Sub-weakly Embedded Singular and Degenerate Polar Spaces

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Abstract

We show that every sub-weak embedding of any singular (degenerate or not) orthogonal or unitary polar space of non-singular rank at least three in a projective space $\mathbf{PG}(d, \mathbb{K})$, $\mathbb{K}$ a commutative field, is the projection of a full embedding in some subspace $\mathbf{PG}(\overline{d}, \mathbb{F})$ of $\mathbf{PG}(\overline{d}, \mathbb{K})$, where $\mathbf{PG}(\overline{d}, \mathbb{K})$ contains $\mathbf{PG}(d, \mathbb{K})$ and $\mathbb{F}$ is a subfield of $\mathbb{K}$. The same result is proved in the symplectic case under the assumption that the field over which the polarity is defined is perfect if the characteristic is two and if each secant line of the embedded polar space Γ contains exactly two points of Γ. This completes the classification of all sub-weak embeddings of orthogonal, symplectic and unitary polar spaces (singular or not; degenerate or not) of non-singular rank at least three and defined over a commutative field $\mathbb{F}'$, where in the characteristic 2 case $\mathbb{F}'$ is perfect if the polar space is symplectic and the degree of the embedding is two; see Thas & Van Maldeghem [5] for the non-singular case.

1 Introduction and Statement of the Results

In this paper we always assume that $\mathbb{K}$ and $\mathbb{F}$ are commutative fields.

A *weak embedding* of a point-line geometry Γ with point set $\mathcal{S}$ in a projective space $\mathbf{PG}(d, \mathbb{K})$ is a monomorphism θ of Γ into the geometry of points and lines of $\mathbf{PG}(d, \mathbb{K})$ such that

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(WE1) the set $\mathcal{S}^\theta$ generates $\mathbf{PG}(d, \mathbb{K})$,

(WE2) for any point x of Γ, the subspace generated by the set $X = \{y^\theta \parallel y \in \mathcal{S}$ is collinear with $x\}$ meets $\mathcal{S}^\theta$ precisely in X,

(WE3) if for two lines L_1 and L_2 of Γ the images L_1^θ and L_2^θ meet in some point x, then x belongs to $\mathcal{S}^\theta$.

In such a case we say that the image Γ^θ of Γ is weakly embedded in $\mathbf{PG}(d, \mathbb{K})$.

A *full embedding in* $\mathbf{PG}(d, \mathbb{K})$ is a weak embedding with the additional property that for every line L, all points of $\mathbf{PG}(d, \mathbb{K})$ on the line L^θ have an inverse image under θ. In that case (WE3) is trivially satisfied and also (WE2) can be proved, see BUEKENHOUT & LEFEVRE [1].

Weak embeddings were introduced by LEFEVRE-PERCSY [2, 3]. When Γ is a polar space, then we can view Γ as a point-line geometry and so we can consider weak embeddings of polar spaces. In that case, LEFÈVRE-PERCSY [2] shows that the number δ of points of Γ^θ on a line of $\mathbf{PG}(d, \mathbb{K})$ containing at least two points of Γ^θ which are not collinear in Γ^θ, is a constant. We call δ the *degree* of the embedding. If Γ is non-degenerate (i.e., no point of Γ is collinear in Γ with all other points of Γ), then THAS & VAN MALDEGHEM [5] prove that the condition (WE3) is superfluous and they classify all — finite and infinite — weakly embedded non-singular polar spaces of rank at least 3 of orthogonal, symplectic or unitary type defined over a commutative field $\mathbb{F}$, i.e., which have a standard full embedding in some projective space $\mathbf{PG}(d, \mathbb{F})$, where in the characteristic 2 case $\mathbb{F}$ is perfect if the polar space is symplectic and the degree of the embedding is two. The classification of all generalized quadrangles weakly embedded in a finite projective space can be found in THAS & VAN MALDEGHEM [6].

We call a monomorphism θ from the point-line geometry of a polar space Γ with point set $\mathcal{S}$ to the point-line geometry of a projective space $\mathbf{PG}(d, \mathbb{K})$ a *sub-weak embedding* if it satisfies conditions (WE1) and (WE2). Usually, we simply say that Γ is weakly or sub-weakly embedded in $\mathbf{PG}(d, \mathbb{K})$ without referring to θ, that is, we identify the points and lines of Γ with their images in $\mathbf{PG}(d, \mathbb{K})$. In such a case the set of all points of Γ on a line L of Γ will be denoted by L^*.

If the polar space Γ arises from a quadric it is called *orthogonal*, if it arises from a hermitian variety it is called *unitary*, and if it arises from a symplectic polarity it is called *symplectic*. In these cases Γ is called *non-singular* either if the hermitian variety is non-singular, or if the symplectic polarity is non-singular, or if the quadric is non-singular (in the sense that the quadric Q, as algebraic hypersurface, contains no singular point over the algebraic closure of the ground field over which Q is defined); in the symplectic and hermitian case this is equivalent to assuming that the corresponding matrix is non-singular. In the orthogonal case with characteristic not 2, in the symplectic case and in the hermitian case, Γ is non-singular if and only if it is non-degenerate; in the orthogonal case with characteristic 2, non-singular implies non-degenerate, but when not every element of $\mathbb{K}$ is a square, a non-degenerate quadric may be singular.

Suppose that Γ is an orthogonal, symplectic or unitary polar space naturally embedded in the projective space $\mathbf{PG}(d,\mathbb{F})$. Denote by $\overline{\Gamma}$ the extension of Γ to $\mathbf{PG}(d,\overline{\mathbb{F}})$, where $\overline{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$, i.e., for Γ orthogonal or unitary the equation of Γ with coefficients in $\mathbb{F}$ is considered as an equation over $\overline{\mathbb{F}}$ and for Γ symplectic the matrix over $\mathbb{F}$ defining Γ is taken as the matrix over $\overline{\mathbb{F}}$ defining $\overline{\Gamma}$. The *rank* $\mathrm{Rank}(\Gamma)$ of Γ is one more than the (projective) dimension of any maximal singular subspace of Γ; the *absolute rank* $\mathrm{Rank}_A(\Gamma)$ of Γ is the rank of $\overline{\Gamma}$; the *non-singular corank* $\mathrm{Rank}^s(\Gamma)$ of Γ is one more than the dimension of the unique projective subspace $S(\Gamma)$ of Γ consisting of all points x of Γ which are collinear in Γ with all other points of Γ; the *non-singular rank* $\mathrm{Rank}^n(\Gamma)$ of Γ is equal to $\mathrm{Rank}(\Gamma) - \mathrm{Rank}^s(\Gamma)$; the *absolute non-singular corank* $\mathrm{Rank}^s_A(\Gamma)$ is the non-singular corank of $\overline{\Gamma}$ and the *absolute non-singular rank* $\mathrm{Rank}^n_A(\Gamma)$ is the non-singular rank of $\overline{\Gamma}$.

The elements of $S(\Gamma)$ are called the *singular points* of Γ; all other points of Γ are *non-singular points*.

Our Main Result reads as follows.

Main Result. *Let the orthogonal, symplectic or unitary polar space Γ of non-singular rank at least 3 be sub-weakly embedded in the projective space $\mathbf{PG}(d,\mathbb{K})$, where for Γ symplectic and having degree 2 we assume that the field over which the polarity is defined is perfect in the characteristic 2 case. Then there is a projective space $\mathbf{PG}(\overline{d},\mathbb{K})$ containing $\mathbf{PG}(d,\mathbb{K})$ such that Γ is the projection from a $\mathbf{PG}(\overline{d}-d-1,\mathbb{K}) \subseteq \mathbf{PG}(\overline{d},\mathbb{K})$ into $\mathbf{PG}(d,\mathbb{K})$ of a polar space $\widetilde{\Gamma}$ which is fully embedded in some subspace $\mathbf{PG}(\overline{d},\mathbb{F})$ of $\mathbf{PG}(\overline{d},\mathbb{K})$, for some subfield $\mathbb{F}$ of $\mathbb{K}$ (in particular, if $d = \overline{d}$, then Γ is fully embedded in some subspace $\mathbf{PG}(d,\mathbb{F})$ of $\mathbf{PG}(d,\mathbb{K})$).*

A finite polar space is automatically of one of the three types mentioned in the Main Result. So, in view of Thas & Van Maldeghem [5], we have the following corollary.

Corollary. *Let Γ be a polar space of non-singular rank at least 3 sub-weakly embedded in the finite projective space $\mathbf{PG}(d,q)$. Then there is a projective space $\mathbf{PG}(\overline{d},q)$, $\overline{d} \geq d$, containing $\mathbf{PG}(d,q)$ such that Γ is the projection into $\mathbf{PG}(d,q)$ of a polar space $\widetilde{\Gamma}$ which is fully embedded in some subspace $\mathbf{PG}(\overline{d},q')$ of $\mathbf{PG}(\overline{d},q)$, for some subfield $\mathbf{GF}(q')$ of $\mathbf{GF}(q)$.*

We remark that for symplectic polar spaces sub-weakly embedded of degree two, the condition that the field over which the polarity is defined is perfect in the characteristic 2 case cannot be dispensed with in view of a counterexample in THAS & VAN MALDEGHEM [5] for the non-degenerate case.

2 Proof of the Main Result

In the sequel, for any point x of a polar space, we adopt the notation $x^\perp$ for the set of all points collinear with the point x in the polar space. We remark that polar spaces are *Shult spaces*, i.e., for every point x and every line L, $x^\perp$ either contains all points of L or exactly one point of L (we will call that property the *Buekenhout-Shult axiom*).

From now on, we suppose that Γ is an orthogonal, symplectic or unitary singular polar space sub-weakly embedded in $\mathbf{PG}(d,\mathbb{K})$. We will prove the Main Result by means of an induction on $r(\Gamma) =: \mathrm{Rank}^s_A(\Gamma) - \mathrm{Rank}^s(\Gamma)$. First we show two lemmas.

Lemma 1 *If L is a line of the sub-weakly embedded polar space Γ, then the only points of Γ on L are the points of L^*.*

Proof. See THAS & VAN MALDEGHEM [5], Lemma 1. □

Lemma 2 *If L_1 and L_2 are two lines of Γ which intersect in a point of Γ, then L_1^* and L_2^* are lines of a unique projective subplane $\mathbf{PG}(2,\mathbb{F})$, with $\mathbb{F}$ a subfield of $\mathbb{K}$. The subfield $\mathbb{F}$ does not depend on the choice of L_1, L_2.*

Proof. This follows immediately from Lemma 1 and a connectedness argument; cf. THAS & VAN MALDEGHEM [5], part (a) of the proof of Lemma 5. □

Lemma 3 *Let the point-line geometry Ω of a projective space $\mathbf{PG}(k',\mathbb{F})$ be sub-weakly embedded in a projective space $\mathbf{PG}(k,\mathbb{K})$. Then for any $b \geq -1$ there exists a projective space $\mathbf{PG}(k'+b+1,\mathbb{K})$ containing $\mathbf{PG}(k,\mathbb{K})$, a subspace $\mathbf{PG}(k'-k+b,\mathbb{K})$ of $\mathbf{PG}(k'+b+1,\mathbb{K})$ and a subspace $\mathbf{PG}(k',\mathbb{F}')$ of $\mathbf{PG}(k'+b+1,\mathbb{K})$ over a subfield $\mathbb{F}'$ of $\mathbb{K}$, with $\mathbb{F}'$ isomorphic to $\mathbb{F}$, such that Ω is the projection of $\mathbf{PG}(k',\mathbb{F}')$ from $\mathbf{PG}(k'-k+b,\mathbb{K})$ into $\mathbf{PG}(k,\mathbb{K})$.*

Proof. If $b = -1$, then this is due to LIMBOS [4], who proved it in the finite case. But her proof can also be used in the infinite case without any notable change. If $b \geq 0$, then we first apply the result for $b = -1$ to obtain a projective space $\mathbf{PG}(k', \mathbb{K})$ containing $\mathbf{PG}(k, \mathbb{K})$, a subspace $\mathbf{PG}(k' - k - 1, \mathbb{K})$ of $\mathbf{PG}(k', \mathbb{K})$ and a subspace $\mathbf{PG}(k', \mathbb{F}')$ of $\mathbf{PG}(k', \mathbb{K})$ over a subfield $\mathbb{F}'$ of $\mathbb{K}$, with $\mathbb{F}'$ isomorphic to $\mathbb{F}$, such that Ω is the projection of $\mathbf{PG}(k', \mathbb{F}')$ from $\mathbf{PG}(k' - k - 1, \mathbb{K})$ into $\mathbf{PG}(k, \mathbb{K})$. We now extend $\mathbf{PG}(k', \mathbb{K})$ to a $\mathbf{PG}(k' + b + 1, \mathbb{K})$ and we extend $\mathbf{PG}(k' - k - 1, \mathbb{K})$ inside $\mathbf{PG}(k' + b + 1, \mathbb{K})$ to a $\mathbf{PG}(k' - k + b, \mathbb{K})$ such that $\mathbf{PG}(k' - k + b, \mathbb{K}) \cap \mathbf{PG}(k', \mathbb{K}) = \mathbf{PG}(k' - k - 1, \mathbb{K})$. The result now follows easily. $\square$

We now prove the Main Result for $\mathrm{Rank}^s(\Gamma) = \mathrm{Rank}^s_A(\Gamma)$. So suppose $r(\Gamma) = 0$.

First we note that, as Γ contains planes, for every line L of Γ the set L^* is a subline of $\mathbf{PG}(d, \mathbb{K})$ over some subfield $\mathbb{F}$ of $\mathbb{K}$ and $\mathbb{F}$ does not depend on the particular line by Lemma 2.

Let Γ' be a polar space isomorphic to Γ, embedded in a standard way in some projective space $\mathbf{PG}(d', \mathbb{F}')$, $\mathbb{F}'$ a commutative field isomorphic to $\mathbb{F}$. Put $k' = \mathrm{Rank}^s(\Gamma)$ and let S' be the space of singular points of Γ'. With S' there corresponds a projective space S of $\mathbf{PG}(d, \mathbb{K})$ sub-weakly embedded in some subspace $\mathbf{PG}(k, \mathbb{K})$ of $\mathbf{PG}(d, \mathbb{K})$, $k \leq k'$. In $\mathbf{PG}(d', \mathbb{F}')$ we consider a projective space $T' \cong \mathbf{PG}(d' - k' - 1, \mathbb{F}')$ skew to S'. Let $T' \cap \Gamma' = \Gamma'_1$; then Γ'_1 is non-singular. With Γ'_1 there corresponds Γ_1 in Γ and, by THAS & VAN MALDEGHEM [5], Γ_1 is fully embedded in a $\mathbf{PG}(d_1, \mathbb{F})$. Let $\mathbf{PG}(d_1, \mathbb{K})$ be the extension over $\mathbb{K}$ of $\mathbf{PG}(d_1, \mathbb{F})$. Clearly the spaces $\mathbf{PG}(k, \mathbb{K})$ and $\mathbf{PG}(d_1, \mathbb{K})$ generate $\mathbf{PG}(d, \mathbb{K})$. Putting $\mathbf{PG}(d_1, \mathbb{K}) \cap \mathbf{PG}(k, \mathbb{K}) = \mathbf{PG}(b, \mathbb{K})$ implies that $b = d_1 - d + k$.

We show that $b \in \{-1, 0\}$. Let x be any point of Γ_1. Since Γ_1 is fully embedded in $\mathbf{PG}(d_1, \mathbb{F})$, the set $x^\perp \cap \mathbf{PG}(d_1, \mathbb{K})$ lies in a unique hyperplane $\mathbf{PG}(d_1 - 1, \mathbb{K})$ of $\mathbf{PG}(d_1, \mathbb{K})$. Since x is collinear in Γ with every point of S, the subspace of $\mathbf{PG}(d, \mathbb{K})$ spanned by $x^\perp$ contains $\mathbf{PG}(k, \mathbb{K})$. Let b^* be the dimension of $\mathbf{PG}(d_1 - 1, \mathbb{K}) \cap \mathbf{PG}(b, \mathbb{K}) = \mathbf{PG}(d_1 - 1, \mathbb{K}) \cap \mathbf{PG}(k, \mathbb{K})$. Since by (WE2) $\mathbf{PG}(d_1 - 1, \mathbb{K})$ and $\mathbf{PG}(k, \mathbb{K})$ span at most a $(d - 1)$-dimensional space, we have $b^* \geq (d_1 - 1) + k - (d - 1) = d_1 - d + k$. Hence $b = b^*$ and every tangent hyperplane of Γ_1 contains $\mathbf{PG}(b, \mathbb{K})$. Since Γ_1 is non-singular, this implies that $b = -1$ or $b = 0$; if $b = 0$, then the characteristic of $\mathbb{F}$ is equal to 2, d_1 is even and the unique element of $\mathbf{PG}(b, \mathbb{K})$ lies in $\mathbf{PG}(d_1, \mathbb{F})$, but does not belong to Γ_1.

We extend $\mathbf{PG}(d, \mathbb{K})$ to $\mathbf{PG}(\overline{d}, \mathbb{K})$, with $\overline{d} = b + d + k' - k + 1$. Let U be a $(k' + b + 1)$-dimensional projective subspace of $\mathbf{PG}(\overline{d}, \mathbb{K})$ containing $\mathbf{PG}(k, \mathbb{K})$ and meeting $\mathbf{PG}(d_1, \mathbb{K})$ precisely in $\mathbf{PG}(b, \mathbb{K})$. It follows from Lemma 3 that S is the projection into $\mathbf{PG}(k, \mathbb{K})$ of a certain k'-dimensional subspace $\widetilde{S} \subseteq U$ over $\mathbb{F}$ from a $(k' - k + b)$-dimensional subspace $C \subseteq U$. The space $\widetilde{S}$ can be chosen not to contain $\mathbf{PG}(b, \mathbb{K})$.

If $b = 0$ and the unique element of $PG(b, \mathbb{K})$ belongs to Γ, then let c denote that unique element; otherwise let c be any element of S. Let U_1 be the projective space over $\mathbb{K}$ generated by the points of Γ_1 and c. Denote by $\tilde{c}$ the unique point of $\widetilde{S}$ which is projected from C onto c. Let $\{v\} = C \cap c\tilde{c}$. Let z be an arbitrary but fixed point in Γ_1. The line $(cz)^*$ is a subline of cz over $\mathbb{F}$, hence the points x on $\tilde{c}z$ such that $vx \cap cz$ belongs to Γ (or equivalently to $(cz)^*$) also form a subline $L \cong \mathbf{PG}(1, \mathbb{F})$ of $\tilde{c}z$ over $\mathbb{F}$ containing the points $\tilde{c}$ and z. The points of $\widetilde{S}$, L and Γ_1 now define a unique projective subspace $\mathbf{PG}(\overline{d}, \mathbb{F})$ of $\mathbf{PG}(\overline{d}, \mathbb{K})$. Let $\widetilde{\Gamma}$ be the unique polar space isomorphic to Γ embedded in $\mathbf{PG}(\overline{d}, \mathbb{F})$, containing Γ_1 and having as set of singular points the set $\widetilde{S}$. The points of $\tilde{c}z$ belonging to $\widetilde{\Gamma}$ and the points of $(cz)^*$ determine a unique plane π over $\mathbb{F}$ which contains v.

Let y be any point of Γ_1 collinear in Γ_1 with z. Then π and $(yz)^*$ determine a unique 3-dimensional subspace over $\mathbb{F}$ which contains all elements of $(cy)^*$ and all points of $\widetilde{\Gamma}$ lying on $\tilde{c}y$. Hence v lies in the plane over $\mathbb{F}$ determined by $(cy)^*$ and the points of $\widetilde{\Gamma}$ on $\tilde{c}y$. But this implies that the projection from v of all points of $\widetilde{\Gamma}$ on $\tilde{c}y$ is precisely the set $(cy)^*$. If u is any point of Γ_1, then u is collinear with at least one point w collinear with z. Substituting u for y and w for z, we easily see with the previous argument that $(cu)^*$ is the projection from v of the set of points of $\widetilde{\Gamma}$ on $\tilde{c}u$. We conclude that $\Gamma \cap U_1$ is the projection from v into U_1 of the polar subspace of $\widetilde{\Gamma}$ containing Γ_1 and having $\tilde{c}$ as unique singular point.

Now let $\tilde{a}$ be an arbitrary element of $\widetilde{S} \setminus \{c\}$ and let a be its projection from C into S. Let $(au)^*$ be the set of points of Γ on the line au with $u \in \Gamma_1$ arbitrary; let $(\tilde{a}u)^{**}$ be the set of points of $\widetilde{\Gamma}$ on $\tilde{a}u$. The set $(au)^*$ belongs to the plane π_1 over $\mathbb{F}$ determined by the elements of $(cu)^*$ and $(ac)^*$. Similarly, the set $(\tilde{a}u)^{**}$ belongs to the plane π_2 over $\mathbb{F}$ determined by the points of $\widetilde{\Gamma}$ on $\tilde{c}u$ and on $\tilde{a}\tilde{c}$. By assumption the points of $\widetilde{\Gamma}$ on $\tilde{a}\tilde{c}$ are projected, from C into S, onto the points of $(ac)^*$; by the previous paragraph the points of $\widetilde{\Gamma}$ on $\tilde{c}u$ are projected, from C into U_1, onto the points of $(cu)^*$. Hence π_1 is the projection from C of π_2 into the projective space generated by $\mathbf{PG}(d_1, \mathbb{K})$ and S. This implies that $(\tilde{a}u)^{**}$ is projected from C onto $(au)^*$, showing that Γ is the projection from C into $\mathbf{PG}(d, \mathbb{K})$ of $\widetilde{\Gamma}$.

This completes the proof of the case $r(\Gamma) = 0$.

Now suppose the Main Result holds for orthogonal, symplectic and unitary polar spaces Γ^* with commutative underlying field $\mathbb{F}'$ and with $r(\Gamma^*) < r$, $r > 0$. Suppose $r = r(\Gamma)$; since $r > 0$, Γ is necessarily orthogonal, but not symplectic.

Let $\Gamma' \cong \Gamma$ be embedded in the standard way in some $\mathbf{PG}(d', \mathbb{F}')$, so that the point set of Γ' is a quadric, and let $\overline{\Gamma'}$ be the extension of Γ' over the algebraic closure $\overline{\mathbb{F}'}$ of $\mathbb{F}'$. Let

$\mathbf{PG}(d', \overline{\mathbb{F}'})$ be the corresponding extension of $\mathbf{PG}(d', \mathbb{F}')$. Let S' be a maximal projective subspace over $\mathbb{F}'$ entirely contained in Γ'. We now show that S' can be chosen in such a way that there exists a hyperplane H' of $\mathbf{PG}(d', \mathbb{F}')$ containing S' and such that the extension of H' over $\overline{\mathbb{F}'}$ does not contain all singular points of $\overline{\Gamma'}$. Let us denote the extension of a subspace X of $\mathbf{PG}(d', \mathbb{F}')$ to $\mathbf{PG}(d', \overline{\mathbb{F}'})$ by $\overline{X}$. Factoring out the space of all singular points of Γ', we obtain a non-degenerate polar space Φ'. With S' there corresponds a maximal projective space R' over $\mathbb{F}'$ on Φ'. It suffices to show that R' can be chosen such that $\overline{R'}$ does not contain all singular points of $\overline{\Phi'}$. If, on the contrary, for some R' the space $\overline{R'}$ does contain all singular points of $\overline{\Phi'}$, then we can choose a hyperplane T' in R' with the property that $\overline{T'}$ does not contain all singular points of $\overline{\Phi'}$. Let R'_1 be a second maximal projective space over $\mathbb{F}'$ on Φ', and let $R' \cap R'_1 = T'_1$. Further let T'' be a subspace of T' which has the same dimension as T'_1. By the transitivity of the automorphism group of Φ' on the set of *chambers* of Φ' (viewed as a building of rank ≥ 3, see Tits [7]), there exists an automorphism of Φ' fixing R' and mapping T'_1 onto T''. If R'' is the image of R'_1, then $R' \cap R'' = T'' \subseteq T'$. It follows that $\overline{R''}$ does not contain all singular points of $\overline{\Phi'}$. With R'' there corresponds a maximal projective space S'' over $\mathbb{F}'$ on Γ'. Now there exists a hyperplane H' of $\mathbf{PG}(d', \mathbb{F}')$ containing S'' and such that $\overline{H'}$ does not contain all singular points of $\overline{\Gamma'}$. Then $H' \cap \Gamma' = \Gamma'_1$ is a polar space with $r(\Gamma'_1) = r - 1$ and with $\mathrm{Rank}^n(\Gamma'_1) = \mathrm{Rank}^n(\Gamma')$ ($\mathrm{Rank}^s(\Gamma'_1) = \mathrm{Rank}^s(\Gamma')$ because H' is not a tangent hyperplane).

With Γ'_1 there corresponds a sub-polar space Γ_1 of Γ sub-weakly embedded in a subspace $\mathbf{PG}(d_1, \mathbb{K})$ of $\mathbf{PG}(d, \mathbb{K})$, $d_1 \leq d$. Since Γ' is linearly generated by Γ'_1 and one single point of $\Gamma' \setminus \Gamma'_1$, we see that $d_1 = d$ or $d_1 = d - 1$. By the induction hypothesis there exists a projective space $\mathbf{PG}(\overline{d} - 1, \mathbb{K})$ containing $\mathbf{PG}(d_1, \mathbb{K})$ such that Γ_1 is the projection from a $(\overline{d} - d_1 - 2)$-dimensional subspace C_1 of $\mathbf{PG}(\overline{d} - 1, \mathbb{K})$ into $\mathbf{PG}(d_1, \mathbb{K})$ of a polar space $\widetilde{\Gamma}_1$ fully embedded in a subspace $\mathbf{PG}(\overline{d} - 1, \mathbb{F})$ of $\mathbf{PG}(\overline{d} - 1, \mathbb{K})$, where $\mathbb{F}$ is a subfield of $\mathbb{K}$ isomorphic to $\mathbb{F}'$. Now the idea of the proof is the following. If $d_1 = d - 1$, then $\mathbf{PG}(\overline{d} - 1, \mathbb{K})$ is extended to the space $\mathbf{PG}(\overline{d}, \mathbb{K})$ which contains $\mathbf{PG}(d, \mathbb{K})$. If $d_1 = d$, then we extend C_1 to a $(\overline{d} - d - 1)$-dimensional space C not contained in $\mathbf{PG}(\overline{d} - 1, \mathbb{K})$ and we extend $\mathbf{PG}(\overline{d} - 1, \mathbb{K})$ to the space $\mathbf{PG}(\overline{d}, \mathbb{K})$ containing C. The first case is easier than the second one (but both are based on the same ideas) and so we will give the detailed proof in the second case, leaving the first case to the reader.

Clearly we may assume that $\widetilde{\Gamma}_1$ contains a non-singular point $\tilde{z}$ which does not belong to Γ_1. Let z be the corresponding point in Γ_1. Let L_1 be any line of $\Gamma \setminus \Gamma_1$ through z and let $\widetilde{\Gamma}$ be a polar space containing $\widetilde{\Gamma}_1$, which is isomorphic to Γ and which is fully embedded in a $\mathbf{PG}(\overline{d}, \mathbb{F})$ which contains $\mathbf{PG}(\overline{d} - 1, \mathbb{F})$ as a hyperplane. Consider an isomorphism γ from $\widetilde{\Gamma}$ to Γ such that the restriction of γ to $\widetilde{\Gamma}_1$ is the projection of $\widetilde{\Gamma}_1$ from C_1 into

$\mathbf{PG}(d,\mathbb{K})$. Let $\widetilde{L}_1$ be the line of $\widetilde{\Gamma}$ for which $\widetilde{L_1^*}^\gamma = L_1^*$. Now consider a plane $\widetilde{\beta}$ of $\widetilde{\Gamma}$ containing $\widetilde{L}_1^*$. The restriction of γ to $\widetilde{\beta}$ is a semi-linear map; the restriction of γ to $\widetilde{\beta}\cap\widetilde{\Gamma}_1$ is a linear map (it is a projection), hence the restriction of γ to $\widetilde{L}_1^*$ is a linear isomorphism. Hence the lines joining corresponding points of $\widetilde{L}_1^*$ and L_1^* belong to a regulus. The line $z\tilde{z}$ is a line of that regulus, hence we can consider the line M_1 containing $z\tilde{z}\cap C_1$ of the complementary regulus. Let C be the space containing M_1 and C_1. Then the points of $\tilde{L}_1^*$ are projected from C onto the elements of L_1^*. Let $\tilde{x}$ be any point of $\widetilde{\Gamma}\setminus\widetilde{\Gamma}_1$ collinear in $\widetilde{\Gamma}$ with all points of $\widetilde{\Gamma}$ on $\tilde{L}_1$. The plane $\widetilde{\pi}$ of $\widetilde{\Gamma}$ containing $\tilde{x}$ and the points of $\tilde{L}_1^*$ meets $\widetilde{\Gamma}_1$ in a point set $\widetilde{N}_1^*$, with $\widetilde{N}_1$ a line of $\mathbf{PG}(\overline{d}-1,\mathbb{K})$. Let N_1^* be the projection of $\widetilde{N}_1^*$ from C into $\mathbf{PG}(d,\mathbb{K})$; by assumption this is the point set of a line N_1 of Γ. Clearly the projection π of $\widetilde{\pi}$ from C into $\mathbf{PG}(d,\mathbb{K})$ is a plane of Γ (since it contains all points of Γ on two coplanar lines of Γ); hence $\tilde{x}$ is projected from C onto a point x of Γ. Clearly every point x of $\Gamma\setminus\Gamma_1$ collinear with all points of L_1^* is covered this way. Now let $\tilde{y}$ be any point of $\widetilde{\Gamma}\setminus\widetilde{\Gamma}_1$. Then $\tilde{y}$ is collinear with all points of $\widetilde{\Gamma}$ on a line $\widetilde{R}_1$, $\tilde{y}\notin\widetilde{R}_1$, of $\widetilde{\pi}$. Let R_1 be the line of $\mathbf{PG}(d,\mathbb{K})$ containing the projections from C into $\mathbf{PG}(d,\mathbb{K})$ of all points of $\widetilde{\pi}$ on $\widetilde{R}_1$. If R_1 does not belong to Γ_1, then we substitute in the previous argument R_1 for L_1, $\widetilde{R}_1$ for $\widetilde{L}_1$ and $\tilde{y}$ for $\tilde{x}$. So we see that the projection from C of $\tilde{y}$ into $\mathbf{PG}(d,\mathbb{K})$ is a point y of Γ. Suppose now that R_1 is contained in Γ_1 or, equivalently, that $\widetilde{R}_1$ is contained in $\widetilde{\Gamma}_1$. Since $\tilde{z}$ is not a singular point, there exists some point $\tilde{y}_0\in\widetilde{\Gamma}\setminus\widetilde{\Gamma}_1$ not collinear in $\widetilde{\Gamma}$ with $\tilde{z}$. Let $\tilde{y}_1$ be the unique point of $\widetilde{R}_1^*$ collinear in $\widetilde{\Gamma}$ with $\tilde{y}_0$. All elements of $(\tilde{y}_0\tilde{y}_1)^*$ are projected from C into $\mathbf{PG}(d,\mathbb{K})$ onto points of Γ (by substituting in the previous argument y_0 for y). Clearly the set of points of $\widetilde{\Gamma}$ collinear in $\widetilde{\Gamma}$ with $\tilde{y}_0$ and lying in the plane of $\widetilde{\Gamma}$ determined by $\tilde{y}$ and $\widetilde{R}_1$ forms a point set $\widetilde{R}_0^*$ which is not contained in $\widetilde{\Gamma}_1$. Now we see that $\tilde{y}$ is collinear with all points of $\widetilde{R}_0^*$ and the result follows from a previous argument (by interchanging the roles of $\widetilde{\pi}$ and the plane of $\widetilde{\Gamma}$ defined by $\tilde{y}_0$ and $\widetilde{R}_0^*$). It is also clear that every point y of $\Gamma\setminus\Gamma_1$ arises in this way.

The Main Result is proved.

Remark. Also in the case of a full embedding, proper projections arise. For instance consider a non-singular quadric Q in $\mathbf{PG}(d,\mathbb{K})$ which defines a polar space of rank at least 3. Suppose that Q has a kernel c. Now extend $\mathbf{PG}(d,\mathbb{K})$ to a projective space $\mathbf{PG}(d+1,\mathbb{K})$ and let Q_1 be the cone projecting Q from some arbitrary point c_1 in $\mathbf{PG}(d+1,\mathbb{K})\setminus\mathbf{PG}(d,\mathbb{K})$. Let c' be any point on the line cc_1, $c\neq c'\neq c_1$. Now project Q_1 from c' into $\mathbf{PG}(d,\mathbb{K})$. Then we obtain a full embedding Q_1' of the polar space defined by Q_1 in $\mathbf{PG}(d,\mathbb{K})$. Note that $\mathbb{K}$ is perfect if and only if all points of $\mathbf{PG}(d,\mathbb{K})$ are points of Q_1'.

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