Ovoids and Spreads arising from Involutions

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Abstract

In this paper, we give a new construction of the hermitian spreads in $H(q)$ without using the standard embedding in $\mathbf{PG}(6, q)$, without using the group $U_3(q)$, but using some geometric properties of the hexagon and an involution. Remarking that a similar construction holds in certain quadrangles of order s, with s a power of 2, we obtain ovoids in quadrangles of type $T_2(O)$. We also survey a few recent constructions of new ovoids and spreads in the finite Moufang hexagons of order (q, q).

1 Introduction and definitions

A *generalized polygon* or *generalized n-gon*, $n \in N$, $n \geq 2$, is a point-line incidence geometry with an incidence graph of diameter n and girth $2n$ (or gonality n). For finite generalized quadrangles, we refer to Payne & Thas [5]. The only known examples of finite generalized hexagons (6-gons) are defined in Tits [9] and they satisfy the so-called Moufang condition, see Tits [10]. They arise from the Chevalley groups $G_2(q)$ and ${}^3D_4(q)$. We will be concerned with the class arising from G_2 and sometimes called the *split Cayley hexagons*, because they can be constructed using a split Cayley algebra. We will give two other constructions below: one due to Tits [9], the other using (intrinsic) coordinates, see De Smet & Van Maldeghem [3].

It is common to call a generalized polygon *thick* if every element is incident with at least three other elements. It is well-known that for thick generalized polygons the number $s + 1$ of points on a line is a constant, and, dually, the number $t+1$ of lines incident with a point is a constant. In this case, the pair (s, t) is called the *order* of the polygon.

An *ovoid* of a generalized quadrangle Γ is a set $\mathcal{O}$ of points such that every line is incident with a unique element of $\mathcal{O}$. It follows readily that all points of $\mathcal{O}$

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are mutually at distance 4 (distances measured in the incidence graph) and also, $|\mathcal{O}| = 1 + q^2$ if the quadrangle has order (q, q). An *ovoid* in a generalized hexagon is a set of points such that every point is at distance ≤ 2 from a unique element of the ovoid (distances again measured in the incidence graph). It follows readily that all points of an ovoid are at distance 6 from each other, and that there are $1 + q^3$ elements in an ovoid if the hexagon has order (q, q). A *spread* is the dual notion of an ovoid.

Let $\Gamma = H(q)$ be the generalized hexagon of order (q, q) arising from $G_2(q)$. For an element u of Γ, we denote by $\Gamma_i(u)$ the set of points and lines of Γ at distance i from u. We fix the duality class of $H(q)$ by requiring that all points of $H(q)$ are *regular*, i.e., for every three points x, y, z such that $y, z \in \Gamma_6(x)$, the inequality $|\Gamma_i(x) \cap \Gamma_{6-i}(y) \cap \Gamma_{6-i}(z)| \geq 2$ implies $|\Gamma_i(x) \cap \Gamma_{6-i}(y) \cap \Gamma_{6-i}(z)| = q + 1$, for $i = 2, 3$ (see Ronan [6]). We will use that property along with a certain involution to construct a spread $\mathcal{S}$ in a subhexagon $H(\sqrt{q})$ of $H(q)$, and we show that $\mathcal{S}$ is isomorphic to the so-called hermitian spread, as contructed by Thas [7]. We also give a survey of all known ovoids and spreads in $H(q)$. Finally, we show that the method above can also be applied to quadrangles and we give some non-classical examples.

2 Hermitian spreads of $H(q)$

The generalized hexagon $H(q)$ can be constructed as follows (see Tits [9]). Consider in $\mathbf{PG}(6, q)$ the quadric $Q(6, q)$ with equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$. The points of $H(q)$ are all the points of $Q(6, q)$ and the lines of $H(q)$ are the lines of $Q(6, q)$ the Grassmann coordinates of which satisfy the following six linear equations:

$$p_{12} = p_{34}, \qquad p_{54} = p_{32}, \qquad p_{20} = p_{35},$$
$$p_{65} = p_{30}, \qquad p_{01} = p_{36}, \qquad p_{46} = p_{31}.$$

One can deduce all above equations from the first one by consecutively applying the following rule: if $p_{ij} = p_{3k}$ is in the list, then so are $p_{(i\pm4)k} = p_{3j}$ and $p_{k(j\pm4)} = p_{3i}$, where in ± 4, one should choose the appropriate sign in order to obtain a number between 0 and 7. Incidence is inherited from $\mathbf{PG}(6, q)$. Now consider a hyperplane H of $\mathbf{PG}(6, q)$ that intersects $Q(6, q)$ in an elliptic quadric. Then the lines of H which also belong to $H(q)$ form a spread $\mathcal{S}$ in $H(q)$, called the *hermitian spread*, see Thas [7]. The spread $\mathcal{S}$ has the following property. Let L, M be 2 lines of $\mathcal{S}$, then

$\mathrm{H}_{L,M}$ every line of $H(q)$ at distance 3 from every point of $H(q)$ which is itself at distance 3 from both L and M, is contained in $\mathcal{S}$.

By the regularity mentioned above, the number of lines in $H(q)$ at distance 3 from all points at distance 3 from both L and M is equal to $q + 1$. Note that Bloemen, Thas & Van Maldeghem [1] show that, whenever a spread of $H(q)$

has the property $\mathrm{H}_{L,M}$, for all lines L and at least 2 lines M, then the spread is a hermitian spread.

Now consider $H(q)$ and embed $H(q)$ in $H(q^2)$. Let θ be an involution in $H(q^2)$ fixing $H(q)$ pointwise. Such an involution always exists (apply the field automorphism $x \mapsto x^q$ on the above representation of $H(q^2)$ in $\mathbf{PG}(6, q^2)$). Let L and M be two opposite lines of $H(q)$. Let p be a point of $H(q^2)$ incident with L, but not fixed by θ. Let p' be the projection of p^θ onto M (the point of M nearest to p^θ). By RONAN [6], there exists a unique subhexagon Γ of order $(1, q^2)$ through p and p', and Γ is isomorphic to the incidence graph of the projective plane $\mathbf{PG}(2, q^2)$. Let $\mathcal{S}$ be the set of lines of Γ fixed by θ, or in other words, $\mathcal{S}$ is the intersection of the set of lines of Γ with the set of lines of $H(q)$. Then we claim:

With the above notation, the set $\mathcal{S}$ of lines is a spread of $H(q)$.

PROOF. Since θ fixes L and M, it maps p' to the projection of p onto M. Hence both p^θ and p'^θ belong to Γ and hence θ preserves Γ. Note that no point of Γ is a point of $H(q)$. Indeed, every point of Γ is either at distance 4 from p or at distance 4 from p^θ. Hence if a point w of $H(q)$ would belong to Γ, then, since L belongs to $H(q)$, also the point p or p^θ would belong to $H(q)$, a contradiction. Since Γ is the incidence graph of $\mathbf{PG}(2, q^2)$, the involution θ induces in $\mathbf{PG}(2, q^2)$ a polarity (which we also denote by θ). Let x be the unique point of Γ collinear with both p and p'^θ, and let y be the unique point collinear with both p^θ and p'. By the regularity in $H(q^2)$, there are $q + 1$ lines of $H(q)$ at distance 3 from both x and y, hence belonging to $H(q)$. Without loss of generality, we may assume that x represents a point of $\mathbf{PG}(2, q^2)$, and y represents a line of $\mathbf{PG}(2, q^2)$ not incident with x. Then we have shown that the polarity θ in $\mathbf{PG}(2, q^2)$ contains exactly $1+q$ absolute points incident with y (and equivalently, $1+q$ absolute lines incident with x). Hence θ is a unitary polarity in $\mathbf{PG}(2, q^2)$ and hence it contains $1 + q^3$ absolute points. If z is such a point, then $\{z, z^\theta\}$ represents a collinear pair of points in $H(q^2)$ and the line zz^θ is fixed by θ, hence it belongs to $H(q)$. So we have found $1 + q^3$ lines in the intersection of Γ and $H(q)$. Clearly, no two of these lines are at distance ≤ 4 from each other, because this would imply that the shortest path connecting these lines lies in both Γ and $H(q)$, and hence Γ and $H(q)$ would share at least one point, a contradiction. So $\mathcal{S}$ is a set of $1 + q^3$ lines mutually at distance 6 from each other. By CAMERON, THAS & PAYNE [2], $\mathcal{S}$ is a spread of $H(q)$. □

It is clear that $\mathcal{S}$ is a hermitian spread. Indeed, if two lines belong to that spread (and we may take L and M), then all lines at distance 3 from two points at distance 3 from L and M belong to $\mathcal{S}$, as follows directly from the above proof. At the same time, $\mathcal{S}$ can be viewed as a hermitian curve in $\mathbf{PG}(2, q^2)$, motivating the name for this spread.

3 Some other spreads of $H(q)$

We now review briefly some classes of spreads of $H(q)$. Therefore, we need a second description of $H(q)$.

Let us relabel the points and lines of the quadric $Q(6,q)$ (defined in the previous section) which belong to $H(q)$ according to Table 1. Then, according to De Smet & Van Maldeghem [3], incidence in $H(q)$ is given by

$$[k,b,k',b',k''] \,\mathbf{I}\, (k,b,k',b') \,\mathbf{I}\, [k,b,k'] \,\mathbf{I}\, (k,b) \,\mathbf{I}\, [k] \,\mathbf{I}\, (\infty) \,\mathbf{I}$$

$$[\infty] \,\mathbf{I}\, (a) \,\mathbf{I}\, [a,l] \,\mathbf{I}\, (a,l,a') \,\mathbf{I}\, [a,l,a',l'] \,\mathbf{I}\, (a,l,a',l',a''),$$

for all $a,a',a'',b,b',k,k',k'',l,l' \in \mathbf{GF}(q)$, and by

$$(a,l,a',l',a'') \,\mathbf{I}\, [k,b,k',b',k'']$$

$$\Updownarrow$$

$$\left\{\begin{array}{lcl} b &=& a''-ak, \\ a' &=& a^2k+b'+2ab, \\ l &=& k''-ka^3-3ba^2-3ab', \\ k' &=& k^2a^3+l'-kl-3a^2a''k-3a'a''+3aa''^2 \end{array}\right.$$

This provides a complete and explicit description of $H(q)$.

1. If $q=3^{2h+1}$, then $H(q)$ admits a polarity, and the set of absolute lines (lines incident with their image) forms a spread of $H(q)$, the Ree-Tits spread, see Cameron, Thas & Payne [2].

2. If $q=3^e$, then $H(q)$ is self-dual, and we may apply a duality to any spread $\mathcal{S}$. This gives us an ovoid $\mathcal{O}$ of $H(q)$. We may then consider the image $\mathcal{O}^\sigma$ of $\mathcal{O}$ under an automorphism σ of $Q(6,q)$ which does not preserve $H(q)$, and interpret the set $\mathcal{O}^\sigma$ again in $H(q)$. We obtain a new ovoid $\mathcal{O}^\sigma$ in $H(q)$. Then we can again apply a duality to obtain a new spread $\mathcal{S}'$ of $H(q)$. One special case is worth mentioning. By Bloemen, Thas & Van Maldeghem [1], it is possible to start with a hermitian spread $\mathcal{S}$ and to choose σ such that the spread $\mathcal{S}'$ has a line L for which property $\mathrm{H}_{L,M}$ holds, for all lines M, $M \neq L$, of $\mathcal{S}'$. We say that $\mathcal{S}'$ is *locally hermitian in* L. If we consider a point x on L and the set of $1+q^2$ lines of $Q(6,q)$ meeting exactly $1+q$ lines of $\mathcal{S}'$, then this set of $1+q^2$ lines constitutes in the residue $Q(4,q)$ of $Q(6,q)$ an ovoid of the generalized quadrangle associated with $Q(4,q)$. Ovoids thus arising are isomorphic to the ones of Thas & Payne [8], see again Bloemen, Thas & Van Maldeghem [1].

3. It is calculated in Bloemen, Thas & Van Maldeghem [1] that, using the coordinates above, the set

$$\{[\infty]\} \cup \{[\gamma b', -\gamma k'', k', b', k''] | k', b', k'' \in \mathbf{GF}(q)\},$$

POINTS	
Coordinates in $H(q)$	Coordinates in $\mathbf{PG}(6,q)$
(∞)	$(1,0,0,0,0,0,0)$
(a)	$(a,0,0,0,0,0,1)$
(k,b)	$(b,0,0,0,0,1,-k)$
(a,l,a')	$(-l-aa',1,0,-a,0,a^2,-a')$
(k,b,k',b')	$(k'+bb',k,1,b,0,b',b^2-b'k)$
(a,l,a',l',a'')	$(-al'+a'^2+a''l+aa'a'',-a'',-a,-a'+aa'',$ $1,l+2aa'-a^2a'',-l'+a'a'')$

LINES	
Coordinates in $H(q)$	Representation in $\mathbf{PG}(6,q)$
$[\infty]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,0,1)\rangle$
$[k]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,1,-k)\rangle$
$[a,l]$	$\langle(a,0,0,0,0,0,1),(-l,1,0,-a,0,a^2,0)\rangle$
$[k,b,k']$	$\langle(b,0,0,0,0,1,-k),(k',k,1,b,0,0,b^2)\rangle$
$[a,l,a',l']$	$\langle(-l-aa',1,0,-a,0,a^2,-a'),$ $(-al'+a'^2,0,-a,-a',1,l+2aa',-l')\rangle$
$[k,b,k',b',k'']$	$\langle(k'+bb',k,1,b,0,b',b^2-b'k),$ $(b'^2+k''b,-b,0,-b',1,k'',-kk''-k'-2bb')\rangle$

Table 1: Coordinatization of $H(q)$.

for any non-square γ, is a hermitian spread in $H(q)$. A little distortion now yields new spreads for $q\equiv 1$ mod 3, namely, the set

$$\mathcal{S}_{[9]}=\{[\infty]\}\cup\{[9\gamma b',-\gamma k'',k',b',k'']|k',b',k''\in\mathbf{GF}(q)\}$$

is a spread of $H(q)$, not isomorphic to a previous mentioned one, see *loc.cit.*, where it is also shown that $\mathcal{S}_{[9]}$ is locally hermitian in $[\infty]$.

4 Some ovoids of non-classical quadrangles

We now apply the method of section 2 to generalized quadrangles. Dualizing the situation, there is the following result.

Let Γ be a generalized quadrangle having a subquadrangle Γ' with the following properties:

(*i*) every point of Γ' is incident with exactly two lines of Γ';

(*ii*) every point of Γ incident with a line of Γ' belongs to Γ';

(*iii*) every line of Γ is incident with at least one point of Γ'.

Suppose moreover that there is an involution θ of Γ which preserves Γ' and which has the following properties:

(a) there exist two points x_1, x_2 of Γ' such that θ interchanges the two lines through x_i, for each $i = 1, 2$;

(b) θ fixes a thick subquadrangle Γ''.

Then the set of points of Γ' fixed under θ forms an ovoid of Γ'', or in other words, the intersection of the point sets of Γ' and Γ'' is an ovoid in Γ''.

PROOF. By (iii), every line L of Γ'' is incident with a unique point x of Γ' (unique indeed because otherwise L lies in Γ', contradicting (a), which asserts that L is not fixed in this case). Since θ fixes L and Γ', it fixes x, hence x belongs to Γ''. The result follows. $\square$

In the finite case, conditions (i), (ii) and (iii) are equivalent with saying that the order of Γ is (s, s) and that the order of Γ' is $(s, 1)$, for some integer $s \geq 2$ (see Payne & Thas [5](2.2.1)). Putting $\Gamma \cong Q(4, q^2)$, the generalized quadrangle arising from a non-degenerate quadric in $\mathbf{PG}(4, q^2)$, and $\Gamma'' \cong Q(4, q)$, we obtain an ovoid isomorphic to $Q^-(3, q)$ in $Q(4, q)$. So for q even, the two known ovoids in $Q(4, q)$, $q = 2^{2h+1}$, arise either from a polarity (Suzuki-Tits ovoid), or from an involution. So one could say that they are both phenomena related to order 2 elements of the correlation group of $Q(4, q)$ (a similar remark holds for the Ree-Tits spreads and hermitian spreads in $H(3^{h+1})$ above).

Now we apply the above theorem to non-classical quadrangles of type $T_2(O)$. We describe a certain class of them algebraically. Let Γ be a geometry whose points are (∞), (a), (k, b) and (a, l, a'), for $a, a', k, l \in \mathbf{GF}(2^{2e})$, whose lines are $[\infty]$, $[k]$, $[a, l]$ and $[k, b, k']$, for $k, k', a, b \in \mathbf{GF}(2^{2e})$, and incidence is given by

$$[k, b, k'] \,\mathbf{I}\, (k, b) \,\mathbf{I}\, [k] \,\mathbf{I}\, (\infty) \,\mathbf{I}\, [\infty] \,\mathbf{I}\, (a) \,\mathbf{I}\, [a, l] \,\mathbf{I}\, (a, l, a'),$$

for all $a, a', b, k, k', l \in \mathbf{GF}(2^{2e})$, and by

$$(a, l, a') \,\mathbf{I}\, [k, b, k']$$

$$\Updownarrow$$

$$\begin{cases} a' &= k^{2^h} a + b, \\ k' &= ka + l. \end{cases}$$

It is an elementary calculation to verify that this defines a generalized quadrangle, using the results of Hanssens & Van Maldeghem [4], if and only if $(h, 2e) = 1$. Since in this case $(h, e) = 1$, we see that restricting coordinates to $\mathbf{GF}(q)$, we obtain a subquadrangle Γ'' which can be seen as the fix point structure of the involution θ obtained by applying the field automorphism $x \mapsto x^{2^e}$ on each coordinate of each element (and fixing (∞) and $[\infty]$). It is also an elementary calculation, using the description above of Γ to verify that there is a unique subquadrangle Γ' of order $(2^{2e}, 1)$ through any pair of lines $\{[k], [k]^\theta\}$, for which $k \in \mathbf{GF}(2^{2e}) \setminus \mathbf{GF}(2^e)$. Hence we can apply the previous theorem and obtain an ovoid $\mathcal{O}$ of Γ''. The explicit form of the ovoid is, after calculation,

$$\mathcal{O} = \{(\infty)\} \cup \{(a, l, l(k + k^{2^e})^{2^h-1} + a\frac{k^{2^e+2^h} + k^{1+2^{e+h}}}{k + k^{2^e}})|a, l \in \mathbf{GF}(2^e)\}.$$

The construction of ovoids via involutions is in fact inspired by the situation in the classical case: the intersection of a standard embedded quadrangle with a non-tangent hyperplane yields either a subquadrangle or an ovoid. But in a quadratic extension, we always get a subquadrangle. This is the quadrangle Γ' *of the last theorem. The idea is to reverse the procedure, and start with* Γ'*, then restrict coordinates in* Γ *with the aid of an involution and obtain an ovoid in the subquadrangle* Γ'' *over the subfield. A similar argument holds in case of hexagons.*

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