MOUFANG AFFINE BUILDINGS HAVE MOUFANG SPHERICAL BUILDINGS AT INFINITY

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Abstract. We show in a direct and elementary way that the spherical building at infinity of every rank 3 affine building which satisfies Tits' Moufang condition, is itself a Moufang building. This result is also true for higher rank affine buildings by Tits' classification [**4**].

1. Introduction. The Moufang condition for—not necessarily spherical—buildings was introduced by Tits [**5**]. It generalizes the usual Moufang condition for spherical buildings, see Tits [**3**], which on its turn was a generalization of the Moufang condition for projective planes. The Moufang condition seems to be the most natural condition under which a classification of certain classes of buildings is possible. For spherical buildings of rank ≥ 3, and for affine building of rank ≥ 4, this is trivially true for all those buildings are classified without any supplementary condition. For spherical buildings of rank 2, Tits [**3**] announces such classification and partial results have been published. There seems to be no further explicit classification of Moufang affine buildings of rank 3 in the literature. In this short note, we show that such a classification can be reduced to checking the Moufang property in the "known classical buildings". Our method uses the building at infinity of the affine building. The definition of Moufang affine building does not imply ipso facto that the building at infinity also satisfies the Moufang condition. We will show that this is however a consequence, using elementary techniques. So our main result reads:

MAIN RESULT. *The building at infinity of an irreducible Moufang rank* 3 *affine building is a Moufang rank* 2 *spherical building.*

COROLLARY. *The irreducible Moufang rank* 3 *affine buildings are amongst the affine buildings arising from an algebraic group of relative rank* 2 *defined over a field with discrete valuation* (*with respect to which the field is complete*), *which is invariant under the field involution that is possibly needed to define the group; also*, *the local field has equal characteristic.*

2. Preliminaries. In this paper, we take the original viewpoint of Tits [**2**]. So buildings are thick simplicial chamber complexes endowed with a set of thin subcomplexes (these subcomplexes are called *apartments*) such that every two simplices are contained in a common apartment, and such that for any pair of apartments (Σ, Σ'), there exists an isomorphism $\theta : \Sigma \to \Sigma'$ fixing every simplex contained in the intersection of Σ and Σ'. It turns out that the apartments of a building Δ are Coxeter complexes and that the Coxeter diagram of this complex is the same as the Buekenhout diagram of the geometry associated to Δ (see Tits [**2**]). In this paper, we are concerned with the (irreducible) rank 3 affine cases, i.e. the types $\tilde{A}_2$, $\tilde{C}_2$ and $\tilde{G}_2$. We briefly describe the Coxeter complexes of each of these types.

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($\tilde{A}_2$) Consider the triangulation of the real euclidian plane $\mathbb{E}$ with equilateral triangles. The vertices of the Coxeter complex are the vertices of the triangles, the edges form simplices of dimension 1 and triangles itself define simplices of dimension 2. This Coxeter complex can also be defined as the barycentric subdivision of the tiling of $\mathbb{E}$ into regular hexagons.

($\tilde{C}_2$) Here we triangulate $\mathbb{E}$ with isosceles right triangles. Or we consider the barycentric subdivision of the tiling of $\mathbb{E}$ into regular octagons and squares.

($\tilde{G}_2$) Here we triangulate $\mathbb{E}$ with right triangles having an angle of 30°. Or we consider the barycentric subdivision of the tiling of $\mathbb{E}$ into regular 12-gons, hexagons and squares.

A *special vertex* in these complexes is a vertex lying in a maximal number of chambers (these numbers are respectively 6, 8 and 12). A *ray* is the set of vertices lying on a half line in $\mathbb{E}$ starting at a special vertex and containing edges of triangles of the respective triangulation. A *sector* is the set of all vertices belonging to the convex closure in $\mathbb{E}$ of two rays starting at the same special vertex and forming a minimal angle (i.e. 60°, 45° and 30° in the three respective cases). A *special vertex*, a *ray* or a *sector* in a rank 3 affine building Δ is a special vertex, a ray of a sector respectively in some apartment of Δ. Also, a *panel* is a simplex consisting of two vertices.

Let Δ be a rank 3 affine building. Suppose that we endow Δ with a maximal set of apartments (this is always possible by Tits [**4**]). Note that the metric of $\mathbb{E}$ induces a metric in Δ. The relation R: "... *is at bounded distance from*..." is an equivalence relation in the set of all rays. We define the following simplicial complex Δ_∞. The vertices of Δ_∞ are the equivalence classes of rays with respect to R; the simplices of dimension 1 (also the chambers) are the pairs of such classes for which there exist respective representatives which lie in a common sector. It can be shown that Δ_∞ is a rank 2 spherical building and every automorphism of Δ also preserves the structure of Δ_∞ (see Tits [**4**]). Note that, by Tits [**4**], each equivalence class of rays has a (unique) representative starting at any special vertex.

The union of two rays in an apartment Σ of a rank 3 affine building Δ which start from the same vertex and which form an angle of 180° is called a *wall*; each wall w in Σ divides Σ in two half apartments α and $-\alpha$ which share the vertices in w. We denote w by $\partial\alpha$. *Parallel walls* are walls which lie on parallel lines in some apartment. Each half apartment is called a *root*. Let Φ be the set of roots in a given apartment Σ of Δ. Given $\alpha \in \Phi$, we denote, as above, by $-\alpha$ the complementary root in Σ (so α and $-\alpha$ meet in the wall $\partial\alpha = \partial(-\alpha)$). For two roots α and β, we write, following Tits [**5**],

$$[\alpha,\beta] = \{\gamma \in \Phi : \alpha \cap \beta \subseteq \gamma \text{ and } (-\alpha)\cap(-\beta) \subseteq (-\gamma)\}.$$

We call Δ a *pre-Moufang* building if, for some apartment Σ, there is a family of automorphism groups $(U_\alpha)_{\alpha\in\Phi}$ (where Φ is defined as above) of Δ satisfying the following two conditions:

(pM1) For each $\alpha \in \Phi$ and each panel p in $\alpha \cap (-\alpha)$, the group U_α fixes every vertex of α and acts transitively on the set of chambers containing p and not contained in α.

(pM2) If $\alpha, \beta \in \Phi$ and $\partial\alpha$ and $\partial\beta$ are not parallel, then the commutator $[U_\alpha, U_\beta]$ is contained in the group generated by all U_γ, where $\gamma \in [\alpha,\beta]\setminus\{\alpha,\beta\}$.

A *Moufang* building satisfies two further properties (and stronger versions of both (pM1) and (pM2)), but we will not need these here. It suffices to remark that every Moufang rank 3 affine building is a pre-Moufang building, and that the Moufang condition requires that the transitive action in (pM1) is simply-transitive. This follows immediately from Tits [**5**]; see also Ronan [**1**].

Finally, we define a Moufang rank 2 building, as introduced by Tits [**2, 3**]. Let Γ be a rank two spherical building and let Σ be some apartment in Γ. We can view Σ as a $2n$-path (connected vertices form a panel; such vertices are also called *adjacent*), for fixed $n \in \mathbb{N}\backslash\{0,1,2\}$ (and Γ corresponds to a so-called *generalized n-gon*). If for each n-path $(x_0, x_1, \ldots, x_n)$ in Σ, the group of automorphisms of Γ fixing all vertices of Γ adjacent to one of $x_1, x_2, \ldots, x_{n-1}$ acts transitively on the set of vertices of Γ adjacent to x_0 but distinct from x_1, then Γ is called a *Moufang rank 2 building*.

3. Proof of the main result. From now on, we suppose that Δ is a pre-Moufang rank 3 affine building containing an apartment Σ (with corresponding set of roots Φ) and such that there exists a family of groups $(U_\alpha)_{\alpha\in\Phi}$ which satisfy (pM1) and (pM2). We consider the rank 2 spherical building Δ_∞ and we let s be any special vertex in Σ. By the previous section, we may identify the vertices of Δ_∞ with the rays starting at s. It follows from Tits [**4**] that adjacent vertices correspond to rays forming a minimal angle.

The rays in Σ starting in s define a $2n$-path Π of $\Delta_\infty(s)$, $n = 3, 4, 6$ for respectively Δ of type $\tilde{A}_2$, $\tilde{C}_2$, $\tilde{G}_2$. Note that Π does not depend on s. The rays starting in s and corresponding to an n-path π contained in Π all lie in a unique well-defined root π_s with $s \in \partial\pi_s$. Let $\pi = (x_0, x_1, \ldots, x_n)$, then we show that every element of U_{π_s}, for arbitrary s, fixes every vertex of Δ_∞ adjacent to x_1.

Indeed, let r be the ray starting in s and representing the vertex x of Δ_∞ which is adjacent to x_1. Let r_i be the ray starting in s and representing x_i. Since x is adjacent to x_1, the ray r "leaves" the apartment Σ at a vertex s' of r_0 or r_2 (i.e. the vertex s' is the common vertex of r and Σ at maximal distance from s). By considering the intersection of Σ with a sector containing r and r_1 (and noting that this intersection is convex in the sense of Tits [**2**]), one sees that s' is a special vertex (because it is contained in two rays forming a minimal angle).

From now on, the term "distance", denoted by δ, applies to the natural distance in the adjacency graph of Δ. We show by induction on m, that the vertex v_m on r at distance m from s' and distance $m + \delta(s, s')$ from s is fixed by every element θ of U_{π_s}. So let $\theta \in U_{\pi_s}$. In view of (pM1) the claim holds for $v_0 = s'$, i.e. for $m = 0$. Now let v_m be arbitrary on r and suppose $\delta(s', v_m) = m > 0$. Let π' be the n-path $(x_1, x_2, \ldots, x_n, x_{n+1})$ (with x_{n+1} in Π). By (pM1), there exists $\theta' \in U_{\pi'_{s'}}$ mapping v_1 in Σ. By the induction hypothesis, $v_m^{\theta'}$ is fixed by θ (even if θ' does not preserve s) and hence we obtain that $v_m^{\theta'\theta\theta'^{-1}} = v_m$. But (pM2) implies $\theta'\theta\theta'^{-1} = \theta$, hence the claim. So we have shown that θ preserves r and hence x.

Now suppose that x is a vertex of Δ_∞ adjacent to x_2. In order to show that $\theta \in U_{\pi_s}$ fixes the ray r representing x, we can copy the above proof up to the very last point, i.e. we obtain $v_m^{\theta'\theta\theta'^{-1}} = v_m$, with v_m defined similarly as before as the vertex in Δ at distance m from the vertex s' (r "leaves" Σ at s') and distance $m + \delta(s, s')$ from s, and with $\theta' \in U_{\pi'_{s'}}$, where $\pi' = (x_2, x_3, \ldots, x_{n+1}, x_{n+2})$ is a sub-n-path of Π. But now (pM2) implies $\theta'\theta\theta'^{-1} \in U_{\pi''_{s''}} . \theta$, where $\pi'' = (x_1, x_2, \ldots, x_n, x_{n+1})$ and s'' is the intersection of $\partial\pi_s$ and $\partial'_{s'}$. This notation includes the case where s'' is not a special vertex, and then one must put $U_{\pi''_s} = \{1\}$. But in this case the result follows similarly as before. Hence, we may assume that s'' is a special vertex. Substituting π'' for π and s'' for s in the previous claim one sees that each element of $U_{\pi''_s}$ fixes v_n. Hence the result again follows.

Similarly, one shows that each element of U_{π_s} fixes every vertex of Δ_∞ adjacent to x_i, $i \in \{1, 2, \ldots, n-1\}$ (actually, by symmetry, there is only one case left: $(n, i) = (6, 3)$).

Now we define V_π to be the automorphism group of Δ_∞ generated by all U_{π_s}, for all special vertices s in Σ. The group V_π cannot be the right choice for U_π since it certainly cannot act transitively on the set S of vertices of Δ_∞ adjacent to x_0 and different from x_1. Indeed, S has a higher cardinal number than V_π, e.g. if Δ is locally finite, then V_π is countable and S is not (that is because we are considering the maximal set of apartments for Δ).

Let s be any special vertex in Σ and let V_{π_s} be the automorphism group of Δ generated by all $U_{\pi_{s'}}$, for all s' such that the root $\pi_{s'}$ contains s as a vertex. It is clear that V_{π_s} preserves the sphere of radius m in Δ centered at s, for all positive integers m. When restricting these spheres to vertices on rays starting in s, and by defining a new suitable—but obvious—adjacency relation, one sees easily that Δ_∞ can be identified with the inverse limit of all these restricted spheres. Now we denote by $\bar{U}_{\pi_s}$ the group of all automorphisms of Δ_∞ obtained by considering all possible inverse limits of automorphisms of these spheres induced by elements of V_{π_s}. Finally, we define U_π as the group generated by all $\bar{U}_{\pi_s}$ for all special vertices s in Σ. We show that U_π acts transitively on S (and from our proof, it will follow that U_π is in fact the *union* of all $\bar{U}_{\pi_s}$).

Let $x_{-1} \in S$ be the vertex in Π adjacent to x_0 and different from x_1. Let $x \in S$ be arbitrary. As before we consider a ray r respectively r_i representing x respectively x_i and starting at some special vertex in Σ, $i = -1, 0$. Again the vertex s where r leaves Σ is special and we can assume without loss of generality that r, r_0 and r_{-1} all start in s. Denote by v_m respectively v_{-m} the vertex of r respectively r_{-1} at a distance m from s. Condition (pM1) implies the existence of an element $\theta_1 \in U_{\pi_s}$ mapping v_1 to v_{-1}. If v_k, $v_k \neq s$, is the special vertex of r closest to s, then as before one sees by considering the intersection of Σ with any sector containing r and r_0 that $v_k^{\theta_1} = v_{-k}$. Let $\theta_2 \in U_{\pi_{s_{-k}}}$ be such that it maps $v_{k+1}^{\theta_1}$ to v_{-k-1}. It is now clear how to continue and to define θ_l, for all positive integers l. It is also clear that the inverse limit of $(\theta_1, \theta_1\theta_2, \theta_1\theta_2\theta_3, \ldots)$ maps r to r_{-1}. Hence the Main Result is proved.

Remarks. It is easily seen in Δ_∞ that U_π acts simply-transitively on S. Hence U_π is indeed the (non-disjoint) union of all $\bar{U}_{\pi_s}$ by the last paragraph of the proof.

The restricted spheres of radius m that turn up above are in case of type $\tilde{A}_2$ *Hjelmslev-planes of level m*, see Van Maldeghem [6]. These geometries were used by Van Maldeghem in various papers to characterize affine buildings of type $\tilde{A}_2$ and $\tilde{C}_2$, for instance in terms of valuations on their spherical building at infinity [7].

If we endow Δ with a *symmetric* system of apartments (see Tits [4]), then our main result still holds since the building at infinity of such a system of apartments is a subbuilding of the one obtained from the full system of apartments.

4. Proof of the corollary. The corollary follows immediately from section 14 of Tits [4]. By Tits [2], this determines all irreducible rank 3 affine buildings. The assertion about the residue field follows directly from the fact that the root groups in Δ are subgroups of root groups in Δ_∞. Since for a Moufang rank 3 affine building, the root groups of Δ act simply-transitive on an "affine line" in the residue of a special vertex, the characteristics in question must be the same. However, it is not clear to us whether this condition is also sufficient to imply the Moufang condition. In fact, this is an open question for affine buildings of arbitrary rank.

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