Orthogonal, Symplectic and Unitary Polar Spaces Sub-weakly Embedded in Projective Space

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Abstract

We show that every sub-weak embedding of any non-singular orthogonal or unitary polar space of rank at least 3 in a projective space $\mathbf{PG}(d,\mathbb{K})$, $\mathbb{K}$ a commutative field, is a full embedding in some subspace $\mathbf{PG}(d,\mathbb{F})$, where $\mathbb{F}$ is a subfield of $\mathbb{K}$; the same theorem is proved for every sub-weak embedding of any non-singular symplectic polar space of rank at least 3 in $\mathbf{PG}(d,\mathbb{K})$, where the field $\mathbb{F}'$ over which the symplectic polarity is defined is perfect in the case that the characteristic of $\mathbb{F}'$ is two and the secant lines of the embedded polar space Γ contain exactly two points of Γ. This generalizes a result announced by LEFÈVRE-PERCSY [5] more than ten years ago, but never published. We also show that every quadric defined over a subfield $\mathbb{F}$ of $\mathbb{K}$ extends uniquely to a quadric over the groundfield $\mathbb{K}$, except in a few well-known cases.

1 Introduction and Statement of the Results

In this paper we always assume that $\mathbb{K}$ and $\mathbb{F}$ are commutative fields. Any polar space considered in this paper is assumed to be non-degenerate (which means that no point of the polar space is collinear with all points of the polar space), unless explicitly mentioned otherwise.

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A *weak embedding* of a point-line geometry Γ with point set $\mathcal{S}$ in a projective space $\mathbf{PG}(d,\mathbb{K})$ is a monomorphism θ of Γ into the geometry of points and lines of $\mathbf{PG}(d,\mathbb{K})$ such that

(WE1) the set $\mathcal{S}^\theta$ generates $\mathbf{PG}(d,\mathbb{K})$,

(WE2) for any point x of Γ, the subspace generated by the set $X = \{y^\theta \parallel y \in \mathcal{S}$ is collinear with $x\}$ meets $\mathcal{S}^\theta$ precisely in X,

(WE3) if for two lines L_1 and L_2 of Γ the images L_1^θ and L_2^θ meet in some point x, then x belongs to $\mathcal{S}^\theta$.

In such a case we say that the image Γ^θ of Γ is weakly embedded in $\mathbf{PG}(d,\mathbb{K})$.

A *full embedding in* $\mathbf{PG}(d,\mathbb{K})$ is a weak embedding with the additional property that for every line L, all points of $\mathbf{PG}(d,\mathbb{K})$ on the line L^θ have an inverse image under θ.

Weak embeddings were introduced by Lefevre-Percsy [3, 5]; in these papers she announced the classification of all weakly embedded finite polar spaces (clearly the polar spaces are considered here as point-line geometries) having the additional property that there exists a line of $\mathbf{PG}(d,\mathbb{K})$ which does not belong to Γ^θ and which meets $\mathcal{S}^\theta$ in at least three points. Only the case $d = 3$, $|\mathbb{K}| < \infty$ and rank(Γ)=2 was published [4]. The question arose again in connection with full embeddings of generalized hexagons (see Thas & Van Maldeghem [7]) and a proof seemed desirable. In the present paper, we will first show that the condition (WE3) is superfluous and then classify all — finite and infinite — weakly embedded non-singular polar spaces of rank at least 3 of orthogonal, symplectic or unitary type, assuming that for the symplectic type the field $\mathbb{F}'$ over which the symplectic polarity is defined is perfect in the case that $\mathbb{F}'$ has characteristic two and no line of $\mathbf{PG}(d,\mathbb{K})$ which does not belong to Γ^θ intersects $\mathcal{S}^\theta$ in at least three points. The classification of all generalized quadrangles weakly embedded in finite projective space can be found in Thas & Van Maldeghem [8].

We call a monomorphism θ from the point-line geometry of a polar space Γ with point set $\mathcal{S}$ to the point-line geometry of a projective space $\mathbf{PG}(d,\mathbb{K})$ a *sub-weak embedding* if it satisfies conditions (WE1) and (WE2). Usually, we simply say that Γ is weakly or sub-weakly embedded in $\mathbf{PG}(d,\mathbb{K})$ without referring to θ, that is, we identify the points and lines of Γ with their images in $\mathbf{PG}(d, \mathbb{K})$. In such a case the set of all points of Γ on a line L of Γ will be denoted by L^*.

If the polar space Γ arises from a quadric it is called *orthogonal*, if it arises from a hermitian variety it is called *unitary*, and if it arises from a symplectic polarity it is called *symplectic*. In these cases Γ is called *non-singular* either if the hermitian variety is non-singular, or if the symplectic polarity is non-singular, or if the quadric is non-singular (in the sense that the quadric Q, as algebraic hypersurface, contains no singular point over the algebraic closure of the ground field over which Q is defined); in the symplectic and hermitian case this is equivalent to assuming that the corresponding matrix is non-singular. In the orthogonal case with characteristic not 2, in the symplectic case and in the hermitian case, Γ is non-singular if and only if it is non-degenerate; in the orthogonal case with characteristic 2, non-singular implies non-degenerate, but when not every element of $\mathbb{K}$ is a square, and only then, a non-degenerate quadric may be singular.

Our main results read as follows.

Theorem 1 *Let Γ be a non-singular polar space of rank at least 3 arising from a quadric, a hermitian (unitary) variety or a symplectic polarity, where for Γ symplectic the polarity is defined over a perfect field $\mathbb{F}'$ in the case that $\mathbb{F}'$ has characteristic two and the secant lines of Γ contain exactly two points of Γ, and let Γ be sub-weakly embedded in the projective space $\mathbf{PG}(d, \mathbb{K})$. Then Γ is fully embedded in some subspace $\mathbf{PG}(d, \mathbb{F})$ of $\mathbf{PG}(d, \mathbb{K})$, for some subfield $\mathbb{F}$ of $\mathbb{K}$.*

If Γ is finite, then it is automatically of one of the three types mentioned. Moreover, it is non-degenerate if and only if it is non-singular. Combining this with Thas & Van Maldeghem [8], we have

Corollary 1 *(i) Let Γ be a non-degenerate polar space sub-weakly embedded in the finite projective space $\mathbf{PG}(d, q)$. Then Γ is fully embedded in some subspace $\mathbf{PG}(d, q')$ of $\mathbf{PG}(d, q)$, for some subfield $\mathbf{GF}(q')$ of $\mathbf{GF}(q)$, unless Γ is the unique generalized quadrangle of order $(2, 2)$ universally embedded in $\mathbf{PG}(4, q)$ with q odd.*

(ii) Let Γ be a finite non-degenerate polar space of rank at least 3 sub-weakly embedded in the projective space $\mathbf{PG}(d, \mathbb{K})$. Then Γ is fully embedded in some subspace $\mathbf{PG}(d, q)$ of $\mathbf{PG}(d, \mathbb{K})$, for some subfield $\mathbf{GF}(q)$ of $\mathbb{K}$.

Our second main result might belong to folklore but we give a full proof here.

Theorem 2 (i) *Let Q be a non-degenerate non-empty quadric of $\mathbf{PG}(d, \mathbb{F})$, $d \geq 2$, and let $\mathbb{K}$ be a field containing $\mathbb{F}$. Then in the corresponding extension $\mathbf{PG}(d, \mathbb{K})$ of $\mathbf{PG}(d, \mathbb{F})$ there exists a unique quadric containing Q, except if $d = 2$ and $\mathbb{F} \in \{\mathbf{GF}(2), \mathbf{GF}(3)\}$, or $d = 3$, $\mathbb{F} = \mathbf{GF}(2)$ and Q is of elliptic type.*

(ii) *Let Γ be a non-singular symplectic polar space defined by a symplectic polarity in $\mathbf{PG}(d, \mathbb{F})$, $d \geq 3$, and let $\mathbb{K}$ be a field extending $\mathbb{F}$. Then in the corresponding extension $\mathbf{PG}(d, \mathbb{K})$ of $\mathbf{PG}(d, \mathbb{F})$, there exists a unique symplectic polarity whose corresponding polar space contains Γ.*

(iii) *Let H be a non-singular non-empty hermitian variety of $\mathbf{PG}(d, \mathbb{F})$, $d \geq 2$, with associated $\mathbb{F}$-involution σ, and let $\mathbb{K}$ be a field containing $\mathbb{F}$ admitting a $\mathbb{K}$-involution τ the restriction of which to $\mathbb{F}$ is exactly σ. Then in the corresponding extension $\mathbf{PG}(d, \mathbb{K})$ of $\mathbf{PG}(d, \mathbb{F})$ there exists a unique hermitian variety with associated field involution τ and containing H.*

Remark. It is now easy to extend Theorem 2 to the singular cases with at least one non-singular point over $\mathbb{F}$. Again the extension of the polar space Γ is unique, except for Γ orthogonal and $\mathbb{F} \in \{\mathbf{GF}(2), \mathbf{GF}(3)\}$.

2 Proof of Theorem 1

In the sequel, we adopt the notation $x^\perp$ for the set of all points collinear with the point x in a polar space. After having coordinatized $\mathbf{PG}(d, \mathbb{K})$, we denote by e_i, $1 \leq i \leq d+1$, the point with coordinates $(0, \ldots, 0, 1, 0 \ldots, 0)$, where the 1 is in the ith position. By generalizing this, we denote by e_J the point with every coordinate equal to 0 except in each position belonging to the set J, $J \subseteq \{1, 2, \ldots, d+1\}$, where the coordinate equals 1. We also remark that polar spaces are *Shult spaces*, i.e. for every point x and every line L, $x^\perp$ contains either all points of L or exactly one point of L (we will call that property the *Buekenhout-Shult axiom*).

We prove Theorem 1 in a sequence of lemmas.

Lemma 1 *If L is a line of the sub-weakly embedded polar space Γ, then the only points of Γ on L are the points of L^*.*

PROOF. Let x be a point of Γ on L with $x \notin L^*$. By the Buekenhout-Shult axiom L^* contains a point y collinear with x. So the lines xy and L of Γ coincide in $\mathbf{PG}(d, \mathbb{K})$, contradicting the fact that θ is a monomorphism. $\square$

Lemma 2 *Every sub-weak embedding of a non-degenerate polar space is also a weak embedding.*

PROOF. Let Γ be a polar space sub-weakly embedded in $\mathbf{PG}(d, \mathbb{K})$ for some field $\mathbb{K}$. Let L_1 and L_2 be two lines of Γ meeting in a point x of $\mathbf{PG}(d, \mathbb{K})$ which does not belong to $\mathcal{S}$, the point set of Γ. If some point y of Γ is collinear with all points of L_1^*, then $y^\perp$ contains a triangle of the plane L_1L_2 of $\mathbf{PG}(d, \mathbb{K})$ ($y^\perp$ contains some point of L_2^* by the Buekenhout-Shult axiom). Hence (WE2) implies that y is collinear with all points of L_2^*. If we let y vary on L_1^*, then we see that all points of L_1^* are collinear with all points of L_2^*, in other words, L_1^* and L_2^* span a 3-dimensional singular subspace S of Γ. Since Γ is non-degenerate, no point of S is collinear with all other points of Γ, hence there exists a point z of Γ not collinear with all points of S. It is easily seen that $z^\perp$ meets S in the point set of a plane π of Γ. Since any two lines of Γ in π generate the plane L_1L_2, the points of π span the plane L_1L_2 of $\mathbf{PG}(d, \mathbb{K})$. By (WE2), $z^\perp$ must contain all points of S (since they all lie in L_1L_2), a contradiction. $\square$

Let L be any line of $\mathbf{PG}(d, \mathbb{K})$ containing at least two points of Γ which are not collinear in Γ. Then we call L a *secant line*. By Lemma 1, no secant line contains two collinear points. The following result is due to Lefevre-Percsy [3].

Lemma 3 *The number of points of* Γ *on a secant line is a constant.*

We put that number equal to δ (δ is possibly an infinite cardinal) and call it the *degree* of the embedding.

We now prepare the proof of the case $\delta = 2$ by first proving a lemma which certainly belongs to folklore.

A *kernel* of a non-empty non-singular quadric in a projective space is any point belonging to every tangent hyperplane of the quadric. As the quadric is non-singular a kernel does not belong to the quadric. The subspace of all kernels is sometimes called the *radical* of the quadric.

Lemma 4 *Every non-empty non-singular quadric has at most one kernel.*

PROOF. Suppose that the non-singular non-empty quadric Γ of $\mathbf{PG}(d,\mathbb{K})$ has a radical V of dimension at least one. Extend Γ over the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$ to the non-singular quadric $\overline{\Gamma}$. Then $\overline{\Gamma}\cap\overline{V}$, with $\overline{V}$ the corresponding extension of V, is a non-empty quadric. Let x be a point of it. Every line xp with $p\in\Gamma$, $p\neq x$, is a tangent line of $\overline{\Gamma}$ and all these lines generate the whole projective space $\mathbf{PG}(d,\overline{\mathbb{K}})$. This yields a contradiction as all tangent lines of $\overline{\Gamma}$ at x lie in the tangent hyperplane of $\overline{\Gamma}$ at x. $\square$

Lemma 5 *Let Γ be a non-singular polar space of rank at least 3 arising from a quadric, a hermitian (unitary) variety or a symplectic polarity, where for Γ symplectic the polarity is defined over a perfect field $\mathbb{F}'$ in the characteristic two case, and let Γ be sub-weakly embedded of degree 2 in the projective space $\mathbf{PG}(d,\mathbb{K})$. Then Γ is fully embedded in some subspace $\mathbf{PG}(d,\mathbb{F})$ of $\mathbf{PG}(d,\mathbb{K})$, for some subfield $\mathbb{F}$ of $\mathbb{K}$.*

PROOF. We label the steps of the proof for future reference.

(a) Let Γ be a non-singular orthogonal polar space sub-weakly embedded in $\mathbf{PG}(d,\mathbb{K})$, $d\geq 3$, and suppose that Γ has rank at least 3. We identify the points and lines of Γ with the corresponding points and lines of $\mathbf{PG}(d,\mathbb{K})$. Let π be any plane of Γ. Three non-concurrent lines of π span a unique plane π' of $\mathbf{PG}(d,\mathbb{K})$. Any other line of π meets these three lines in at least two points, hence we see that π' is uniquely determined by π; moreover, the points and lines of π determine a unique subplane of π'. Hence π is isomorphic to a projective plane over some subfield $\mathbb{F}$ of $\mathbb{K}$. Moreover, since Γ is residually connected (as a polar space or a building, see e.g. BUEKENHOUT [1]), $\mathbb{F}$ is independent from π. Hence, if we coordinatize $\mathbf{PG}(d,\mathbb{K})$, then every re-coordinatization by means of a linear transformation (so without using a field automorphism) which maps the points e_1,e_2,e_3 and $e_{\{1,2,3\}}$ onto points of π, defines a subfield $\mathbb{F}$ of $\mathbb{K}$ which is independent of the choice of π and where $\mathbb{F}$ is equal to the set of possible coordinates (in the new coordinate system) for points of π. This implies that the set of all points of Γ on any line of Γ is uniquely determined in $\mathbf{PG}(d,\mathbb{K})$ by any three of its points; indeed, re-coordinatize so that these points become e_1,e_2 and

$e_{\{1,2\}}$, and then all points of the line are obtained by taking all linear combinations of the vectors $(1, 0, \ldots, 0)$ and $(0, 1, 0, \ldots, 0)$ over $\mathbb{F}$. All this shows that not only the isomorphism type of $\mathbb{F}$ is fixed, but also the subfield $\mathbb{F}$ itself.

(b) Now consider a line L_1 of Γ and a point x_1 of Γ on it. Through x_1 there is a line M_1 of Γ with the property that L_1 and M_1 are not in a common plane of Γ. Now we take a point y_1 of Γ not collinear with x_1 and we consider the unique line L_2 of Γ passing through y_1 and meeting M_1 in a point of Γ. Now we show that in Γ no point on L_2 is collinear with all points of L_1. The point x_1 is not collinear with y_1, and as L_1 and M_1 are not in a common plane of Γ the point $M_1 \cap L_2$ is not collinear with all points of L_1. As x_1 is not collinear with y_1, it is not collinear with two distinct points of L_2; hence no point of L_2 different from y_1 and $M_1 \cap L_2$ is collinear with all points of L_1. Similarly, in Γ no point on L_1 is collinear with all points on L_2. If L_1 and L_2 would span a plane L_1L_2, then every point of L_2 is in the space spanned by $x^\perp$ for every $x \in L_1^*$, since there is at least one point of $x^\perp$ on L_2^*. So by (WE2) the point $x \in L_1^*$ is collinear with every point of L_2^*, a contradiction. Hence L_1 and L_2 generate a 3-space U of $\mathbf{PG}(d, \mathbb{K})$. In Γ the lines L_1, L_2 and their points generate a polar space Ω; Ω corresponds to a hyperbolic quadric Q_3^+ (of a 3-space) on the non-singular quadric from which Γ arises. The point set of Ω will also be denoted by Q_3^+, and the sets of lines of Ω corresponding to the reguli of Q_3^+ will also be called the reguli of Ω. Since all points of Ω lie on lines meeting both L_1 and L_2, we see that Ω is entirely contained in U. Let $M_2 \neq M_1$ belong to the regulus of Ω defined by M_1. Put $x_2 = L_1 \cap M_2$, $x_3 = L_2 \cap M_1$ and $x_4 = L_2 \cap M_2$. Let x_5 be one further point of Ω not on one of the lines L_1, L_2, M_1, M_2 and let L_3, respectively M_3, be the line of Ω through x_5 and belonging to the regulus defined by L_1, respectively M_1. No four of the points $\{x_1, x_2, x_3, x_4, x_5\}$ are coplanar, so they determine a unique subspace V of U over $\mathbb{F}$.

(c) We claim that Ω is fully embedded in V, that is, we claim that all points of Ω are contained in V. Indeed, the points on L_1 in V are uniquely determined by the three points x_1, x_2 and $M_3 \cap L_1$. But as remarked above, these points are precisely all points of Γ on L_1. Similarly for L_2, M_1 and M_2. Let M_4 be a line of Ω meeting L_1, L_2 in points of Ω, so of V, with $M_1 \neq M_4 \neq M_2$; then M_4 is a line of V. As L_3 is a line of V, also $L_3 \cap M_4$ is a point of V. It follows that the points of M_4 in V are exactly the points of M_4 in Ω. Similarly, for any line L_4 of Ω meeting M_1, M_2 in points of Ω, the points of L_4 in V are exactly the points of L_4 in Ω. If y is any point of Ω, then the line of Γ through y meeting L_1, L_2, respectively M_1, M_2, contains at least two points of V, and hence the intersection y of these two lines also belongs to V. This shows our claim.

(d) Next we prove that no other point of Γ belongs to U. Indeed, suppose the point z of Γ lies in U, but is not contained in Ω. Then z does not belong to V since the unique line M in V through z meeting both L_1 and L_2 contains three points of Γ, say z, x_1, x_4, hence belongs to Γ, contradicting the fact that z does not belong to Ω. In Γ the points of Ω collinear with z either are all the points of Ω, or are the points of a point set $\mathcal{C}$ of Ω corresponding to a non-singular conic of the hyperbolic quadric Q_3^+, or are the points of Ω on two lines of Ω, say L_1 and M_1. Noticing that for every point y of Ω, the space generated by $y^\perp$ in $\mathbf{PG}(d, \mathbb{K})$ meets U in a plane (by axiom (WE2)), we see that in the first case z must lie in every plane containing two lines of Ω. This yields a contradiction since these planes have no intersection point in V, hence neither in U. In the second case z must lie in the planes tangent to Q_3^+ at points of $\mathcal{C}$. These planes meet in at most one point, which lies in V, a contradiction. In the third case z must lie in all planes of V containing L_1 or M_1, hence $z = x_1$, a contradiction. This proves our claim.

(e) An orthogonal subspace of Γ containing lines is called s-dimensional if the corresponding subquadric on the quadric from which Γ arises generates an $(s+1)$-dimensional space. Now suppose that any $(c-1)$-dimensional non-singular orthogonal subspace Ω' of Γ containing lines is fully embedded in a c-dimensional projective subspace over $\mathbb{F}$ of $\mathbf{PG}(d, \mathbb{K})$, $3 \leq c \leq d-1$. We show that, if Ω is a c-dimensional non-singular orthogonal subspace of Γ containing lines, then Ω is fully embedded in some $(c+1)$-dimensional projective subspace $\mathbf{PG}(c+1, \mathbb{F})$ of $\mathbf{PG}(c+1, \mathbb{K})$. Since Ω is non-singular, it contains some $(c-1)$-dimensional non-singular orthogonal subspace Ω' containing lines. By assumption Ω' is contained in a c-dimensional projective space V' over

$\mathbb{F}$. Let U' be the extension of V' over $\mathbb{K}$. We first show that U' does not contain any point of $\Omega \setminus \Omega'$. Let the point x of $\Omega \setminus \Omega'$ belong to U'. Then $x^\perp$ and the point set of Ω' intersect in a point set Q'' which corresponds to a non-singular subquadric of the quadric from which Γ arises. By (WE2) Q'' is contained in a $(c-1)$-dimensional subspace V'' of V'. Assume that Q'' does not generate V''. Then Ω' contains a point u of V'' not on Q''. Every line of Ω' through u contains a point of $x^\perp$, so every line of Ω' through u contains a point of Q''. Hence V'' contains all lines of Ω' through u. Analogously, V'' contains all lines of Ω' through u', with $u' \neq u$ a second point of Ω' in $V'' \setminus Q''$. So the tangent hyperplanes of the point set of Ω' at u and u' coincide with V'', a contradiction. We conclude that Q'' generates V''. The extension of V'' over $\mathbb{K}$ will be denoted by U''. If $x \notin U''$, then $x^\perp \cap U'$ spans U', hence by (WE2) all points of Ω' are collinear with x, a contradiction. So $x \in U''$. Let y be a point of Q'' and let V'_y be the tangent hyperplane of Ω' at y; the extension of V'_y to $\mathbb{K}$ is denoted by U'_y. If $x \notin U'_y$, then the space generated by x and U'_y is U', so by (WE2) $y^\perp$ contains all points of Ω', a contradiction. Hence $x \in U'_y$. Let V''_y be the tangent hyperplane of Q'' at y, and let U''_y be the extension of V''_y to $\mathbb{K}$; then $V''_y = V'_y \cap V''$ and $U''_y = U'_y \cap U''$. As $x \in U''$, we have $x \in U''_y$ for every point y of Q''. This implies that $x \in V''$ and that x is the unique kernel of Q'' in V''. Since Q'' has a unique kernel, the dimension $c-1$ of the space generated by Q'' is even and the matrix defined by Q'' has rank equal to $c-1$. If x is also kernel of Ω', then as $c+1$ is even Ω' admits at least a line L of kernels. Over the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$ the extension $\overline{L}$ of L contains a point r of the extension $\overline{\Omega}'$ of Ω'. The point r is singular for $\overline{\Omega}'$, hence Ω' is singular, a contradiction. Consequently x is not a kernel for Ω'. Hence there is a line N of V' containing x and two distinct points y_1, y_2 of Ω'. Since the degree of the weak embedding is equal to 2, N is a line of Γ, so $y_1 = y_2 \in Q''$, a contradiction. It follows that U' does not contain any point of $\Omega \setminus \Omega'$.

(f) Let x_1 be any point of $\Omega \setminus \Omega'$ and let L_1 be any line of Ω through x_1. Evidently, L_1 meets Ω' in a unique point y. Let L_2 be any line of Ω' such that L_1, L_2 and their points in Ω' generate a polar space in Ω with as point set a hyperbolic quadric $Q = Q_3^+$. Take any point $x_2 \neq x_1$ on L_1^* with $x_2 \neq y$. The space V' together with the two points x_1, x_2 defines a unique $(c+1)$-dimensional subspace V over $\mathbb{F}$, which contains x_1, x_2

and y and hence all points of Ω on L_1. Also, V contains all points of Ω on L_2 and all points of the line of Ω' containing y and concurrent with L_2. Similarly as in (c), one now shows that Q_3^+ is completely contained in a 3-dimensional subspace over $\mathbb{F}$ which clearly belongs to V.

(g) We now show that all points of Ω belong to V. Let z be any point of $\Omega \setminus \Omega'$. First suppose that z is not collinear with y. Consider a line M_1 on Ω' through y and such that L_1 and M_1 are not contained in a plane of Ω. Let L_3 be the unique line of Ω through z meeting M_1 in a point of Ω. Then clearly L_1 and L_3 define a hyperbolic quadric Q' over $\mathbb{F}$ on Ω. We show that the polar subspace of Ω with point set Q' has two different lines M_1 and L_2' in common with Ω'. If we identify the point set of Ω with a quadric in some $\mathbf{PG}(c+1, \mathbb{F})$, then the 3-space of Q' and the hyperplane defined by Ω have a plane ζ in common, which intersects Q' in two distinct lines. Hence Q' has two different lines M_1 and L_2' in common with Ω'. Interchanging roles of L_2 and L_2', we now see that z also belongs to the space V. Now suppose that the point z of $\Omega \setminus \Omega'$, $z \neq y$, is collinear with y. Let L_3 and L_4, with $L_3 \neq yz \neq L_4$, be two distinct lines of Ω through z for which yL_3 and yL_4 are not planes of Ω. By the foregoing all points of $L_3^* \setminus \{z\}$ and $L_4^* \setminus \{z\}$ belong to V. Hence also the intersection of L_3 and L_4, that is z, belongs to V. So we conclude that each of the points of Ω belongs to V, and consequently Ω is fully embedded in the space V over $\mathbb{F}$.

(h) Applying consecutively the previous paragraphs for $c = 3, 4, \ldots, d-1$, we finally obtain that Γ is fully embedded in some $\mathbf{PG}(d, \mathbb{F})$.

(i) Now let Γ be a non-singular hermitian polar space sub-weakly embedded in $\mathbf{PG}(d, \mathbb{K})$, $d \geq 3$, and suppose that the degree is 2. On the non-singular hermitian variety $\overline{\mathcal{H}}$ from which Γ arises we consider a non-singular hermitian variety $\mathcal{H}'$, where $\mathcal{H}'$ generates a 3-dimensional space. The corresponding point set on Γ will be denoted by $\mathcal{H}$ and the corresponding polar subspace of Γ by Ω. Let L, M be two non-intersecting lines of Ω. In $\mathbf{PG}(d, \mathbb{K})$, the lines L and M generate a 3-dimensional subspace $U = \mathbf{PG}(3, \mathbb{K})$, which contains all points of $\mathcal{H}$ (Ω is generated by L, M and their points in Ω). Now consider two points x and y in $\mathcal{H}$ which are not collinear in $\mathcal{H}$. Let $\mathcal{H}_x$ and $\mathcal{H}_y$ be the set of points of $\mathcal{H}$ collinear in $\mathcal{H}$ with x and y respectively. Clearly neither $\mathcal{H}_x$ nor $\mathcal{H}_y$ can be contained in a line of U. Also, by condition (WE2), neither $\mathcal{H}_x$ nor $\mathcal{H}_y$ generates U. Hence $\mathcal{H}_x$ and $\mathcal{H}_y$ define unique planes U_x and U_y respectively. These planes meet in a unique line N of U. Clearly N contains all points of $\mathcal{H}$ collinear in Ω with both x and y. Assume that z is any point of Γ on N. Further, let $u, v \in N \cap \mathcal{H}$, $u \neq v$. Then z is collinear in Γ with all points of $u^\perp \cap v^\perp$. Let u', v', z' be the points of $\overline{\mathcal{H}}$ which correspond to u, v, z respectively. As z' is collinear in $\overline{\mathcal{H}}$ with all points of $u'^\perp \cap v'^\perp$, it belongs to $\overline{\mathcal{H}} \cap u'v' = \mathcal{H}' \cap u'v'$. Hence z belongs to $\mathcal{H} \cap uv$. It follows that the set of *all* points of Γ on N corresponds to the point set $\overline{\mathcal{H}} \cap u'v' = \mathcal{H}' \cap u'v'$. As N meets Γ in more than 2 points, we are in contradiction with $\delta = 2$.

(j) Finally let Γ be a non-singular symplectic polar space sub-weakly embedded in $\mathbf{PG}(d, \mathbb{K})$, $d \geq 3$. Let $\mathbb{F}'$ be the ground field over which the symplectic polarity ζ from which Γ arises is defined.
If the characteristic of $\mathbb{F}'$ is not two, then a similar proof as for the hermitian case leads to a contradiction; here the secant line N will contain $|\mathbb{F}'| + 1$ points (note that the secant lines of Γ correspond (bijectively) to the non-isotropic lines of the symplectic polarity ζ).
If the characteristic of $\mathbb{F}'$ is two, then $\mathbb{F}'$ is perfect, hence Γ is also orthogonal. Now it follows from (a) – (h) that Γ is fully embedded in some $\mathbf{PG}(d, \mathbb{F})$. □

The next lemma is a result similar to Theorem 1 for projective spaces. A *sub-n-space* of a projective space $\mathbf{PG}(n, \mathbb{K})$ is any space $\mathbf{PG}(n, \mathbb{F})$, $\mathbb{F}$ a subfield of $\mathbb{K}$, obtained from $\mathbf{PG}(n, \mathbb{K})$ by restricting coordinates to $\mathbb{F}$ (with respect to some coordinatization). Note that, for many fields $\mathbb{K}$ and positive integers n, there exist subsets $\mathcal{S}$ of the point set of $\mathbf{PG}(n, \mathbb{K})$ such that the linear space induced in $\mathcal{S}$ by the lines of $\mathbf{PG}(n, \mathbb{K})$ is the point-line space of a $\mathbf{PG}(m, \mathbb{F})$ with $m > n$. The following result gives a necessary and sufficient condition for such a structure to be a sub-n-space. These conditions are basically (WE1) and some analogue of (WE2).

Lemma 6 *Let $\mathcal{S}$ be a generating set of points in the projective space $\mathbf{PG}(n, \mathbb{K})$, $\mathbb{K}$ a skewfield and let $\mathcal{L}$ be the collection of all intersections of size > 1 of $\mathcal{S}$ with lines of $\mathbf{PG}(n, \mathbb{K})$. Suppose $(\mathcal{S}, \mathcal{L})$ is the point-line space of some projective space $\mathbf{PG}(m, \mathbb{F})$, for some skewfield $\mathbb{F}$ and some positive integer m. Then $\mathbb{F}$ is a subfield of $\mathbb{K}$, $m = n$ and $\mathcal{S}$ and $\mathcal{L}$ are the point set and line set respectively of some sub-n-space $\mathbf{PG}(n, \mathbb{F})$ of $\mathbf{PG}(n, \mathbb{K})$ if and only if there* *exists a dual basis of hyperplanes in* $\mathbf{PG}(m, \mathbb{F})$ *such that each element* H *of that basis is contained in a hyperplane* H' *of* $\mathbf{PG}(n, \mathbb{K})$ *with* $H' \cap \mathcal{S} = H$.

PROOF. It is clear that the given condition is necessary. Now we show that it is also sufficient. If $m+1$ points of $\mathcal{S}$ generate $\mathbf{PG}(m, \mathbb{F})$, then by the condition that lines of $\mathbf{PG}(m, \mathbb{F})$ are line intersections of $\mathbf{PG}(n, \mathbb{K})$ with $\mathcal{S}$, these $m+1$ points must also span $\mathbf{PG}(n, \mathbb{K})$ (otherwise $\mathcal{S}$ is contained in some proper subspace of $\mathbf{PG}(n, \mathbb{K})$). Hence $m \geq n$. Now let $\{H_i : i = 0, 1, \ldots, m-1, m\}$ be a collection of hyperplanes of $\mathbf{PG}(m, \mathbb{F})$ meeting the requirements of the lemma. Put $S_i = H_0 \cap H_1 \cap \ldots \cap H_i$, $i = 0, 1, \ldots, m$. Suppose that S_j generates the same space as S_{j+1} in $\mathbf{PG}(n, \mathbb{K})$ for some j, $0 \leq j \leq m-1$. Let H_i be contained in the hyperplane H_i' (not necessarily unique at this point) of $\mathbf{PG}(n, \mathbb{K})$, $i = 0, 1, \ldots, m$. If x is a point of S_j not lying in S_{j+1} (x exists by the assumptions on H_i), then in $\mathbf{PG}(n, \mathbb{K})$ x is not generated by the points of H_{j+1}, since H'_{j+1} meets $\mathcal{S}$ precisely in H_{j+1}. But $S_{j+1} \subseteq H_{j+1}$, hence in $\mathbf{PG}(n, \mathbb{K})$ x is not generated by S_{j+1}, a contradiction. So S_j generates a space in $\mathbf{PG}(n, \mathbb{K})$ which is strictly larger than S_{j+1}. That means that we have a chain of $m+1$ subspaces of $\mathbf{PG}(n, \mathbb{K})$ consecutively properly contained in each other and all contained in H_0'; hence $n \geq m$. We conclude that $n = m$.

Now if we choose a basis of $\mathbf{PG}(n, \mathbb{F})$ (this is also a basis of $\mathbf{PG}(n, \mathbb{K})$), then is is clear that the corresponding coordinatization of $\mathbf{PG}(n, \mathbb{F})$ is the restriction of the coordinatization of $\mathbf{PG}(n, \mathbb{K})$ to the field $\mathbb{F}$. The result follows. $\square$

Lemma 7 *Let* Γ *be a non-singular polar space of rank at least* 3 *arising from a quadric, a symplectic polarity or a hermitian variety, and let* Γ *be sub-weakly embedded of degree* $\delta > 2$ *in the projective space* $\mathbf{PG}(d, \mathbb{K})$*. Then* Γ *is fully embedded in some subspace* $\mathbf{PG}(d, \mathbb{F})$ *of* $\mathbf{PG}(d, \mathbb{K})$*, for some subfield* $\mathbb{F}$ *of* $\mathbb{K}$*.*

PROOF. Let $\mathbb{F}'$ be the field underlying Γ.

(1) First, let the characteristic of $\mathbb{F}'$ be odd and let Γ be a non-singular symplectic polar space. By (j) in the proof of Lemma 5, secant lines of Γ correspond (bijectively) with non-isotropic lines of the symplectic polarity ζ from which Γ arises. Now the space Ω with point set $\mathcal{S}$,

the point set of Γ, and line set $\{L^* : L \text{ is a line of } \Gamma\} \cup \{S \cap \mathcal{S} : S \text{ is}$ the point set in $\mathbf{PG}(d, \mathbb{K})$ of a secant line of $\Gamma\}$ is a projective space. Every hyperplane H in that projective space Ω is the set of points of $\mathcal{S}$ collinear in Γ with some fixed point x of $\mathcal{S}$. It is easy to see that, as $\mathcal{S}$ is a generating set of $\mathbf{PG}(d, \mathbb{K})$, the hyperplane H of Ω generates a hyperplane H' of $\mathbf{PG}(d, \mathbb{K})$. Now by (WE2) the assumptions of Lemma 6 are satisfied and the result follows.

Next, assume that the characteristic of $\mathbb{F}'$ is two and let Γ be a non-singular symplectic polar space. Let ζ be again the symplectic polarity from which Γ arises. If ζ is defined in $\mathbf{PG}(d', \mathbb{F}')$, then we consider a subspace $\mathbf{PG}(3, \mathbb{F}')$ of $\mathbf{PG}(d', \mathbb{F}')$ in which ζ induces a non-singular symplectic polarity η. The polar space defined by ζ is Γ', and the polar space defined by η is Ω'. With Ω' corresponds the polar subspace Ω of Γ. Let L, M be two non-intersecting lines of Ω and let L', M' be the corresponding lines of Ω'. Let x be a point of Ω on L and y a point of Ω on M, where x and y are not collinear in Ω. The points of $\mathbf{PG}(3, \mathbb{F}')$ which correspond to x, y are denoted by x', y' respectively. As $\delta > 2$ the line xy contains a third point z of Γ. As, by (WE2), z is collinear in Γ to all points of $x^\perp \cap y^\perp$, the corresponding point z' of $\mathbf{PG}(d', \mathbb{F}')$ is collinear in Γ' to all points of $x'^\perp \cap y'^\perp$. Hence z' belongs to the line $x'y'$, so belongs to Ω'. It follows that z belongs to Ω. As Ω' is generated by z', L', M' and all points of L' and M', also Ω is generated by z, L, M and all points of L and M. Hence Ω is contained in a subspace $\mathbf{PG}(3, \mathbb{K})$ of $\mathbf{PG}(d, \mathbb{K})$. Then a similar argument as in (i) of Lemma 5 shows that the secant lines of Γ correspond (bijectively) to the non-isotropic lines of ζ. Now, analogously as in the odd characteristic case, the result follows.

(2) Now suppose that Γ is of orthogonal type. Let Γ' be the image of a natural full embedding of Γ in a projective space $\mathbf{PG}(d', \mathbb{F}')$ where the point set of Γ' is a non-degenerate quadric Q' of $\mathbf{PG}(d', \mathbb{F}')$. Denote by x' the element of Γ' corresponding to any element x of Γ. Let M be a secant line in $\mathbf{PG}(d, \mathbb{K})$. Let p_1, p_2, p_3 be three points of Γ on M. Consider a point r of Γ collinear with both p_1 and p_2. By (WE2) all points of Γ on M are collinear with r. If the lines $r'p_1', r'p_2', r'p_3'$ lie in a plane of $\mathbf{PG}(d', \mathbb{F}')$, then this must be a plane of Γ' and hence M is a line of Γ, a contradiction. Consequently r', p_1', p_2', p_3' generate a 3-dimensional subspace $\mathbf{PG}(3, \mathbb{F}')$ of $\mathbf{PG}(d', \mathbb{F}')$. Let $\mathbf{PG}(4, \mathbb{F}') \supseteq \mathbf{PG}(3, \mathbb{F}')$ intersect

Q' in a non-singular quadric Q'_1. Suppose the characteristic of $\mathbb{F}'$ is not 2. Then there is a unique second point s' of Q'_1 collinear with p'_1, p'_2, p'_3. So s is collinear with p_1, p_2, p_3. Since s and r are not collinear in Γ, s is not in the plane $rp_1p_2p_3$ by (WE2). Let N be a line of Γ concurrent with rp_1 and sp_2 in Γ, but not incident with r or s. The line R of Γ through p_3 meeting N^* lies in the 3-dimensional space $srp_1p_2p_3$. By (WE2) R is in the plane p_3rs. Let w be the unique point of R^* collinear with p_1; then w is also collinear with p_2 (by (WE2)). Clearly $w' \in Q'_1$, a contradiction. Hence the characteristic of $\mathbb{F}'$ is equal to 2.

Let p'_1, p'_2, p'_3 and r' be as above, and let $p'_1p'_2p'_3 \cap Q' = C'$; further let Q'_1 be as above. Let $s' \neq r'$ be a point of Q'_1 collinear with p'_1, p'_2 (s' exists since Q'_1 defines itself a polar space). By (WE2), s' is also collinear with p'_3. As in the previous paragraph, we construct the line R and the point w. Let V' be a line on Q'_1 through w', not containing p'_1, p'_2. There is a line L' meeting $r'p'_1$, $s'p'_2$ and V', thus implying that V belongs to the space $rswp_1p_2 = rsp_1p_2$. By (WE2), V is contained in the plane wp_1p_2. Let W be a line of Γ containing r and meeting V^*. Then W is in the plane $rp_1p_2 \neq wp_1p_2$, hence $V \cap W$ is on M. So M contains all the points x such that x' is on the conic C'. Note that the kernel k' of C' coincides with the kernel of Q'_1 (as all tangents $k'r'$, $k's'$ and $k'p'$ with $p' \in C'$ generate the 4-space of Q'_1). We now show that for any point x of Γ on M, the point x' belongs to C'. By (WE2), each point of Γ on M lies in $(\{p_1, p_2\}^\perp)^\perp$. But $(\{p'_1, p'_2\}^\perp)^\perp$ is the intersection of Q' with either a line (and this happens if and only if d' is odd) or a plane π (and this happens if and only if d' is even) containing the kernel k' of Q'. The first case contradicts $\delta > 2$, hence only the latter case occurs. But clearly π must meet Q' in C' and our claim follows.

Note that the argument of the previous paragraph also shows that all points of every conic on Q' lying in a plane which contains the kernel k' of Q' correspond to the points of intersection of Γ with some secant line M. Also, every two non-collinear points of Q' lie in such a unique plane. Projecting Γ' from the kernel k' onto some hyperplane $\mathbf{PG}(d'-1, \mathbb{F}')$ not containing k', we obtain an embedding of Γ' into $\mathbf{PG}(d'-1, \mathbb{F}')$ such that secant lines of Γ correspond with secant lines of the image Γ'' of Γ' in $\mathbf{PG}(d'-1, \mathbb{F}')$. Note that if $\mathbb{F}'$ is perfect, in particular when $\mathbb{F}'$ is finite, then Γ'' is a non-singular symplectic space and the result follows from the first part of the proof.

(3) Remark that in (1) and (2) the proof does not depend on the rank of Γ, as long as it is at least 2.

From now on we use the fact that the rank of the orthogonal polar space Γ is at least 3. By the last part of (2) we may assume that the field $\mathbb{F}'$ is not perfect. As in paragraph (a) of the proof of Lemma 5, one shows that any set L^*, with L a line of Γ, is a subline of L over a subfield $\mathbb{F}$ of $\mathbb{K}$ which is independent of L (and clearly $\mathbb{F}$ is isomorphic to $\mathbb{F}'$). We now proceed in the same style as in the proof of Lemma 5, adapting the arguments to our present case $\delta > 2$.

We denote by x'' the element of Γ'' in $\mathbf{PG}(d'-1,\mathbb{F}')$ corresponding to any element x of Γ in $\mathbf{PG}(d,\mathbb{K})$. Let L_1 and L_2 be two lines of Γ such that in $\mathbf{PG}(d',\mathbb{F}')$ L_1' and L_2' span a 3-space which intersects Q' in a non-singular quadric Q^+. Let Q_1' be the intersection of Q' with the 4-dimensional subspace of $\mathbf{PG}(d',\mathbb{F}')$ generated by L_1', L_2' and the kernel k' of Q'; note that Q_1' is non-singular. Let Ω be the polar subspace of Γ which corresponds with the quadric Q^+. As in paragraphs (b) and (c) of the proof of Lemma 5, one shows that Ω is fully embedded in a unique 3-dimensional subspace V over $\mathbb{F}$ of the 3-dimensional subspace U (over $\mathbb{K}$) of $\mathbf{PG}(d,\mathbb{K})$ generated by L_1 and L_2. Let V'' be the 3-dimensional subspace of $\mathbf{PG}(d'-1,\mathbb{F}')$ generated by L_1'' and L_2'' (where L_1'' and L_2'' are the respective projections of L_1' and L_2'). Let x'' be any point of Γ'' in V''. Then $x' \in Q_1'$ and since Q_1' is non-singular, x is not collinear with all points of L_i^*, $i = 1, 2$. Suppose x' does not lie on Q^+ and let y be the unique point on L_1 collinear with x in Γ. Let x_1, x_2 be two other points of Γ on L_1. Let L be the line of Γ containing y and concurrent with L_2. The lines $x'y'$, L' and L_1' define a cone on Q_1' and consequently there is a unique conic C_i' on that cone with kernel k' and containing x' and x_i', $i = 1, 2$. These conics correspond with the respective secant lines M_1 and M_2 of Γ. Hence M_i, $i = 1, 2$, contains x_i and another point y_i of Γ on L. But $x_i, y_i \in V$, hence M_i defines a line of V, $i = 1, 2$. Since x is the intersection of M_1 and M_2, it belongs to V. So we obtain a full embedding of the polar subspace of Γ determined by Q_1'.

Now let z be any other point of Γ contained in U. If z belongs to V then there is a unique line M in V meeting both L_1 and L_2 and containing z.

The extension of M to $\mathbb{K}$ is a secant line of Γ and hence it corresponds with a conic on Q'_1; hence z' belongs to Q'_1, a contradiction.

Suppose now $z \in U \setminus V$. Considering the polar subspace of Γ generated by L_1, L_2 and their points in Γ, one shows as in paragraph (d) of the proof of Lemma 5 that $z \in V$, a contradiction. Hence the only points x of Γ in U satisfy $x' \in Q'_1$.

As in paragraphs (e), (f), (g) and (h) of the proof of Lemma 5 we use an inductive argument. The assumption is that any $(2c-1)$-dimensional non-singular orthogonal subspace Γ_1 of Γ, whose corresponding subspace V'_1 in $\mathbf{PG}(d', \mathbb{F}')$ contains k', is fully embedded in a $(2c-1)$-dimensional projective subspace V_1 over $\mathbb{F}$ of $\mathbf{PG}(d, \mathbb{K})$, $2 \leq c < \frac{d}{2}$. We want to show that every $(2c+1)$-dimensional non-singular orthogonal subspace Γ_2 of Γ, whose corresponding subspace of $\mathbf{PG}(d', \mathbb{F}')$ contains k', is fully embedded in a $(2c+1)$-dimensional projective subspace over $\mathbb{F}$ of $\mathbf{PG}(d, \mathbb{K})$.

Let Γ_2 be a $(2c+1)$-dimensional non-singular subspace of Γ, whose corresponding subspace V'_2 of $PG(d', \mathbb{F}')$ contains k', $2 \leq c < \frac{d}{2}$. Further, let Γ_1 be a $(2c-1)$-dimensional non-singular subspace of Γ_2, whose point set corresponds to the set of all points of Γ'_2 collinear to two given non-collinear points u' and v' of Γ'_2. Then the subspace V'_1 of $\mathbf{PG}(d', F')$ containing Γ'_1, also contains the kernel k'. Hence Γ_1 is fully embedded in a $(2c-1)$-dimensional projective subspace V_1 over $\mathbb{F}$ of $\mathbf{PG}(d, \mathbb{K})$.

First, suppose there is a point x of $\Gamma_2 \setminus \Gamma_1$ with the property that the subspace V'_3 of $\mathbf{PG}(d', \mathbb{F}')$ generated by V'_1 and x' meets the point set of Γ'_2 in a non-degenerate quadric Q'_3, i.e. the singular point of Q'_3 lies in a proper extension of V'_3 over some extension field $\mathbb{F}_1$ of $\mathbb{F}$, but not in V'_3 itself. Let U_1 be the extension of V_1 over $\mathbb{K}$. We first show that U_1 does not contain any point of $\Gamma_3 \setminus \Gamma_1$, where Γ_3 is the polar subspace of Γ which corresponds to Q'_3. Let the point z of $\Gamma_3 \setminus \Gamma_1$ belong to U_1. Since Γ_3 is generated by Γ_1 and z, all points of Γ_3 belong to U_1. All points of Γ_1 are collinear with u. Since the point set of Γ_1 generates U_1, by (WE2) all points of Γ_3 are collinear with u. As Γ_3 is non-degenerate the point u does not belong to Γ_3, and so the set of all points of Γ_3 collinear with u is just the point set of Γ_1. This yields a contradiction. Consequently no point of $\Gamma_3 \setminus \Gamma_1$ is contained in U_1. Similarly to parts

(f) and (g) of the proof of Lemma 5 we can now show that Γ_3 is fully embedded in a subspace $\mathbf{PG}(2c, \mathbb{F})$ of $\mathbf{PG}(d, \mathbb{K})$. Let $\mathbf{PG}(2c, \mathbb{K})$ be the extension of $\mathbf{PG}(2c, \mathbb{F})$ over $\mathbb{K}$. Assume, by way of contradiction, that $\mathbf{PG}(2c, \mathbb{K})$ contains a point r of $\Gamma_2 \setminus \Gamma_3$. Since Γ_2 is generated by Γ_3 and r, all points of Γ_2 belong to $\mathbf{PG}(2c, \mathbb{K})$. Hence u belongs to $\mathbf{PG}(2c, \mathbb{K})$. By (WE2) the points u and v belong to the $(2c-1)$-dimensional space U_1. Since Γ_2 is generated by Γ_1, u and v, the polar space Γ_2 belongs to U_1. Hence Γ_3 belongs to U_1, a contradiction. Consequently no point of $\Gamma_2 \setminus \Gamma_3$ is contained in $\mathbf{PG}(2c, \mathbb{K})$. Similarly to parts (f) and (g) of the proof of Lemma 5 we now show that Γ_2 is fully embedded in a subspace $\mathbf{PG}(2c+1, \mathbb{F})$ of $\mathbf{PG}(d, \mathbb{K})$.

Next, suppose that for each point x of $\Gamma_2 \setminus \Gamma_1$ the subspace V_3' of $\mathbf{PG}(d', \mathbb{F}')$ generated by V_1' and x' meets the point set of Γ_2' in a degenerate quadric Q_3', that is, the singular point y' of Q_3' belongs to V_3'. The set of all singular points y' is a non-singular conic C' with kernel k'. Let L' be any line through k' in the plane π' of C'. Then the $(2c+1)$-dimensional space generated by V_1' and L' intersects the point set of Γ_2' in a degenerate quadric with singular point on C' and L'. It follows that each line L' in π' through k' contains a point of C'. Consequently the field $\mathbb{F}'$ is perfect, a contradiction.

As in (h) of the proof of Lemma 5, induction now shows that $d = d' - 1$ and that Γ is fully embedded in a subspace $\mathbf{PG}(d, \mathbb{F})$ of $\mathbf{PG}(d, K)$.

(4) Finally suppose that Γ is a non-singular unitary polar space of rank at least 3 arising from some hermitian variety $\mathcal{H}' = H(d', \mathbb{F}', \sigma)$ in $\mathbf{PG}(d', \mathbb{F}')$ with σ an involutory field automorphism of $\mathbb{F}'$. Again we can copy part (a) of the proof of Lemma 5. As in (b) of that proof we can choose two lines L_1 and L_2 of Γ generating a 3-space U of $\mathbf{PG}(d, \mathbb{K})$. In Γ the lines L_1 and L_2 and their points generate a non-singular polar space Ω which corresponds to a hermitian surface $\mathcal{H}_3'$ (of a 3-space) on $\mathcal{H}'$. Now L_1 and L_2 (but not all their points) are contained in a polar subspace Ω_0 corresponding to a symplectic space $W(3, \mathbb{F}_\sigma')$ in a 3-dimensional subspace $\mathbf{PG}(3, \mathbb{F}_\sigma')$ of $\mathbf{PG}(d', \mathbb{F}')$ over the field $\mathbb{F}_\sigma'$ which consists of all elements of $\mathbb{F}'$ fixed by σ. By part (1) of this proof we know that there exists a subfield $\mathbb{F}_\sigma$ of $\mathbb{K}$ isomorphic to $\mathbb{F}_\sigma'$ and a 3-dimensional subspace V_σ of $\mathbf{PG}(d, \mathbb{K})$ over $\mathbb{F}_\sigma$ such that Ω_0 is fully embedded in V_σ. We also know that for any line L of Γ the set

L^* is a projective subline of L in $\mathbf{PG}(d,\mathbb{K})$ over some field $\mathbb{F}$, which is independent of L. Evidently $\mathbb{F}$ contains $\mathbb{F}_\sigma$. Let V be the extension of V_σ over $\mathbb{F}$. Let L be a line of Ω_0 and let x be a point on L belonging to $\Omega \setminus \Omega_0$. Then clearly x lies in V. We will show that every point x of Ω lies on a line of Ω_0.

Let x be an arbitrary point of $\Omega \setminus \Omega_0$ and let x' be the corresponding point of $\mathcal{H}'_3$. Since $\mathbf{PG}(3,\mathbb{F}'_\sigma)$ is a Baer subspace of $\mathbf{PG}(3,\mathbb{F}')$, there is a unique line L' of $\mathbf{PG}(3,\mathbb{F}'_\sigma)$ containing x'. If L' were not a line of $W(3,\mathbb{F}'_\sigma)$, then it would meet $\mathcal{H}'_3$ in a subline of L' over $\mathbb{F}'_\sigma$, hence x' would be a point of $\mathbf{PG}(3,\mathbb{F}'_\sigma)$, a contradiction. So L' is a line of $\mathcal{H}'_3$ (alternatively, this can be easily seen by considering the dual generalized quadrangle). The corresponding line L of Ω is incident with x and belongs to Ω_0. Hence Ω is fully embedded in V and U is the extension of V over $\mathbb{K}$.

Now we show that no other point of Γ belongs to U. Suppose, by way of contradiction, that the point z of Γ lies in U but is not contained in Ω. Let z' be the corresponding point of $\mathcal{H}'$. If $\mathcal{T}'$ is the set of all points of $\mathcal{H}'_3$ collinear with z', then either $\mathcal{H}'_3 = \mathcal{T}'$, or $\mathcal{T}'$ is a non-singular hermitian curve, or $\mathcal{T}'$ is a singular hermitian curve. Let $\mathcal{T}$ be the corresponding point set of Ω. First, let $\mathcal{H}'_3 = \mathcal{T}'$. Noticing that for every point y of Ω, the space generated by $y^\perp$ in $\mathbf{PG}(d,\mathbb{K})$ meets U in a plane (by axiom (WE2)), we see that z must lie in every plane containing two intersecting lines of Ω. Hence the extensions over $\mathbb{K}$ of all tangent planes of the unitary polar space Ω (the point set of Ω is a hermitian variety of V) have a common point, clearly a contradiction. Hence $\mathcal{H}'_3 \neq \mathcal{T}'$. Then, by (WE2), $\mathcal{T}$ and z are contained in a common plane $\mathbf{PG}(2,\mathbb{K})$. Assume that $\mathcal{T}'$ is a singular hermitian curve, with singular point u'. Let $r' \in \mathcal{T}' \setminus \{u'\}$. As r is collinear with u and z in Γ, by (WE2) it is collinear in Γ with all points of $\mathcal{T}$, clearly a contradiction. Finally, let $\mathcal{T}'$ be a non-singular hermitian curve. Let s be any point of $\mathcal{T}$, and let M_1, M_2 be any two distinct lines of Ω through s. By (WE2) the lines M_1, M_2, zs are contained in a common plane, which is the extension over $\mathbb{K}$ of the tangent plane of the unitary polar space Ω at s. Hence z belongs to the extensions of all tangent planes of Ω at points of $\mathcal{T}$, so z belongs to V. It follows that all tangent lines of the hermitian curve $\mathcal{T}$ concur at z, a contradiction. We conclude that the only points of Γ in U are the points of Ω.

As in paragraphs (e), (f), (g) and (h) of the proof of Lemma 5 (and as in (3) of the present proof) we use an inductive argument. Let Γ_1 be the polar subspace of Γ arising from a non-degenerate hermitian subvariety $\mathcal{H}'_1$ of $\mathcal{H}'$ containing lines, and obtained from $\mathcal{H}'$ by intersecting it with a c-dimensional subspace W'_1 of $\mathbf{PG}(d', \mathbb{F}')$, $3 \leq c < d'$. Suppose that Γ_1 is fully embedded in a c-dimensional subspace V_1 over $\mathbb{F}$ of $\mathbf{PG}(d, \mathbb{K})$. Let Γ_2 be the polar subspace of Γ arising from a non-degenerate hermitian subvariety $\mathcal{H}'_2$ of $\mathcal{H}'$ obtained from $\mathcal{H}'$ by intersecting it with a $(c+1)$-dimensional subspace W'_2 of $\mathbf{PG}(d', \mathbb{F}')$ containing W'_1. Then we will show that Γ_2 is fully embedded in some $(c+1)$-dimensional subspace V_2 over $\mathbb{F}$ of $\mathbf{PG}(d, \mathbb{K})$. Let x be a point of $\Gamma_2 \setminus \Gamma_1$. Let U_1 be the extension of V_1 over $\mathbb{K}$. Suppose by way of contradiction that x belongs to U_1. The points of Γ_1 collinear with x in Γ_2 form a point set $\mathcal{H}_3$ corresponding to a non-singular hermitian subvariety $\mathcal{H}'_3$ of $\mathcal{H}'_1$ obtained by intersecting $\mathcal{H}'_1$ with a hyperplane of W'_1. By (WE2), x must belong to the extension over $\mathbb{K}$ of every hyperplane of V_1 tangent to Γ_1 at a point of $\mathcal{H}_3$. Also by (WE2), x and $\mathcal{H}_3$ are contained in a common hyperplane W_3 of U_1. As the polar space with point set $\mathcal{H}'_1$ is generated by $\mathcal{H}'_3$ and any point of $\mathcal{H}'_1 \setminus \mathcal{H}'_3$, also Γ_1 is generated by $\mathcal{H}_3$ and any point of Γ_1 not in $\mathcal{H}_3$. Hence $\mathcal{H}_3$ generates a hyplerplane R_3 of V_1. Clearly W_3 is the extension over $\mathbb{K}$ of the hyperplane R_3. It follows that the extensions over $\mathbb{K}$ of the tangent hyperplanes of Γ_1 at points of $\mathcal{H}_3$ intersect in a unique point which belongs to $V_1 \setminus R_3$. Hence $x \notin W_3$, a contradiction. Consequently no point of $\Gamma_2 \setminus \Gamma_1$ belongs to U_1. Let L be any line of $\Gamma_2 \setminus \Gamma_1$; then L^* defines a projective subline over $\mathbb{F}$ and hence there is a unique $(c+1)$-dimensional subspace V_2 over $\mathbb{F}$ of $\mathbf{PG}(d, \mathbb{K})$ containing V_1 and all elements of L^*. We now show that all points of Γ_2 are contained in V_2. Let x be any point of Γ_2. Clearly we may assume that x does not belong to Γ_1 nor to L^*.

In the sequel, we again denote the corresponding element in $\mathbf{PG}(d', \mathbb{F}')$ of an element e of Γ by e'.

First suppose that x is collinear in Γ_2 with a point $y \in L^*$ which does not belong to Γ_1. All points of the line $x'y'$ belong to $\mathcal{H}'_2$ and hence there is a unique point z' of $x'y'$ in $\mathcal{H}'_1$. Let w be the unique point of Γ_1 on L^*. The line wz is either a line of Γ_1 or a secant line. In the first case the points of Γ_2 in the plane xwz of $\mathbf{PG}(d, \mathbb{K})$ form a projective subplane over $\mathbb{F}$ sharing all points of at least two lines with V_2. Hence all points of that subplane belong to V_2 and so does x. In the second case let u be any point of Γ_1 on wz, $w \neq u \neq z$ (this is possible by the assumption $\delta > 2$). By Proposition 4 of LEFÈVRE-PERSCY [5] the line xu meets L in a point of Γ. Hence both xu and xz are lines of V_2 and the result follows.

Now suppose that x is not collinear in Γ_2 with an element of L^* not belonging to Γ_1. By the Buekenhout-Shult axiom x is collinear in Γ_2 with the unique point w of L^* in Γ_1. Let $y \in L^*$, $y \neq w$. It is easy to see that there is at most one point on the line $y'w'$ collinear in $\mathcal{H}_2'$ to all points of $\mathcal{H}_1'$ which are collinear to x' (since all such points belong to a secant line of $\mathcal{H}_2'$). So there is a point $y_1 \neq w$ on L^* and a point r of Γ_1 collinear with y_1 in Γ_2, but not collinear with x in Γ_2. By the Buekenhout-Shult axiom, there exists a unique line M of Γ_2 incident with x and containing a point s of Γ_2 on the line ry_1. By assumption $s \neq r$, so s does not belong to Γ_1. By the previous paragraph, all points of Γ on ry_1 belong to V_2. Interchanging the roles of ry_1 and L, we now see that x belongs to V_2. We conclude that Γ_2 is fully embedded in a $(c+1)$-dimensional subspace over $\mathbb{F}$ of $\mathbf{PG}(d, \mathbb{K})$. Applying this for $c = 3, 4, \ldots, d'-1$, we finally obtain that Γ is fully embedded in some $\mathbf{PG}(d', \mathbb{F})$ from which immediately follows that $d' = d$.

This completes the proof of the lemma. $\square$

The previous lemmas prove Theorem 1. $\square$

Remarks.

1. When Γ arises from a non-degenerate but singular quadric (and that can only happen if the characteristic of the ground field $\mathbb{F}'$ is equal to 2), Theorem 1 is not valid. For example consider in $\mathbf{PG}(7, \mathbb{F}')$, where $\mathbb{F}'$ is a non-perfect field with characteristic 2, the quadric Q with equation

$$X_0^2 + X_1^2 + X_0X_1 + X_2^2 + aX_3^2 + X_4^2 + X_5^2 + X_4X_5 + X_6X_7 = 0,$$

where $a \in \mathbb{F}'$ is a non-square. Let $\mathbb{K}$ be the algebraic closure of $\mathbb{F}'$ and let $\mathbf{PG}(7, \mathbb{K})$ be the corresponding extension of $\mathbf{PG}(7, \mathbb{F}')$. The point $x(0, 0, \sqrt{a}, 1, 0, 0, 0, 0)$ is the unique singular point of Q. If we project Q from x onto a hyperplane $\mathbf{PG}(6, \mathbb{K})$ of $\mathbf{PG}(7, \mathbb{K})$ which does not contain x, then we obtain a weakly embedded polar space which is not fully embedded in any subspace $\mathbf{PG}(6,\mathbb{F})$, for any subfield $\mathbb{F}$ of $\mathbb{K}$. In a forthcoming paper, we will classify sub-weakly embedded singular polar spaces, degenerate or not, arising from quadrics, symplectic polarities or hermitian varieties.

2. When Γ has $\delta = 2$ and arises from a non-singular symplectic polar space of rank at least three over a non-perfect field of characteristic two, then Theorem 1 is not valid. We give an example. Let $\mathbb{K}$ be a field of characteristic two for which the subfield $\mathbb{F}$ of squares is not perfect. Then also $\mathbb{K}$ is not perfect. Now consider in $\mathbf{PG}(6,\mathbb{K})$ the set $\mathcal{S}$ of points $(x_0, x_1, \ldots, x_6)$ with $x_0, x_1, \ldots, x_5 \in \mathbb{F}$, $x_6 \in \mathbb{K}$, and lying on the quadric Q with equation

$$X_0X_3 + X_1X_4 + X_2X_5 = X_6^2.$$

Then $\mathcal{S}$, provided with lines and planes induced by Q, is a polar space Γ isomorphic to the non-singular symplectic polar space $W(5,\mathbb{F})$ in $\mathbf{PG}(5,\mathbb{F})$ by projecting $\mathcal{S}$ from $(0,0,0,0,0,0,1)$ into the subspace U with equation $X_6 = 0$ over $\mathbb{F}$. Clearly Γ is sub-weakly embedded in $\mathbf{PG}(6,\mathbb{K})$. Let e_i, $0 \leq i \leq 5$, be the point of $\mathbf{PG}(6,\mathbb{K})$ with all coordinates 0 except the $(i+1)$th coordinate, which is equal to 1. Let e be the point all coordinates of which are equal to 1 and let e_{01} be the point with coordinates $(1,1,0,0,0,0,0)$. Then it is easy to see that the set V of points of $\mathcal{S}$ on the lines e_ie_{i+1}, $i \in \{0,1,\ldots,4\}$, on e_0e_5 and on ee_{01} generates the subspace $\mathbf{PG}(6,\mathbb{F})$ of $\mathbf{PG}(6,\mathbb{K})$ consisting of all points with coordinates in $\mathbb{F}$. Hence, if $\mathcal{S}$ were fully embedded in a subspace of $\mathbf{PG}(6,\mathbb{K})$ over a subfield of $\mathbb{K}$, then this subspace would be $\mathbf{PG}(6,\mathbb{F})$. As $\mathcal{S}$ contains the point $(0,0,1,0,0,a^2,a)$, $a \in \mathbb{K} \setminus \mathbb{F}$, which does not belong to $\mathbf{PG}(6,\mathbb{F})$, the polar space Γ is not fully embedded in a subspace of $\mathbf{PG}(6,\mathbb{K})$.

3 Proof of Theorem 2

(i) First suppose that the non-degenerate quadric Q does not contain lines. Since by assumption the points of Q span $\mathbf{PG}(d,\mathbb{F})$, we may assume that $e_i = (0,\ldots,0,1,0,\ldots,0)$, where the 1 is in the ith position, lies on Q for every i. The plane $e_ie_je_k$, $1 \leq i < j < k \leq d+1$, meets Q in a non-singular non-empty conic. Assume that the coefficient of $X_\ell X_m$ in a fixed equation for Q over $\mathbb{F}$ is $a_{\ell m} = a_{m\ell}$. Let the quadric Q' of $\mathbf{PG}(d, \mathbb{K})$, with $\mathbb{K}$ an extension of $\mathbb{F}$ and $\mathbf{PG}(d, \mathbb{K})$ the corresponding extension of $\mathbf{PG}(d, \mathbb{F})$, contain Q. The coefficient of $X_\ell X_m$ in a fixed equation for Q' over $\mathbb{K}$ is denoted by $a'_{\ell m} = a'_{m\ell}$. If $|\mathbb{F}| \geq 4$, then, either $e_ie_je_k \cap Q'$ is a non-singular non-empty conic or the plane $e_ie_je_k$ itself. As a non-singular non-empty conic is uniquely defined by any five of its points, we have $a'_{\ell m} = c_{\{i,j,k\}} a_{\ell m}$ with $\ell, m \in \{i, j, k\}$ and $c_{\{i,j,k\}} \in \mathbb{K}$ (as $e_ie_je_k \cap Q$ is non-singular we have $a_{\ell m} \neq 0$). By fixing i and j we see that $c_{\{i,j,k\}} = c_{\{i,j,k'\}}$, for every k, k' and now it is easy to see that $c_{\{i,j,k\}}$ is a constant c; it is clear that $c \neq 0$, whence the result for $|\mathbb{F}| \geq 4$. Suppose now $|\mathbb{F}| = 3$. As Q does not contain lines we have $d \in \{2, 3\}$. For $d = 2$, there are indeed distinct conics in $\mathbf{PG}(2, \mathbb{K})$, where $\mathbb{K}$ is a field of characteristic 3 with $|\mathbb{K}| > 3$, containing the four points of a conic in a subplane isomorphic with $\mathbf{PG}(2, 3)$, and the same remark holds for $|\mathbb{F}| = 2$ and $d = 2$. If $d = 3$ and $|\mathbb{F}| = 3$, then a direct and straightforward computation shows that the ten points of Q are on a unique quadric in every extension $\mathbf{PG}(3, \mathbb{K})$. For $|\mathbb{F}| = 2$ and $d = 3$, the five points of Q are contained in several non-singular quadrics over every proper extension of $\mathbb{F}$. This completes the case where Q does not contain lines.

Now suppose that Q contains lines. Let Q' be a quadric in $\mathbf{PG}(d, \mathbb{K})$ containing Q, with $\mathbb{K}$ an extension of $\mathbb{F}$ and $\mathbf{PG}(d, \mathbb{K})$ the corresponding extension of $\mathbf{PG}(d, \mathbb{F})$. Again we can assume that $e_i \in Q$ for all i. Let $a_{ij} = a_{ji}$ respectively $a'_{ij} = a'_{ji}$ be the coefficient of X_iX_j in the equation of Q respectively Q'. The tangent hyperplane U_i of Q at e_i is spanned by all lines through e_i contained in Q. If e_i is not singular for Q', then also the tangent hyperplane U'_i of Q' at e_i is spanned by all lines through e_i contained in Q'; in such a case the hyperplane U_i is necessarily a subhyperplane of U'_i. The equation of U_i is $\sum_j a_{ij} X_j = 0$ (note that $a_{ii} = a'_{ii} = 0$ for all i). If e_i is not singular for Q', then the equation of U'_i is $\sum_j a'_{ij} X_j = 0$; if e_i is singular for Q', then $a'_{ij} = 0$ for all j. From the foregoing it follows that $a'_{ij} = c_i a_{ij}$ for all j, with $c_i \in \mathbb{K}$. Hence if $a_{ij} = 0$, then also $a'_{ij} = 0$. Now consider $1 \leq i < j \leq d + 1$ and $1 \leq k < \ell \leq d + 1$ with $\{i, j\} \cap \{k, \ell\} = \emptyset$ and suppose that $a_{ij} \neq 0 \neq a_{k\ell}$. From the preceding it immediately follows that if a_{ik},

$a_{i\ell}$, a_{jk} and $a_{j\ell}$ are not all zero, then

$$\frac{a'_{ij}}{a_{ij}} = \frac{a'_{k\ell}}{a_{k\ell}}.$$

On the other hand, if $a_{ik} = a_{i\ell} = a_{jk} = a_{j\ell} = 0$, then the same equality follows from considering the tangent hyperplane of Q at the point $e_{ik} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$, with the 1 in the ith and the kth position, from considering the tangent hyperplane of Q' at e_{ik} if this point is not singular for Q' (if this point is singular for Q', then $a'_{ij} = a'_{k\ell} = 0$), and from considering the coefficients of X_j and X_ℓ in the equations of these hyperplanes. Now it immediately follows that Q' is uniquely determined by Q.

(ii) The proof is similar to the last part of (i) and in fact it can be simplified a great deal because we can immediately use standard equations.

(iii) First suppose that the non-singular non-empty hermitian variety H does not contain lines. Since the points of H span $\mathbf{PG}(d, \mathbb{F})$, $d \geq 2$, we may assume that $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the ith position, lies on H for every i. The plane $e_i e_j e_k$, $1 \leq i < j < k \leq d+1$, meets H in a non-singular non-empty hermitian curve C. Assume that the coefficient of $X_\ell X_m^\sigma$ in a fixed equation for H over $\mathbb{F}$ is $a_{\ell m}$. Let $\mathbb{K}$ be a field containing $\mathbb{F}$ admitting a $\mathbb{K}$-involution τ the restriction of which to $\mathbb{F}$ is σ, let $\mathbf{PG}(d, \mathbb{K})$ be the corresponding extension of $\mathbf{PG}(d, \mathbb{F})$, and let the hermitian variety H' of $\mathbf{PG}(d, \mathbb{K})$ contain H. The coefficient of $X_\ell X_m^\tau$ in a fixed equation for H' over $\mathbb{K}$ is denoted by $a'_{\ell m}$. The intersection of C with the line $e_i e_j$ is determined by the equation $a_{ij} X_i X_j^\sigma + a_{ji} X_j X_i^\sigma = 0$ (as C is non-singular we have $a_{ij} \neq 0$). For each point of that intersection also the equation $a'_{ij} X_i X_j^\sigma + a'_{ji} X_j X_i^\sigma = 0$ is satisfied. Let $(0, \ldots, 0, 1, 0, \ldots, 0, u, 0, \ldots, 0)$ be a point of $C \cap e_i e_j$ with $u \neq 0$. Then $a_{ij} u^\sigma + a_{ji} u = a'_{ij} u^\sigma + a'_{ji} u = 0$. Hence

$$\frac{a'_{ij}}{a_{ij}} = \frac{a'_{ji}}{a_{ji}}.$$

Let us now consider a point $(0, \ldots, 0, 1, 0, \ldots, 0, u, 0, \ldots, 0, v, 0, \ldots, 0)$ of $C \cap e_i e_j e_k$ with the u as above and $v \neq 0$. Then $a_{ik} v^\sigma + a_{ki} v + a_{jk} u v^\sigma + a_{kj} v u^\sigma = a'_{ik} v^\sigma + a'_{ki} v + a'_{jk} u v^\sigma + a'_{kj} v u^\sigma = 0$. As

$$\frac{a'_{ik}}{a_{ik}} = \frac{a'_{ki}}{a_{ki}} \qquad \text{and} \qquad \frac{a'_{jk}}{a_{jk}} = \frac{a'_{kj}}{a_{kj}},$$

we have

$$a_{ik}v^\sigma + a_{ki}v + a_{jk}uv^\sigma + a_{kj}vu^\sigma = b(a_{ik}v^\sigma + a_{ki}v) + c(a_{jk}uv^\sigma + a_{kj}vu^\sigma) = 0,$$

with $b, c \in \mathbb{K}$. Assume, by way of contradiction, that

$$\begin{cases} a_{ij}u^\sigma + a_{ji}u & = & 0, \\ a_{ik}v^\sigma + a_{ki}v & = & 0, \\ a_{jk}uv^\sigma + a_{kj}vu^\sigma & = & 0. \end{cases}$$

Then it readily follows that $a_{ij}a_{jk}a_{ki} + a_{ji}a_{ik}a_{kj} = 0$. As C is non-singular, we have $a_{ij}a_{jk}a_{ki} + a_{ji}a_{ik}a_{kj} \neq 0$, a contradiction. Hence $a_{ik}v^\sigma + a_{ki}v$ and $a_{jk}uv^\sigma + a_{kj}vu^\sigma$ are not both zero, so that $b = c$. Hence

$$\frac{a'_{ik}}{a_{ik}} = \frac{a'_{ki}}{a_{ki}} = \frac{a'_{jk}}{a_{jk}} = \frac{a'_{kj}}{a_{kj}}.$$

Now it readily follows that H' is uniquely determined by H.

Now suppose that H contains lines. If the line e_ie_j, $i \neq j$, does not belong to H, then as in the first part of (iii) we obtain

$$\frac{a'_{ij}}{a_{ij}} = \frac{a'_{ji}}{a_{ji}}.$$

If the line e_ie_j, $i \neq j$, belongs to H, then $a_{ij} = a_{ji} = a'_{ij} = a'_{ji} = 0$. Now we proceed as in the second part of the proof of (i). □

Remark. In the finite case, any $\mathbf{GF}(q^2)$ contains a unique involution. But in the infinite case, examples arise where distinct choices for τ can be made. For instance, one can extend the unique involution $x \mapsto x^q$ of $\mathbf{GF}(q^2)$, q odd, to the involutions $\sum a_it^i \mapsto \sum a_i^q t^i$ and $\sum a_it^i \mapsto \sum a_i^q(-t)^i$ of $\mathbf{GF}(q^2)(t)$.

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