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Elation generalized quadrangles of order (p, t), p prime, are classical

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Abstract

Elation generalized quadrangles of order (p, t), p prime, are classical.

1. Introduction and definitions

1.1. Generalized quadrangles

A finite generalized quadrangle (GQ) of order (s, t) is a finite incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (nonempty) sets of objects called points and lines respectively, and for which I is a symmetric point–line incidence relation satisfying the following properties:

GQ1. Each point is incident with $t + 1$ lines $(t \geqslant 1)$ and two distinct points are incident with at most one line.

GQ2. Each line is incident with $s + 1$ points $(s \geqslant 1)$ and two distinct lines are incident with at most one point.

GQ3. If (x, L) is a nonincident point–line pair, then there is a unique point–line pair (y, M) for which $x I M I y I L$.

If $s = t$, then $\mathcal{S}$ is said to have order s. For any point x of $\mathcal{S}$, the set of all points lying on the lines through x is denoted by $x^{\perp}$. Generalized quadrangles were introduced by Tits (1959). For terminology, notation, results, etc., concerning finite GQ, see the monograph by Payne and Thas (1984).

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The classical GQ

1. Consider a nonsingular quadric Q of projective index 1 of the projective space $\mathrm{PG}(d, q)$, with $d = 3, 4$ or 5. Then the points of Q together with the lines of Q (which are the subspaces of maximal dimension on Q) form a GQ $Q(d, q)$ of order $(q, 1)$ when $d = 3$, order (q, q) when $d = 4$, and order (q, q^2) when $d = 5$.
2. Let H be a nonsingular hermitian variety of the projective space $\mathrm{PG}(d, q^2)$, $d = 3$ or 4. Then the points of H together with the lines on H form a GQ $H(d, q^2)$ of order (q^2, q) when $d = 3$ and order (q^2, q^3) when $d = 4$.
3. The points of $\mathrm{PG}(3, q)$, together with the totally isotropic lines with respect to a symplectic polarity, form a GQ $W(q)$ of order q.

Isomorphisms. The GQ, $Q(4, q)$ is isomorphic to the dual of $W(q)$, and $Q(4, q)$ (and hence $W(q)$) is self-dual if and only if q is even. The GQ $H(3, q^2)$ is isomorphic to the dual of $Q(5, q)$.

1.2. Whorls, elations and translations

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a GQ of order (s, t), $s \neq 1$ and $t \neq 1$. A collineation θ of $\mathcal{S}$ is a *whorl about the point* x *if* θ fixes each line incident with x. A whorl θ about x is said to be an *elation about* x if θ is the identity or if θ fixes no point of $\mathcal{P} - x^{\perp}$. The set of elations about a point does not necessarily form a group (see Payne, 1985). If there is a group G of elations about x acting regularly on $\mathcal{P} - x^{\perp}$, we say $\mathcal{S}$ is an *elation generalized quadrangle* (EGQ) with *elation group* G and *base point* x. Briefly we say $(\mathcal{S}^{(x)}, G)$ is an EGQ. If the group G is abelian, then $(\mathcal{S}^{(x)}, G)$ is a *translation generalized quadrangle* (TGQ) with *translation group* G and *base point* x; in such a case G is the set of all elations about x. (See Payne and Thas, 1984.)

Example. The classical GQ $Q(4, q)$, $W(q)$, $Q(5, q^2)$ are EGQ with base point x, for any point x.

It follows that any classical GQ of order (p, t), $t \neq 1$ and p prime, is an EGQ.

In this paper we will prove the converse.

Main result. *Any elation generalized quadrangle of order* (p, t), *p prime and* $t \neq 1$, *is classical.*

1.3. Elation generalized quadrangles as group coset geometries

Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s, t), $s \neq 1 \neq t$, and let y be a fixed point of $\mathcal{P} - x^{\perp}$. Let $L_0, L_1, \ldots, L_t$ be the lines incident with x. For each L_i, $i \in \{0, 1, \ldots, t\}$, there is a unique point–line pair (z_i, M_i) for which $yIM_iIz_iIL_i$. Define S_i, S_i^* and J as follows: $S_i = \{\theta \in G \| M_i^\theta = M_i\}$, $S_i^* = \{\theta \in G \| z_i^\theta = z_i\}$ and $J = \{S_i \| 0 \leqslant i \leqslant t\}$. The groups G, S_i, S_i^* have respective orders s^2t, s and st. They also have the following two properties:

K1: $S_iS_j \cap S_k = \{1\}$ for distinct i, j, k,
K2: $S_i^* \cap S_j = \{1\}$ for distinct i, j.

It was first shown by Kantor (1980) that the converse is also true, i.e., given a group G of order s^2t $(s > 1, t > 1)$ with $1 + t$ subgroups S_i of order s and $1 + t$ subgroups $S_i^* \supset S_i$ of order st satisfying properties K1 and K2, then one constructs as follows an EGQ $\mathcal{S}(G, J)$, with $J = \{S_i \| 0 \leqslant i \leqslant t\}$. There are three kinds of points:
(i) the elements of G,
(ii) the right cosets S_i^*g, $g \in G$, $i \in \{0, 1, \ldots, t\}$,
(iii) a symbol (∞).
There are two kinds of lines:
(a) the right cosets S_ig, $g \in G$, $i \in \{0, 1, \ldots, t\}$,
(b) symbols $[S_i]$, $i \in \{0, 1, \ldots, t\}$.
A point g of type (i) is incident with each line S_ig, $0 \leqslant i \leqslant t$. A point S_i^*g of type (ii) is incident with $[S_i]$ and with each line S_ih contained in S_i^*g. The point (∞) is incident with each line $[S_i]$ of type (b). There are no further incidences. Then $\mathcal{S}(G, J)$ is an EGQ of order (s, t) with base point (∞). Each EGQ $(S^{(x)}, G)$ with J defined as above is isomorphic to $\mathcal{S}(G, J)$.

2. Elation generalized quadrangles of order (p, t), $t \neq 1$ and p prime, are classical

Let $\mathcal{S} = (\mathcal{S}^{(x)}, G) = (\mathcal{P}, \mathcal{B}, I)$ be an EGQ of order (p, t), $t \neq 1$ and p prime. We will use the notation of the previous sections.

Proposition 1. *If $t = p^2$, then $\mathcal{S} \cong Q(5, p)$.*

Proof. Consider a triple $\{x, x_1, x_2\}$ of pairwise noncollinear points. As $t = p^2$ the set $\{x, x_1, x_2\}^{\perp\perp} = \{x, x_1, \ldots, x_r\}$ contains at most $p + 1$ points. Let δ_i be the elation of G mapping x_1 onto x_i, $i = 1, 2,, \ldots, r$. As δ_i fixes $\{x, x_1, x_2\}^\perp$, it also fixes $\{x, x_1, x_2\}^{\perp\perp}$. It is clear that $G_1 = \{\delta_1, \delta_2, \ldots, \delta_r\}$ is the subgroup of G fixing $\{x, x_1, x_2\}^{\perp\perp}$. As $|G| = p^4$, either $|G_1| = 1$ or $p \mid |G_1| = r$. As $p \geqslant r \geqslant 2$, we necessarily have $r = p$. So the triple $\{x, x_1, x_2\}$ is 3-regular, and consequently x is 3-regular (cf. 1.3 of Payne and Thas (1984). If $p = 2$, then $\mathcal{S}$ is the unique GQ $Q(5, 2)$ of order $(2, 4)$; if $p > 2$, then by 5.3.3(i) of Payne and Thas (1984), the GQ $\mathcal{S}$ is isomorphic to $Q(5, p)$. □

Proposition 2. *We have $p \leqslant t$. For given lines L_i, L_j, with $i \neq j$, the pair $\{L_j, M'\}$ is regular for every line M' with $x\,IM'$ and $L_i \sim M'$, if $|\{L_j, M, N\}^\perp| \geqslant 3$ for at least two* *lines M, N of $L_i^{\perp}$ with $x \not{I} M, x \not{I} N, M \not\sim N$; if $\{L_j, M'\}$ is not regular (that is, if $|\{L_j, M, N\}^{\perp}| \leqslant 2$ for every two lines M, N of $L_i^{\perp}$, with $x \not{I} M$, $x \not{I} N$, $M \not\sim N$) then $|\{L_j, M, N\}^{\perp}| = 2$ for every two lines M, N of $L_i^{\perp}$, with $x \not{I} M$, $x \not{I} N$, $M \not\sim N$, if and only if $p = t$.*

Proof. Let M, N be lines of $L_i^{\perp}$ with $x \not{I} M$, $x \not{I} N$, $M \not\sim N$. Assume that $U \in \{L_j, N, M\}^{\perp}$, $U \neq L_i$. If $\{L_j, N, M\}^{\perp}$ contains a third line U', then put $U' I u' I M$ and $U I u I M$, and let δ be the element of G mapping u onto u'. The subgroup of G fixing M is the group $\langle \delta \rangle$ of order p. This group fixes the point $z = L_i M$ and acts on the set V of points of L_i different from x and z. As $|V| = p - 1 < p = |\langle \delta \rangle|$, the group $\langle \delta \rangle$ fixes every point of L_i. It easily follows that $N^{\delta} = N$. Hence the elements of $\langle \delta \rangle$ fix N and map U onto all lines of $\{M, L_j\}^{\perp} - \{L_i\}$. It follows that $N \in \{L_j, M\}^{\perp\perp}$. Analogously every line of $\{L_j, M\}^{\perp} - \{L_i\}$ belongs to $\{U, L_i\}^{\perp\perp}$. Consequently $\{L_i, U\}$ and $\{L_j, M\}$ are regular. Let $M' \in L_i^{\perp}$, $x \not{I} M'$, $M \neq M'$. As there is an element of G mapping M onto M', also the pair $\{L_j, M'\}$ is regular.

If $\{L_j, M\}$, $M \in L_i^{\perp}$, $x \not{I} M$, is regular, then $p \leqslant t$ by 1.3.6 (i) of Payne and Thas (1984). Now suppose that $\{L_j, M\}$ is not regular, and count in two ways the ordered triples $\{U', N', M'\}$ with $z I L_i I u$, $x \neq z \neq u \neq x$, $z I M'$, $u I N'$, $U' \in \{M', N', L_j\}^{\perp} - \{L_i\}$. We obtain $pt \leqslant t^2$, i.e., $p \leqslant t$, with equality if and only if $|\{L_j, M', N'\}^{\perp}| = 2$ for every choice of M' and N'. □

Proposition 3. *If $t = p$, then $\mathcal{S} \cong Q(4, p)$ or $\mathcal{S} \cong W(p)$.*

Proof. Let $p = t$. First assume that for no pair (L_i, L_j), $i \neq j$, we have $|\{L_j, M, N\}^{\perp}| \leqslant 2$ for each pair (M, N) in Proposition 2. Then each line L_i is regular, $i = 0, 1, \ldots, p$. Now by 8.3.3 of Payne and Thas (1984) $(\mathcal{S}^{(x)}, G)$ is a TGQ of order p. Then by 8.7.3 of Payne and Thas (1984) we have $\mathcal{S} \cong Q(4, p)$.

Next assume that for at least one pair (L_i, L_j), $i \neq j$, we have $|\{L_j, M, N\}^{\perp}| \leqslant 2$ for each pair (M, N) in Proposition 2. By Proposition 2 $|\{L_j, M, N\}^{\perp}| = 2$ for every two such lines M, N. If $p = t = 2$, then $\mathcal{S} \cong Q(4, 2)$, so each line is regular, a contradiction. Hence p is odd. Now we introduce the following incidence structure $\mathcal{A} = (\mathcal{P}', \mathcal{B}', I')$. The elements of $\mathcal{P}'$ are the lines of $L_i^{\perp}$ not incident with x. The elements of $\mathcal{B}'$ are

(i) the points of L_i different from x, and

(ii) the lines of $L_j^{\perp}$ not incident with x.

An element of $\mathcal{P}'$ is incident (I') with an element of type (i) of $\mathcal{B}'$ if it is incident with it in $\mathcal{S}$; an element of $\mathcal{P}'$ is incident (I') with an element of type (ii) of $\mathcal{B}'$ if it is concurrent with it in $\mathcal{S}$. Then $\mathcal{A}$ is a $2-(p^2, p, 1)$ design, that is, an affine plane of order p.

Let M and N be as before, let $U \in \{L_j, M, N\}^{\perp} - \{L_i\}$, and let $U I m I M$ and $U I n I N$. If δ is the element of G mapping m onto n, then δ fixes all points of L_j and induces a translation in the affine plane $\mathcal{A}$. It follows easily that $\mathcal{A}$ is a translation plane. As p is prime, the translation plane $\mathcal{A}$ is desarguesian.

Let C be a line not concurrent with L_i or L_j. The elements of G fixing C form a group $\langle\sigma\rangle$ of order p. In $\mathscr{A}$ the group $\langle\sigma\rangle$ induces an automorphism group Σ of order p fixing the parallel class consisting of lines of A of type (i). If the matrix A represents a generator of Σ, then we may take for A the matrix.

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 1 \end{bmatrix},$$

with $A^p = \ell\mathscr{I}$. It follows that $a = d = \ell = 1$, and so

$$A^r = \begin{bmatrix} 1 & rb & rc + \dfrac{r(r-1)}{2}be \\ 0 & 1 & re \\ 0 & 0 & 1 \end{bmatrix}, \quad r = 0,1,\ldots,p-1$$

Since x is the only point of L_i or L_j which is fixed by σ, the group Σ fixes no line of $\mathscr{A}$ and fixes exactly one point at infinity of $\mathscr{A}$. Hence $e \neq 0 \neq b$. The orbit under Σ of the point $(x_0, y_0, 1)$ of $\mathscr{A}$ is the parabola

$$\tfrac{1}{2}(Y - y_0)[(Y - y_0) - e]b + (Y - y_0)(c + by_0) - e(X - x_0) = 0.$$

Its point at infinity is exactly the point at infinity of the lines of $\mathscr{A}$ of type (i). The parabola having as elements the p lines $M \in \{L_i, C\}^{\perp}$, with $x \not{I} M$, will be denoted by $\tilde{C}$.

Now we consider distinct nonconcurrent lines C and C', which are not concurrent with L_i. First suppose that $C' \sim L_j \sim C$. As $\{L_i, C\}$ is not regular we necessarily have $|\{L_i, C, C'\}^{\perp}| \leqslant 2$. Next, let $C' \not\sim L_j \sim C$. As the line C of $\mathscr{A}$ contains at most two points of the parabola $\tilde{C}'$, we have $|\{L_i, C, C'\}^{\perp}| \leqslant 2$. Finally, let $C' \not\sim L_j \not\sim C$. Then the parabolas $\tilde{C}$ and $\tilde{C}'$ either coincide or have at most two points in common. Hence, either for some k we have $C \sim L_k \sim C'$ with $\{C, L_i\}$ regular and $C' \in \{C, L_i\}^{\perp\perp}$ or $|\{L_i, C, C'\}^{\perp}| \leqslant 2$. Assume by way of contradiction that $\tilde{C} = \tilde{C}'$. If $w\, I L_j$, $w \neq x$, then the line W defined by $w\, I W \sim C$ is a tangent line of $\tilde{C}$. In this way we obtain the p tangent lines of $\tilde{C}$. As $\tilde{C} = \tilde{C}'$, W is also a tangent line of $\tilde{C}'$, hence $W \sim C'$. It follows that $L_j \in \{C, C'\}^{\perp\perp}$. Hence $L_i = L_j$, a contradiction. So again we have $|\{L_i, C, C'\}^{\perp}| \leqslant 2$. We conclude that the line L_i is antiregular.

The affine plane $\mathscr{A}$ is the affine plane $\pi(L_i, L_j)$ defined by the antiregular line L_i and the line L_j; cf. 1.3.2 of Payne and Thas (1984). As $\pi(L_i, L_j) = \mathscr{A}$ is desarguesian, then by 5.2.7 of Payne and Thas (1984), the GQ $\mathscr{S}$ is isomorphic to $Q(4,p)$. □

Proposition 4. *We have* $t \in \{p, p^2\}$.

Proof. We already know that $p \leqslant t \leqslant p^2$.

Let us show that p divides t. By way of contradiction assume that p does not divide t. Consider the subgroup S_0^* of G which fixes the common point z_0 of L_0 and M_0. We have $|S_0^*| = pt$, and as $p \nmid t$ the group S_0^* has $1 + kp$ Sylow p-subgroups of order p, with $t = r(1 + kp)$. Let V be the set of all lines through z_0, but different from L_0. If M, M' are lines of V and if S_M, $S_{M'}$ are the subgroups of S_0^* fixing M, M' respectively, then $S_M = \sigma S_{M'} \sigma^{-1}$ with σ any element of S_0^* which maps M onto M'. Also, any Sylow p-subgroup of S_0^* is of the form S_M with M some line of V. For given $M \in V$ the number α of lines $M' \in V$ for which $S_M = S_{M'}$ is the size of the normalizer $N(S_M)$ of S_M is S_0^* divided by the order of S_M. As $|N(S_M)|(1 + kp) = |S_0^*| = pt$, we have $|N(S_M)| = pr$, and so $\alpha = r$. Let $u I L_0$ with $x \neq u \neq z_0$. The group S_0 fixes u, and as $p \nmid t$ it also fixes at least one line U through u, $U \neq L_0$. Let $M_1, M_2, \ldots, M_r$ be the lines of V fixed by S_0. Then $\{U, M_j\}^{\perp}$ is fixed by S_0, and all lines of $\{U, M_j\}^{\perp}$ are concurrent with some line L_{i_j}, $i_j \neq 0$. Now it follows from Proposition 2 that $\{L_{i_j}, M_j\}$ is regular, $j = 1, 2, \ldots, r$. Notations are chosen in such a way that $i_j = j$, $j = 1, 2, \ldots, r$. By Proposition 2 all pairs $\{L_j, M\}$ are regular, M any line with $x \not{I} M$ and $L_0 \sim M$. Now it is clear that every line of $\{L_j, M_i\}^{\perp\perp}$, $i, j = 1, 2, \ldots, r$, is fixed by S_0. The r^2 sets $\{L_j, M_i\}^{\perp\perp}$, $i, j = 1, 2, \ldots, r$, contain in total exactly $r(1 + p)$ lines (through each point of L_0 different from x, there are r lines ($\neq L_0$) fixed by S_0). It easily follows that each pair $\{M', N'\}$, $M' \sim L_0 \sim N'$, $x \not{I} M'$, $x \not{I} N'$, $M' \not\sim N'$, M' and N' fixed by S_0, belongs to exactly one of the sets $\{L_j, M_i\}^{\perp\perp}$; also, each pair $\{L_j, M'\}$, $M' \sim L_0$, $L_j \not\sim M'$, $j \in \{1, 2, \ldots, r\}$ and M' fixed by S_0, belongs to exactly one of the sets $\{L_j, M_i\}^{\perp\perp}$. Then by 2.3.1 of Payne and Thas (1984), the set $\mathcal{P}'$ consisting of the points on the lines of the sets $\{L_j, M_i\}^{\perp\perp}$, together with the lines of $\mathcal{S}$ containing at least two (and then exactly $p + 1$) points of $\mathcal{P}'$, form a subquadrangle $\mathcal{S}'$ of order (p, r) of $\mathcal{S}$. As $\mathcal{S}'$ also contains a subquadrangle of order $(p, 1)$, by 2.2.2 of Payne and Thas (1984), we have $r = t$ or $r = 1$ or $(r, t) = (p, p^2)$. But $p \nmid t$, and so $r \in \{1, t\}$. If $r = 1$, then $t = 1 + kp$. By 1.2.2 of Payne and Thas (1984), $p(1 + k) + 1$ divides $p(1 + kp)(p + 1)(2 + kp)$, so divides $p^2 - 1$, so divides $kp(p - 1)$, so divides $k(p - 1)$. Hence $p(1 + k) + 1 \leqslant k(p - 1)$, a contradiction. Consequently, we may assume that S_i^* has exactly one Sylow p-subgroup, and so S_i fixes all lines of $L_i^{\perp}$, $i = 0, 1, \ldots, t$. It follows that $\mathcal{S}$ is a translation GQ (cf. 8.2 of Payne and Thas, (1984)) and so $p \mid t$ by 8.5.2 of Payne and Thas (1984). We conclude that $t = np$, $1 \leqslant n \leqslant p$.

Assume that $n < p$. The elation group G of order $p^3 n$ has $1 + bp$ Sylow p-subgroups of order p^3, where $1 + bp$ divides n. Hence $b = 0$, and so G has a unique Sylow p-subgroup G' of order p^3. If $y_1 \sim y_2$, $y_1 \not\sim x \not\sim y_2$, then the elation $\delta \in G$ mapping y_1 onto y_2 has order p. Hence $\langle \delta \rangle \leqslant G'$. Let u_1 ad u_2 be any two noncollinear points of $\mathcal{P} - x^{\perp}$. If $x \notin \{u_1, u_2\}^{\perp\perp}$, then choose u_3 with $u_1 \sim u_3 \sim u_2$, $u_3 \notin x^{\perp}$. If $u_1^{\delta_1} = u_3$, $u_3^{\delta_2} = u_2$, with δ_1, $\delta_2 \in G$, then δ_1, $\delta_2 \in G'$. Hence the elation of G mapping u_1 onto u_2 belongs to G'. If $x \in \{u_1, u_2\}^{\perp\perp}$, then, as $s \neq 1$, we can choose points u_3, u_4 in $\mathcal{P} - x^{\perp}$ such that $u_1 \sim u_3 \sim u_4 \sim u_2$. Again it is clear that the elation of G mapping u_1 onto u_2 belongs to G'. Consequently $G \leqslant G'$, that is, $n = 1$. So $t = p$.

We conclude that $t \in \{p, p^2\}$. □

Notice that Frohardt (1988) already proved the previous statement using group-theoretical arguments only.

Theorem 5. *Elation generalized quadrangles of order (p, t), $t \neq 1$ and p prime, are classical.*

Proof. This follows immediately from Propositions 1, 3 and 4. □

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