quad becomes a $PG(3,k)$ inside Σ^* and meets Γ^* at a plane, that intersects L in a point or contains it. □

The map $p \mapsto p^*$, $p \in \mathcal{P}$ induces an isomorphism between Γ and the hyperplane Γ^* of Θ. It extends uniquely to an isomorphism between $\mathcal{S}$ and the hyperplane $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L} \cup \mathcal{H}^*)$ of Σ^*. Here a line H in $\mathcal{H}$ is mapped to the line $H^* = \{p^* \mid p \in H\}$ in $\mathcal{H}^*$.

The above not only shows that Σ^* is a D_4 polar space, it also describes the various subspace of this space.

The points of Σ^* are the points of Θ, i.e. the points in $\mathcal{P}^*$ together with the set of all nonfocal planes of $\mathcal{S}$, say $\mathcal{R}^*$. The lines of Σ^* are the lines in $\mathcal{L}$, $\mathcal{H}$, $\mathcal{H}^*$ and $\mathcal{H}^{**}$. The maximal singular subspaces of Σ^* come in three types: subspaces induced on quads of Θ, the set of points collinear with some given point of Θ, as well as subspaces containing only points at distance 2 in Θ.

Each quad of Θ is a point of $\mathcal{S}$ and hence is in $\mathcal{P}$. Furthermore, each other maximal singular subspace of Σ^* containing some line of $\mathcal{L}$ consists of all the points at distance ≤ 2 in Θ from some fixed point of $\mathcal{P}^*$, respectively, $\mathcal{R}^*$. We denote this set of subspaces by $\mathcal{P}^{**}$, respectively, $\mathcal{R}^{**}$. Finally, the remaining maximal singular subspaces of Σ^* form a class that we denote by $\mathcal{R}$. These subspaces intersect the quads in a line or are disjoint from them. The polar space with point set $\mathcal{P} \cup \mathcal{R}$ will be denoted by Σ, and that with point set $\mathcal{P}^{**} \cup \mathcal{R}^{**}$ by Σ^{**}.

Each point $p^* \in \mathcal{P}^*$ is collinear inside Θ to the points of a singular plane inside the polar subspace of Σ^* induced on $\mathcal{P}^*$. This plane is contained in unique elements $p \in \mathcal{P}$ and $p^{**} \in \mathcal{P}^{**}$. Clearly p^{**} is the maximal singular subspace consisting of all points collinear inside Θ to p^*, and p is the unique quad of Θ containing p^*, i.e. p is the focus of the focal plane p^*. If $r^* \in \mathcal{R}^*$, then the lines of Σ^* on r^* intersects $\mathcal{P}^*$ in a D_3 polar subspace which we call Δ. The lines of Θ on r^* meet Δ in some plane containing only hyperbolic lines. By r we denote the maximal singular subspace of Σ^* containing this plane but no line from Θ. The quads of Θ on r^* meet Δ in a plane containing the center of the quad and all the lines through this center. These centers are all at distance 4 in Θ and hence form a plane with only hyperbolic lines. This plane is contained in two maximal singular subspaces of Σ^*. Let r^{**} be the maximal singular subspace on this plane containing some line of Θ.

Now we can define a triality map τ on Σ in the following way:

$$\tau : \ p \mapsto p^* \mapsto p^{**} \mapsto p$$

for all points $p \in \mathcal{P} \cup \mathcal{R}$.

The map $p^* \mapsto p$ as well as the map $p^* \mapsto p^{**}$, $p \in \mathcal{P}^*$, induce isomorphisms between Γ^* and Γ, respectively, $\Gamma^{**} = (\mathcal{P}^{**}, \mathcal{L})$. In particular, τ induces isomorphisms between the three hexagons Γ, Γ^* and Γ^{**}. Just as we saw before that $\tau(H) \in \mathcal{H}^*$ for all $H \in \mathcal{H}$, one can easily check that $\tau(H^*) \in \mathcal{H}^{**}$ and $\tau(H^{**}) \in \mathcal{H}$ for all $H^* \in \mathcal{H}^*$ and $H^{**} \in \mathcal{H}^{**}$. This implies that the above isomorphisms extend to isomorphisms between $\mathcal{S}$, $\mathcal{S}^*$ and $\mathcal{S}^{**} = (\mathcal{P}^{**}, \mathcal{L} \cup \mathcal{H}^{**})$, and also between Σ, Σ^* and Σ^{**}. Hence τ is a triality automorphism of the D_4 geometry induced on Σ. The lines fixed by τ are precisely the lines in $\mathcal{L}$, while $\mathcal{P}$ is the set of self conjugate points (i.e. points p with $p \in \tau(p)$). Thus Γ is isomorphic to the generalized hexagon whose lines are the lines fixed by a triality automorphism of Σ. If p is a point of $\mathcal{P}$, then all lines on p inside the plane $p^* \cap p^{**}$ of the polar space Σ are in

$\mathcal{L}$. Thus by [15] the triality map τ is of type I_{Id} and Γ is isomorphic to the G_2 hexagon $H(k)$ as constructed by Tits. This finishes our second proof of Theorem 4.1.

5. Distance-3 hyperbolic lines in generalized hexagons

By Theorem 2.9 we know that a generalized n-gon whose distance-$n/2$ hyperbolic lines are long is either a generalized quadrangle or a generalized hexagon. In Section 3 we have given the classification of all generalized quadrangles with only long distance-$n/2$ hyperbolic lines, in this section we classify such hexagons.

Note that from Theorem 2.9 follows that, if all distance-3 hperbolic lines of a generalized hexagon Γ are long, then all points are distance-2 regular. From Ronan's characterization of the Moufang hexagons, it follows that Γ has the Moufang property, see [12]. By inspection, Γ is isomorphic to $H(k)$ for some field k of even characteristic. We will now avoid the yet unpublished classification of the Moufang hexagons to prove that statement.

Theorem 5.1 *Suppose $\Gamma = (\mathcal{P}, \mathcal{L})$ is a thick generalized hexagon and all distance-3 hyperbolic lines of Γ are long. Then Γ is isomorphic to the generalized hexagon $H(k)$ for some field k of even characteristic.*

Proof. We prove this theorem in a number of steps. Let $\mathcal{H}$ denote the set of distance-3 hyperbolic lines of Γ, and consider the space $\mathcal{C} = (\mathcal{P}, \mathcal{H})$.

Step 1. The space $\mathcal{C}$ is a connected copolar space; its diameter is 2.
Proof. Let $H \in \mathcal{H}$ be a hyperbolic line on the two points x and y. Let $L \in \mathcal{L}$ be at distance 3 from both x and y. All the points of H are at distance 3 from L. Moreover, each point on L is at distance 2 from a point of H. Now suppose z is a point in $\mathcal{P}$. If $z^{\perp}$ meets H in at least 2 points, then

by defintion, H is contained in $z^\perp$. It remains to show that $z^\perp \cap H$ is not empty. Suppose the contrary. Then the distance between z and L is 5. Let $(z, L_1, z_1, L_2, z_2, L)$ be a path from z to L. Then L_1 is as distance 5 from all points of H. On H there is a unique point at distance 2 from z_2. Without loss of generality we may assume that x is that point. But then the distance between z_1 and x is 4 and between z_1 and y is 6. Hence, as H is long, every point of L_1, and thus in particular z, is at distance 4 from a point on H. A contradiction.

For any two points x and y at distance $2j$, $j \neq 3$ let v be an element of Γ at distance j from both x and y. Then, as Γ is thick, we can find a point at distance $6 - j$ from v, that is at distance 6 from both x and y. Thus $\mathcal{C}$ is connected of diameter 2.□

Step 2. Let x and y be two points collinear in $\mathcal{C}$. Then $x^\perp \cap y^\perp$ is a connected subspace of $\mathcal{C}$.

Proof. The set $x^\perp \cap y^\perp$ is clearly a subspace of $\mathcal{C}$. Denote this space by S.

Let L and M be two opposite lines of Γ at distance 3 from x and y. All points of L and M are in $x^\perp \cap y^\perp$. Moreover, each point of L is at distance 6 to all but one of the points of M and vice versa. In particular, inside S there is a path from any point of L to any point of M. Now suppose that z is an arbitrary point of $x^\perp \cap y^\perp$. Without loss of generality we can assume z to be at distance 4 from x. This implies that z is at distance 6 from the point of L or of M that is at distance 2 from x. In particular, z is in the same connected component of S as the points of L and M. Hence S is connected.

□

Step 3. $\mathcal{C}$ satisfies Pasch's axiom.

Proof. Let H_1 and H_2 be two hyperbolic lines in $\mathcal{H}$ meeting at a point a.

Let b_1 are b_2 be points on H_1 and let c_1 and c_2 be points on H_2, all different from a. Suppose that b_i and c_i are collinear in $\mathcal{C}$, $i \in \{1, 2\}$. We will prove that the lines $H(b_1, c_1)$ and $H(b_2, c_2)$ of $\mathcal{H}$ intersect. Clearly we may assume that $b_1 \neq b_2$ and $c_1 \neq c_2$.

Let x be a point in $S = b_1^\perp \cap c_1^\perp$, but not in $a^\perp$. (Such a point exists, since $H(b_1, c_1) \neq H_1$.) Then x and $a^\perp \cap b_1^\perp \cap c_1^\perp$ are in S. Moreover, x and $S_a = a^\perp \cap S$ generate S. For, if $y \in S$ is collinear in $\mathcal{C}$ to x, then the distance-3 hyperbolic line $H(x, y)$ meets $a^\perp$ and thus is in the subspace of S generated by x and S_a. If H is a distance-3 hyperbolic line of S on y then x is collinear with at least 2 points of that line. Thus H is also in $\langle x, S_a \rangle$. By connectedness of S we find that $\langle x, S_a \rangle = S$.

Let d be the unique point on $H(b_2, c_2)$ that is also in $x^\perp$. Then $d \in x^\perp \cap S_a^\perp$, hence $d \in S^\perp = \{b_1, c_1\}^{\perp\perp} = H(b_1, c_1)$. □

A consequence of the above is that any two intersecting lines from $\mathcal{C}$ generate a dual affine plane. Each point of such a plane is contained in a unique maximal coclique of the collinearity graph of the plane called *transversal coclique.* For a point p outside a dual affine plane π the intersection of π with $p^\perp$ is either a line, a transversal coclique or the whole plane π. This clearly implies that for any two points x and y in a transversal coclique T we have $T \subseteq \{x, y\}^{\perp\perp}$.

Step 4. Distance-2 hyperbolic lines in Γ are long.

Proof. Let x and y be two points at distance 4 in Γ, and H the hyperbolic line on x and y.

Suppose L is a line of Γ on the point x and at distance 5 from y. Let z and z' be two points on L different from x and let π be the dual affine plane in $\mathcal{C}$ generated by y, z and z'. By the remark preceding this step, we find that the transversal coclique of π on z and z' is contained in L. Denote by T the transversal coclique of π containing y. If u is a point in $x^\perp \cap y^\perp$, then either $L \in u^\perp$, and $\pi \subset u^\perp$ or $u^\perp \cap L = \{x\}$ and $u^\perp \cap \pi = T$. In any case $T \subseteq u^\perp$ and hence $T \subseteq H$.

If w is some arbitrary point of Γ opposite x, then $w^\perp$ meets L in a point different from x. Without loss of generality we could have chosen z to be that point. Since w is opposite x, we find $w^\perp \cap \pi$ to be a line in $\mathcal{H}$. In particular, $w^\perp$ meets T and hence H. Hence for all points v of Γ we have $v^\perp$ meeting H nontrivially. This clearly implies H to be long. □

Step 5. Γ is isomorphic to $G_2(k)$ hexagon for some field k of even characteristic.

Proof. By Theorem 4.1 Γ is isomorphic to the $G_2(k)$ hexagon $H(k)$ for some field k. This hexagon has long hyperbolic lines only if the hyperbolic lines of the orthogonal polar space on the singular and distance-2 hyperbolic lines has long hyperbolic lines. But that is only the case for fields of even characteristic. □

Remark. Instead of referring to Theorem 4.1 in the proof of the above theorem, we could also have used the results of [5] and [7] on copolar spaces to find that $\mathcal{C}$ is the geometry of hyperbolic lines in a 5 dimensional symplectic space. A direct proof of the fact that the singular and hyperbolic lines of Γ form a projective space on which $\perp$ induces a nondegenerate symplectic polarity can also be obtained easily.

6. On generalized polygons with parameters and regular points

In this final section we prove Corollary 1.5. Thus assume that Γ is a thick finite or compact connected topological generalized n-gon of order (s,t), with all points distance-2 regular.

By the Feit-Higman Theorem for finite n-gons and a similar result by Knarr [8] and Kramer [9] on topological n-gons we can assume n to be 4, 6 or 8, respectively, 4 or 6.

Lemma 6.1 *If Γ contains a distance-2 regular point p, then $s \geq t$. Moreover, if $s = t$, then the point p is projective.*

Proof. For topological n-gons, this is proved in [14]. There it is also remarked that the same result also holds for finite n-gons. A check is left to the reader.
$\square$

By the assumption of Corollary 1.5, Proposition 2.10 and the above lemma we find that Γ is either a generalized quadrangle or a generalized hexagon with all points projective. Lemma 2.8 and Theorem 1.2 and 1.3, imply that Γ is isomorphic to either $W(k)$ or $H(k)$, where k is a finite field or topological. As in [13] we can conclude that k is either finite or isomorphic to $\mathbb{R}$ or $\mathbb{C}$. This proves 1.5.

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