Embedded Thick Finite Generalized Hexagons in Projective Space

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Abstract

We show that every embedded finite thick generalized hexagons $\mathcal{H}$ of order (q,t) in $PG(n,q)$ which satisfies the conditions

(i) *the set of all points of* $\mathcal{H}$ *generates* $PG(n,q)$,
(ii) *for any point* x *of* $\mathcal{H}$, *the set of all points collinear in* $\mathcal{H}$ *with* x *is contained in a plane of* $PG(n,q)$,
(iii) *for any point* x *of* $\mathcal{H}$, *the set of all points of* $\mathcal{H}$ *not opposite* x *in* $\mathcal{H}$ *is contained in a hyperplane of* $PG(n,q)$,

is necessarily the standard representation of $H(q)$ in $PG(6,q)$ (on the quadric $Q(6,q)$), the standard representation of $H(q)$, q even, in $PG(5,q)$ (inside a symplectic space), or the standard representation of $H(q,\sqrt[3]{q})$ in $PG(7,q)$ (where the lines of $\mathcal{H}$ are the lines fixed by a triality on the quadric $Q^+(7,q)$). This generalizes a result by Cameron & Kantor [3], which is used in our proof.

1 Introduction

A *generalized n-gon*, $n \geq 2$, or a *generalized polygon*, is a rank 2 point-line geometry the incidence graph of which has diameter n (i.e. two elements are at most at distance n) and girth $2n$ (i.e. the length of any shortest circuit is $2n$). A *thick* generalized polygon is a generalized polygon for which each element is incident with at least three elements.

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In this case, the number of points on a line is a constant, say $s+1$, and the number of lines through a point is also constant, say $t+1$. The pair (s,t) is called the *order* of the polygon. As is immediately clear from the definition, there is a point-line duality for generalized polygons.

These objects were introduced by TITS [16] to study classical, exceptional and twisted Chevalley groups (of relative rank 2). Since then, they have become research objects in their own right for many geometers.

A generalized polgygon Γ is *embedded* in the projective space $PG(d,\mathbb{K})$ if its points are points of $PG(d,\mathbb{K})$ which generate $PG(d,\mathbb{K})$, if its lines are lines of $PG(d,\mathbb{K})$ and if all points of Γ incident with any line L of Γ are exactly all points of $PG(d,\mathbb{K})$ on the line L. One of the many beautiful geometric results is the classification of all generalized quadrangles embedded in projective space $PG(d,\mathbb{K})$, a project which was started by BUEKENHOUT & LEFÈVRE [2], who treated the finite case, and finished by DIENST in the infinite case [4]. A similar result for thick generalized hexagons seems at this moment not within reach since there are many "exotic" examples obtained by projecting (from a point) a given embedded generalized hexagon into a hyperplane. Several of these "exotic" examples have the property that for some point x (but not necessarily all points) of the hexagon the set of points not opposite x in the hexagon span the projective space generated by the hexagon (*opposite* points are points at distance 6 in the incidence graph). So a first natural condition to ask is that this never happens. But also under that assumption, it is not clear whether a classification is possible since examples may be constructed in high dimensional spaces as images on grassmannians of the classical embeddings. But these have the property that all points collinear with any given point in the hexagon generate at least a three dimensional space. So one may impose as a second condition that this never happens. Explicit inspection of the classical hexagons arising from trialities reveals that they satisfy the two conditions stated above. Hence the question may be asked whether they are the only ones. The present paper answers this question affirmatively in the finite case.

2 Statement of the Main Result

In [16], TITS defines two classes of thick generalized hexagons arising from trialities on the hyperbolic quadric in projective 7-dimensional space over a commutative field. Here we are only interested in the finite case, and then the two classes are related to Dickson's simple group $G_2(q)$ respectively the triality group ${}^3D_4(q)$ (which was discovered in that same paper [16] by TITS); we call them the *classical generalized hexagons*. The generalized hexagon $\mathcal{H}$ corresponding to a triality τ on the quadric $Q^+(7,q)$ is defined as follows: the points of $\mathcal{H}$ are the points p of $Q^+(7,q)$ which are mapped by τ onto a maximal singular subspace incident with p, the lines of $\mathcal{H}$ are the lines which are fixed by τ and incidence is the natural one. TITS [16] shows that there are two cases in which thick generalized hexagons arise; in the first case one obtains a hexagon of order $(q, \sqrt[3]{q})$, here denoted by $H(q, \sqrt[3]{q})$, in the second case one obtains a hexagon of order (q,q) which lies in a hyperplane of $PG(7,q)$, and denoted by $H(q)$. So the latter can be represented on a quadric $Q(6,q)$ in 6-dimensional projective space (its points are all the points of $Q(6,q)$ while its lines are some lines of $Q(6,q)$; see TITS [16] for more details). In the even characteristic case, the polar space $Q(6,q)$ is isomorphic to the symplectic polar space $W(5,q)$ and hence in this case one obtains a representation of $H(q)$ in 5-dimensional projective space. We call these three different representations of the classical generalized hexagons the *natural embeddings*. They all satisfy conditions (i) up to (iv) below, as follows immediately from TITS [16]; in (i) up to (iv), $\mathcal{H}$ is a classical hexagon naturally represented in d-dimensional projective space, $d = 5, 6, 7$:

(i) *The points incident with a line in $\mathcal{H}$ are all points of some line in $PG(d,q)$.*

(ii) *The points of $\mathcal{H}$ span $PG(d,q)$.*

(iii) *The points collinear (in $\mathcal{H}$) with any given point in $\mathcal{H}$ are coplanar in $PG(d,q)$.*

(iv) *The points not opposite any given point in $\mathcal{H}$ are contained in a hyperplane of $PG(d,q)$.*

If we call *any* representation of *any* thick generalized hexagon satisfying the properties (i), (ii), (iii) and (iv) a *regular* or *ideal embedding* in $PG(d,q)$, then our Main Result states that every regular embedding is natural.

Main Result. *Let $\mathcal{H}$ be a finite thick generalized hexagon, regularly embedded in some projective space. Then $\mathcal{H}$ is a natural embedding of a classical generalized hexagon.*

It should be mentioned that in the proof of the Main Result we rely on the following result on embeddings of generalized hexagons due to CAMERON & KANTOR [3].

Theorem (Cameron & Kantor [3],3.1, 3.2). *Suppose that the finite thick generalized hexagon $\mathcal{H}$ satisfies properties (i), (ii) together with*

(iii)' the set of all points collinear (in $\mathcal{H}$) with any given point in $\mathcal{H}$ is a plane of $PG(d,q)$,
(iv)' the points not opposite any given point x in $\mathcal{H}$ are contained in a hyperplane π of $PG(d,q)$ and π does not contain points opposite x.

Then $\mathcal{H}$ is a natural embedding in $PG(5,q)$ or $PG(6,q)$ of a classical generalized hexagon.

Remark. In this theorem the order of $\mathcal{H}$ is necessarily (q,q).

3 Digression

Before we prove the Main Result, we motivate the term *regular* or *ideal embedding* by the following observations. The (finite) generalized quadrangles admitting a regular embedding are precisely those with all points regular (in the sense of PAYNE & THAS [10]). As a first step, we will show below that every generalized hexagon admitting a regular embedding must have ideal lines (in the sense of RONAN [11]). The notions of "regular points" for generalized quadrangles and "ideal lines" for generalized hexagons were unified for generalized polygons by VAN MALDEGHEM [19] as "distance-2 regular" points in generalized polygons. One might wonder whether regular embeddings of finite thick generalized octagons exist (octagons are 8-gons and according to FEIT & HIGMAN [5] $n = 8$ is the only value distinct from 3,4 and 6 for which there exist finite thick generalized n-gons). In the appendix we show that any finite thick generalized octagon admitting a regular embedding must be distance-2 regular and hence by VAN MALDEGHEM [19] this cannot occur. In view of these remarks, an alternative formulation of our Main Result is:

Main Result – Second version. *If a finite thick generalized n-gon $\mathcal{G}$, $n \geq 4$, is regularly embedded, then $n = 4$ or 6 and we have one of the following cases:*

(a) *$\mathcal{G}$ is the symplectic quadrangle $W(q)$ naturally embedded in $PG(3,q)$;*

(b) *$\mathcal{G}$ is the Hermitian quadrangle $H(3,q^2)$ naturally embedded in $PG(3,q^2)$;*

(c) *$\mathcal{G}$ is the classical hexagon $H(q)$, q even, naturally embedded in $PG(5,q)$;*

(d) *$\mathcal{G}$ is the classical hexagon $H(q)$ naturally embedded in $PG(6,q)$;*

(e) *$\mathcal{G}$ is the classical hexagon $H(q,\sqrt[3]{q})$ naturally embedded in $PG(7,q)$.*

4 Proof of the Main Result

We prove the Main Result in a sequence of lemmas. The main idea is first to find the upper bound $d = 7$ and lower bound $d = 5$ for the dimension of the projective space, and then to consider the cases $d = 5, 6, 7$ separately in detail.

In the sequel, let $\mathcal{H}$ be a thick generalized hexagon of order (q, t), with pointset $\mathcal{P}$ and lineset $\mathcal{L}$, regularly embedded in $PG(d, q)$. For the sake of convenience, we will call collinearity in $\mathcal{H}$ *polycollinearity* and keep the notion of collinearity strictly for $PG(d, q)$. Note that concurrency of lines is the same in the hexagon as in the projective space, that coplanarity is only defined in the projective space, and that oppositeness will only be considered in the hexagon; hence these notions do not cause confusion.

For two points x and y at distance 4 in the generalized hexagon $\mathcal{H}$, we denote by $x * y$ the unique point polycollinear with both.

4.1 General lemmas

An *apartment* in $\mathcal{H}$ is a set of six points and six lines forming a circuit. A *full subhexagon* of the generalized hexagon $\mathcal{H}$ of order (q, t) is a subhexagon (i.e. a generalized hexagon whose pointset and lineset are subsets of the pointset and lineset of $\mathcal{H}$ respectively and for which the incidence is just the restriction of the incidence relation in $\mathcal{H}$) of order (s, t), for some $s \geq 1$. It is called *thin* if $s = 1$.

Lemma 1 *Let U be a subspace of $PG(d, q)$ containing an apartment of $\mathcal{H}$. Then all points of $\mathcal{H}$ contained in U and incident with at least two lines of $\mathcal{H}$ in U, together with the lines of $\mathcal{H}$ in U and the natural incidence, form a full subhexagon $\mathcal{H}'$ of $\mathcal{H}$.*

PROOF. Let x and L be respectively a point and a line of $\mathcal{H}$ in U and suppose that there exist at least two lines of $\mathcal{H}$ through x in U. Then all lines of $\mathcal{H}$ through x lie in U by condition (iii). So there remains to show, by a well-known result (see e.g. WALKER [20]) , that the unique chain in $\mathcal{H}$ joining x with L lies in U. If L contains a point polycollinear with x, then this is obvious. Now suppose that the distance in $\mathcal{H}$ of x and L is 5. Then there are points y, z and lines M, N in $\mathcal{H}$ such that $x \,\mathbf{I}\, M \,\mathbf{I}\, y \,\mathbf{I}\, N \,\mathbf{I}\, z \,\mathbf{I}\, L$. Since L and M are in U, both z and y are and hence N is. This proves the lemma. $\square$

Remark 2 Lemma 1 also holds if condition (iv) is deleted and condition (i) is replaced by the weaker condition:

> $(i)'$ *The points incident with a line in $\mathcal{H}$ are points (not necessaily all) of some line in $PG(d, q)$.*

If U and $\mathcal{H}'$ are as in the above lemma, then we say that $\mathcal{H}'$ is *induced by* U.

Consider any $x \in \mathcal{P}$. The points not opposite x span a subspace which we denote by ξ_x. By assumption (iv), $\xi_x \neq PG(d,q)$ for all $x \in \mathcal{P}$.

Lemma 3 *For every $x \in \mathcal{P}$ the space ξ_x has dimension $d-1$ and does not contain any point opposite x.*

PROOF. Let $x \in \mathcal{P}$ and suppose that U is a hyperplane of $PG(d,q)$ containing ξ_x and a point u opposite x. Clearly U contains any apartment through x and u. So U induces a subhexagon $\mathcal{H}'$. But also all points polycollinear with x belong to $\mathcal{H}'$ (by condition (iii)), and so the order of $\mathcal{H}'$ is (q,t); hence $\mathcal{H}' = \mathcal{H}$, contradicting condition (ii). So if the dimension of ξ_x is less than $d-1$, then any point opposite x lies in some hyperplane U containing ξ_x. Hence ξ_x is a hyperplane. Putting $\xi_x = U$ in the first part of the proof, the result follows. $\square$

Corollary 4 *For $x, y \in \mathcal{P}$, $x \neq y$, we have $\xi_x \neq \xi_y$.*

PROOF. The hyperplane ξ_x contains a point of $\mathcal{H}$ opposite y. By Lemma 3 we have $\xi_x \neq \xi_y$. $\square$

For any $x \in \mathcal{P}$, we denote by π_x the unique plane in $PG(d,q)$ spanned by all points polycollinear with x. The next lemma shows that there are no other points of $\mathcal{H}$ in π_x.

Lemma 5 *For every $x \in \mathcal{P}$, the plane π_x does not contain points of $\mathcal{H}$ not polycollinear with x.*

PROOF. Let $u \in \mathcal{P} \cap \pi_x$ be not polycollinear with x. If u is not opposite x, then the unique line of $\mathcal{H}$ through u nearest to x lies in π_x, so it meets every line of $\mathcal{H}$ through x, a contradiction. Hence u is opposite x, but then $u \in \pi_x \subseteq \xi_x$, contradicting Lemma 3. $\square$

Lemma 6 *$\mathcal{H}$ is a classical generalized hexagon. Hence also every thick full subhexagon of $\mathcal{H}$ is classical.*

PROOF. If x and y are opposite points of $\mathcal{H}$, then π_x is not contained in ξ_y by Lemma 3. Hence the set x^y of points of $\mathcal{H}$ polycollinear with x and at distance 4 from y is contained in the line $\xi_y \cap \pi_x$, and hence x^y is equal to $\pi_x \cap \xi_y \cap \mathcal{P}$. Clearly, any such set is determined uniquely by any two of its points. This shows that $\mathcal{H}$ has ideal lines (in the sense of RONAN [11]) and so by RONAN [11] it is classical. It is clear that also every full subhexagon of $\mathcal{H}$ has ideal lines, so is classical if it is thick. $\square$

The set $\pi_x \cap \xi_y \cap \mathcal{P}$ is called an *ideal line* in RONAN [11], or a *distance-2 trace* in VAN MALDEGHEM [19].

Lemma 7 *We have $5 \leq d \leq 7$. Also, if $\mathcal{H} \cong H(q)$, then $d \neq 7$. If $\mathcal{H} \cong H(q, \sqrt[3]{q})$, then no subhexagon of $\mathcal{H}$ isomorphic with $H(\sqrt[3]{q})$ is contained in a $PG(d-2, q)$.*

PROOF. Clearly $d \geq 3$. If $d \leq 4$ then $\xi_x \cap \xi_y$ is a plane or a line for opposite points x and y, and consequently every two lines at distance 3 from both x and y intersect, respectively coincide, a contradiction. Hence $d \geq 5$.

Consider an apartment Σ in $\mathcal{H}$ and a line L of $\mathcal{H}$ concurrent with exactly one line of Σ. Let L and Σ generate a $PG(m, q)$. Then $m \leq 6$. Let $\mathcal{H}'$ be the full subhexagon induced by $PG(m, q)$. Then the order of $\mathcal{H}'$ is (s, t), with $2 \leq s \leq q$. If $q = s$, then $m = d \leq 6$ and we are done. So suppose $s < q$. Then there is a line M of $\mathcal{H}$ which does not lie in $PG(m, q)$, but which contains a point on a line of Σ. Let M and $PG(m, q)$ generate $PG(m+1, q)$ and let $PG(m+1, q)$ induce a full subhexagon $\mathcal{H}''$ of order (s', t), $s < s' \leq q$. Note that $\mathcal{H}'$ is a full subhexagon of $\mathcal{H}''$. If $s' = q$, then $d = m + 1 \leq 7$ and we are done again. If $s' < q$, then it follows from THAS [14] that $q \geq s'^2 t$, $s' \geq s^2 t$, and hence $q \geq s^4 t^3$. Now from HAEMERS & ROOS [6] we deduce $q \leq t^3$. This implies $q = t^3$ and $s = 1$, a contradiction.

Alternatively, we can argue as follows: we know that $\mathcal{H}$ is classical, hence if it contains a proper full subhexagon, then necessarily $\mathcal{H}$ is $H(q, \sqrt[3]{q})$ and $\mathcal{H}'$ is $H(\sqrt[3]{q})$. But then $\mathcal{H}''$ must coincide with one of them.

Assume that $\mathcal{H} \cong H(q)$. By THAS [14], either $s = q$ or $q \geq s^2 q$. As $s \geq 2$, necessarily $s = q$ and so $m = d \leq 6$.

Now let $\mathcal{H}^*$ be a proper thick full subhexagon of $\mathcal{H}$, contained in a $PG(d-2, q)$. The subhexagon induced by the subspace $PG(d-1, q)$ generated by this $PG(d-2, q)$ and any line of $\mathcal{H}$ not in $\mathcal{H}^*$ but concurrent with a line of $\mathcal{H}^*$ must coincide with $\mathcal{H}$ (as above), a contradiction. In particular, if $\mathcal{H} \cong H(q, \sqrt[3]{q})$, then no subhexagon of $\mathcal{H}$ isomorphic with $H(\sqrt[3]{q})$ is contained in a $PG(d-2, q)$. $\square$

A point-line geometry Ω with pointset $\mathcal{P}$ is *sub-weakly embedded* in a projective space $PG(d, q)$ if the following conditions are satisfied:

(a) *$\mathcal{P}$ is a pointset of $PG(d, q)$ which generates $PG(d, q)$;*

(*b*) *the points incident with a line in* Ω *are points (not necessarily all points) of some line in* $PG(d,q)$*; distinct lines of* Ω *define distinct lines of* $PG(d,q)$*;*

(*c*) *for any point* x *of* Ω*, the subspace generated by the set* $\{y \parallel y \in \mathcal{P}$ *is collinear with* $x\}$ *meets* $\mathcal{P}$ *precisely in that set;*

Theorem 8 *Let the finite (thick) polar space* Ω *of rank* $r \geq 3$ *be sub-weakly embedded in* $PG(d,q)$*,* $d \geq 3$*, and assume that any line* L *of* $PG(d,\mathbb{K})$ *which is not a line of* Ω *intersects* Ω *in at most two points. If the lines of* Ω *contain* $q'+1$ *points, then* $GF(q')$ *is a subfield of* $GF(q)$ *and* Ω *is embedded in a subspace* $PG(d,q')$ *of* $PG(d,q)$*.*

PROOF. The proof is due to THAS & VAN MALDEGHEM [15]. □

We now start our case-by-case study depending on d.

4.2 The case $d = 5$

By Lemma 6, $\mathcal{H}$ is classical and hence $t \in \{q, \sqrt[3]{q}\}$.

Lemma 9 *If* $t = q$*, then* q *is even and* $\mathcal{H} \cong H(q)$ *is naturally embedded in* $PG(5,q)$*.*

PROOF. As $|\mathcal{P}| = q^5 + q^4 + q^3 + q^2 + q + 1$, it is immediate that $\mathcal{P}$ is the pointset of $PG(5,q)$. By CAMERON & KANTOR [3], Theorem 3.2, q is even, $x \mapsto \xi_x$ defines a symplectic polarity θ of $PG(5,q)$ and $\mathcal{H} \cong H(q)$ is naturally embedded in $PG(5,q)$. □

We denote by δ the distance function in the incidence graph of $\mathcal{H}$.

Lemma 10 *The case* $t = \sqrt[3]{q}$ *cannot occur.*

PROOF. Consider two opposite lines L and M in $\mathcal{H}$. Let $x_1, y_1 \in L$, $x_1 \neq y_1$, and $x_2, y_2 \in M$, $x_2 \neq y_2$, with $\delta(x_1,x_2) = \delta(y_1,y_2) = 4$. All points at distance 3 from both L and M are in $\xi_{x_1} \cap \xi_{y_1}$ and in $\xi_{x_2} \cap \xi_{y_2}$. The space $\xi_{x_1} \cap \xi_{y_1}$ is 3-dimensional (by Corollary 4) and has no point in common with M (such a point would be at distance 4 from each of x_1, y_1, a contradiction). So $\xi_{x_1} \cap \xi_{y_1} \cap \xi_{x_2} \cap \xi_{y_2}$ is a line R of $PG(5,q)$. Hence R is the set of all points at distance 3 from L and M. So if $\delta(x,y) = 6$, then the line xy of $PG(5,q)$ consists of $q+1$ mutually opposite points of $\mathcal{H}$. Also, it is readily seen using Lemma 5 that a line of $PG(5,q)$ containing 2 points at mutual distance 4 contains exactly $\sqrt[3]{q}+1$

points of $\mathcal{H}$. Hence $\mathcal{P}$ is a set of type $(0, 1, \sqrt[3]{q}+1, q+1)$ in $PG(5,q)$. By LEFÈVRE-PERCSY [9], $\mathcal{P}$ has either a plane section consisting of a line and a maximal arc, or a plane section which is the complement of a maximal arc (and the maximal arc is respectively a $((\sqrt[3]{q}-1)(q+1)+1; \sqrt[3]{q})$-arc or a $((q-\sqrt[3]{q}-1)(q+1)+1; q-\sqrt[3]{q})$-arc). In the latter case $q-\sqrt[3]{q}$ must divide q (see e.g. HIRSCHFELD [7]), a contradiction. Hence there is a plane section consisting of a line L and a maximal arc K. Let x, y, z be a triangle in K. Then $\delta(x,y) = \delta(y,z) = \delta(z,x) = 4$ (as no line xy, yz, zx of $PG(5,q)$ is contained in the plane section). Since x, y, z form a triangle, the points $x * y$, $y * z$, $z * x$, which are not contained in the plane of K, are distinct, and hence x, y, z define a unique apartment in $\mathcal{H}$. As $\mathcal{H}$ is classical, this apartment is contained in a unique subhexagon $\mathcal{H}'$ of order $(1, \sqrt[3]{q})$. Remark that $\mathcal{H}'$ contains the ideal lines respectively defined by x and y, by y and z, and by z and x. These ideal lines consist precisely of the points of the plane section on the respective lines xy, yz, zx. Hence also the points z', x', y' on L and respectively xy, yz, zx belong to $\mathcal{H}'$. But all points of $\mathcal{H}'$ in the plane xyz are at mutual distance 4, while all points of L are at mutual distance 2 or 6, a contradiction. □

4.3 The case $d = 6$

By Lemma 6 we conclude again that $\mathcal{H}$ is classical and hence $t \in \{q, \sqrt[3]{q}\}$.

Lemma 11 *If $t = q$, then $\mathcal{H} \cong H(q)$ is embedded naturally in the quadric $Q(6,q)$ of $PG(6,q)$.*

PROOF. This follows immediately from CAMERON & KANTOR [3], Theorem 3.2. □

So from now on in this subsection, we assume that $t = \sqrt[3]{q}$.

Lemma 12 *Let $\mathcal{H}'$ be any subhexagon of order (t,t) of $\mathcal{H}$. Then $\mathcal{H}'$ is naturally embedded in a subspace $PG(5, \sqrt[3]{q})$ or $PG(6, \sqrt[3]{q})$.*

PROOF. By Lemma 7, the points of $\mathcal{H}'$ either span $PG(6,q)$ or a subspace $PG(5,q)$. Suppose first that $\mathcal{H}'$ spans $PG(5,q)$. Consider two opposite lines L and M of $\mathcal{H}'$. Let x_1, y_1 be distinct points of $\mathcal{H}'$ on L, let x_2, y_2 be distinct points of $\mathcal{H}'$ on M, and suppose that $\delta(x_1, x_2) = \delta(y_1, y_2) = 4$. All points of $\mathcal{H}'$ at distance 3 from both L and M are in $\xi_{x_1} \cap \xi_{y_1}$ and in $\xi_{x_2} \cap \xi_{y_2}$. By Lemma 3 the spaces $\xi_{x_i} \cap PG(5,q)$ and $\xi_{y_i} \cap PG(5,q)$ are 4-dimensional, and as $\xi_{x_i} \cap PG(5,q)$ contains a point of $\mathcal{H}'$ opposite y_i we have $\xi_{x_i} \cap PG(5,q) \neq \xi_{y_i} \cap PG(5,q)$ and so $\xi_{x_i} \cap \xi_{y_i} \cap PG(5,q)$ is 3-dimensional, $i = 1, 2$. Also, the space $\xi_{x_1} \cap \xi_{y_1} \cap PG(5,q)$ has no point in common with M (such a point would be at distance 4 from each of x_1, y_1, a contradiction). So $\xi_{x_1} \cap \xi_{y_1} \cap \xi_{x_2} \cap \xi_{y_2} \cap PG(5,q)$ is a line R. Hence R contains the $\sqrt[3]{q}+1$ points of $\mathcal{H}'$ at distance 3 from L and M (clearly R does not contain other points of $\mathcal{H}'$). So if x, y are points of $\mathcal{H}'$ with $\delta(x,y) = 6$, then the line xy of $PG(5,q)$ contains $\sqrt[3]{q}+1$ points of $\mathcal{H}'$. Hence the points of $\mathcal{H}'$ together with the lines of $PG(5,q)$ which contain at least 2 points (and hence $\sqrt[3]{q}+1$ points) of $\mathcal{H}'$ is a $2-((\sqrt[3]{q})^5 + (\sqrt[3]{q})^4 + \ldots + \sqrt[3]{q}+1, \sqrt[3]{q}+1, 1)$ design.

Suppose that two blocks of the design are disjoint in $\mathcal{H}'$, but that the corresponding lines of $PG(5,q)$ intersect in $PG(5,q)$. The plane defined by these blocks will be denoted by π. In π a subdesign with more than $(\sqrt[3]{q})^2 + \sqrt[3]{q} + 1$ points is induced. Let z be a point of $\mathcal{H}'$, but not in π. Through z we take a block M whose corresponding line in $PG(5,q)$ is disjoint from π (this is possible as $\mathcal{H}'$ generates $PG(5,q)$). We now consider the planes generated by M and the points of $\mathcal{H}'$ not on M; at least one of these planes, say π', is disjoint from π (again as $\mathcal{H}'$ generates $PG(5,q)$). Now we join the points of $\mathcal{H}'$ in π to the points of $\mathcal{H}'$ in π'. These lines contain more than

$$(\sqrt[3]{q}-1)((\sqrt[3]{q})^2+\sqrt[3]{q}+1)^2+2((\sqrt[3]{q})^2+\sqrt[3]{q}+1) = (\sqrt[3]{q})^5+(\sqrt[3]{q})^4+(\sqrt[3]{q})^3+(\sqrt[3]{q})^2+\sqrt[3]{q}+1$$

points of $\mathcal{H}'$, a contradiction.

Hence the design satisfies the axiom of Veblen and so it is the design of points and lines of $PG(5, \sqrt[3]{q})$. This implies that Γ is embedded in a $PG(5, \sqrt[3]{q})$; hence by CAMERON & KANTOR [3], Theorem 3.1, $\mathcal{H}'$ is naturally embedded in $PG(5, \sqrt[3]{q})$.

Now suppose that $\mathcal{H}'$ spans the whole space $PG(6,q)$. We consider the polar space Γ consisting of the points of $\mathcal{H}'$, the lines of $\mathcal{H}'$ and the ideal lines of $\mathcal{H}'$. We show that Γ is sub-weakly embedded in $PG(6,q)$. Clearly conditions (a) and (b) are satisfied. Condition (c) follows from Lemma 3 by simply remarking that the points of $\mathcal{H}'$ not opposite a given point x of $\mathcal{H}'$ are precisely those points which are collinear in Γ with x. We now show that any line L of $PG(6,q)$ which is not a line of Γ intersects Γ in at most two points.

Let L be a line of $PG(6,q)$ which is not a line of Γ, and assume that L contains at least three points x_1, x_2, x_3 of $\mathcal{H}'$. Since L does not belong to Γ, these three points are mutual opposite. Let $PG(5,q)$ be the projective 5-space generated by an arbitrary apartment Σ in $\mathcal{H}'$ containing x_1 and x_2. As $\delta(x_1,x_3) = \delta(x_2,x_3) = 6$ the point x_3 does not belong to the full subhexagon of order $(1,t)$ defined by Σ. Let u be any point of Σ polycollinear with x_1 (there are two distinct choices for u). Since both x_1 and x_2 are not opposite u, Lemma 3 implies that x_3 is not opposite u. Hence there exists a line M of $\mathcal{H}'$ through x_3 meeting a line of $\mathcal{H}'$ through u. Since we have two choices for u, we obtain two such lines through x_3 which must clearly belong to $PG(5,q)$. Hence $PG(5,q)$ induces $\mathcal{H}'$ (since the induced hexagon contains the full subhexagon of order $(1, t)$ defined by Σ plus an extra point). This contradicts our assumption.

Consequently, by Theorem 8, Γ is embedded in a $PG(6, \sqrt[3]{q})$; hence by CAMERON & KANTOR [3], Theorem 3.2, $\mathcal{H}'$ is naturally embedded in $PG(6, \sqrt[3]{q})$. $\square$

Lemma 13 *There exists no regular embedding of $H(q, \sqrt[3]{q})$ in $PG(6, q)$.*

PROOF. Consider a thick proper full subhexagon $\mathcal{H}'$ of $\mathcal{H} \cong H(q, \sqrt[3]{q})$, where $\mathcal{H}$ is regularly embedded in $PG(6, q)$. Then $\mathcal{H}'$ has order $(\sqrt[3]{q}, \sqrt[3]{q})$, and by Lemma 12 $\mathcal{H}'$ is embedded in either a $PG(5, \sqrt[3]{q})$ or a $PG(6, \sqrt[3]{q})$. Suppose each such subhexagon is embedded in a 5-dimensional subspace. Let Σ be an apartment with points $x_1, x_2, \ldots, x_6$, where x_i is polycollinear with x_{i+1} (subscripts taken modulo 6). By Lemma 5 the plane π_{x_1} does not contain x_3. By Lemma 3, the line x_4x_5 does not contain a point of the 3-space $x_1x_2x_3x_6$ (such a point would be in $\xi_{x_1} \cap \xi_{x_2}$). Hence Σ generates a 5-space. If $\mathcal{H}'$ and $\mathcal{H}''$ are distinct subhexagons of order $(\sqrt[3]{q}, \sqrt[3]{q})$ containing Σ, then, by assumption, they are both induced by the 5-space generated by Σ, a contradiction. Hence we may assume that at least one subhexagon $\mathcal{H}'$ isomorphic to $H(\sqrt[3]{q})$ is naturally embedded in a quadric $\Gamma' = Q(6, \sqrt[3]{q})$. Extend Γ' to $\Gamma = Q(6, q)$. The set of all points of $Q(6, q)$ not in $Q(6, \sqrt[3]{q})$ will be denoted by $\tilde{Q}$.

1. Suppose $\mathcal{H}$ contains a point x of Γ not on a line of $\mathcal{H}'$. Let $L_0, \ldots, L_{\sqrt[3]{q}}$ be the lines of $\mathcal{H}$ containing x. Each line L_i contains one point y_i of a line M_i of $\mathcal{H}'$, see THAS [14], and remark that y_i is not contained in $\mathcal{H}'$. All the lines of $\mathcal{H}'$ containing a point of $\mathcal{H}'$ on M_i lie in a $PG(4, q)$, which is the tangent space of Γ at M_i. The space $PG(4, q)$ contains all lines of $\mathcal{H}$ intersecting M_i (since it is the intersection of ξ_u and ξ_w for two distinct points u and w of $\mathcal{H}'$ on M_i), so it contains L_i. As x belongs to Γ, also the line L_i belongs to Γ. Hence all points polycollinear with x belong to Γ. On the set of points of $\mathcal{H}$ not on a line of $\mathcal{H}'$, a graph G is induced by the point graph of $\mathcal{H}$; by BROUWER [1], the graph G is connected. Consequently all points of $\mathcal{H}$ belong to Γ. Let t_i be the number of planes π_p containing a point x_i of Γ not in $\mathcal{H}$. If $\bar{t}$ is the average of the numbers t_i, then

$$\bar{t} = \frac{(q^2(\sqrt[3]{q})^2 + q\sqrt[3]{q} + 1)(q+1)[q^2 + q + 1 - (\sqrt[3]{q} + 1)q - 1]}{q^5 + q^4 + q^3 + q^2 + q + 1 - (q+1)(q^2(\sqrt[3]{q})^2 + q\sqrt[3]{q} + 1)}$$

$$= \frac{q^2(\sqrt[3]{q})^2 + q\sqrt[3]{q} + 1}{q^2 + q\sqrt[3]{q} + 1} > 1.$$

Hence there is a point x_i for which $t_i > 1$. Let π_p and $\pi_{p'}$ be two planes containing a common point x, x in Γ but not in $\mathcal{H}$. Necessarily $\delta(p,p') \geq 4$. If $\delta(p,p') = 4$, then π_p and $\pi_{p'}$ generate a $PG(3,q)$ and there arises a thick proper full subhexagon $\mathcal{H}''$ of order $(\sqrt[3]{q}, \sqrt[3]{q})$ in a $PG(5,q)$ (by considering 5 points in $PG(3,q)$ which lie in an apartment in some $PG(4,q)$). If $\delta(p,p') = 6$, then π_p and $\pi_{p'}$ are contained in a 4-space and again there arises a thick proper full subhexagon $\mathcal{H}''$ of order $(\sqrt[3]{q}, \sqrt[3]{q})$ in a $PG(5,q)$ (by considering any apartment through p and p'). Consequently Γ contains a $PG(5, \sqrt[3]{q})$, a contradiction.

2. We may assume by the foregoing that the points of $\mathcal{H}$ in Γ are the points of Γ on the lines of $\mathcal{H}'$. Let L, M be two lines of $\mathcal{H}'$ at distance 6. Let x, y be two points of $\mathcal{H}'$ on L. Then ξ_x (respectively ξ_y) is the tangent hyperplane of Γ at x (respectively y), which follows from the fact that $\mathcal{H}'$ is embedded in $\Gamma' \subset \Gamma$. So $\xi_x \cap \xi_y$ is 4-dimensional. Let x', y' be two distinct points of M in $\mathcal{H}'$. Then $\xi_{x'} \cap \xi_{y'}$ is also 4-dimensional. The space $\xi_x \cap \xi_y$, respectively $\xi_{x'} \cap \xi_{y'}$, intersects Γ in $q+1$ planes through L, respectively M. Since $\mathcal{H}'$ is embedded in Γ', $\xi_x \cap \xi_y \cap \xi_{x'} \cap \xi_{y'}$ is a plane β.

   In $\mathcal{H}'$ the points at distance 3 from L and M form a conic $\mathcal{C}$ over $GF(\sqrt[3]{q})$ of β; in $\mathcal{H}$ the points at distance 3 from L and M form a set $\mathcal{O}$ of $q+1$ points.

   (1) Suppose q is odd. If some three points of $\mathcal{O}$ are collinear, then these three points together with the lines L and M generate a 5-dimensional space which induces a thick full proper subhexagon of order $(\sqrt[3]{q}, \sqrt[3]{q})$. By above considerations, it must be embedded in a $PG(5, \sqrt[3]{q})$, hence q is even. Therefore no three points of $\mathcal{O}$ are collinear and so, by Segre's theorem (see HIRSCHFELD [7]), $\mathcal{O}$ is a conic containing $\mathcal{C}$. The tangent of $\mathcal{O}$ at $z \in \mathcal{C}$ is the intersection of β and ξ_z, that is, the intersection of β and the tangent hyperplane of Γ at z. The conic $\Gamma \cap \beta$ contains $\mathcal{C}$ and the tangent lines of it at points of $\mathcal{C}$ are the tangent lines of $\mathcal{O}$ at these points. Consequently $\mathcal{O} = \Gamma \cap \beta$. Let $u \in \mathcal{O} \setminus \mathcal{C}$. By 1., the point u is on a line N of $\mathcal{H}'$. On M there is a point v, not in $\mathcal{H}'$, at distance 2 from u. Hence each point of M in $\mathcal{H}'$ is at distance 6 from each point of N in $\mathcal{H}'$, a contradiction as M and N are lines of $\mathcal{H}'$.

   (2) Suppose that q is even. Let x be a point of $\mathcal{H}$ not in Γ. By THAS [14] and 1., there are $\sqrt[3]{q}+1$ points in Γ polycollinear with x, which are points on $\sqrt[3]{q}+1$ lines of Γ'. Also, they form an ideal line (see e.g. RONAN [11]). So the plane π_x intersects Γ in a line. This is also the case if $x \in \Gamma \setminus \Gamma'$ (then the common line of Γ and π_x belongs to $\mathcal{H}$). It easily follows that each line xz, with $x \in \mathcal{H} \setminus \Gamma$, $z \in \mathcal{H}$, and $\delta(x,z) \leq 4$, is a tangent line of Γ (if $\delta(x,z) = 4$, look in π_{x*z}). Hence

ξ_x is generated by tangent lines of Γ through x. So ξ_x contains the kernel k of Γ. It is clear that also ξ_y, $y \in \mathcal{H} \cap \Gamma'$, contains k. Consider now a point $w \in \mathcal{H} \cap \Gamma$, $w \notin \Gamma'$. Then w belongs to a line W of $\mathcal{H}'$. If w_1 and w_2 are distinct points of $\mathcal{H}'$ on W, then $\xi_{w_1} \cap \xi_{w_2}$ intersects Γ' in $\sqrt[3]{q}+1$ planes and Γ in $q+1$ planes. The hyperplane ξ_w contains these $\sqrt[3]{q}+1$ planes, hence contains $\xi_{w_1} \cap \xi_{w_2}$. As $\xi_{w_1} \cap \xi_{w_2}$ contains k, also ξ_w contains k. Let W' be a line of $\mathcal{H}'$ at distance 6 from W, and let w' be the point of W' at distance 4 from w. Then by the foregoing, ww' is a line of Γ. Hence w' belongs to the tangent hyperplane ζ of Γ at w. As $w' \notin \xi_{w_1} \cap \xi_{w_2}$, we have $\zeta = \langle w', \xi_{w_1} \cap \xi_{w_2} \rangle$. As also $\xi_w = \langle \xi_{w_1} \cap \xi_{w_2}, w' \rangle$, we have $\zeta = \xi_w$. If k belongs to $\mathcal{H}$, then all points of $\mathcal{H}$ are at distance at most 4 from k, a contradiction. Hence $k \notin \mathcal{H}$. If u, v, k with $u, v \in \mathcal{H}$ and $u \neq v$ are collinear, then the hyperplanes ξ_u, ξ_v generated by all tangent lines of Γ at respectively u, v coincide, a contradiction. So we can injectively project $\mathcal{H}$ from k into a $PG(5,q)$ not containing k; it is immediately checked that we thus obtain a regular embedding of $H(q, \sqrt[3]{q})$ in $PG(5,q)$, contradicting Lemma 10.

This completes the proof of the lemma. $\square$

4.4 The case $d = 7$

By Lemma 6 and Lemma 7 we have $t = \sqrt[3]{q}$.

Lemma 14 *Every subhexagon $\mathcal{H}'$ of $\mathcal{H}$ isomorphic to $H(\sqrt[3]{q})$ is naturally embedded in a quadric $Q(6, \sqrt[3]{q})$.*

PROOF. Let $\mathcal{H}'$ be a subhexagon of $\mathcal{H}$ isomorphic to $H(\sqrt[3]{q})$. By Lemma 7 the subhexagon $\mathcal{H}'$ generates either a $PG(6,q)$ or a $PG(7,q)$. Let Σ be an apartment in $\mathcal{H}'$. The subhexagon of $\mathcal{H}'$ of order $(1, \sqrt[3]{q})$ containing Σ will be denoted by $\mathcal{H}''$. The subhexagon $\mathcal{H}''$ is contained in the space $PG(m,q)$ generated by Σ; clearly $m \leq 5$. Let L be a line of $\mathcal{H}'$ not in $PG(m,q)$, but containing a point of one of the lines of Σ. Then the space $PG(m+1,q)$ containing $PG(m,q)$ and L induces a subhexagon $\mathcal{H}'''$ of $\mathcal{H}$ of order $(\sqrt[3]{q}, \sqrt[3]{q})$. As Σ and L are contained in a unique subhexagon of $\mathcal{H}$ of order $(\sqrt[3]{q}, \sqrt[3]{q})$, we have $\mathcal{H}' = \mathcal{H}'''$. So $\mathcal{H}'$ is contained in a $PG(6,q)$. This result is also easily obtained from Remark 2.

Similarly as in the proof of Lemma 12 we now show that $\mathcal{H}'$ is naturally embedded in a $Q(6, \sqrt[3]{q})$. $\square$

Lemma 15 *The pointset $\mathcal{P}$ of $\mathcal{H}$ is contained in a non-singular hyperbolic quadric.*

PROOF. Fix a thick proper full subhexagon $\mathcal{H}'$ (necessarily of order $(\sqrt[3]{q}, \sqrt[3]{q})$) of $\mathcal{H}$. By Lemma 14 it is contained in a unique quadric $\Gamma' = Q(6, \sqrt[3]{q})$. By Theorem 1 of THAS & VAN MALDEGHEM [15], there is a unique quadric $\Gamma = Q(6, q)$ containing Γ'. If $\mathcal{H}$ contains a point $x \notin \mathcal{H}'$ of the hyperplane $U \supset \Gamma$, $U = PG(6, q)$, not on a line of $\mathcal{H}'$, then, as each line of $\mathcal{H}$ through x contains a point of Γ' (see THAS [14]), each line of $\mathcal{H}$ through x is in $PG(6, q)$. So the subhexagon induced by U is $\mathcal{H}$ itself, a contradiction.

Now consider any apartment Σ in $\mathcal{H}'$. The space $V' = PG(5, \sqrt[3]{q})$ in $U' = PG(6, \sqrt[3]{q})$, with $\Gamma' \subset U' \subset U$, generated by the points of Σ intersects Γ' in a hyperbolic quadric $\Delta' = Q^+(5, \sqrt[3]{q})$ (Δ' contains a plane through each of the six points of Σ). By Theorem 1 of THAS & VAN MALDEGHEM [15] there is a unique quadric $\Delta = Q^+(5, q)$ containing Δ'; clearly $\Delta = \Gamma \cap PG(5, q)$, with $PG(5, q) \supset V'$. All points of $\mathcal{H}$ in $V = PG(5, q)$ are the points of Δ on the lines of $\mathcal{H}'$ in Δ, that is, on lines of Δ intersecting two skew planes π_1 and π_2 of Δ' (these planes are the *ideal planes*, cf. RONAN [11], corresponding with the unique subhexagon of order $(1, \sqrt[3]{q})$ containing Σ).

Let $M \in \mathcal{L}$, with M concurrent with a line of Σ, but not incident with a point of $\mathcal{H}'$. Then Σ and M define a subhexagon $\mathcal{H}'_*$, uniquely embedded in a quadric $\Gamma'_* = Q_*(6, \sqrt[3]{q})$. As in the previous paragraphs, we define Γ_*, U_*, $V_* = V$ and $\Delta_* = \Delta$.

Coordinates can be chosen in such a way that

$$U : X_7 = 0, \qquad U_* : X_6 = 0,$$

and hence

$$V = V_* : X_6 = X_7 = 0.$$

Let

$$\Delta : F(X_0, X_1, \ldots, X_5) = X_6 = X_7 = 0,$$

$$\Gamma : X_6 G(X_0, X_1, \ldots, X_5) + F(X_0, X_1, \ldots, X_5) = X_7 = 0,$$

$$\Gamma_* : X_7 G_*(X_0, X_1, \ldots, X_5) + F(X_0, X_1, \ldots, X_5) = X_6 = 0.$$

Consider the quadric

$$Q_\alpha : F(\ldots) + X_6 G(\ldots) + X_7 G_*(\ldots) + \alpha X_6 X_7 = 0, \qquad \alpha \in GF(q) \cup \{\infty\}$$

($Q_\infty : X_6 X_7 = 0$). Then $Q_\alpha \cap U = \Gamma$ and $Q_\alpha \cap U_* = \Gamma_*$.

Now we show that $\mathcal{H}$ lies on Q_α, for some $\alpha \in GF(q)$ (as the number of points of $\mathcal{H}$ is greater than the number of points of $\mathcal{H}$ in $U \cup U_*$, we have $\alpha \neq \infty$).

Let S be a line of $\mathcal{H}$ not in Γ but containing a point of $\mathcal{H}$ in $\Gamma \setminus \Delta$. Let r be a point of S, $r \notin U$, $r \notin U_*$. Let Q be the unique quadric Q_α containing r. By the choice of r we already have $\alpha \neq \infty$. Let R be any line of $\mathcal{H}$ through r, but not intersecting V; since S does not intersect V, there are at least $\sqrt[3]{q}$ such lines. Then R has a point in common with $\mathcal{P} \cap \Gamma$ and also with $\mathcal{P} \cap \Gamma_*$. As these points are distinct, R contains at least three points of Q, so R is contained in Q. The $\sqrt[3]{q}+1$ lines of $\mathcal{H}$ through r intersect Γ in collinear points (an ideal line), so these points all lie on a line of Γ. Hence all lines R (including S) lie in a plane of Q and this implies that the possible line $\tilde{R}$ through r intersecting V also lies on Q. Let s and s' be the points of S in Γ and Γ_*, respectively. Then every point m at distance 3 from S, but not polycollinear with s or s' lies in Q. Let T be a line of $\mathcal{H}$ containing s, $T \neq S$ and T not in Γ. As S is in Q, as the line X of $\mathcal{H}$ through s in Γ is in Q, and as T contains a point of $\Gamma_* \subset Q$, the plane SX is on Q, hence also $T \subset SX$ is on Q. It follows that each point of $\mathcal{H}$ at distance 3 from S is in Q.

Now let m be a point at distance 5 from S. We may assume without loss of generality that m and r are opposite. Let Y be at distance 2 from S and at distance 3 from m. If Y is not in Γ nor in Γ_*, and if it does not intersect a line of $\mathcal{H}$ in Δ, then, interchanging the roles of Y and S (which both uniquely define Q), we see that m is in Q. Now assume that Y is not in Γ nor in Γ_*, but intersects a line of $\mathcal{H}$ in Δ. If the line Y' at distance 2 from Y and incident with m, does not intersect a line of $\mathcal{H}$ in Δ, then Y' belongs to Q, so m belongs to Q. So assume that Y' intersects a line of Δ. The common points b, b' of respectively Y, Y' and Δ are on a line T contained in Δ (they define a unique ideal line). Let $N \neq S$ be a line through r not intersecting Δ. Let n be a point on N, but such that n is opposite every point of the ideal line defined by b and b' (this is possible since at most $\sqrt[3]{q}+1$ points of N can be at distance 4 from at least one point of the ideal line defined by b and b'). Choose n also outside Γ and Γ_*. Let N' be any line of $\mathcal{H}$ through n, $N' \neq N$, not intersecting Δ. The intersection y of Y and Y' is at distance 5 from N' (as can be checked easily). Consider the point u at distance 2 from y and 3 from N'. As n is in Q, also N' is in Q, so u is in Q. Clearly u is not on the line Y. If u is in Δ, then u is at distance 4 from n, a contradiction. So u is not on the line bb'. Hence the plane YY' is in Q because bb', Y and u are all in Q. This shows that m is indeed in Q. Finally assume that Y is in Γ (the case Y inside Γ_* is similar). Let Y' be as above (i.e. $\delta(Y, Y') = 2$ and $\delta(Y', m) = 1$). Let A be any line of $\mathcal{H}$ not meeting Δ and distinct from Y'. There are $q-1$ points a on A at distance 5 from S such that the line at distance 3 from a meeting S does not intersect S in a point of Γ or Γ_*. By the preceding, all these points belong to Q, hence A lies on Q and so does m.

Assume that the quadric Q admits a singular point k. Suppose, by way of contradiction, that $\mathcal{H}$ contains distinct points u, v with u, v, k collinear. Let $\tilde{Q}(6, q)$ be the non-singular quadric defined by any full thick proper subhexagon $\tilde{\mathcal{H}}$ of $\mathcal{H}$ containing u, v. The quadric $\tilde{Q}(6, \sqrt[3]{q})$ in which $\tilde{\mathcal{H}}$ is embedded extends to a unique non-singular quadric of the space $PG(6, q) \supset \tilde{Q}(6, \sqrt[3]{q})$. Hence $\tilde{Q}(6, q) = Q \cap PG(6, q)$. Consequently $k \in \tilde{Q}(6, q)$, so $\tilde{Q}(6, q)$ is singular, a contradiction. It also follows that $k \notin \mathcal{H}$ (otherwise, take $v = k$). Further, for any point u of $\mathcal{H}$ the hyperplane ξ_u is the tangent hyperplane of Q at u (since it is spanned by lines on Q through u). Hence ξ_u contains k. Consequently, by projecting $\mathcal{H}$ from k into a $PG(6, q)$ not containing k, we obtain a regular embedding of $H(q, \sqrt[3]{q})$ into $PG(6, q)$, contradicting Lemma 13.

Now we show that Q is of hyperbolic type. Suppose by way of contradiction that $Q = Q^-(7, q)$. Let $S = \{x_i | i = 1, 2, \ldots, N\}$, $N \in \mathbb{N}$, be the set of points of Q orthogonal to all points of at least one line of $\mathcal{H}$, but not lying in a plane which contains at least two lines of $\mathcal{H}$. We count in two different ways the ordered pairs (x, L), where $x \in S$, L is a line of $\mathcal{H}$ and x is orthogonal to all points of L. Denoting by t_i the number of lines of $\mathcal{H}$ all points of which are orthogonal to x_i, $1 \leq i \leq N$, we obtain

$$\sum_{i=1}^{N} t_i = (1 + \sqrt[3]{q})(1 + q\sqrt[3]{q} + (q\sqrt[3]{q})^2)(q^2 - q)q^2.$$

Now we count in two ways the number of triples (x, L, M), where $x \in S$, L and M are distinct lines of $\mathcal{H}$, and x is orthogonal to all points of both L and M; as $Q^-(7, q)$ does not contain 3-spaces, such lines L and M are not concurrent. Next we show that L and M are opposite. Suppose N is a line of Γ meeting both L and M. Then x is orthogonal to all points of N, hence x is orthogonal to all points of the planes LN and MN. As Q does not contain 3-spaces, x is a point of the plane LN, in contradiction with the definition of the set S. Hence L and M are opposite. Now we show that no point on L is orthogonal (on Q) to all points of M. Suppose by way of contradiction that z is incident with L and that z is orthogonal (on Q) to all points of M. Let L' be the unique line of $\mathcal{H}$ through z at distance 4 from M. Let y be the unique point on L' at distance 3 from M. Since every point on M is orthogonal to y and to z, the lines M and L' are contained in a plane; hence they meet in Q and also in Γ, a contradiction. Now let π be a plane through L contained in Q. If π does not contain a line of Γ other than L, then by the foregoing there is a unique point in π, not on L, orthogonal to all points of M. Since there are $q^2 - q$ choices for π, we obtain

$$\sum_{i=1}^{N} t_i(t_i - 1) = (1 + \sqrt[3]{q})(1 + q\sqrt[3]{q} + (q\sqrt[3]{q})^2)q^3(q^2 - q).$$

Expressing that $N \cdot \sum t_i^2 - (\sum t_i)^2 \geq 0$, one obtains

$$N \geq \frac{q^3(q-1)(1+\sqrt[3]{q})(1+q\sqrt[3]{q}+(q\sqrt[3]{q})^2)}{1+q}.$$

Now it is easily checked that the right hand side is stricktly bigger than $(1+q^4)(1+q+q^2)$ whenever $q \geq 8$, which is the total number of points on Q. Hence the result. $\square$

Lemma 16 *The subgroup of $P\Gamma L(8,q)$ fixing $\mathcal{H}$ and the hyperbolic quadric Q containing $\mathcal{H}$ admits Tits' triality group ${}^3D_4(\sqrt[3]{q})$ as subgroup.*

PROOF. By the preceding lemma, we know that $\mathcal{H}$ lies entirely on a hyperbolic quadric Q. Let Σ be an apartment of $\mathcal{H}$. Then Σ generates a 5-dimensional space V which meets Q in a hyperbolic quadric $\Delta = Q^+(5,q)$. There are exactly two points p, p' of Q such that Δ lies in the tangent hyperplane of Q at both p and p'. These two points together with the six points of Σ determine a unique apartment Ω' of the building Q. Let us denote the points of Σ by $e_0, \ldots, e_5$, where e_i is polycollinear with e_{i+1} (with subscripts taken modulo 6). In Q these 6 points form an octahedron Ω in which the points e_0, e_3, the lines e_2e_4, e_1e_5 (of Q) and the planes $e_0e_1e_5$, $e_3e_1e_5$, $e_3e_2e_4$ and $e_0e_2e_4$ form a wall W (for the building-terminology, see e.g. Tits [17] or Ronan [12]) of that octahedron. Now W is contained in a unique wall W' of Ω', where W' is obtained by adding the points p, p' to W together with the spaces generated by p respectively p' and each of the spaces of W. Let L be any line of $\mathcal{H}$ through e_0, $L \neq e_0e_1$. Then L is contained in the plane $e_0e_1e_5$. Hence by Tits [17], Addenda, there exists a unique collineation θ of Q mapping e_0e_5 onto L and fixing all singular subspaces of Q incident with at least two singular subspaces in the half apartment $\overline{\Omega}'$ of Ω' containing e_1e_2 and bounded by W'. In particular, θ fixes all points of $\mathcal{H}$ on the lines e_0e_1, e_1e_2, e_2e_3 and all lines of $\mathcal{H}$ through e_1 and e_2. Moreover, the planes $e_0e_1e_2$ and $e_1e_2e_3$ are fixed pointwise, hence all points of $\mathcal{H}$ polycollinear with e_1 or e_2 are fixed. Let $\mathcal{H}'$ be any thick proper full subhexagon containing Σ. Let $\Gamma = Q(6,q)$ be the corresponding quadric on Q. We show that θ induces an automorphism of Γ. Take for Q the equation $X_0X_3 + X_1X_4 + X_2X_5 + X_6X_7 = 0$ and for θ the map defined by $x_i \mapsto x_i$ with $i \in \{0,3,4,5,6,7\}$, $x_1 \mapsto x_1 - ax_5$ and $x_2 \mapsto x_2 + ax_4$, $a \in GF(q)$. Let e_i be the point with 1 in the $(i+1)$th position and with 0 elsewhere, $i = 0, 1, \ldots, 5$. Then the corresponding points p and p' have coordinates $(0, \ldots, 0, 1, 0)$ and $(0, \ldots, 0, 1)$. Since Γ lies in a hyperplane with equation $bX_6 + cX_7 = 0$, $b, c \in GF(q)$, it is clearly fixed by θ. Also θ fixes all singular subspaces of Q incident with at least two singular subspaces in the half apartment $\overline{\Omega}'$ and maps e_0e_5 onto the line $L : X_2 = X_3 = X_4 = X_1 + aX_5 = 0$ of the plane $e_0e_1e_5$. Let $\Gamma' = Q(6, \sqrt[3]{q})$ be the quadric whose set of points is exactly the set of points of $\mathcal{H}'$. Define θ' on Γ' similarly as we defined θ on Q, where now the role of W' is taken over by W. Then θ' extends to an automorphism θ' of Γ and $\theta'\theta^{-1}$ acts as the identity on Γ (by TITS [17], Theorem 4.1.2). Hence θ preserves Γ'. Now let θ'' be the axial root elation in $\mathcal{H}'$ fixing all points at distance 1 and 3 from e_1e_2 and mapping e_0e_5 onto L. Then θ'' extends to the polar space Γ' (by TITS [17], Addenda) and, again by TITS [17], Theorem 4.1.2, $\theta''\theta^{-1}$ is the identity on Γ'. Hence θ preserves $\mathcal{H}'$. As each line of $\mathcal{H}$ meeting Δ (necessarily in a point on a line of Δ which is also a line of the quadric $\Delta' = Q^+(5, \sqrt[3]{q})$ containing the subhexagon of order $(1, \sqrt[3]{q})$ defined by Σ) is contained in a thick proper full subhexagon containing Σ, θ maps lines of $\mathcal{H}$ meeting Δ onto lines of $\mathcal{H}$. Now let M be a line of $\mathcal{H}$ not meeting Δ. Then M meets a line of every thick proper full subhexagon containing Σ (this common point of M and the line of the subhexagon is not contained in the subhexagon); there are exactly $(\sqrt[3]{q})^2 + \sqrt[3]{q} + 1$ such subhexagons, giving rise to $(\sqrt[3]{q})^2 + \sqrt[3]{q} + 1$ distinct points on M. Hence M^θ contains at least that many points of $\mathcal{H}$, from which it follows that M^θ is a line of $\mathcal{H}$ (otherwise two points x and y of $\mathcal{H}$ on M^θ must be at distance 4 and hence M^θ lies in π_{x*y} and meets $\mathcal{H}$ in exactly $\sqrt[3]{q}+1$ points). Hence θ preserves $\mathcal{H}$ and is a long root elation. So every axial root elation of $\mathcal{H}$ extends to an automorphism of Q. As no non-trivial automorphism of Q fixes $\mathcal{H}$ pointwise, any automorphism of $\mathcal{H}$ extends to at most one automorphism of Q. Since these root elations generate ${}^3D_4(\sqrt[3]{q})$, the lemma is proved. $\square$

Lemma 17 *The generalized hexagon $\mathcal{H} \cong H(q, \sqrt[3]{q})$ is naturally embedded in $PG(7, q)$.*

PROOF. Let Q be as in the preceding lemma. Then Q is the triality quadric $Q^+(7, q)$. Its automorphism group contains only one conjugacy class of groups isomorphic to ${}^3D_4(\sqrt[3]{q})$ by KLEIDMAN [8], Theorem 2.3 (proved without using the classification of the finite simple groups). As the naturally embedded generalized hexagon $H(q, \sqrt[3]{q})$ is uniquely defined by a subgroup of $P\Gamma O^+(8, q)$ isomorphic to ${}^3D_4(\sqrt[3]{q})$, it now follows immediately that $\mathcal{H}$ is naturally embedded in $PG(7, q)$. $\square$

5 Appendix

As promised in Section 3, we show that no thick finite generalized octagon admits a regular embedding.

Suppose $\mathcal{O}$ is a thick finite generalized octagon regularly embedded in $PG(d, q)$ for some $d \in \mathbb{N}$. Then $s = q > t$. Let x and y be opposite points of $\mathcal{O}$. Let π_x be the plane spanned by all points polycollinear with x and let ξ_y be the subspace of $PG(d, q)$ generated by all points not opposite y. We claim that π_x is not contained in ξ_y. Indeed, suppose π_x is contained in ξ_y and let z be a point opposite y and at distance 2 from x. Since z and all points polycollinear with z not opposite y are in $\pi_z \cap \xi_y$, and since these points determine uniquely π_z, it follows that $\pi_z \subset \xi_y$. On the set of points at distance 8 from y a graph $\mathcal{G}$ is induced by the point graph of $\mathcal{O}$; by BROUWER [1], Theorem 1.1 (recall that $t < s$), the graph $\mathcal{G}$ is connected. Hence all points of $\mathcal{O}$ belong to ξ_y, a contradiction. So the plane π_x is not contained in ξ_y.

The set of points polycollinear with x and at distance 6 from y is contained in $\pi_x \cap \xi_y$. By the previous paragraph $\pi_x \cap \xi_y$ is a line of $PG(d, q)$, so is determined by any two of its points. Hence $\mathcal{O}$ is distance-2 regular in the sense of VAN MALDEGHEM [19], contradicting Theorem 2.3 of that paper. This completes the proof of the second version of the Main result.

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