A Geometric Characterization of the Perfect Ree-Tits Generalized Octagons.

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Dedicated to Prof. J. Tits on the occasion of his 65th birthday

1 Introduction

The world of *Tits-buildings*, created by J. Tits, is for many mathematicians a very fascinating world, with a lot of interesting applications. One of the most remarkable buildings is undoubdly the generalized octagon related to the Ree groups of characteristic 2, as defined over a perfect field by REE [11] and generalized by TITS [23]. This building does not fit in the family of so-called *classical* buildings, but neither does it belong to the family of *exceptional* buildings, although it shares with the classical and exceptional polygons (or more generally, buildings) the *Moufang condition*. Geometrically though, it is very different from the other Moufang polygons, see JOSWIG & VAN MALDEGHEM [8] and VAN MALDEGHEM [27]. Responsible for that strange "mixed" behaviour is the automorphism group, which is not an algebraic group, but "very close" to one. The literature on generalized octagons, in particular on the Moufang octagons, is almost non-existent. To the papers already mentioned, one can add SARLI [13] and that is about it. The most important paper though remains TITS [23], where J. Tits classifies all Moufang generalized octagons. In this paper, which we dedicate to J. Tits, we want to make a contribution to the geometric counterpart of Tits' work.

ACKNOWLEDGEMENT. On this occasion, I express my profound thanks to J. Tits, not only as a mathematician but also as a person, for his beautiful lectures in Paris, his interest in my work and the valuable time he sacrifices from time to time for my sake.

A *generalized octagon* is an element of the larger class of *generalized polygons*, or *generalized n-gons*, corresponding to the case $n = 8$. A generalized polygon is essentially a building of rank 2, although sometimes it is required to have certain *parameters* or a

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certain *order*. These different points of view do not matter much for the present paper, but in order to try to establish a general convention, we will follow the definitions of a book that is now being prepared by the author.

Let $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ be a connected rank 2 incidence geometry with point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation I. A *path* of *length* d is a sequence $v_0, \ldots, v_d \in \mathcal{P} \cup \mathcal{L}$ with $v_i I v_{i+1}$, $0 \leq i < d$. Define a function $\delta : (\mathcal{P} \cup \mathcal{L}) \times (\mathcal{P} \cup \mathcal{L}) \to \mathbb{N}$ by $\delta(v, v') = d$ if and only if d is the minimum of all $d' \in \mathbb{N}$ such that there exists a path of length d' joining v and v' (d always exists by connectedness).

Then Γ is a *generalized n-gon, $n \in \mathbb{N} \setminus \{0, 1, 2\}$*, or a *generalized polygon* if it satisfies the following conditions:

[GP1] *There is a bijection between the sets of points incident with two arbitrary lines. There is also a bijection between the sets of lines incident with two arbitrary points.*

[GP2] *The image of* $(\mathcal{P} \cup \mathcal{L}) \times (\mathcal{P} \cup \mathcal{L})$ *under* δ *equals* $\{0, \ldots, n\}$. *For* $v, v' \in \mathcal{P} \cup \mathcal{L}$ *with* $\delta(v, v') = d < n$ *the path of length* d *joining* v *and* v' *is unique.*

[GP3] *Each* $v \in \mathcal{P} \cup \mathcal{L}$ *is incident with at least* 2 *elements.*

We will occasionally omit the word "generalized" in the term "generalized polygon", especially when another adjective is used. From the definition, it follows that there are constants s and t such that there are $s + 1$ points incident with every line and $t + 1$ lines incident with every point. The pair (s, t) is called *the order* of Γ. If $s, t \geq 2$, then the polygon is called *thick*. A geometry satisfying [GP2] and [GP3], but not having an order is called a *weak polygonal geometry*.

Throughout the paper, we will use the following terminology.

Let x be a point of the generalized n-gon Γ and $i \in \{1, 2, 3, \ldots, n\}$. We denote by $\Gamma_i(x)$ the set of elements of Γ at distance i from x. Elements at distance n are called *opposite*. It follows from Axiom [GP2] that in a generalized polygon, whenever two elements G and H are not opposite, then there exists a unique element K incident with G and at distance $\delta(G, H) - 1$ from H. We call K *the projection of H onto G*. An *apartment* is a circuit of length $2n$. Let x and y be opposite points and $2 \leq i \leq n/2$. The set

$$x^y_{[i]} = \Gamma_i(x) \cap \Gamma_{n-i}(y)$$

is called a *distance-i-trace with center x and direction y*. For $i = 2$, we sometimes omit the prefix "distance-2"and the subscript "[2]". For lines a similar definition holds if one adds the adjective "dual" to it. For instance, a dual distance-2-trace is simply a dual trace which is a certain set of lines meeting a fixed line.

Generalized polygons were introduced by TITS [16]. They appeared later as the fundamental building bricks of the *Tits-buildings* and they are the natural geometries of the Chevalley groups of relative rank 2. In fact, these natural geometries were the first examples of generalized polygons and they are all due to Tits, except for the case $n = 3$, the generalized triangles, which are the projective planes. There are a lot of classes of generalized quadrangles known — also many so-called non-classical ones, i.e. quadrangles not arising from Chevalley groups. Relatively few generalized hexagons are known, and only in the infinite case are there any non-classical examples known besides the free constructions. There is at present only one class of generalized octagons known (besides the free constructions) and it arises from the Ree groups of type ${}^2F_4$ over any field of characteristic 2 in which the Frobenius map has a square root, i.e. in which there exists an endomorphism whose square is the Frobenius map, see TITS [23]. For $n \neq 3, 4, 6, 8$, no Moufang n-gon exists (by a result of WEISS [30] or TITS [19, 21]) and only free constructions are known (see TITS [20]).

The classical polygons as mentioned above are characterized by a condition on the group of automorphisms, the so-called *Moufang condition*, introduced by TITS [18]. This characterization follows in the finite case by FONG & SEITZ from their Theorem D of [7], but a proof of the general result is now being written up by TITS & WEISS in a book in preparation, see also TITS [24]. The case $n = 8$ however was treated separately more than 10 years ago by TITS [23]. It looks as if this is the easiest case to handle. Surprisingly and contrary to that observation, KANTOR [9] remarks that, though there are geometric and combinatorial characterizations of the classical quadrangles and hexagons — and more characterizations were discovered since [9] appeared — there was (in 1986) no geometric characterization of the classical octagons. In the meantime, I proved in [25] a configurational characterization of *all* classical generalized polygons. But a separate geometric characterization which clearly shows how the geometry of the Ree-Tits octagons differs from the geometry of the other polygons is still unknown. I want to fill that gap in the present paper. Nevertheless I shall only be concerned with *perfect octagons*, i.e. the case where the field is perfect. I consider the geometry in the other case twice as hard and only half as important.

The characterization that I will present here is much more complicated than the beautiful characterization of all classical hexagons by RONAN [12]. It is worth recalling that characterization in the terminology of my paper [27]. A generalized n-gon is called *point-distance-i-regular*, $2 \leq i \leq n/2$, if every distance-i-trace with a point as center is determined by any two of its elements. A generalized hexagon $\mathcal{H}$ is classical if and only if $\mathcal{H}$ or its dual is point-distance-2-regular. As a step in the proof, one shows that any point-distance-2-regular hexagon is also point-distance-3-regular (which is equivalent to line-distance-3-regular). In [27], it is pointed out that a similar characterization of the classical octagons is impossible since neither point-distance-2-regular octagons nor point-distance-3-regular ones exist. We will show however that our conditions imply point-distance-4-regularity.

As we already remarked, there does not exist an extensive bibliography on generalized octagons, and so it will already be worthwhile to prove that the perfect Ree-Tits octagons satisfy the axioms below. This will mainly follow from my recent paper [26]. Also, every property derived from the axioms is a property valid in the perfect Ree-Tits octagons (and proved in a geometric fashion). Thus we obtain an unexpected number of geometric properties of the perfect Ree-Tits octagons, which one can compare with the properties of the classical quadrangles obtained in PAYNE & THAS [10] for example. It certainly makes the perfect Ree-Tits octagons look less mysterious!

One of the future tasks will be the reduction of the axioms in the general perfect case (and first we think of deleting the axiom [RT4], see below), and also in the finite case. This appears to be a non-trivial problem. In any case, the present paper will be a solid base for such work. In fact, the point of the present paper is not only the characterization itself, but also the fact that there exists a geometric characterization. And once one gets used to the geometry involved in the Ree-Tits octagons, the axioms given below are no longer surprising nor complicated.

Roughly speaking, I shall introduce a set of axioms which will allow us to reconstruct the building of type F_4 and the polarity defining the perfect Ree-Tits octagon. This is the content of Sections 3 up to 6. In Section 7, I will show that every perfect Ree-Tits octagon satisfies the given axioms. In Section 8, I treat the finite case.

2 Main Results

The main tools of our characterization are the *traces*, as already announced in [26]. Basically, we will take as axioms the properties of the perfect Ree-Tits octagons that are used to define a *derived geometry* (for definitions, see below or VAN MALDEGHEM [26]) in each point and to show that this derived geometry is a generalized quadrangle. Then we impose additional axioms which give relations between properties of traces with distinct centers.

For the rest of this paper, we let $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ be a thick generalized octagon — not necessarily finite! By a perfect Ree-Tits octagon, we understand the generalized octagon naturally associated to a Ree group of type ${}^2F_4$ over a perfect field of characteristic 2 (in which the Frobenius automorphism has a square root) in such a way that the point stabilizer acts on the lines incident with that point as the Suzuki group on the corresponding Suzuki-Tits ovoid (see TITS [23]). The latter condition is only there to fix the names 'points' and 'lines' for the two kinds of vertices of the associated rank 2 building. If Γ is a finite Ree-Tits octagon, then it follows that Γ has order $(2^{2e+1}, 2^{4e+2})$. Our main result is:

Main Result. *A generalized octagon Γ is isomorphic to a perfect Ree-Tits octagon if and only if Γ satisfies the axioms* [RT1], [RT1′], [RT2], [RT3] *and* [RT4] *stated below.*

In the finite case, one can reduce the number of axioms, and also simplify some of the remaining.

Main Result — Finite Case. *A finite generalized octagon Γ of order (s,t), $s,t>1$, is isomorphic to a Ree-Tits octagon if and only if Γ satisfies the Axioms* [RT1f], [RT1′], [RT3] *and one of the equivalent conditions $t=s^2$ or Axiom* [RT2f] *(the Axioms* [RT1f] *and* [RT2f] *are stated below).*

In order to state our axioms, we introduce some more terminology.

A *Suzuki-Tits inversive plane* Ω, or briefly an STi-plane, is a rank 2 geometry — in which the blocks are called *circles* and incidence is denoted by "$\in$"— together with a map ∂ from the set of circles to the set of points of Ω satisfying Axioms [MP1], [MP2], [CH1], [CH2], [P], [ST1] and [ST2] below. Roughly speaking, the axioms express that

(1) we are dealing with "characteristic 2",

(2) for any circle C, the point ∂C belongs to C and plays the role of a double condition for the uniqueness conclusion in (1).

In the following, *touching circles* are circles having exactly one point in common. We label the axioms as in Van Maldeghem [28].

[MP1] *Every circle contains at least three points and every triple of (distinct) points determines a unique circle.*

[MP2] *For every triple (C,x,y) consisting of a circle C, a point $x\in C$ and a point $y\notin C$, there exists a unique circle D touching C in x and containing y.*

[CH1] *For every triple (C,x,y) consisting of a circle C and two distinct points $x,y\notin C$, either there exists a unique circle D touching C and containing both x and y or all circles through both x and y touch C.*

[CH2] *There do not exist three circles touching each other two by two in different points.*

[P] *For every two triples $\{C_i,D_i,E_i\}$, $i=1,2$, of pairwise disjoint circles, it follows that E_1 touches E_2 whenever both C_i and D_i touch the three circles C_j,D_j,E_j, for all $\{i,j\}=\{1,2\}$.*

[ST1] *For every circle C, one has that $\partial C\in C$ and C is uniquely determined by ∂C and any further point of C, in other words, for every pair of points (x,y), there exists a unique circle C such that $\partial C=x$ and $y\in C$.*

[ST2] *For every pair* (C, x) *consisting of a circle* C *and a point* $x \notin C$*, there exists a unique circle* D *containing* x *and* ∂C *and such that* $\partial D \in C$*.*

We note that in the finite case Axioms [CH1], [CH2] and [P] can be substituted by the condition that the order of Ω is even, see Van Maldeghem [28].

Now we write down some important consequences of these axioms. Their proofs are contained in [28], but they can easily be reconstructed.

First of all, the axioms imply that Ω is a *Möbius* circle plane in the classical sense. That means for instance that

[MC1] *for each point* x*, the set of points of* Ω *distinct from* x *together with the circles through* x *and the induced incidence relation, forms an affine plane, denoted by* $\mathrm{Res}(x)$*.*

We can now reformulate [ST1] and [ST2] as follows:

[MC2] *The map* ∂ *is surjective and for all points* x *the inverse image* $\partial^{-1}x$ *is a class of parallel lines in* $\mathrm{Res}(x)$*).*

[MC3] *For any circle* C*, the set of circles* D *such that* $\partial D \in C$*,* $\partial C \in D$ *and* $\partial C \neq \partial D$ *forms a parallel class of lines in* $\mathrm{Res}(\partial C)$*).*

Secondly, we remark that we will not use Axiom [P]. By abuse of language, we will from now on call an *STi-plane* every rank 2 incidence geometry satisfying [MP1], [MP2], [CH1], [CH2], [ST1] and [ST2]. The above corollaries remain valid. The next corollaries follow from [28], where it is proved that, under condition [P], an STi-plane Ω is isomorphic (as a Möbius plane) to the circle plane obtained from a Suzuki-Tits ovoid (as defined by Tits [17]) by plane sections, over a perfect field of characteristic 2 in which the Frobenius automorphism has a square root. But we will not use that result in the present paper. The following properties can be easily proved as an exercise without referring to [28].

[MC4] *For every triple* (C, C', x) *consisting of two distinct non-disjoint circles* C *and* C'*, and a point* $x \notin C \cup C'$*, there exists a unique circle* D *through* x *touching both* C *and* C'*.*

[MC5] *For every pair* (C, x) *consisting of a circle* C *and a point* $x \notin C$*, there exists a unique circle* D *touching* C *such that* $\partial D = x$*.*

The point ∂C for any circle C is called the *corner* of C. Suppose now that, still under the assumption that Ω is an STi-plane, C and C' are two circles of Ω intersecting each other in the two distinct points ∂C and $\partial C'$. The set of circles touching both C and C' is called a *transversal partition with extremities ∂C and $\partial C'$* (note that ∂C and $\partial C'$ indeed determine C and C' uniquely), see [26]. We now claim that the set of corners of the elements of the transversal partition with extremities $x = \partial C$ and $x' = \partial C'$ coincides with the set of corners of the circles through x and x' different from C and C'.

First remark that if two circles touch in a point z, then z is either the corner of both circles, or of neither of them. Indeed, let E and E' touch in z, and suppose $\partial E = z \neq \partial E'$. Let z' be any point of $E' \setminus E$. Consider a circle E'' with corner z and containing z', existing, unique and different from E by [ST1]. Both circles E' and E'' go through $z' \notin E$ and touch E in z, contradicting [MP2].

Now let y be a point of a circle D touching both C and C'. By [CH1], all circles through both ∂C and $\partial C'$ touch D. Hence the unique circle E through y, ∂C and $\partial C'$ has y as its corner if and only if $y = \partial D$, by the previous remark. This proves our claim.

We return to the generalized octagon Γ. Our first set of axioms basically says that the set of lines through any point x admits the structure of an STi-plane, and this structure is completely determined by the mutual position of traces with center x.

Let x be any point of Γ. Let S be a set of points in $\Gamma_2(x)$. Then the set of lines through x containing a point of S is called the *back up* of S on x. A back up on x is called trivial either if it consists of one element, or if it consists of all the lines through x (so note that the empty back up set is not treated as trivial!). The *block geometry* ${}^x\Gamma$ in x is the geometry with point set the set of lines through x and block set the non-trivial back ups on x of trace intersections with center x.

DIGRESSION. For the classical generalized polygons, the block geometries are very different. Some of them are completely trivial (there are no blocks except possibly the empty one; this is the case when the center is a regular point), others are the classical Möbius plane, or the linear space determined by a hermitian unital, or the geometry of sublines over a subfield of a line over an extension field, or the geometry of 2- and 3-subsets of a set. In fact, Ronan's characterization of all Moufang hexagons can be phrased as follows:

> *A generalized hexagon $\mathcal{H}$ satisfies the Moufang condition if and only if the block geometry in every point of $\mathcal{H}$ or of the dual of $\mathcal{H}$ has only the empty block or does not have any block at all.*

So Ronan's characterization prescribes the precise structure of all block geometries. This is exactly what we are going to do in the octagon case, except that the block geometries are much more complex and harder to describe. They involve the Suzuki-Tits ovoid as can be seen group-theoretically from the stabilizer of a point.

Our first axiom tells us how a block geometry looks, if we do not look further away from x than distance 2.

[RT1] *If we call blocks circles, then the block geometry in every point of* Γ *satisfies the conditions* [MP1], [MP2], [CH1] *and* [CH2].

In particular, there are no empty blocks; that means that every two traces meet in at least one point. Note that already this observation is false for non-perfect Ree-Tits octagons (see Section 7).

Now we have to introduce axioms to handle the corners so that we obtain an STi-plane. For this purpose, we have to look at distance 3 from a point. Let x again be any point of Γ. Let S be a subset of $\Gamma_2(x)$ contained in some trace with center x. For $z \in S$, we define the *gate set of S through z* as the set of all lines incident with z which lie on a shortest path from x to a point y with $S \subseteq x^y$. A gate set through some point z is called *trivial* if it contains all lines through z. Note that a gate set of some set $S \subseteq \Gamma_2(x)$ through some point z contains at least two lines one of which is xz. We will only consider this case when dealing with gate sets.

[RT1′] *Let x be any point of Γ and let C be any block in ${}^x\Gamma$. Let S be any intersection of traces with center x with back up C. Then there exists a unique point z of S such that the gate set of S through z is non-trivial. Moreover, the line xz is independent of S and we denote it by ∂C. The geometry ${}^x\Gamma$ with mapping ∂ just defined satisfies the axioms* [ST1] *and* [ST2] *(again calling blocks circles).*

The next axiom will allow an explicit construction of the transversal partitions. It will also imply that, if three traces with common center meet pairwise in unique points, then the three unique points thus obtained coincide (see next section).

[RT2] *Let x be any point of Γ and let z_1, z_2 be two points collinear with x, with $xz_1 \neq xz_2$. Let X be any trace centered at x containing z_1 and z_2. Let Z_1 and Z_2 be two traces such that $X \cap Z_i = \{z_i\}$, $i = 1, 2$. Then the back up of $Z_1 \cap Z_2$ is a member of the transversal partition of ${}^x\Gamma$ with extremities xz_1 and xz_2.*

The next important issue is the gate sets. We already know that the gate set of an intersection of two traces with center x the back up of which is a circle C in ${}^x\Gamma$, through the point lying on ∂C, is non-trivial. The next axiom precisely tells us what this set is.

[RT3] *Let x be any point of Γ and let Z be the non-trivial intersection of two distinct traces x^u and x^v, where u and v are points opposite x. Let z be the unique point of Z through which the gate set of Z is non-trivial and let y be any other point of Z. Then the lines at distance* 5 *from u and v respectively, incident with z (respectively y) are contained (respectively not contained) in a circle C of ${}^z\Gamma$ (respectively ${}^y\Gamma$) with $\partial C = xz$ (respectively $\partial C = xy$).*

Another axiom concerns the gate sets of a full trace. We will prove that through every point of a trace the gate set is non-trivial and in fact corresponds to a circle in the block geometry of that point. But we will need the following more technical axiom too, which in the finite case is a consequence of the others.

[RT4] *Let x and y be two opposite points of Γ and $L \in \Gamma_5(x) \cap \Gamma_3(y)$. Let $\{z\} = \Gamma_2(x) \cap \Gamma_3(L)$. Let $z', z'' \in x^y$ be such that the lines xz, xz', xz'' do not lie in a circle of ${}^x\Gamma$ with corner xz. Let L' and L'' be any lines of the gate set of x^y through z' and z'' respectively. Then there exists a point y' opposite x at distance* 3 *from L and at distance* 5 *from L' and L''. Also, for every y'' opposite x, such that $x^{y''} = x^y$, $\delta(y, y'') = 6$ and $\delta(y'', L) = 3$, we have $|x^y_{[3]} \cap x^{y''}_{[3]}| > 1$.*

This completes the set of axioms in the general case. In the finite case, we have the following weaker versions of [ST1] and [ST2], respectively.

[RT1f] *If we call blocks circles, then the block geometry in every point of Γ satisfies the conditions* [MP1] *and* [MP2].

[RT2f] *If three traces with common center meet pairwise trivially, then they all share a common point.*

In the next section we prepare the construction of a building of type F_4 in which we will embed Γ. We will denote that building by $\mathcal{M}(\Gamma)$ and refer to it as a *metasymplectic space*; in fact we will not only define its points and lines, but also its planes and hyperlines.

3 Auxiliary Results

The first two lemmas tell us something about the existence and uniqueness of traces with certain properties.

Consider any point x of Γ. Consider four points z_i, $i = 1, 2, 3, 4$, collinear with x and with distinct back ups on x. Suppose that the lines xz_i, $i = 1, 2, 3, 4$, are not contained in one block of ${}^x\Gamma$. In that case we say that z_1, z_2, z_3 and z_4 are in general position with respect to x. It follows trivially from [RT1] that there is at most one trace passing through all four of these points. The first lemma assures that there is exactly one.

Lemma 1 *There is at least one trace containing* 4 *points in general position with respect to any point.*

PROOF. Let z_1, z_2, z_3, z_4 be four points in general position with respect to a point x of Γ. Let C be any circle of ${}^x\Gamma$ containing xz_1 and xz_2, but not containing xz_3 (it is easily seen that C exists). Considering a point y opposite x and at distance 6 from both z_1 and z_2, we obtain a trace $X = x^y$ containing z_1, z_2. Suppose X does not contain z_3. We construct a trace X' containing z_1, z_2 and z_3. Consider in ${}^x\Gamma$ two circles through xz_3 touching C in distinct points. By Axiom [CH2], these circles meet in a second line L, L incident with x. Hence C is an element of the transversal partition with extremities L and xz_3. Let z be the unique point of Γ incident with L and contained in X. Let L_y be the projection of z onto y and let y' be the projection of z_3 onto L_y. We have $x^{y'} \cap X = \{z\}$. Similarly we construct a trace X' containing z_1 and such that $X' \cap x^{y'} = \{z_3\}$. By Axiom [RT2] the back up C' onto x of $X \cap X'$ is a member of the transversal partition with extremities xz_3 and $xz = L$. But $xz_1 \in C \cap C'$, hence $C = C'$ and X' contains also z_2. By substituting X' for X and z_4 for z_3 in the previous argument, we now see that there exists a trace X'' with center x containing z_1, z_2, z_3, z_4. The lemma is proved.

If X is some point of Γ and L is a line at distance 7 from x, then the points on L opposite x define a set of traces which meet pairwise in the unique point of $\Gamma_2(x) \cap \Gamma_5(L)$. So for a given trace x^y, y opposite x, a given point z of x^y and a given point u collinear with x but not on the line xz, nor on the trace x^y, there exists at least one trace containing u and meeting x^y in exactly z. The next lemma says that this trace is unique. For convenience, we say that two traces with common center *meet trivially* if their intersection is a point.

Lemma 2 *For every point x of Γ, every point y opposite x and every pair of non-collinear points $(z, u) \in x^y \times (\Gamma_2(x) \setminus x^y)$, there is a unique trace with center x containing both z and u and meeting x^y trivially.*

PROOF. By the preceding remarks we already know that there is at least one trace X' containing z, u and meeting x^y trivially. Suppose the trace X'' also contains z, u and meets x^y trivially. Suppose by way of contradiction that $X' \neq X''$. Since $\{z, u\} \subseteq X' \cap X''$, Axiom [RT1] implies that the back up of $X' \cap X''$ is a circle C in ${}^x\Gamma$. Let L be any line of Γ through x, $L \notin C$. Let v be the unique point incident with L and belonging to x^y. Again we can construct a trace Y meeting x^y trivially in v and containing u. By Axiom [RT2] the back ups D' and D'' of respectively $Y \cap X'$ and $Y \cap X''$ are elements of the transversal partition of ${}^x\Gamma$ with extremities xz and L. Since $xu \in D' \cap D''$, we must have $D' = D''$. Hence $X' \cap X''$ contains $Y \cap X'$. This implies that $D' \subseteq C$, so $D' = C$, a contradiction since $xz \in C$ and $xz \notin D'$. This completes the proof of the lemma.

So for each trace X with center x and each point $z \in X$, there is a unique set $\mathcal{T}$ of traces containing z, meeting two by two trivially in z and such that every point of $\Gamma_2(x) \setminus \Gamma_2(z)$

is contained in a member of $\mathcal{T}$. We call such a set a *pencil of traces based at* z, or with *base point* z.

It follows that we can rephrase Axiom [RT2] as follows.

Lemma 3 *For every point x of Γ, every point y opposite x and every pair of distinct points z_1 and z_2 in x^y, the set of back ups of the intersections of elements different from x^y of the pencil based at z_1 and containing x^y with elements different from x^y of the pencil based at z_2 and containing x^y constitutes the transversal partition with extremities the lines xz_1 and xz_2 in ${}^x\Gamma$.*

Remark. These lemmas already allow us to define a derived geometry Γ_x at every point and to prove that it is a generalized quadrangle (see [26]). Though we will not use this result, let us for the sake of completeness mention how that quadrangle is defined. The points of Γ_x are the traces with center x together with the elements of $\Gamma_2(x) \cup \{x\}$. The lines of Γ_x are the pencils of traces (with center x) together with the lines through x. Incidence is defined as follows: a pencil of traces is incident with its own elements and with its base point; a line through x is incident with all points incident with it in Γ.

The next lemma tells us exactly when two traces coincide.

Lemma 4 *Let x be a point of Γ, and let y_1 and y_2 be two points opposite x. Then $x^{y_1} = x^{y_2}$ if and only if x^{y_1} and x^{y_2} have at least two points in common, say x_1 and x_2, and for each $j \in \{1, 2\}$, the lines L_{ij} at distance 5 from y_i and incident with x_j are contained in a circle of ${}^{x_j}\Gamma$ with corner the line xy_j, $i = 1, 2$.*

As a consequence we have:

Lemma 5 *The gate set of a trace with center x through any of its points z is a circle in ${}^z\Gamma$ with corner xz.*

PROOF. Let y and z be points opposite x such that there are two points $x_1, x_2 \in x^y \cap x^z$ with the property that the projections of y and z onto x_i are contained in a circle of ${}^{x_i}\Gamma$ with corner xx_i, for $i = 1, 2$. By Axiom [RT3], x^y and x^z cannot be distinct, hence the circle in ${}^{x_1}\Gamma$ containing the projection of y onto x_1 and having xy as corner, is contained in the gate set of x^y through x_1.

Conversely, suppose $x^y = x^z$ and let $u \in x^y$. Axiom [RT1$'$] and Lemma 1 imply the existence of a trace X meeting x^y in a set Y the back up of which is a circle in ${}^x\Gamma$ with corner xu. Let $v \in \mathcal{P}$ be such that $X = x^v$. Let L, M, N be the projections onto u of y, z, v respectively. By Axiom [RT3] applied twice, L, M and N belong to the same circle in ${}^u\Gamma$ with corner xu. The lemma is proved.

The next result essentially tells us that the knowledge of one ${}^x\Gamma$ suffices to determine all ${}^y\Gamma$, y any point of Γ.

Lemma 6 *Let x and y be two opposite points of Γ. Then the projection of any circle C of ${}^x\Gamma$ onto y is a circle D in ${}^y\Gamma$ and the projection of ∂C is exactly ∂D.*

PROOF. Let L_i, $i = 1, 2, 3$, be three distinct lines through x and let M_i be the projection of L_i onto y. Clearly it suffices to show that, if L_1 is the corner of a circle in ${}^x\Gamma$ containing L_2 and L_3, then M_1 is the corner of a circle in ${}^y\Gamma$ containing M_2 and M_3. So let L_1 be the corner of the circle in ${}^x\Gamma$ containing L_1, L_2 and L_3. Let z be the projection of L_1 onto M_1 and let u be the projection of M_2 onto L_2. Let L' be any line through u not belonging to the circle, say C, of ${}^u\Gamma$ with corner L_2 and containing the projection L of M_2 onto u. Let M' be the projection of L' onto z and let v' be the projection of L' onto M'. The traces x^y and $x^{v'}$ both contain u and the projection x' of z onto L_1. Since the projections of y and v' onto x' coincide, we conclude by Axiom [RT3] and Lemma 4 that x^y and $x^{v'}$ meet on the lines of the circle in ${}^x\Gamma$ with corner L_1 and containing L_2, hence have a point v on L_3 in common, by our assumption. We can do the same reasoning if L' does belong to the circle C of ${}^u\Gamma$. Hence M' runs through all lines incident with z, except the projection of x onto z. This means that $z^u = z^v$ and by Lemma 4, we conclude that M_2 and M_3 lie in a circle of ${}^y\Gamma$ with corner M_1. The lemma is proved.

The last case concerning gate sets and mutual positions of traces is the case where two traces meet trivially. Strictly speaking, we cannot talk about gate sets here; nevertheless the following result is similar to the previous lemma and to Axiom [RT3].

Lemma 7 *Let x be any point of Γ and let z_1 and z_2 be two points opposite x. Suppose that x^{z_1} and x^{z_2} meet trivially in the point y. Let L_i be the line at distance 5 from z_i and incident with y, $i = 1, 2$. Then L_1 and L_2 belong to a circle in ${}^y\Gamma$ with corner xy.*

PROOF. Let M_1 be the unique line incident with z_1 and at distance 5 from y. Let y_2 be any point of x^{z_2} distinct from y. Let z_2' be the projection of y_2 onto M_1. Clearly $x^{z_2'}$ meets x^{z_1} trivially in y (otherwise there is circuit of length 14 in the incidence graph) and it contains y_2, hence by Lemma 2 it coincides with x^{z_2}. The lemma now follows from Lemma 5 applied to $x^{z_2} = x^{z_2'}$ in the point y.

The next lemmas aim at a property of the dual traces.

Lemma 8 *Let x be any point of Γ, let z_1 and z_2 (respectively y_1 and y_2) be distinct collinear points collinear with but distinct from x and suppose $xz_1 \neq xy_1$. Let Y_i (respectively X_i), $i = 1, 2$, be a trace with center x containing z_1 (respectively z_2) and y_i and suppose that Y_1 and Y_2 meet trivially. If X_1 and Y_1 respectively X_2 and Y_2 meet trivially, then so do X_1 and X_2.*

PROOF. By Axiom [RT2] the traces Y_1 and X_2 meet in a set Z the back up of which is a member of the transversal partition with extremities xz_1 and xy_2. By Lemma 2 there is a unique trace X_2' meeting X_1 trivially in z_2 and containing some given point of Z. By Axiom [RT2] again, Y_1 and X_2' meet in a set Z' the back up of which is again a member of the transversal partition with extremities $xz_2 = xz_1$ and $xy_1 = xy_2$, so the back up of Z' coincides with that of Z, hence $Z = Z'$ and $X_2' \cap X_2$ contains Z and an extra point z_2. This implies by Lemma 1 that $X_2 = X_2'$. The lemma is proved.

Lemma 9 *Let x and z be any two distinct but collinear points of Γ. Let X_1 and X_2 (respectively Y_1 and Y_2) be two traces with center x meeting trivially in z. If the back ups onto x of $X_1 \cap Y_1$ and $X_2 \cap Y_2$ have at least two elements in common, then they coincide.*

PROOF. Clearly the circles C_1 and C_2 in ${}^x\Gamma$ defined by the back ups of $X_1 \cap Y_1$ and $X_2 \cap Y_1$ respectively touch in xz. Similarly, the circles D_1 and D_2 in ${}^x\Gamma$ defined by the back ups of $X_1 \cap Y_2$ and $X_2 \cap Y_2$ respectively touch in xz. Suppose now that C_1 and D_2 meet in at least two elements xz and, say, L. We have to show that $C_1 = D_2$. Suppose by way of contradiction that $C_1 \neq D_2$. Then, by Axiom [MP2], D_2 does not touch C_2 and hence it meets C_2 in a second element, say M. That implies that $X_2 \cap Y_1$ and $X_2 \cap Y_2$ have a point of M in common, hence $Y_1 \cap Y_2$ contains a point of M. We conclude that $M = xz$, a contradiction. The lemma is proved.

We now state a very important proposition. It will enable us to define a certain type of point in the metasymplectic space. It also explains the behaviour of the dual traces. We use the same notation for dual traces as for traces.

Proposition 10 *Let L be any line of Γ and let M_1 and M_2 be two lines opposite to L. For any point x incident with L, consider the following property $P(L; M_1, M_2)$: the projections of M_1 and M_2 onto x lie in a common circle of ${}^x\Gamma$ with corner L. If at least two points incident with L have the property $P(L; M_1, M_2)$, then all points do.*

PROOF. The proof goes in several steps, converging to the general mutual position of M_1 and M_2.

STEP 1.
Suppose that M_1 and M_2 meet, say in the point y. Clearly the projection x of y onto L has property $P(L; M_1, M_2)$. Without loss of generality we may assume that $x_1 \neq x$ and we may assume $x_2 = x$. Let M be the projection of L onto y. Then by Lemma 6, M_1 and M_2 are contained in a circle of ${}^y\Gamma$ with corner M. If u is any point of L distinct from x_1 and x_2, then the result follows by applying once again Lemma 6.

STEP 2.
Suppose that there is a line M at distance 6 from L meeting both M_1 and M_2. As above,

we can assume that the projection of M onto L is x_1. Let y_i be the intersection of M and M_i, $i = 1, 2$. Let N be the projection of M_1 onto x_2 and let y be the projection of M_1 onto N. Let y' be the projection of y_2 onto N and let M_2' be the projection of y' onto y_2. Let x, $x_1 \neq x \neq x_2$, be any point on L and let L_1, L_2, L_2' be the projections onto x of respectively M_1, M_2 and M_2'. By Step 1, the projections of M_2 and M_2' onto x are contained in a circle of ${}^x\Gamma$ with corner L. So it remains to show that the projection of M_1 onto x is also contained in that circle; it suffices to show that the projections of M_1 and M_2' onto x are contained in a circle of ${}^x\Gamma$ with corner L. Let z_1 and z_2' be the projections of x onto M_1 and M_2' respectively. Now, the traces $x_2^{y_1}$ and $x_2^{y_2}$ meet trivially in x_1 (because y_1 and y_2 are collinear); the traces $x_2^{y_2}$ and $x_2^{z_2'}$ meet trivially in y' (similar reason); the traces $x_2^{y_1}$ and $x_2^{z_1}$ meet trivially in y (similar reason) and $x \in x_2^{z_1} \cap x_2^{z_2'}$. According to Lemma 8, the traces $x_2^{z_1}$ and $x_2^{z_2'}$ meet trivially in x. The result follows from Lemma 7.

STEP 3.
Suppose that there is a point y at distance 3 from M_1 and M_2 and at distance 5 from L. Again we can assume that the projection of y onto L is x_1. Denote by N_i the projection of M_i onto y, $i = 1, 2$. Let $\{u\} = \Gamma_2(x_2) \cap \Gamma_5(M_1)$ and let M_2' be the projection of u onto N_2. Let y_1 respectively y_2 be the intersection of M_1 and N_1 respectively M_2' and N_2. Let x, $x_1 \neq x \neq x_2$, be arbitrary on L and let L_1, L_2, L_2' be the projections onto x of M_1, M_2, M_2' respectively. By Step 2 the lines L_2 and L_2' are contained in a circle of ${}^{x_2}\Gamma$ with corner L. As in the previous step, it suffices to show that L_1 and L_2' are contained in such a circle. Therefore, let z_1 respectively z_2' be the projection of x onto M_1 respectively M_2'. The traces $x_2^{y_1}$ and $x_2^{z_1}$ respectively $x_2^{y_2'}$ and $x_2^{z_2'}$ meet trivially in u. If $x_2^{y_1} = x_2^{y_2'}$, then by Lemma 2 also $x_2^{z_1} = x_2^{z_2'}$ and the result follows from Lemma 4. Assume now that $x_2^{y_1} \neq x_2^{y_2'}$. But then $x_2^{y_1}$ and $x_2^{y_2'}$ meet in a set Y the back up of which on x_2 is a circle in ${}^{x_2}\Gamma$ with corner L (indeed, this follows from the fact that these traces meet in at least two points (x_1 and u); from the fact that the projections of y_1 and y_2' onto x_1 coincide and from Axiom [RT3]). By Lemma 9, $x_2^{z_1}$ and $x_2^{z_2'}$ meet in a set Z the back up of which equals the back up of Y, hence by Axiom [RT3] the result follows.

STEP 4.
Suppose that there is a line M at distance 4 from L, M_1 and M_2. We may assume that the projection of M onto L is x_1; let y_1 be the projection of M onto M_1 and let y be the projection of M_2 onto M. Suppose $\{N\} = \Gamma_3(x_2) \cap \Gamma_4(M_1)$. Let $\{M_2'\} = \Gamma_3(y) \cap \Gamma_4(N)$ and let y_2' be the projection of y onto M_2'. Let x, $x_1 \neq x \neq x_2$, be any point of L and let L_1, L_2, L_2' be the projections onto x of respectively M_1, M_2, M_2'. As before, it suffices by Step 3 to show that L_1 and L_2' are contained in a common circle of ${}^{x_2}\Gamma$ with corner L. Let z_1 and z_2' be the projection of x onto M_1 and M_2' respectively. By Lemma 4 the traces $x_2^{y_1}$ and $x_2^{y_2'}$ coincide; since y_1 and z_1 respectively y_2' and z_2' are collinear the traces $x_2^{z_1}$ and $x_2^{z_2'}$ both meet $x_2^{y_1}$ trivially in u, where u is the projection of x_2 onto N. But $x_2^{z_1}$

and $x_2^{z_2'}$ also meet in x, hence they coincide and the result follows from Lemma 4.

STEP 5.
Now consider the general case. Let $\{y\} = \Gamma_4(x_1) \cap \Gamma_3(M_2)$, let $\{N\} = \Gamma_4(M_1) \cap \Gamma_3(x_2)$ and let $\{M_2'\} = \Gamma_4(N) \cap \Gamma_3(y)$. Then the result follows from Step 3 applied to M_2 and M_2' and from Step 4 applied to M_1 and M_2'.

This completes the proof of the proposition.

Let L and M be opposite lines of Γ. For each point x of L, there is a unique line L_x incident with x and at distance 6 from M. The line L_x defines in ${}^x\Gamma$ a unique circle C_x with $\partial C_x = L$. We call the set of all circles C_x, for x ranging over all points incident with L, the *Suzuki trace* with focus line L and direction M, and we denote it by $L_{\bigcirc}^{M}$.

We have defined a Suzuki trace as a set of circles. These circles are the *elements* of the Suzuki trace. The elements of these circles (which are lines of Γ) will be called the *fringes* of the Suzuki trace.

The preceding proposition can now be reformulated as follows:

Corollary 11 *(i) A Suzuki trace is determined by any two of its elements.*
(ii) A Suzuki trace is determined by any two of its fringes lying in different elements.

Later on, the Suzuki traces will be one type of point in the metasymplectic space $\mathcal{M}(\Gamma)$ that we will define.

Lemma 12 Γ *is point-distance-*4*-regular.*

PROOF. Let x_1, x_2 and x_3 be three distinct points opposite each other and suppose that $y_1, y_2 \in (x_1)_{[4]}^{x_2} \cap (x_1)_{[4]}^{x_3}$, $y_1 \neq y_2$. Let y_3 be any point at distance 4 from both x_1 and x_2, $y_1 \neq y_3 \neq y_2$. We have to show that $\delta(x_3, y_3) = 4$. Let L_i be the projection of y_i onto x_1; let M_i be the projection of y_i onto x_2, $i = 1, 2, 3$. Let L_0 be the corner of the circle of ${}^{x_1}\Gamma$ containing L_1, L_2 and L_3. Let M_0 be the projection of L_0 onto x_2 and let y_0 be the point at distance 3 from both L_0 and M_0. We want to show that $\delta(y_0, x_3) = 4$. Therefore, suppose first that $L_1 \neq L_0 \neq L_2$.

Let L_i' be the projection of x_i onto y_1, $i = 1, 2, 3$. First suppose that L_1' is the corner of the circle C' of ${}^{y_1}\Gamma$ containing L_1', L_2' and L_3'. Let u_i be the projection of x_2 onto L_i and let v_i be the projection of x_1 onto M_i. By Lemma 4 we have $\delta(x_3, u_0) = \delta(x_3, v_0) = 6$; let N_{01} respectively N_{02} be the projection of u_0 respectively v_0 onto x_3. If $N_{01} = N_{02}$, then there is a circuit of length ≤ 14 passing through x_3 and y_0 unless $\delta(y_0, x_3) = 4$; so suppose $N_{01} \neq N_{02}$. For the same reason the projection of x_3 onto u_0 is distinct from u_0y_0 and the projection of x_3 onto v_0 is distinct from v_0y_0. Let w_1, w_2, w_{01} and w_{02} be the projection of x_1 onto respectively N_1, N_2, N_{01} and N_{02}. By Lemma 4, these are exactly the projections of x_2 on the respective lines. Hence the projections of x_1 and x_2 onto w_{01} are contained in a circle of ${}^{w_{01}}\Gamma$ with corner N_{01}. This implies by Lemma 4 that $x_3^{y_0} = x_3^{x_2}$ (consider w_{01} and w_{02}). Hence $\delta(w_1, y_0) = 6$. Let N_1' be the projection of N_1 onto y_0. Now note that by Lemma 6 applied to x_1 and x_3, the line N_{01} is the corner of a circle C of ${}^{x_3}\Gamma$ containing N_1 and N_2; applied to x_2 and x_3, we obtain from the same lemma that also the line N_{02} is the corner of a circle of ${}^{x_3}\Gamma$ containing N_1 and N_2. Hence, by Axiom [ST1], N_{02} is not contained in C. By Lemma 6 applied to x_3 and y_0 we now know that the circle D of ${}^{y_0}\Gamma$ containing v_0y_0 and having u_0y_0 as corner does not contain N_1'. Noting that $\delta(L_3', N_1') = 6$, and projecting D onto y_1, Lemma 6 implies that L_3' does not belong to the circle of ${}^{y_1}\Gamma$ with corner L_1' and containing L_2', which is C', contradicting our assumption. Hence as remarked above, we must have $\delta(x_3, y_0) = 4$. By symmetry, the result also follows if we switch the roles of L_1' and L_2' in our assumption. Now assume that L_3' is the corner of C'. Then the result follows similarly from the fact that N_1 cannot be the corner of a circle of ${}^{x_3}\Gamma$ containing N_{01} and N_{02} (that would contradict Axiom [ST2]).

Hence we may assume that the corner of C' is some line L_0' distinct from L_i' for $i = 1, 2, 3$. Let x_0 be the point at distance 3 from L_0' and at distance 4 from y_2. Let v_0 be as above, then by Lemma 4 $\delta(x_0', v_0) = 6$. Let y_0' be the point collinear with v_0 and at distance 4 from x_0'. By the preceding paragraph both x_3 and x_1 are at distance 4 from y_0'. If $y_0 \neq y_0'$, then there is a circuit of length ≤ 12 passing through x_1, v_0, y_0 and y_0', a contradiction. Hence we have shown that x_3 is at distance 4 from y_0.

Now let x_3' be at distance 4 from y_3 and 3 from L_3' (which is the projection of x_3 onto y_1). Switching the roles of y_2 and y_3, we see that x_3' is at distance 4 from y_0. But now there is a circuit of length ≤ 14 passing through x_3', x_3 and y_0, unless $x_3 = x_3'$. The result follows.

If $L_0 = L_3$, then the result follows simply by putting L_3 equal to L_0 in the paragraph preceding the last one. This completes the proof of the lemma.

Remark. The perfect Ree-Tits octagons are also line-distance-4-regular. It seems to be hard to prove this directly from our axioms. We will not need this however to prove our main result. So in a way, line-distance-4-regularity is not an essential property of the Ree-Tits octagons; it is rather there “incidentally”.

Lemma 13 *Let x be a point of Γ and let y and z be two points opposite x such that there exists a line L at distance 3 from both y and z and at distance 5 from x, and such that there exists a line M at distance 5 from both y and z and at distance 3 from x. Then the set of lines N through x such that the projections of y and z onto the unique point of $x^y = x^z$ on N coincide is a circle of ${}^x\Gamma$ with the projection of L onto x as corner.*

PROOF. Note that the lemma makes sense since by Lemma 4 the traces x^y and x^z indeed coincide. Now let L_x and M_x be the projections onto x of L, M respectively. Let

N be a line incident with x. Suppose first that N is contained in the circle of ${}^x\Gamma$ with corner L_x and containing M_x, and let u be the unique point of $x^y = x^z$ on N. We show that the projections N_z and N_y onto u of respectively z and y coincide. Suppose by way of contradiction that they do not coincide. Let y_1 and z_1 be the points at distance 4 from u and collinear with y and z respectively. Then y_1 and z_1 are opposite. We show that this leads to a contradiction. Let z_0 be the point collinear with z and at distance 3 from M. By Lemma 4 the traces y^x and y^{z_0} coincide, hence $\delta(z_0, y_1) = 6$. Denote the circle of ${}^x\Gamma$ with corner L_x and containing M_x by C. Projecting C onto y and then projecting the resulting circle onto z_0, we see by Lemma 6 that in ${}^{z_0}\Gamma$ there is a circle with corner zz_0 containing the projections onto z_0 of y_1 and x. Using Lemma 4 again, we now see that $z^x = z^{y_1}$, hence $\delta(y_1, z_1) = 6$ (since $z_1 \in z^x$), a contradiction if $N_y \neq N_z$. We conclude that $N_y = N_z$.

Now suppose that N does not belong to the circle of ${}^x\Gamma$ with corner L_x and containing M_x. If N_y coincides with N_z, then, with the notation of the previous paragraph, $\delta(y_1, z_1) = 6$ and again $z^x = z^{y_1}$. But this is impossible by Axiom [RT3] since now there is no circle of ${}^{z_0}\Gamma$ having corner zz_0 and containing the projections onto z_0 of y_1 and x.

This completes the proof of the lemma.

A kind of dual statement can also be proved, and in fact it is a consequence of the previous lemma.

Lemma 14 *Let L be a line of Γ and let M and N be two lines opposite L such that there exists a point x at distance 3 from both M and N and at distance 5 from L, and such that there exists a point y at distance 5 from both M and N and at distance 3 from L. Then $L^M_{[3]} = L^N_{[3]}$ if and only if the circle of ${}^x\Gamma$ containing the projections of M, N and L has the projection of the latter as corner if and only if the circle of ${}^y\Gamma$ containing the projections of M, N and L has the latter projection as corner. If one of these equivalent conditions is not satisfied, then $|L^M_{[3]} \cap L^N_{[3]}| = 2$.*

PROOF. Let z' be any point of L distinct from the projections x' and y' of x and y respectively on L. Consider the point z collinear with z' and at distance 5 from M. Applying Lemma 13 to x, y, z, L and M (with corresponding symbols in the statement of the lemma), we see that the point z is at distance 5 from N if and only if the circle of ${}^x\Gamma$ containing the projections onto x of respectively M, N and L has the latter projection as corner. The other equivalence follows directly from Lemma 6. The lemma is now clear.

Our next goal is to prepare for the definition of the third kind of point of our future metasymplectic space $\mathcal{M}(\Gamma)$. It would be nice if traces of Γ would represent points in $\mathcal{M}(\Gamma)$, but in fact they will represent a kind of hyperbolic or imaginary lines. More precisely, in the parabolic quadric model (in 6-dimensional projective space) of the C_3-subgeometry of $\mathcal{M}(\Gamma)$, they represent conics which are non-singular lines in the symplectic model (in 5-dimensional projective space) of this C_3-residue. So we must find a way to distinguish the points on such a line.

Lemma 15 *Let x and z be two distinct collinear points of Γ. Let C be a circle of ${}^x\Gamma$ with corner xz. On the set of traces through z with center x, we define the following relation denoted by $\sim$: $X \sim Y$ if and only if either $X = Y$, or X meets Y trivially in z, or the back up of $X \cap Y$ is a circle of ${}^x\Gamma$ the corner of which belongs to $C \setminus \{xz\}$. Then "$\sim$" is an equivalence relation.*

PROOF. Clearly only the transitivity property of the relation "$\sim$" needs to be checked. So suppose X, Y and Z are three traces with center x and containing z. Suppose $X \sim Y \sim Z$. We may assume that they are all distinct. Hence there are essentially three cases to consider.

Case 1 *X meets Y trivially and Y meets Z trivially.*
By Lemma 2, X, Y and Z are contained in a pencil of traces, hence $X \sim Z$.

Case 2 *X meets Y trivially and the back up of $Y \cap Z$ on x is a circle D of ${}^x\Gamma$.*
It is convenient to look in the affine plane $\mathrm{Res}(xz)$ of ${}^x\Gamma$. By Lemma 2 we know that X and Z do not meet trivially, hence they meet in a set the back up of which is a circle D' of ${}^x\Gamma$. Since X and Y meet trivially, D and D' touch in xz, hence they represent parallel lines in $\mathrm{Res}(xz)$. The result now follows immediately from [MC3].

Case 3 *The back ups of both $X \cap Y$ and $Y \cap Z$ are circles of ${}^x\Gamma$.*
Denote these back ups respectively by D and D'. Since the corners of both these circles lie in C and are distinct from xz, it follows from [MC3] that D and D' represent parallel lines in $\mathrm{Res}(xz)$. But that implies that either $X \cap Z$ is trivial or its back up on x is a circle touching both D and D'. The result follows from [MC3] again.

This completes the proof of the lemma.

To be more precise, the equivalence relation "$\sim$" of the previous lemma is called (x, z, C)-*equivalence.*

Consider now two opposite points x and y in Γ. Every point u of y^x defines in every point z of x^y (except in the unique point of x^y at distance 4 from u) a trace z^u which contains x. If C_z is the circle of ${}^z\Gamma$ containing the projection of y onto z and with corner xz, then it is easily seen that, for fixed $z \in x^y$, all these traces are (z, x, C_z)-equivalent. We call the set of all such traces the *crown of the pair* (x, y), and the elements of the crown with fixed center z will be called the *crown-traces with center* z. Consider now a point y' opposite x such that $x^y = x^{y'}$. Let $z \in x^y$. In the following lemmas we give a description of mutual positions of y and y' that allows one to decide whether the crown-traces with center z of the pair (x,y) are (z,x,C_z)-equivalent (with C_z as above) with those of the pair (x,y') or not. At the same time we aim at proving the property that they are equivalent for one center $z \in x^y$ if and only if they are equivalent for all centers $z \in x^y$.

Lemma 16 *Let x be any point of Γ. Let y and y' be two distinct points opposite x such that $x^y = x^{y'}$. Let z_1 and z_2 be two distinct elements of x^y. Let $y \mathbf{I} L_i \mathbf{I} u_i \mathbf{I} M_i \mathbf{I} v_i \mathbf{I} N_i \mathbf{I} z_i$ and $y' \mathbf{I} L'_i \mathbf{I} u'_i \mathbf{I} M'_i \mathbf{I} v'_i \mathbf{I} N'_i \mathbf{I} z_i$, $i = 1,2$. If y is not opposite y', and if*

$$|\{u_1,u_2,M_1,M_2,v_1,v_2,N_1,N_2\} \cap \{u'_1,u'_2,M'_1,M'_2,v'_1,v'_2,N'_1,N'_2\}| = 4,$$

then for every $z \in x^y$, the crown-traces with center z belonging to the crown of (x,y) are (z,x,C_z)-equivalent with those belonging to the crown of (x,y') (with C_z as above or below).

PROOF. Up to permutation of the indices 1 and 2, there are two cases to distinguish. Set

$$S = \{u_1,u_2,M_1,M_2,v_1,v_2,N_1,N_2\} \cap \{u'_1,u'_2,M'_1,M'_2,v'_1,v'_2,N'_1,N'_2\}.$$

Note that it is enough to prove that for arbitrary $z \in x^y$, one crown-trace with center z belonging to the crown of (x,y) is equivalent with one belonging to the crown of (x,y'). As above, we denote by C_z the circle of ${}^z\Gamma$ with corner xz containing the projections of y and y' onto z.

$\underline{S = \{u_1,M_1,v_1,N_1\}.}$
If $z \neq z_1$, then z^{u_1} belongs to the crown of both (x,y) and (x,y'). If $z = z_1$, then clearly $z^{u_2} = z^{u'_2}$ (by Lemma 4).

$\underline{S = \{M_1,v_1,N_1,N_2\}.}$
If $z = z_1$, then $z^{u_2} = z^{u'_2}$ by Lemma 4. If $z \neq z_1$, then z^{u_1} and $z^{u'_1}$ meet trivially in x since u_1 and u'_1 are collinear.

This completes the proof of the lemma.

We now investigate the case that we left out in Lemma 16, namely when y and y' are opposite.

Lemma 17 *Let x,y and y' be three pairwise opposite points at distance 4 from two given distinct points v_1 and v_2. Let M_i, M'_i and N_i be the projections onto v_i of respectively y, y' and x. Let C_z again be the unique circle of ${}^z\Gamma$ with corner xz and containing the projections of y and y' onto z, $z \in x^y$. Then for any $z \in x^y$, the crown-traces with center z belonging to the crown of (x,y) are (z,x,C_z)-equivalent with those belonging to the crown of (x,y') if and only if there exists a circle of ${}^{v_1}\Gamma$ with corner N_1 containing M_1 and M'_1, which is equivalent to the existence of a circle of ${}^{v_2}\Gamma$ with corner N_2 containing M_2 and M'_2.*
In particular, this condition is independent of z.

PROOF. By Lemma 12, we may assume $z = z_2$. But considering $z_2^{u_1}$ and $z_2^{u_1'}$ (where u_1 and u_1' are collinear with y and v_1, respectively y' and v_1'), the result follows directly from Axiom [RT3], Lemma 4 and Lemma 6. The lemma is proved.

Lemma 18 *Let x be any point of Γ and let y and y' be such that $x^y = x^{y'}$. For $z \in x^y$, we use again the notation C_z defined above. Then either the crown-traces with center z belonging to the crown of (x, y) are (z, x, C_z)-equivalent with those belonging to the crown of (x, y'), for all $z \in x^y$, or this happens for no $z \in x^y$.*

PROOF. Let z_1 and z_2 be two arbitrary distinct elements of x^y. As before, let $y \mathbf{I} L_i \mathbf{I} u_i \mathbf{I} M_i \mathbf{I} v_i \mathbf{I} N_i \mathbf{I} z_i$ and $y' \mathbf{I} L_i' \mathbf{I} u_i' \mathbf{I} M_i' \mathbf{I} v_i' \mathbf{I} N_i' \mathbf{I} z_i$, $i = 1, 2$. By considering the unique path of length 7 connecting u_1 with N_2', we may assume by Lemma 16 that $N_2 = N_2'$. By considering the unique path of length 7 connecting M_1 with v_2', we may assume by the same lemma that $v_2 = v_2'$. Similarly, but now by Lemma 17, we may assume that $M_2 = M_2'$. Again similarly, and using Lemma 16 again, we may assume that $u_2 = u_2'$. But now the result follows from Lemma 16. The lemma follows.

For any $x \in \mathcal{P}$ we can now define an equivalence relation on the set of points opposite x as follows. Two points y and y' are equivalent if $x^y = x^{y'}$ and the crown-traces with center $z \in x^y$ belonging to the respective crowns of (x, y) and (x, y') are (z, x, C_z)-equivalent, for some z, or equivalently, for all $z \in x^y$. An equivalence class $\mathcal{C}$ of this relation together with the trace x^y is called a *trace mark τ (of x^y)*. The trace x^y is said *to belong to τ*. The point y is called a *post* of τ and $\mathcal{C}$ is *<u>the</u> post set* of τ, while it is *<u>a</u> post set* of x^y. We will sometimes denote τ by $x^{[y]}$ (that is, $x^{[y]}$ denotes the trace mark to which the trace x^y belongs and for which y is a post). A trace has several trace marks as Lemma 18 shows.

From Lemma 17 and the proof of Lemma 18 the following criterion to decide whether two points are in the same post set of some trace can be derived.

Lemma 19 *Let x, y and y' be three points of Γ such that both y and y' are opposite x and $x^y = x^{y'}$. Let z_1 and z_2 be two distinct points of x^y. Let v be the unique point collinear with z_1 and at distance 4 from y; let M be the unique line at distance 3 from both y' and z_2. Let M_1 and M_2 be the projections onto v of y and M respectively. Then y and y' belong to the same post set of x^y if and only if the corner of the circle of ${}^v\Gamma$ containing M_1, M_2 and vz_1 is vz_1.*

The next lemma shows us exactly how many trace marks a trace has.

Lemma 20 *Let x and y be opposite points of Γ and let V be the set of points of Γ at distance 4 from all points which lie at distance 4 from both x and y. If y' is a point opposite x and $x^y = x^{y'}$, then there exists an element of V in the same post set of x^y as y'.*

PROOF. Let z be any element of x^y. Let v be at distance 4 from y and collinear with z. Upon replacing y' by another point in the same post set of x^y as y' (for example using Lemma 16), we may assume that $\delta(v, y') = 4$. Let M' be the projection of y' onto v. There exists a unique element y'' of V the projection onto v of which is also M'. The traces $z^{y'}$ and $z^{y''}$ therefore either coincide or meet non-trivially and in the latter case the back up on z of the intersection is a circle of ${}^z\Gamma$ with corner M' (by Axiom [RT3]). The lemma is proved.

Now we must prepare the definition of lines of our future metasymplectic space $\mathcal{M}(\Gamma)$. This is the motivation for the next few lemmas.

Lemma 21 *Let x be any point of Γ and let $\mathcal{C}$ be a post set of some trace X with center x. Let $z \in X$, let $y \in \mathcal{C}$ and let L be the projection of y onto z. Then the set of lines at distance 2 from L and at distance 3 from an element of $\mathcal{C}$ which itself lies at distance 5 from L, together with the lines of the unique circle of ${}^z\Gamma$ with corner L and containing xz, is precisely the set of all fringes of a Suzuki trace with focus line L.*

PROOF. Let w be any point of L, $w \neq x$. By Lemma 19, the projection onto w of all points of $\mathcal{C}$ at distance 4 of w is a circle of ${}^w\Gamma$ with corner L. We have to show that for variable w, all these circles constitute a Suzuki trace. Therefore let z' be any point of X, $z \neq z'$, and let M be the line at distance 3 from both y and z'. Every point u at distance 3 from M and 5 from L is in $\mathcal{C}$ by Lemma 16. Since the line at distance 3 from u and 2 from L belongs to L^M, and since also xz is contained in L^M, the lemma follows.

We use the notation of the previous lemma. Let C be the circle of ${}^z\Gamma$ with corner xz and containing L. Then every point y' of $\mathcal{C}$ is at distance 5 from a unique element L' of C. Applying Lemma 21 to y', one obtains a set of Suzuki traces with focus lines the elements of C distinct from xz, and the unique circle of ${}^z\Gamma$ belonging to such a given Suzuki trace contains the line xz. We call such a set a *Suzuki cycle with origin x, center z and rotation C (defined by y)*. We now prove that a Suzuki cycle with origin x, center z and rotation D is completely determined by one of its elements (which are Suzuki traces), hence is independent of the given trace X.

Lemma 22 *Let γ be a Suzuki cycle with origin x, center z and rotation D constructed with a trace X with center x. Then γ is completely determined by one of its elements and is independent of X.*

PROOF. Let y be such that $x^y = X$, let L be the focus line of an arbitrary element of γ and assume (as we may without loss of generality) that $\delta(y, L) = 5$. Let u be the projection of y onto L. Let L' be any element of D distinct from xz. Let y' be opposite x and such that $\delta(y', u) = 4$. We can always choose y' in that way for a given trace $x^{y'}$.

Suppose that y' defines the same Suzuki trace with focus line L as y does. Let M be any line through x distinct from xz. Let v and v' be at distance 5 from M and collinear with y and y' respectively. Let w and w' be at distance 5 from L' and collinear with v and v' respectively. Then w respectively w' belongs to the same trace mark of x^y respectively $x^{y'}$ as y respectively y'. By Lemma 4 we have $z^v = z^{v'}$. The lemma now follows from Lemma 5.

Lemma 23 *Let x be any point of Γ. Let X be a trace with center x and z a point collinear with x, but not in X. Let $\mathcal{T}$ be a pencil of traces based at z. There exists a unique trace $Z \in \mathcal{T}$ meeting X trivially.*

PROOF. By Lemma 3, there is at most one trace Z with the desired properties. We now prove that there is at least one. In fact, a proof is already implicitly in [26]. Let Y be any element of Λ. If Y meets X trivially, then we are done. So assume the back up of $X \cap Y$ on x is a circle C of ${}^x\Gamma$. In ${}^x\Gamma$, there is a unique circle D_M containing xz and touching C in a given line $M \in C$. For $M \neq M'$ we have that D_M and $D_{M'}$ do not touch (this is a consequence of Axiom [CH3] and [MC4]; indeed, in any STi-plane, the unique circle D_1 through some point p touching two touching circles D_2 and D_3 ($x \notin D_2 \cup D_3$) must contain $D_2 \cap D_3$ for otherwise D_1, D_2, D_3 are three circles touching two by two in different points). Hence D_M and $D_{M'}$ meet in a second line L. But then by Axiom [CH1] all circles D_M for $M \in C$ contain L and we obtain a transversal partition with extremities xz and L and containing C. Let w be the unique point of X incident with L. Let $\mathcal{C}'$ be the pencil of traces with base point w containing X. Let Z be the unique element of $\mathcal{C}'$ containing z and let $\mathcal{C}''$ be the pencil of traces with base point z containing Z. It follows now readily from Lemma 3 that $\mathcal{C}''$ contains Y (by the uniqueness of the transversal partition with given extremities), hence Z is the desired trace. The lemma is proved.

Note that, if Γ is finite, then a simple counting argument proves the preceding result.

We say that a line L *defines* a pencil of traces with center x based at a point z if for each point y of L opposite x the trace x^y belongs to that pencil.

Lemma 24 *Let $\mathcal{T}$ be a pencil of traces with center x and based at z. Let L and M be two lines defining $\mathcal{T}$. If there are two points x_L and x_M on respectively L and M in the same post set of some element of $\mathcal{C}$, then for each point on L, there exists a unique point on M in the same post set of some element of $\mathcal{T}$.*

PROOF. We break the proof up into small steps.

STEP 1.
Suppose that x_L and x_M are collinear with a point u at distance 6 from x. Let N be the projection of u onto x. Now let y_L and y_M be two points on respectively L and M

defining the same element of $\mathcal{T}$ and let z_0 be the projection of both y_L and y_M onto N. We show that y_L and y_M are in the same post set of x^{y_L}. Clearly $x^{y_L} = x^{y_M}$, since also $x^{x_L} = x^{x_M}$. For the same reason we can apply Lemma 14 to obtain that there is a point w collinear with z_0 and at distance 5 from both L and M. Let L_w be the projection of L onto w; let v be the unique point collinear with z and at distance 3 from L; let finally L_v be the projection of L_w onto v. Projecting the circle of ${}^u\Gamma$ containing the respective projections of L, M and x (and having the projection of the latter as corner) onto the point w, and then projecting this circle of ${}^w\Gamma$ onto v, we obtain by Lemma 19 the result.

STEP 2.
Suppose that the respective projections of L and M onto z coincide and that there is a line N at distance 3 from both x_L and x_M and at distance 5 from x. Let L_N be the projection of N onto x. Let y_L on L and y_M on M define the same element of $\mathcal{T}$ and let N' be the line at distance 4 from L_N and 3 from y_M. Let L_z be the line at distance 2 from M and 3 from z; let u be the point collinear with z and at distance 3 from L and let L_u be the projection of N' onto u. Clearly N and N' define the same Suzuki trace with focus line uz, because the lines xz and L_z both meet uz and are at distance 6 from both N and N'. From Lemma 11 follows that, if L'_u is the projection of L onto u, then the circle of ${}^u\Gamma$ containing uz, L_u and L'_u has uz as corner. The result follows now from Lemma 19.

STEP 3.
If there is a point u collinear with z and at distance 3 from both L and M, then the result follows directly from Lemma 19.

STEP 4.
Let L and M now be arbitrary. Let z' be any point of x^{x_L}, $z' \neq z$. Let $z \,\mathbf{I}\, L_z \,\mathbf{I}\, u'$ with $\delta(u', M) = 3$; let $v \,\mathbf{I}\, L_v$ with $\delta(v, x_L) = 2$ and $\delta(l_v, z') = 3$. Let L' be the line at distance 3 from v and at distance 4 from L_z and let M' be the line at distance 3 from u' and at distance 4 from L_v. Then we can apply Step 1 on L and L'; Step 2 on L' and M' and Step 3 on M' and M and the lemma follows.

If $\mathcal{T}$ is a pencil of traces with center x and based at z, then any line L at distance 5 from z all points y of which define a trace x^y belonging to $\mathcal{T}$ is called a *tail of* $\mathcal{T}$. By the previous lemma, the set of equivalence classes of the equivalence relation *"L is equivalent with M if every point of L opposite x is in the same post set of some element of* Π *as some point incident with M"* is in bijective correspondence with the set of trace marks of a given trace. The set of trace marks to which the traces of $\mathcal{T}$ belong and for which the points on some fixed tail are posts will be called a *pencil mark (with center x and base point z)*. So a pencil mark is a set of trace marks; we will however sometimes say that a certain trace X belongs to a pencil mark, or that a certain point y is a post of the pencil mark. By that we will understand that X belongs to a trace mark of the pencil mark and that y is a post of one of the trace marks belonging to that pencil mark.

Next, we look at trace bundles. A *trace bundle (with center x and base circle C)* is the set of all traces with center x which intersect a given trace X with center x precisely on the lines belonging to the circle C of ${}^x\Gamma$. Note that for every point y, $y \neq x$, on any line incident with x and not belonging to C there exists a trace of the bundle containing y. This follows from Lemma 1.

We prove a statement similar to Lemma 24 for bundles.

Lemma 25 *Let $\mathcal{B}$ be a trace bundle with center x and base circle C. Let L be the corner of C and let z be the unique point on L belonging to all traces of the bundle. Let y be opposite x such that x^y belongs to $\mathcal{B}$. Let u be collinear with y and at distance 4 from z. Let M be any line through u at distance 5 from z. Then there is a unique point v on M such that x^v belongs to $\mathcal{B}$. If y' is opposite x and such that $x^y = x^{y'}$, then we can construct similarly a point v'. If $x^v = x^{v'}$, then y and y' are in the same post set of x^y if and only if v and v' are in the same post set of x^v.*

PROOF. The proof is again given in a small series of steps. Note that in view of Axiom [RT3] and Lemma 4, we only need to prove the last statement.

STEP 1.
If y' is collinear with u, then the result follows directly from Lemma 16.

STEP 2.
Let u' be collinear with both y' and v'. Assume that u' is incident with the projection of z onto u. Let z_0 be a point in the intersection of all elements of $\mathcal{B}$, $z \neq z_0$, and let M and N be the respective projections of y and v onto z_0. By Lemma 16, we may assume without loss of generality that $\delta(y', M) = \delta(v', N) = 5$. The result follows directly from Lemma 16 again, noting that v, v', u, u' and N are all in one circuit of length 8.

STEP 3.
Let z_0, M and N be as above. We now assume that u and u' have the same projection w onto M. Since they also have the same projection x onto xz_0, we deduce with Lemma 4 that their projection r onto N is also the same, and that moreover the corner of the circle of ${}^r\Gamma$ containing N and the respective projections of v and v' onto r is exactly N. The result is now a direct consequence of Lemma 19.

STEP 4.
As in the proof of Lemma 24, one deduces the general statement from Steps 1 to 3 above.

This completes the proof of the lemma.

Let $\mathcal{B}$ be a trace bundle with center x and base circle C. Let z be the unique point on the corner of C belonging to every element of $\mathcal{B}$. Let u be any point at distance 4 from z such that the projection of u onto z belongs to the gate set of the intersection of all elements of $\mathcal{B}$. Then on any line through u at distance 5 from z, there is a unique point

y such that x^y belongs to $\mathcal{B}$. Moreover, there is an equivalence relation in the set of all such u defined as follows. Two points u and u' are equivalent if for each element X of $\mathcal{B}$, any two points y and y' collinear with respectively u and u' and such that $x^y = x^{y'} = X$ lie in the same post set of X. By Lemma 25 the set of equivalence classes is in bijective correspondence with the set of trace marks of a given trace. Hence any equivalence class defines in the obvious way a set of trace marks to which the elements of $\mathcal{B}$ belong. This set of trace marks will be called a *bundle mark μ (with center x, base circle C and origin z)*. By abuse of language we will say that an element of $\mathcal{B}$ *belongs to μ*.

Lemma 26 *Let x and y be two opposite points of Γ. Let $z \in x^y$ and suppose $z\,\mathbf{I}\,L\,\mathbf{I}\,u\,\mathbf{I}\,M\,\mathbf{I}\,v\,\mathbf{I}\,N\,\mathbf{I}\,y$. Let ς be the Suzuki trace with focus line M and fringes L and N. Let N' be any fringe of ς not incident with u. Then there is a unique point y' on N' such that $x^y = x^{y'}$. Let w respectively w' be any point collinear with y respectively y', at distance 6 from x and opposite z, then the back up of $z^w \cap z^{w'}$ on z contains the circle C of ${}^z\Gamma$ with corner L containing xz. Moreover, for every point $x_0 \in z^w$, $x_0 \neq u$ and such that x_0 is incident with an element of C, we have $x_0^y = x_0^{y'}$ and y and y' are contained in the same post set of both x^y and x_0^y.*

PROOF. If N' is incident with v, and if w and w' are at distance 4 of the same point of x^y, then all statements follow directly from Lemma 12, and Lemma 16. Now assume that N' is incident with v, but the points z_1 and z_1' of x^y at distance 4 from w and w' respectively are distinct. Then the result follows from Axiom [RT3] and Lemma 4.

So suppose that N' meets M in some point $v' \neq v$. Let R be at distance 3 from both w and x, then we may assume that $\delta(N', R) = 6$ by the first part of the proof and by the fact that M^R precisely defines ς. So w' lies at distance 3 from both R and N' and by Lemma 5 we have $z^w = z^{w'}$. Let $x_0 \in z^w$ be as in the statement of the lemma. Let R' be at distance 3 from both w and x_0. If w_R respectively $w_{R'}$ is the projection of w onto R respectively R', then by Lemma 6 the circle of ${}^w\Gamma$ containing w_R, $w_{R'}$ and wy has wy as corner (projected from z). By Lemma 13, $\delta(y', R') = 5$ and the result follows from Lemma 5 and Lemma 16. The lemma is proved.

The set of trace marks of x_0^y for which y is a post — with the notation of the previous lemma — will be called a *track with focus z and Suzuki direction ς*. The points y and y' are the *tails* of the track, the points x and x_0 the *centers*, and the trace mark of z^w and $z^{w'}$ with posts w and w' respectively are the *supporting trace marks*. We have just shown that a track is completely determind by one of its trace marks and its Suzuki direction.

Let L and M be two opposite lines of Γ. Let $z \in L^M_{[3]}$ and $y\,\mathbf{I}\,M$ with $\delta(y, z) \neq 4$. Then y and z are at distance 6. Let u be at distance 2 from y and 5 from L, then u and z are opposite. By Lemma 5 the trace z^u is independent of u and by Lemma 16 also the trace mark $z^{[u]}$ is independent of u (for fixed z and variable y). Varying z, we obtain a set of trace marks which we call a *curtain*. The line L is called *the rail of the curtain*; the line M *a hem*. We also say that the trace z^u *belongs to* the curtain.

Lemma 27 *A curtain is determined by any two distinct traces belonging to it.*

PROOF. Let η be a curtain with rail L and hem M. Let $z_i \in L^M_{[3]}$, $z_1 \neq z_2$, $i = 1,2$ and let u_i be the point at distance 3 from M and collinear with z_i. Then $z_i^{u_j}$ belongs to η, $\{i,j\} = \{1,2\}$. Let M' be a line opposite L at distance 3 from points u_1' and u_2' contained in $z_1^{u_2}$ and $z_2^{u_1}$ respectively. If L_i is the projection of L onto z_i, $i = 1,2$, then by Axiom [RT3] and Lemma 6, $z_2^{u_1} = z_2^{u_1'}$ if and only if the circle of ${}^{z_1}\Gamma$ containing u_1z_1, $u_1'z_1$ and L_1 has the latter as corner. It then follows by Lemma 19 that $z_2^{[u_1]} = z_2^{[u_1']}$. So we conclude that M' is a hem of some curtain η' to which $z_1^{u_2}$ and $z_2^{u_1}$ belong if and only if the circle of ${}^{z_1}\Gamma$ containing u_1z_1, $u_1'z_1$ and L_1 has L_1 as corner and the circle of ${}^{z_2}\Gamma$ containing u_2z_2, $u_2'z_2$ and L_2 has L_2 as corner. Suppose these equivalent conditions are satisfied and let $z \in L^M_{[3]}$ be distinct from both z_1 and z_2. We have to prove that $z^{[u_1]} = z^{[u_1']}$. Let M'' be at distance 3 from both u_1' and u_2 (M'' exists since $u_2 \in z_2^{u_1'}$). Applying twice Lemma 14, we obtain $\delta(z, M'') = \delta(z, M') = 5$. If N, N' and N'' are the projections onto z of M, M' and M'' respectively, then by projecting ${}^{u_2}\Gamma$ and ${}^{u_1'}\Gamma$ onto ${}^{z}\Gamma$, we deduce from Lemma 6 that the circle of ${}^{z}\Gamma$ with corner the projection of L onto z and containing N also contains N' and N''. Lemma 5 now readily implies that $z^{u_2} = z^{u_1'} = z^{u_2'}$. It also follows from Lemma 16 that u_1' and u_2' are in the same post set of z^{u_1} as u_1 and u_2. Hence $z^{[u_2]} = z^{[u_1']} = z^{[u_2']}$. We conclude that $\eta = \eta'$. The proof of the lemma is complete.

4 Definition of the space $\mathcal{M}(\Gamma)$

In this section, we define the elements of a geometry $\mathcal{M}(\Gamma)$. We will show in the next section that $\mathcal{M}(\Gamma)$ is a metasymplectic space (a building of type F_4) in which the points and lines of Γ are the absolute points and lines of a certain polarity.

4.1 The points of $\mathcal{M}(\Gamma)$

There are three kinds of points.

Type (O) These are the points of the generalized octagon Γ itself.

Type (B) These are the Suzuki traces.

Type (I) These are the trace marks.

4.2 The lines of $\mathcal{M}(\Gamma)$

There are seven kinds of lines.

Type (OO) These are the lines of the generalized octagon Γ itself. Such a line L is incident with the points of type (O) which are incident with L in Γ and no other points of $\mathcal{M}(\Gamma)$ are incident with L.

Type (OB1) These are the sets of Suzuki traces with common focus line L and sharing a fixed circle C of ${}^x\Gamma$ for some $x \, \mathbf{I} \, L$. We call such sets *Suzuki bundles with focus flag* (x, L) *and foundation* C. A Suzuki bundle is incident with all Suzuki traces — points of type (B) — that it contains and with the point x — as point of type (O).

Type (OB2) These are the Suzuki cycles. Given a Suzuki cycle, all its elements — which are Suzuki traces, hence points of type (B) — as well as its unique origin — point of type (O) — are incident with it.

Type (OI) These are the pencil marks. Given a pencil mark, all its trace marks are — as points of type (I) — incident with it, as is its base point — as point of type (O).

Type (IB1) These are the bundle marks. Given a bundle mark, all traces marks belonging to it are incident with it — as points of type (I) —, and the unique Suzuki trace with as focus line the corner of the base circle of the trace bundle, containing the base circle of the bundle mark and the gate set of the intersection of all elements of the corresponding trace bundle is — as a point of type (B) — also incident with it.

Type (IB2) These are the tracks. Given a track, all its trace marks — as points of type (I) — and its Suzuki direction — as points of type (B) — are incident with it.

Type (II) These are the curtains. Incident with a curtain are all trace marks that belong to it — as points of type (I).

Remark. Note that a line λ of $\mathcal{M}(\Gamma)$ of type (OO) or (II) contains only points of type (O) or (I) (respectively). A line λ of type (XY?), where X$\neq$Y and ? is 1,2 or empty, contains a unique point p of type (X) or (Y) and all other points on λ are of type (Y) or (X) respectively. If p has type (U), U$\in$ {X,Y}, then (U) appears before (V), {U,V} = {X,Y}, in the sequence ((O),(B),(I)). This has as a direct consequence that the subgeometry induced on the set of points of type (O) respectively (O) and (B) is a subgeometry of $\mathcal{M}(\Gamma)$. The former will be Γ, the latter is the geometry defined by SARLI [13].

Note also that an alternative way of proving that $\mathcal{M}(\Gamma)$ is a metasymplectic space would be to use COHEN's local characterization of such geometries, see [4]. However, this characterization uses a pentagon of points and lines. In view of the many different types of such objects in $\mathcal{M}(\Gamma)$, this leads to a very long list of possibilities. And then one still has to produce a polarity; so planes and hyperlines are needed anyway.

4.3 The planes of $\mathcal{M}(\Gamma)$

There are seven kinds of planes. Essentially, the types of the planes are the same as these of the lines, but to distinguish them properly we denote them with square brackets. Moreover, as an object of Γ, the planes are identical to the lines, but of course the incidence relation will be different. In the following definitions, we only mention the points incident with a given plane. A line is incident with a plane if and only if all its points are incident with that plane. So in principal, a plane could be incident with no line, but of course in the next section, we will prove that all planes are projective planes.

Type [OO] Let L be a line of Γ, i.e. a line of type (OO) of $\mathcal{M}(\Gamma)$. Then L is a plane of type [OO]. All points of L, viewed as points of type (O), are incident with L, as well as all Suzuki traces with focus line L — as points of type (B).

Type [OB1] Let ζ be a Suzuki bundle with focus flag (x, L) and foundation C. Then ζ is a plane of $\mathcal{M}(\Gamma)$ of type [OB1]. All points of L — as points of type (O) — are incident with ζ, as well as any Suzuki trace with focus line $M \in C \setminus \{L\}$ and fringe L.

Type [OB2] Let γ be a Suzuki cycle with origin x, center z and rotation C. Then γ is a plane of type [OB2]. The point z of type (O) is incident with it; all Suzuki traces with focus line xz and containing C are — as points of type (B) — incident with it and finally all trace marks

* to which belong traces which have center x, which contain z and the gate set through z of which is exactly C, and
* whose post set contains points at distance 3 from the fringes of the Suzuki traces belonging to γ,

are also incident with it — as points of type (I).

Type [OI] Let ρ be a pencil mark with center x and base point z. Each trace mark of ρ defines a Suzuki cycle γ with origin x, center z and rotation the circle C of ${}^{z}\Gamma$ defined by the "gate set" of the trace pencil (see Lemma 7). It is also easily seen that for distinct trace marks the corresponding Suzuki cycles coincide. Also, the set $\mathcal{U}$ of crown-traces with center z is independent of the trace mark of ρ. By definition, ρ is a plane of type [OI] and all points of $\mathcal{M}(\Gamma)$ incident (in $\mathcal{M}(\Gamma)$) with γ are incident with ρ (as points of type (O) and (B)), as well as all trace marks to which traces of $\mathcal{U}$ belong and for which the posts are determined by the gate sets through the points distinct from z of the traces of ρ (points of type (I)). We will show in the next section that everything is well-defined here.

Type [IB1] Let μ be a bundle mark with center x and base circle C. This is a plane of type [IB1]. Now we define which points of $\mathcal{M}(\Gamma)$ are incident with μ. Let L be the corner of C and z the unique point on L contained in every trace of the bundle mark. Let D be the circle of ${}^{z}\Gamma$ equal to the gate set through z of the intersection of all traces belonging to μ. Each trace mark of μ defines the same unique Suzuki cycle γ with origin x, center z and rotation D, by Lemma 22 and Lemma 25. All points — of type (B) and type (O) — of $\mathcal{M}(\Gamma)$ incident in $\mathcal{M}(\Gamma)$ with γ (as a line of type (OB2)) are by definition also incident with the plane μ. Now let X and X' be two traces of μ and let y and y' be corresponding respective posts. We can choose y and y' collinear with a point w at distance 4 from z. But that means that w defines a crown-trace with center any point u of $X \cap X'$, except z, for both the traces X and X'. It is now clear, since the gate sets of X and X' through u are disjoint up to the line xu, that the intersection of the set of crown-traces with center u for X and for X' is a trace pencil defining a unique pencil mark (w is a post of one of the trace marks). By definition every trace mark belonging to such a pencil mark is — as point of type (I) — incident with the plane μ in $\mathcal{M}(\Gamma)$.

Type [IB2] Let ϑ be a track with focus z and Suzuki direction ς, having focus line M. Then ϑ is a plane of type [IB2] incident in $\mathcal{M}(\Gamma)$ with the unique Suzuki trace ς' with focus line the projection L of M onto z, such that ς' contains M and the lines joining z with the centers of ϑ as fringes (this is a point of type (B)), also incident with all supporting trace marks of ϑ — as points of type (I) — and finally incident with all trace marks $y^{[x]}$, where y is an arbitrary tail of ϑ and x an arbitrary center of ϑ (and these are also points of $\mathcal{M}(\Gamma)$ of type (I)).

Type [II] Let η be a curtain with rail L. Then η is a plane of type [II]. Let M be any hem of η. Then the Suzuki trace $L^{M}_{\bigcirc}$ is incident in $\mathcal{M}(\Gamma)$ with η — as point of type (B). Note that $L^{M}_{\bigcirc}$ is independent of M by Corollary 11. The other points incident with η in $\mathcal{M}(\Gamma)$ are the elements — points of type (I) — incident in $\mathcal{M}(\Gamma)$ with the curtains with rail M and hem L.

Remark. SARLI's focal planes in [13] are our planes of type [OO].

4.4 The hyperlines of $\mathcal{M}(\Gamma)$

We first make a very useful observation, the proof of which is left to the reader, since it is merely an inspection of cases. In fact, the poof will follow almost directly from the proofs in section 5.1.

Proposition 28 *Let λ and λ' be two lines of $\mathcal{M}(\Gamma)$. From the definition of lines and planes it follows that we can view λ and λ' also as planes. The following two conditions are equivalent:*

1. *λ is as a line contained in λ' as a plane.*

2. *λ' is as a line contained in λ as a plane.*

This motivates us to define a hyperline as a an object h which is already a point in $\mathcal{M}(\Gamma)$, and which is by definition incident with the plane β if and only if the point h is incident with the line β. A point p of $\mathcal{M}(\Gamma)$ is incident with the hyperline h if it is incident with some plane β which is incident with h. Similarly for lines of $\mathcal{M}(\Gamma)$.

So we have defined all elements of $\mathcal{M}(\Gamma)$ and the incidence relation. In the next section, we will show that $\mathcal{M}(\Gamma)$ is a metasymplectic space, i.e. a building of type F_4.

5 $\mathcal{M}(\Gamma)$ is a metasymplectic space

5.1 All planes are projective planes

5.1.1 Planes of type [OO]

Lemma 29 *Let x be a point of Γ and let z_1 and z_2 be two non-collinear points in $\Gamma_2(x)$. Let $\mathcal{T}_i$ be a pencil of traces based at z_i, $i = 1, 2$ and suppose $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$. Then every line L through x either is the corner of the back up of the intersection of some element of $\mathcal{T}_1$ with some element of $\mathcal{T}_2$, or contains a point which is the (trivial) intersection of some element of $\mathcal{T}_1$ with some element of $\mathcal{T}_2$, or it is the corner of a circle of ${}^x\Gamma$ containing xz_1 and xz_2.*

PROOF. Let $X_i \in \mathcal{T}_i$ contain z_j, $\{i, j\} = \{1, 2\}$. By assumption $X_1 \neq X_2$. We first claim that the back up C onto x of $X_1 \cap X_2$ minus the lines xz_1 and xz_2 is exactly the set of lines through x incident with a point u, u not on xz_1 nor on xz_2, with the following property: the unique elements of $\mathcal{T}_1$ and $\mathcal{T}_2$ through u meet trivially in u. Indeed, suppose first that M is such a line and let $Y_i \in \mathcal{T}_i$, $i = 1, 2$, be the corresponding trivially meeting traces. Then xz_1 and M are the extremities of the transversal partition defined by the back ups onto x of the intersection of Y_2 with all respective elements of $\mathcal{T}_1$ (by Lemma 3). One of the circles touching all elements of that transversal partition is C. Hence $M \in C$. Conversely, suppose now $M \in C$. Consider two distinct circles C_1 and C_2 in ${}^x\Gamma$ containing xz_1 and M, distinct from C (these exist since the order of the affine plane $\mathrm{Res}(M)$ is at least 2). Then $xz_2 \notin C_1 \cup C_2$ and hence by [MC4] there exists a unique circle D containing xz_2 and touching both C_1 and C_2, so also touching C by Axiom [CH1], necessarily in xz_2. Hence D is the back up of the intersection of X_1 with some element Z_2 of $\mathcal{T}_2$ (follows from Axiom [MP2]). By the uniqueness of the transversal partition with a given extremity and containing a given circle, we conclude that the unique element Z_1 of $\mathcal{T}_1$ meeting Z_2 trivially meets the latter on M. Whence our claim.

Now let L be any line through x, $L \notin C$. Consider the unique circle C_L of ${}^x\Gamma$ with corner L and containing xz_1. If C_L touches C (in xz_1), then by considering the unique point y on L contained in Z_2, we see that the back up of the intersection of X_2 with the unique element of $\mathcal{T}_1$ containing y is exactly C_L (by Axiom [MP2]). So suppose that C_L does not touch C. But then it meets C in a second element, say, N. If $N = xz_2$, then there is nothing to prove, so suppose $N \neq xz_2$. Since L is now the corner of an element of the transversal partition with extremities xz_1 and N, the result follows from the previous paragraph.

The lemma is completely proved.

Proposition 30 *The planes of type* [OO] *are projective planes.*

PROOF. The only non-obvious thing to check is that two points of type (B) can always be joined by a line (of type (OB1)). So let L be a line of Γ and let M and M' be two lines opposite L. Let x be any point on L and let M_x and M'_x be the projections onto x of M and M' respectively. We have to show that for some choice of x, the lines M_x and M'_x belong to a circle of ${}^x\Gamma$ with corner L. Let $z \,\mathbf{I}\, L$ be fixed. We may suppose that L is not the corner of the circle of ${}^z\Gamma$ through L, M_z and M'_z (otherwise we can put $x = z$). The lines M and M' define trace pencils $\mathcal{T}$ and $\mathcal{T}'$ respectively, based at the projections u and u' of M and M' respectively onto M_z and M'_z respectively. If $\mathcal{T} \cap \mathcal{T}' \neq \emptyset$, then let $w \,\mathbf{I}\, M$ and $w' \,\mathbf{I}\, M'$ be such that $z^w = z^{w'}$. By Lemma 4, we can take x equal to the projection of both w and w' onto L. So suppose $\mathcal{T} \cap \mathcal{T}' = \emptyset$. We apply Lemma 29. Suppose L is the corner of a circle obtained by the intersection of some trace $x^w \in \mathcal{T}$ with some trace $z^{w'} \in \mathcal{T}'$, where $w \,\mathbf{I}\, M$ and $w' \,\mathbf{I}\, M'$. Then by Axiom [RT3] we can take x equal to the projection of both w and w' onto L. Suppose now L contains a point x' which is the trivial intersection of two traces, one belonging to $\mathcal{T}$ and one belonging to $\mathcal{T}'$. Putting $x = x'$, the result follows from Lemma 7. Finally, L is not the corner of a circle of ${}^z\Gamma$ containing M_z and M'_z by assumption.

The lemma is proved.

5.1.2 Planes of type (OB1)

Lemma 31 *Let x, x' and z be three distinct collinear points of Γ. Let C be a circle of ${}^z\Gamma$ with corner xz. Let γ and γ' be any two Suzuki cycles with origin x and x' respectively, center z and rotation C. Then γ and γ' share exactly one Suzuki trace (with focus line in $C \setminus \{xz\}$).*

PROOF. Fix any line L of $C \setminus \{xz\}$ and a point v on L, $v \neq z$. We can choose a point y at distance 4 from v such that γ is defined by y. Let M be any line incident with y but not at distance 3 from v. Let $M \mathbf{I} w \mathbf{I} R \mathbf{I} u'$ with $\delta(u', x') = 4$. If y' defines γ', then it is easily seen using Lemma 22 that we can choose y' at distance 4 from both v and u'.

If γ and γ' share a Suzuki trace with focus line L, then we even might choose y' equal to y. So let us assume this, and let L' be any element of $C \setminus \{xz, L\}$. To find a point y_1 that also defines γ, but which lies at distance 5 from L' we proceed as follows. Consider the projection u of x onto M; now let y_1 be the unique point collinear with u at distance 5 from L', then y_1 defines γ by Lemma 16. Similarly one defines y'_1 collinear with w at distance 5 from L'; y'_1 defines γ'. Suppose y_1 and y'_1 define the same Suzuki trace with focus line L'. Since the projections of y_1 and y'_1 onto L' do not coincide, that implies that the Suzuki trace with focus line L' belonging to γ (and γ') is defined by M. But that would mean that there is a circle D of ${}^z\Gamma$ with corner L' containing L and xz. But xz, L, L' define exactly one circle and that is C, whose corner is xz, a contradiction. Hence γ and γ' share at most one Suzuki trace.

Suppose now that the Suzuki traces with focus line L of respectively γ and γ' are not equal. Let u be as in the previous paragraph; let w' be collinear with both u' and y'; let N be the projection of x onto u. The line N defines a pencil of traces with center z based at x. The trace $z^{w'}$ does not contain x, hence by Lemma 23 there is a unique point r on N such that z^r and $z^{w'}$ meet trivially in, say, the point s. If s lies on some element L' of C, then the respective points collinear with r and w' and at distance 5 from L' define respectively γ and γ' (by Lemma 16) and they define the same Suzuki trace with focus line L' by Lemma 8. So we may assume that s does not lie on an element of C. Now note that the back up on z of $z^w \cap z^{w'}$ is the circle C of ${}^z\Gamma$ (by Axiom [RT3]). By Axiom [ST2] there exists a unique circle D containing both xz and sz and having its corner, say, L' inside C. Hence there is a unique circle D' with corner L' belonging to the transversal partition with extremities xz and sz. By Lemmas 2 and 3, there is a point r' on N such that the back up of $z^{r'} \cap z^{w'}$ is D'. As in the previous paragraph, the points at distance 5 from L' and collinear with respectively r' and w' define respectively γ and γ' and they define the same Suzuki trace with focus line L'.

This completes the proof of the lemma.

Proposition 32 *The planes of type* [OB1] *are projective planes.*

PROOF. First we show that every two points are joined by a line. Let ζ be a Suzuki bundle with focus flag (x, L) and foundation C. The statement is obvious if one of the points has type (O). Let ς and ς' be two Suzuki traces with focus line $M, M' \in C \setminus \{L\}$ respectively and containing the circle of ${}^x\Gamma$ with corner L. If $M = M'$, then there is a line of type (OB1) joining ς and ς'. All points of $\mathcal{M}(\Gamma)$ of that line belong to ζ (viewed as a plane). If $M \neq M'$, then let N and N' be arbitrary fringes of ς and ς' respectively (not incident with z). Let y be any point at distance 3 from N and 5 from M. Let v be the unique point collinear with y and at distance 5 from N'. Let x be the projection of v onto L. Then it is clear that ς and ς' belong to the same Suzuki cycle with origin x, center z and rotation C.

The fact that every two lines meet in a unique point follows immediately from Lemma 31. The lemma is proved.

From the last part of the first paragraph of the proof of Proposition 32 follows immediately a remarkable configurational property.

Corollary 33 *Let x and z be two collinear points and let M and M' be two lines distinct from xz, incident with x and belonging to the same circle of ${}^x\Gamma$ with corner xz. Let y be any point at distance 6 from z and opposite x. Let $\{N\} = \Gamma_2(M) \cap \Gamma_5(y)$ and $\{N'\} = \Gamma_2(M') \cap \Gamma_5(y)$. Let y' be opposite x and at distance 5 from lines L and L', where L respectively L' is any line of the Suzuki trace with focus line M respectively M', but not incident with x. Then $\delta(y', z) = 6$.*

5.1.3 Planes of type [OB2]

Lemma 34 *Let x be a point of Γ and let z be collinear with x. Let $\mathcal{B}$ and $\mathcal{B}'$ be two trace bundles, both containing z. Let C and C' be base circles of $\mathcal{B}$ and $\mathcal{B}'$, respectively. Suppose that xz is the corner of both C and C'. Suppose also that the gate sets through z of all elements of both $\mathcal{B}$ and $\mathcal{B}'$ coincide. If $C \neq C'$, then there is a unique trace belonging to both $\mathcal{B}$ and $\mathcal{B}'$.*

PROOF. Let the intersection of the elements of $\mathcal{B}$ and $\mathcal{B}'$ be respectively X and X'. Let $w' \in X'$ and consider the unique element Y of $\mathcal{B}$ containing w' (one can construct this by considering w' and three different points of X). Similarly, there is a unique element Y' of $\mathcal{B}'$ containing a point w of X. Since by Axiom [RT3] the gate set of $Y \cap Y'$ is non-trivial, either the back up on x of $Y \cap Y'$ is a circle with corner xz, but that is impossible since it should then touch C or coincide with it — and clearly neither is true — or Y and Y' meet trivially — also impossible since $z, w, w' \in Y \cap Y'$ — or $Y = Y'$. Only the latter survives, which completes the proof of the lemma.

Proposition 35 *The planes of type* [OB2] *are projective planes.*

PROOF. This follows almost immediately from the definition of a Suzuki cycle and the previous lemma. The lines incident with planes of type [OB2] are of type (OB1), (OI) and (IB1). The proof can easily be reconstructed by the reader.

5.1.4 Planes of type [OI]

First we show that the points on such planes are well-defined. This will be an immediate consequence of the following lemma.

Lemma 36 *Let ρ be a pencil mark with center x and base point z. Then the crown-traces with center z of any trace mark of ρ are independent of that trace mark. Let $x^{[y]}$ and $x^{[y']}$ be two trace marks belonging to ρ, $y, y' \in \mathcal{P}$. Let v and v' be collinear with y and y' respectively and at distance 6 from x. Suppose that $z^v = z^{v'}$. Then v and v' are in the same post set of z^v, or, in other words, $z^{[v]} = z^{[v']}$.*

PROOF. For the first statement we only need to show that two posts of two different trace marks define the same trace with center z. Let y and y' be as stated in the lemma. We can always take — by definition — y and y' collinear. Let v and v' be as stated in the lemma, then we may choose v and v' at distance 5 from a line N through x. It is now clear that $z^v = z^{v'}$. This proves the first part.

Now we let y, y', v and v' again be arbitrary, but meeting the conditions of the statement of the lemma. Let u' be at distance 4 from v' and collinear with x. Then $u' \in x^{y'}$. Let M be the projection of z onto y and let N' be the projection of M onto u'. Let N be the projection of v' onto u' and let y'' be the projection of N' onto M. By Lemma 2, $x^{y'} = x^{y''}$ and hence by Lemma 5 there is a circle of ${}^{u'}\Gamma$ containing N and N' with corner xu'. The lemma now follows from Lemma 17.

Proposition 37 *The planes of type* [OI] *are projective planes.*

PROOF. Let ρ be a pencil mark whose traces have center x and all contain the point z. The Suzuki cycle γ defined by each post of any element of ρ is a line incident with the plane τ in $\mathcal{M}(\Gamma)$. It is also clear that each Suzuki trace of γ and every trace mark whose trace is a crown-trace centered at z of some trace belonging to ρ determine a unique line of type (IB1) all other points in $\mathcal{M}(\Gamma)$ of which are trace marks whose corresponding traces are — by definition — crown-traces as above. Two crown traces either meet trivially in x — and are contained in a unique trace pencil; adding the appropriate posts sets as given in the definition, we obtain a line of type (OI) all points of $\mathcal{M}(\Gamma)$ of which again are incident with the plane ρ in $\mathcal{M}(\Gamma)$, — or they define a unique trace bundle — and again we obtain a line, this time of type (IB1), all points in $\mathcal{M}(\Gamma)$ of which are incident in $\mathcal{M}(\Gamma)$ with the plane ρ.

The intersection of each pair of lines above is non empty. This is almost trivial if at most one of the lines is of type (IB1), and if we remark that every trace bundle and every trace pencil we consider share exactly one trace. If both lines are of type (IB1), then the result follows directly from Lemma 34. The proposition is proved.

5.1.5 Planes of type [IB1]

Proposition 38 *Planes of type* [IB1] *are projective planes.*

PROOF. To show that two points are joined by a line, there is only one non-obvious case: the points are trace marks whose traces have distinct centers. So let μ be a bundle mark with center x and base circle C. Let L be the corner of C, and let z be the unique point of L which is contained in all traces belonging to μ. Let D be the circle of ${}^{z}\Gamma$ containing the gate set through z of the intersection U of all traces belonging to μ. Let u and u' be two distinct points of $U \setminus \{z\}$. Let Y and Y' be two traces with respective centers u and u' belonging to trace marks τ and τ' respectively, which are — as points of type (I) — incident with μ in $\mathcal{M}(\Gamma)$. Take any line N of $D \setminus \{L\}$. Then by Lemma 26 there exists a unique track containing τ with Suzuki direction a Suzuki trace with focus line N. We deduce the existence of a point y at distance 5 from N, opposite x such that $x^{[y]}$ is an element of μ and such that for the unique point $w \in \Gamma_2(y) \cap \Gamma_3(N)$, the trace u^w coincides with Y. Let L_y and L'_y be the projections of y onto u and u' respectively. Let v' be the unique point of L'_y belonging to Y'. Let $\{M_y\} = \Gamma_2(L_y) \cap \Gamma_3(y)$. Finally let y' be the unique point at distance 4 from v' and 3 from M_y. Then by Lemma 5, $\delta(y', z) = 6$ and the projection N' of y' onto z lies in the circle of ${}^{z}\Gamma$ containing N with L as corner. Let ς' be the Suzuki trace with focus line N' and with fringes L and the unique element of $\Gamma_3(y') \cap \Gamma_2(N')$, then clearly the track containing τ with Suzuki direction ς contains τ'.

Now we prove that the track defined by two trace marks as above, is unique. Suppose by way of contradiction it is not. Then we have the following configuration. There are points y and y' opposite x at distance 6 from z (which is collinear with x) and at distance 2 from a point w which lies itself at distance 6 from x. We have $x \,\mathbf{I}\, M \,\mathbf{I}\, u \,\mathbf{I}\, L_y$ and $\delta(L_y, u) = 3$. We have $M \neq M' \neq xz$ and $x \,\mathbf{I}\, M' \,\mathbf{I}\, u' \,\mathbf{I}\, L_{y'} \,\mathbf{I}\, w'$ and $\delta(w', y') = 4$. We have by assumption that the projections N and N' of y and y' respectively, onto z are distinct. If $\{v\} = \Gamma_2(y) \cap \Gamma_3(N)$ and $\{v'\} = \Gamma_2(y') \cap \Gamma_3(N')$, then $(u')^{v'} = (u')^{v}$. We also have that the circle of ${}^{x}\Gamma$ respectively ${}^{z}\Gamma$ containing M respectively N with corner xz also contains M' respectively N'. Let R and R' be the projections onto w' of respectively v and v', then by Lemma 5 the circle of ${}^{w'}\Gamma$ containing R, R' and $L_{y'}$ has the latter as corner. Projecting this circle onto y and the result onto x, we see that using Lemma 6, M' should be the corner of the circle in ${}^{x}\Gamma$ containing M, M' and xz, a contradiction since we assumed it is xz.

Now we prove that every two lines meet. Remark that we have lines of type (OB2), (OI) and (IB2). If at least one of the two lines has not type (IB2), then the assertion is obvious. If both lines have type (IB2), then one can check easily that the proof is in fact given in the first paragraph by interchanging the roles of x and z (it is a dual statement reflected here by the swapping of x and z).

This completes the proof of the proposition.

5.1.6 Planes of type [IB2]

The next lemma is a slight generalization of Lemma 13.

Lemma 39 *Let x be a point of Γ and y and y' two points opposite x such that there exists a line M at distance 3 from both of them and 5 from x, such that $x^y = x^{y'}$, and such that $\delta(y, y') = 6$. Then the back up on x of the set of points u of x^y such that the projections onto u of y and y' coincide is a circle of ${}^x\Gamma$ with corner the projection of L onto x.*

PROOF. By Lemma 13, we only need to show that there exists at least one point u with the required property. But that follows directly from Axiom [RT4]. The lemma is proved.

Note that this is the first time that we use Axiom [RT4]. The reason why we did not state the preceding lemma when we stated Lemma 13 is to make it clear that Axiom [RT4] is only needed now. This probably means that [RT4] can be deduced from the other axioms. Although Euclid thought something similar about *his* famous axiom. . .

Proposition 40 *The planes of type* [IB2] *are projective planes.*

PROOF. Let ϑ be a track, viewed as a plane of type [IB2]. Let z be the focus of ϑ and let ς be its Suzuki direction, with M as focus line. Let L be the projection of M onto z. Let x be a center of τ and y any tail. Clearly the Suzuki trace ς' with focus line L and containing M and xz, together with all supporting trace marks forms a line of type (IB1) completely contained in ϑ. Also, the trace marks $(y')^{[x]}$, with $\delta(y, y') = 4$ or $y = y'$, and y' a tail of ϑ, together with ς' form a line of type (IB2) completely contained in ϑ. Now suppose y' is a tail of ϑ at distance 6 from y. We prove that the trace marks $(y')^{[x]}$ and $y^{[x]}$ are contained in a unique curtain η (viewed as line of type (II)) all points in $\mathcal{M}(\Gamma)$ of which lie in ϑ. Indeed, by Axiom [RT4] there is a line M' at distance 3 from x and 5 from both y and y'. If w and w' denote the points at distance 3 from M' and collinear with y and y' respectively, then by Lemma 13, $y^x = y^{w'}$ and $(y')^x = (y')^{w'}$. So M' is a hem of a curtain η as desired. The uniqueness of η follows from Lemma 26.

To prove that two lines of the plane ϑ always intersect, we only treat the non-obvious case of two lines of type (II). So let η and η' be two curtains with rail M all elements of which are incident, as points of type (I), with the plane ϑ of $\mathcal{M}(\Gamma)$. Let u be any point (of Γ) incident with M, but not with L. Let y and y' be the centers of the traces containing u and belonging to η and η' respectively. By the preceding paragraph we can choose a hem H respectively H' of η, respectively η' at distance 3 from x. Let N and N' be the projections onto x of H and H' respectively. If the circle of ${}^x\Gamma$ containing N, N' and xz has xz as corner, then by Lemma 5 we have $z^{[w]} = z^{[w']}$, where $\{w\} = \Gamma_3(H) \cap \Gamma_2(y)$ and $\{w'\} = \Gamma_3(H') \cap \Gamma_2(y')$. Hence η and η' share a trace mark with center z. Suppose now that the circle of ${}^{z}\Gamma$ containing N, N' and xz does not have xz as corner. Then by Axiom [RT4] there exists a point y'' at distance 3 from M and 5 from both H and H'. It follows that both η and η' contain the trace mark $(y'')^{[x]}$.

This completes the proof of the proposition.

5.1.7 Planes of type [II]

Lemma 41 *Let L be a line of Γ and let M and M' be opposite L such that M and M' are hems of the same curtain with rail L. Then there exists a unique point u at distance 5 from L and 3 from both M and M'.*

PROOF. By assumption, there exist points z_1 and z_2 at distance 3 from L and 5 from both M and M'. Let u_i and u'_i be collinear with z_i and at distance 3 from M and M' respectively, $i = 1, 2$. Let L_i be the projection of L onto z_i, $i = 1, 2$. From the proof of Lemma 27 it follows that the circle of ${}^{z_1}\Gamma$ containing L_1, $u_1 z_1$ and $u'_1 z_1$ has L_1 as corner. Noting that $z_1^{u_2} = z_1^{u'_2}$ and defining N_1 and N'_1 as the projection of u_2 onto u_1 and of u'_2 onto u'_1 respectively, Axiom [RT4] guarantees the existence of a point u at distance 3 from M and 5 from both L and N'_1. If $\{z\} = \Gamma_3(L) \cap \Gamma_2(u)$, then by the proof of Lemma 27 we know that $\delta(z, M') = 5$. But $\delta(M', N'_1) = 2$; $\delta(N'_1, u) = 5$ and $\delta(u, z) = 2$, constituting a cycle of length $5 + 2 + 5 + 2 = 14 < 16$. Hence this cycle must collapse somehow. The only way this is possible is that $\delta(u, M') = 3$.

Clearly u is unique since any other point u' satisfying the same conditions would imply a cycle of length 12 containing u, u', M and M'. This completes the proof of the lemma.

Proposition 42 *The planes of type* [II] *are projective planes.*

PROOF. Let η be a curtain with rail L, viewed as a plane of type [II]. By definition there are lots of lines of type (II) in η, namely for every hem M of η, the curtain with rail M and hem L. From the proof of Lemma 27 one deduces easily that every two points of type (I) incident in $\mathcal{M}(\Gamma)$ with two such respective lines lies on one such line. It also follows readily from the definition of lines of type (IB2) that we can join the Suzuki trace $L^M_{\bigcirc}$ to any other point of η in $\mathcal{M}(\Gamma)$ and stay inside η. It is obvious that a line of type (IB2) meets every line of type (II) in η. And two lines of type (II) meet in exactly one point by Lemma 41. The proof of the proposition is complete.

5.2 The diagram of $\mathcal{M}(\Gamma)$ and the polarity σ

The space $\mathcal{M}(\Gamma)$ is a geometry of rank 4 as we have defined it. If we call points, lines, planes and hyperlines elements of type 1,2,3 and 4 respectively, then, with the terminology of Buekenhout [2], we can reconstruct the diagram of $\mathcal{M}(\Gamma)$ almost completely.

Before doing that, we introduce the following notation. We have seen that each element of $\mathcal{M}(\Gamma)$ can be viewed as either an element of type j, $j \in \{1,2,3,4\}$, or as an element of type $5-j$. If λ is an element of type j, then we denote by λ^σ the element λ viewed as element of type $5-j$. Note that $(\lambda^\sigma)^\sigma = \lambda$. We will show that σ is in fact a polarity (an incidence preserving permutation of order 2 of the elements of $\mathcal{M}(\Gamma)$ interchanging the elements of type j and type $5-j$).

Proposition 43 *The map σ is a polarity in $\mathcal{M}(\Gamma)$. Also, the residue in $\mathcal{M}(\Gamma)$ of any flag of type $\{1,2\}$ or $\{3,4\}$ is a projective plane and the residue in $\mathcal{M}(\Gamma)$ of any flag of type $\{1,3\}$, $\{2,3\}$ or $\{2,4\}$ is a generalized digon.*

PROOF. Let λ and κ be two elements of $\mathcal{M}(\Gamma)$ incident with each other. We have to show that λ^σ is incident with κ^σ. This follows from the definition if λ is a plane and κ is a hyperline (or equivalently if λ is line and κ is a point), and this follows from proposition 28 if λ is a line and κ is a plane. Now assume that λ is a line and κ is a hyperline. By definition λ is incident with κ if and only if there exists a plane β incident with both. But this is equivalent with β^σ incident with both λ^σ and κ^σ, which in turn clearly is equivalent with λ^σ incident with κ^σ since planes are projective planes. Now assume that λ is a hyperline incident with a point κ. Then there exists a plane β incident with both. By the foregoing, β^σ is incident with both λ^σ and κ^σ. By definition, there exists a plane α incident with both β^σ and κ^σ. Since λ^σ is incident with β^σ, it is also incident with α (by definition of planes), hence it is incident with κ^σ by definition. This proves that σ is a polarity.

We have already shown above that the planes are projective planes. By the definition of hyperlines and incidence of hyperlines and planes, we readily deduce that the residue of an incident plane-hyperline pair is nothing else than the geometry of points and lines incident with the plane only. Hence, as shown before, this is a projective plane. Applying σ we see that the residues of the flags of type $\{1,2\}$ are also projective planes. Now consider an incident line-hyperline pair (λ, h). Clearly every point incident with λ is also incident with every plane through λ. Hence the residues of flags of type $\{2,4\}$ are generalized digons. Applying σ, we obtain the result for flags of type $\{1,3\}$. Finally, it follows immediately from the definition of incidence between a point and a hyperline that the residue of a flag of type $\{2,3\}$ is a generalized digon.

This proves the proposition.

So all that is missing is the residue of any flag of type $\{1,4\}$. We have to show that this is always a generalized quadrangle. We will do this in the next three propositions. A main tool in doing so will be the following axiom, which is a little stronger than the classical Buekenhout-Shult axiom for polar spaces.

(BSM) *Let h be a hyperline of $\mathcal{M}(\Gamma)$, p a point of $\mathcal{M}(\Gamma)$ in h and λ a line of $\mathcal{M}(\Gamma)$ in h, then either exactly one point of λ is collinear with p, or all points of λ are and p and λ are contained in some plane of $\mathcal{M}(\Gamma)$.*

The original Buekenhout-Shult axiom, adapted to our case, reads as follows.

(BS) *Let h be a hyperline of $\mathcal{M}(\Gamma)$, p a point of $\mathcal{M}(\Gamma)$ in h and λ a line of $\mathcal{M}(\Gamma)$ in h, then either exactly one point of λ is collinear with p, or all points of λ are.*

We will not need to verify (BSM) for all points p and all lines λ of any hyperline h. It will become clear in the proof of Propositions 51, 52 and 53 how we will use (BSM) to prove that the residue of flags of type $\{1,4\}$ are generalized quadrangles.

The verification of (BSM) is in many cases very straigtforward. But a few cases are very tricky and need new lemmas. Below we will give a description of the three kinds of hyperlines in terms of the points they contain and we will skip the proofs of the easy cases of (BSM). We will however consider the other cases in detail.

We first collect the lemmas we will need.

Lemma 44 *Let $\mathcal{T}$ be a pencil of traces in Γ with base point z and center x. Let L be a line at distance 5 from z such that x^y belongs to $\mathcal{T}$ for every point y on L. Let $\{u\} = \Gamma_2(z) \cap \Gamma_3(L)$. Let M' be any line through u contained in the circle of ${}^u\Gamma$ with corner ux and containing M. Let L' be a line concurrent with M', but not incident with u. Then for at least one (and hence for every) point y' on L' the trace $x^{y'}$ belongs to $\mathcal{T}$ if and only if the Suzuki traces with focus line M and M' containing the line ux and the line L respectively L' (as fringes) are incident in $\mathcal{M}(\Gamma)$ with a fixed line of type* (OB2) *which is also incident with the point z viewed as point of type (O) in $\mathcal{M}(\Gamma)$.*

PROOF. Clearly it suffices to establish through each point v' of M' except u a line L' which satisfies the two conditions. To this end, it suffices to find a point y' collinear with v' such that $x^{v'} \in \mathcal{T}$ and such that the line $L' = v'y'$ satisfies the second condition above.

Let z_0 be any point of Γ collinear with x but not on xz. Let X be the trace of $\mathcal{T}$ containing z_0. Let y be incident with L such that $X = x^y$ and let L_0 be the projection of y onto z_0. If v' lies on M, i.e. if $M = M'$, then we put $\{y'\} = \Gamma_2(v') \cap \Gamma_5(L_0)$. By Lemma 4 we have $X = x^{y'}$ and clearly the line $v'y'$ is a fringe of the Suzuki trace $M_{\bigcirc}^{L_0}$. Suppose now $M \neq M'$. Again put $\{y'\} = \Gamma_2(v') \cap \Gamma_5(L_0)$. Let u_0 be the projection of y' onto L_0 and let $\{L''\} = \Gamma_2(M) \cap \Gamma_5(u')$. By Lemma 4, we have $X = x^{y'}$ and since $\delta(u', z) = 6$, it follows from the construction of the line of $\mathcal{M}(\Gamma)$ of type (OB2) containing the Suzuki trace $M_{\bigcirc}^{L_0}$ and the point z that it also contains the Suzuki trace $M'^{L_0}_{\bigcirc}$, which in turn contains the line $v'y'$ (as a fringe).

The proof of the lemma is complete.

Lemma 45 *Let L be a line of Γ and let x, y_1, y_2 be three distinct points on L. Let M be opposite L and let $\{N_i\} = \Gamma_2(M) \cap \Gamma_5(y_i)$, $i = 1, 2$. Let N, $N \neq L$, be any line through x contained in the circle of ${}^x\Gamma$ with corner L and containing the projection M_x of M onto x. Let z, $z \neq x$, be any point on N and let N_1' and N_2' be the projections onto z of N_1 and N_2 respectively. If N_1' and N_2' lie in a circle of ${}^z\Gamma$ with corner N, then $N = M_x$.*

PROOF. Let u_i be the intersection point of M and N_i, $i = 1, 2$. Let v_i be the projection of z onto N_i, $i = 1, 2$. Let v_1' be the projection onto N_1 of the projection of v_2 onto M_x. The traces x^{u_1} and x^{u_2} meet trivially on M_x; the traces x^{u_1} and $x^{v_1'}$, respectively x^{u_2} and x^{v_2}, meet trivially on L, hence by Lemma 8, the traces $x^{v_1'}$ and x^{v_2} meet trivially on M_x. Hence x^{v_1} meets x^{v_2} in a circle C belonging to the transversal partition with extremities L and M_x (by Axiom [RT2]). By the assumption on N_1' and N_2', we deduce from Axiom [RT3] that N is the corner of C. But the corner of any element of the transversal partition is also the corner of a circle through the extremities. If $N \neq M_x$, this contradicts the fact that L is the corner of the circle of ${}^x\Gamma$ through L, M_x and N. Hence $N = M_x$. This proves the lemma.

Lemma 46 *Let L be a line of Γ, let ς be a Suzuki trace with focus line L and let w_1 and w_2 be two points at mutual distance 6 incident in Γ with respective fringes of ς. Let τ_1 be any trace mark with center w_1 and a post y collinear with w_2. Let x be any point collinear with w_1 but not at distance 3 from L. Put $\tau_2 = w_2^{[x]}$. Then τ_1 and τ_2 are contained in the same curtain with rail L if and only if x belongs to τ_1.*

PROOF. First let x belong to τ_1. Then x is at distance 6 from y. So if H is the line at distance 3 fron both x and y, then τ_1 and τ_2 belong to the curtain with rail L and hem H. Suppose now x does not belong to τ_1. Then x is opposite every point of τ_2 except the unique point of τ_2 on L. Hence τ_1 and τ_2 cannot be contained in the same curtain. The lemma is proved.

Now we prove some results in an arbitrary STi-plane.

Lemma 47 *Given two distinct non-touching circles C and D in an STi-plane, and given a point x in C but not in D, there exists a unique circle C' touching C in x and touching D.*

PROOF. Let y be any point of D and consider the unique circle C'' containing y and touching C in x (existing by Axiom [MP2]). If C'' touches D, then the result follows (since C'' must be unique by Axiom [CH2]). So we may assume that C'' and D meet in a second point y'. Let D' be any circle distinct from D and C'' containing y and y'. By [MC4], there is a circle C' touching both D' and D and containing x. Hence C' also touches C'' by Axiom [CH1], necessarily in x, hence C' also touches C in x. The lemma follows.

Lemma 48 *Given two touching circles C and D with different corners in an STi-plane, the unique circle containing ∂C, ∂D and the intersection of C and D has the latter as corner.*

PROOF. Put $\{x\} = C \cap D$. By Axiom [ST2] there exists a unique circle E containing the corners of C and D (which are distinct from x) and having its corner, say e, in C. By [MC3] the circle E' with the same corner as D and containing e touches C in e. But also D touches C and so [MC5] implies that $E' = D$ and $e = x$. The lemma is proved.

Lemma 49 *Let Ω be an STi-plane. Let x, y, z be three points of Ω and let C, D, E be three circles of Ω satisfying the following conditions.*

(i) $\partial C = x$ *and* $y, z \in C$;

(ii) $\partial D = y$ *and* $x \in D$;

(iii) $\partial E = z$ *and* $y \notin E$.

Then there exists a unique pair of points $(d, e) \in D \times E$ such that e is the corner of the circle E' through d, e, z, and d is the corner of the circle D' through d, e, y.

PROOF. If d and e exist, then by [MC3] (or equivalently Axiom [ST2]) the circles D and E' touch in d (since their corners lie in D' and they both contain the corner of D'). Let E'' be any circle containing z and such that $z \neq \partial E'' \in E$. Either E'' touches D or there exists by Lemma 47 a circle E' touching E'' in z and touching D. In the former case we put $E' = E''$; in the latter case, since E' touches E'', we have by [MC3] that the corner of E' lies in E. So if d and e exist, then d must be the intersection of E' and D, and consequently, e must be the corner of E'. By Lemma 48, d is the corner of the circle D' through d, e, y. The proof of the lemma is complete.

Now we write down a lemma which requires a long and tiresome, but straightforward proof. We will skip this proof because we rather spend some more space on the proofs of the propositions to come. As a hint for the proof of the next lemma, one could remark that every plane contains at least two types of lines and so one must consider these two types to reduce the number of possibilities for the point p.

Lemma 50 *If a point p of $\mathcal{M}(\Gamma)$ is collinear with all points of a certain plane β of $\mathcal{M}(\Gamma)$, then p is incident (in $\mathcal{M}(\Gamma)$) with β.*

With the notation of the previous lemma, we remark that in fact, most of the times collinearity to a triangle in β already forces p to be a point of β. We leave the details to the interested reader (but we do not need this remark in the sequel).

We note that we will not need the full strength of the previous lemma, but only for planes of certain types.

A hyperline of type [X] is a hyperline h for which h^σ is a point of type (X).

Proposition 51 *The residue in $\mathcal{M}(\Gamma)$ of an incident point-hyperline pair, where the hyperline has type* [O]*, is a generalized quadrangle.*

PROOF. Let x be a point of Γ. Then $h = x^\sigma$ is a hyperline containing the following points of $\mathcal{M}(\Gamma)$. The points of type (O) incident with x^σ are the elements of $\{x\} \cup \Gamma_2(x)$. The points of type (B) incident with x^σ are the Suzuki traces with an element of $\Gamma_1(x)$ as focus line. The points of type (I) incident with x^σ are the trace marks with center x.

Let us show that (BSM) holds in certain well-defined cases. So let p be a point of $\mathcal{M}(\Gamma)$ in h and let λ be a line of $\mathcal{M}(\Gamma)$ in h. First we remark that (BSM) is straight forward if p has type (O) and λ is arbitrary. Also, if x, as a point of $\mathcal{M}(\Gamma)$, is incident with λ and p is arbitrary, then (BSM) is easy to check. Suppose now that p has type (B). So p is a Suzuki trace ς with focus line L (incident with x). Let C be the circle in ${}^x\Gamma$ belonging to ς. Suppose that λ has type (OB1). Then there is a unique point y of Γ incident with λ as a point of $\mathcal{M}(\Gamma)$. By the preceding remarks we may assume that $y \neq x$. If $L = xy$, then the assertion is clear. So suppose $L \neq xy$. First suppose moreover that xy is not an element of C. By Axiom [ST2], there exists a unique line M through x (in $\mathcal{M}(\Gamma)$) which is in ${}^x\Gamma$ the corner of a circle containing L and xy and which belongs to C. If ς' is the Suzuki trace incident with λ which contains the line M, then by definition ς and ς' belong to the same line of $\mathcal{M}(\Gamma)$ (of type (OB2), namely a Suzuki cycle). Suppose that xy belongs to C, then clearly ς is not collinear with any point of λ of type (B), but it is collinear with y — as a point of type (O).

Next, suppose that λ has type (OB2), hence λ is a Suzuki cycle with center x. Let z be its origin and C its rotation. If $L = xz$, then is it easily seen that p is collinear with only z — as a point of type (B). If L belongs to C, $L \neq xz$, then either xz belongs to ς, in which case p and λ are obviously contained in a plane of $\mathcal{M}(\Gamma)$ of type (OB1), or xz does not belong to ς and then p is only collinear with the Suzuki trace ς' of λ with focus line L (indeed, for a given Suzuki trace ς'', $\varsigma' \neq \varsigma''$, incident with λ as point of type (B), the line xz of Γ is the unique element of ${}^x\Gamma$ contained in the circle belonging to ς'' containing xz which is the corner of a circle containing both L and the focus line of ς'', see Axiom [ST2]). So we may assume that L does not belong to C. If xz belongs to ς, then again p is collinear with z (in $\mathcal{M}(\Gamma)$). Suppose by way of contradiction that ς is collinear with some Suzuki trace ς' incident with λ. Clearly ς and ς' then lie on a line

λ' of type (OB2) of $\mathcal{M}(\Gamma)$. Let N be the line of Γ containing the center and the origin of λ'. Let L' be the focus line of ς'. Then N is the corner of a circle c of ${}^x\Gamma$ containing L and L'. Note that C does not contain xz since otherwise L would be the corner of C. By Axiom [ST2] there is a unique circle of ${}^x\Gamma$ containing xz and N whose corner lies in C. But by assumption both the circles with corner L and L' containing xz also contain N, a contradiction. Hence we may assume that xz does not belong to ς. Let D be the circle of ${}^x\Gamma$ belonging to ς. From the last argument we know that if ς is collinear in $\mathcal{M}(\Gamma)$ with some Suzuki trace ς' of λ, then the circle D' of ${}^x\Gamma$ belonging to ς' touches D. By Lemma 47 there is a unique such circle D' (touching the circle of ${}^x\Gamma$ belonging to an arbitrary Suzuki trace of λ in xz and touching D) with corresponding Suzuki trace ς'. Suppose D and D' touch in K. By Lemma 48 K is the corner of the circle E through K and the corners of D and D', which implies that E is the rotation of some Suzuki cycle containing ς and ς'.

Now let λ be of type (OI). So λ is a pencil mark. Let y be the unique point collinear with x contained in every trace belonging to λ. If $L = xy$, then either the gate set in y of every such trace belongs to ς — and then λ and p lie in some plane of type [OB2] — or else p is only collinear with y, as points of $\mathcal{M}(\Gamma)$. If $L \neq xy$, but xy belongs to ς, then already y is collinear with p in $\mathcal{M}(\Gamma)$. Let M be a line of Γ at distance 5 from y and such that $x^{[z]}$ belongs to λ, for every point z on M opposite x (in Γ). Since $L^M_{\bigcirc}$ contains xy (as a fringe), either it coincides with ς — and then p and λ lie in a plane of type [OI] — or it only meets ς in a circle of ${}^x\Gamma$ — and in this case p is collinear with y, but with no other point on λ (in $\mathcal{M}(\Gamma)$). Finally suppose xy does not belong to ς. This time p is certainly not collinear with y in $\mathcal{M}(\Gamma)$. Let M be as above. Then it follows directly from our assumptions that $L^M_{\bigcirc}$ meets ς in a unique circle of some ${}^u\Gamma$ with u incident with L, $u \neq x$, see Proposition 30. So the unique trace mark of λ with trace through u is collinear with p and the others are not. Hence the result.

Finally let λ have type (IB1). So λ is a bundle mark with center x, with, say, base circle D and origin z. Let ς' be the Suzuki trace incident with λ in $\mathcal{M}(\Gamma)$. As above, (BSM) follows easily if $L = xz$. Suppose now that C touches D in, say, M (necessarily $M \neq xz$). Then ς and ς' are collinear in $\mathcal{M}(\Gamma)$. If the unique point w of the line of type (OB2) joining them lies on all traces of λ, then considering a point u at distance 5 from any line of ς not containing x, and at distance 5 from a line of ς' through z, we know by Corollary 33 that $\delta(u, w) = 6$ and it follows easily that p and all points of λ lie in a plane of type [OI]. If w does not have the above mentioned property, then p cannot be collinear with any point of type (I) on λ, because there is only one Suzuki trace with given focus line collinear with a given trace mark of λ and it must be the one with w as first considered (see (BSM) for points of type (I) and lines of type (OB1) in the next paragraph). If D and C do not touch, then there is a unique circle D' with $\partial D = \partial D'$ touching C. And D' belongs to a unique Suzuki trace ς'' collinear with all points of λ (as we just argued). It follows from Proposition 30 that ς and ς'' share exactly one circle in, say, ${}^v\Gamma$ (and $v \neq x$

by assumption). Hence p is collinear with the unique trace mark of λ passing through v and with no other point of λ. This proves (BSM) for points of type (B).

Now let p be of type (I). If λ is a line of type (OB1) — and note that by the remarks in the beginning of the proof we may assume that λ is not incident with x in $\mathcal{M}(\Gamma)$ — then (BSM) is easily checked. Now let λ have type (OB2). So λ is a Suzuki cycle with center x, some origin z and rotation C. Let z' be the unique point of Γ incident with xz (in Γ) and belonging to the trace corresponding with p. The case $z = z'$ is easily checked; the case $z \neq z'$ follows immediately from the fact that any post of p (as a trace mark) defines a unique line of $\mathcal{M}(\Gamma)$ in the plane of type [OB1] spanned by λ and z' (and then use Proposition 32).

Now we show that, if p is a point of type (I), and if β is a plane of type [OB2] (in h) such that p is collinear with a unique point of type (B) of β, then there exists exactly one point q of type (I) in $\mathcal{M}(\Gamma)$ incident with β and collinear with p such that pq is a line of $\mathcal{M}(\Gamma)$ of type (OI). We refer to this claim by (*).

Indeed, let λ be the unique line of type (OB1) in β; let z be the unique point of type (O) on λ; let X be the trace belonging to p. Since p is collinear with exactly one point of type (B) of β, the unique point y of X on the line xz (in Γ) is distinct from z. Let u be a post of p. Let C be the circle of ${}^{z}\Gamma$ which is the gate set in z of the trace belonging to any trace mark incident with β. By definition of the plane β, every point of Γ at distance 3 from some line M (not incident in Γ with z) belonging to some Suzuki trace of a Suzuki cycle γ with origin x, center z and rotation C is the post of some point of type (I) incident with β, and conversely all posts of every point of type (I) incident with β are obtained in that way. Hence the points q we are after are defined by the points of Γ collinear with u and at distance 3 from some line M as above. The only candidates are the points w collinear with u and at distance 5 from the lines of $C \setminus \{xz\}$. But the lines at distance 3 from such points w and meeting the elements of C determine a Suzuki cycle γ' with origin y, center z and rotation C. By Proposition 32 the Suzuki cycles γ and γ' share exactly one Suzuki trace and hence we obtain the result.

Now we show that, if p is a point of type (I), and if β is a plane of type [OB2] (in h) such that p is collinear with the unique point of type (O) of β, then the set of points of β collinear with p (in $\mathcal{M}(\Gamma)$) has size at least 2 and is a subset of the set of points incident with a line in β, or coincides with the set of all points incident with β. The latter case happens if and only if p is incident with β. We refer to this claim by (**).

Let β be defined by a Suzuki cycle γ with origin x, center z and rotation C (i.e. $\beta = \gamma^\sigma$). Then p is a trace mark with corresponding trace X containing z. If the gate set of X through z is exactly C, then either p is incident with β and the result follows, or p is not incident with β (and p has the "wrong" posts) and then p is collinear only to z and all points of type (B) of β. Now suppose that the gate set of X through z is a circle C' of ${}^{z}\Gamma$ touching C in xz (necessarily). Let y be any post set of p (viewed as trace mark).

Clearly, if a point of type (I) of β is collinear with p, then the line joining p with that point is not of type (OI), because such a line must necessarily contain the point z, which is impossible since the gate sets through z are different. Hence we are looking for posts u of trace marks incident with β at distance 2 from y such that the unique point v collinear with both y and u lies at distance 4 from an element of X. The candidates for v are the elements of y^x. The candidates for u are the points collinear with v and at distance 5 from an element of $C \setminus \{xz\}$. If u is such a point, then it qualifies if the line M at distance 3 from u and concurrent with an element of C belongs to a Suzuki trace incident in $\mathcal{M}(\Gamma)$ with γ. Since $\delta(v,z)=6$, this "qualification" is independent of the element of C. Now note that the circle D of ${}^z\Gamma$ with corner the projection of y onto z and containing xz does not touch C in xz, so D meets C is a second line, say N. Hence on the line N there is a unique point x' such that $y^x = y^{x'}$. The points $v \in x^y$ we are after are obtained by taking the points collinear with y and at distance 5 from a line through x' (in Γ) belonging to the Suzuki trace with focus line N which is incident with γ in $\mathcal{M}(\Gamma)$. Consider two such points v and v' and the corresponding points u and u' (self-explaining notation in view of the u defined above). Since $\delta(y,z)=6$, we deduce from Lemma 44 that x^u and $x^{u'}$ are equal or meet trivially. Hence the claim (**).

Now we remark that the residue of the point x in h is a generalized quadrangle. Indeed, this follows from standard arguments and the observation that we proved (BSM) for any pair $\{p,l\}$ such that p is collinear in h with x and λ is a line inside some plane in h containing also x.

Now consider a point p of type (I) in h. It is never collinear to x, viewed as a point of type (O) of $\mathcal{M}(\Gamma)$. By (BSM) — for the cases we have shown above — there is a bijection Θ from the set of lines in h through x to the set of lines in h through p such that corresponding lines meet in a unique point. Let λ and λ' be two lines of h through x and suppose that λ and λ' lie in a plane of h. Let q and q' be the intersections of λ and λ^Θ, and λ' and ${\lambda'}^\Theta$ (respectively). The lines λ^Θ and ${\lambda'}^\Theta$ both meet the line qq', which is of type (OB1) or (OB2), hence by (BSM), also λ^Θ and ${\lambda'}^\Theta$ lie in a plane. Also the converse is true (by a similar argument). Hence Θ is an isomorphism from the residue of x in h to the residue of p in h. We conclude that the residue of $\{p,h\}$ in $\mathcal{M}(\Gamma)$ is a generalized quadrangle.

Now let p be a point of type (B) in h. Consider a point p' of type (I) in h and suppose that p' is not collinear with p in h (this can always be done). We establish again an isomorphism between the residues of these points. Since (BSM) holds for p, we have a morphism which we call Θ again from the set of lines in h through p' to the set of lines in h through p. We now show that Θ is bijective. Let λ be a line through p in h. If λ has type (OB1) or (OB2), then by (BSM), there is a unique line λ' in h through p' meeting λ. Hence we may suppose that λ has type (IB1). It is easily seen that there is a (unique) plane β of type (OB2) containing λ (the point y of type (O) it contains is the unique point of Γ which lies on all traces of the bundle defined by λ and which also lies on the corner of the base circle of the bundle; the Suzuki cycle defining β has the center of the bundle as origin, the point y as center and the gate set through y of the intersection of all traces in the bundle as rotation — it is furthermore defined by the lines at distance 3 from a post of a trace mark of the bundle and at distance 2 from an element of the rotation). By (*) and (**), there is at least one point q of type (I) incident with β and collinear with p'. Since the residue in h of q is a generalized quadrangle, there is a plane β' containing p, q' and a line λ'' of β. The line λ'' meets λ in a unique point ($\lambda \neq \lambda''$ because p is not collinear with p') q' and hence the line $\lambda' = p'q'$ meets λ. So Θ is surjective. To prove injectivity, we keep the same notation. Now we distinguish between cases (*) and (**). In case (**), we know that all points of β collinear with p' are collinear, hence lie on λ''. So q' is unique in this case. Suppose the assumptions of (*) hold. Let q'' be a point on λ collinear with p'. The type of q'' is necessarily (I). Looking at the residue in h of q'', we see that there is a line λ''' in β containing q'' all of whose points are collinear with p'. By (*), this line must contain q because in the plane $p'\lambda'''$ there is a unique line of type (OI) through p' and this must meet β in q. But λ''', λ'' and the unique line of type (OB1) of β must be concurrent since (BSM) holds for the latter and p'. Hence $\lambda'' = \lambda'''$.

So we have shown that Θ is a bijection which preserves incidence. Hence Θ acts on the set of planes in h through p' and is an injection into the set of planes in h through p. To prove that Θ^{-1} preserves incidence, we now show that Θ is also surjective on the set of planes in h through p'. Indeed, let β be a plane through p. If β has type [OO] or [OB1], then it follows from (BSM) that there exists a plane β' through p' meeting β in a line using standard arguments. If β has type [OB2] or [OI], then since Θ is bijective (on the appropriate sets of lines), there exists at least one point q of type (I) in β collinear with p'. Considering the residue of q in h, we obtain the result.

Hence Θ^{-1} also preserves incidence (because incidence is just containment) and so we conclude that the residue in h of p is a generalized quadrangle.

If p is a point of type (O) distinct from x, then using (BSM), one can show easily that there is an isomorphism from the residue of p in h to the residue of any point p' of type (B) in h, where p' is not collinear with p in h.

This completes the proof of the proposition.

Proposition 52 *The residue in $\mathcal{M}(\Gamma)$ of an incident point-hyperline pair, where the hyperline has type* [B]*, is a generalized quadrangle.*

PROOF. Let ς be a Suzuki trace of Γ with focus line L. Then $h = \varsigma^\sigma$ is a hyperline which contains the following points of $\mathcal{M}(\Gamma)$. As points of type (O), it contains the points of Γ incident with L; as points of type (B), it contains the Suzuki traces with an element of ς as focus line and containing the line L; as points of type (I), it contains the trace marks containing a point x of L, whose center lies on an element of ς, the gate set at x of which contains L and whose posts are determined by ς (in the sense of Proposition 21).

Now we show that (BSM) holds in certain cases.

The verification of (BSM) is very straightforward if the point p has type (O). If the point p has type (I) and the line λ contains a point of type (O), then there is only one non-immediate case, which is the following. Let τ be a trace mark (as point of ς^σ) containing the point x of Γ on L and let c be the center of τ. Let γ be a Suzuki cycle with origin y on L and center some point z of L. Then γ is a line of type (OB2) contained in the hyperline ς^σ. We show that (BSM) holds for the point τ and the line γ. By definition of ς^σ, we can choose a post w of τ at distance 3 from the focus line M of some arbitrary Suzuki trace of γ. Let v be any point at distance 4 from both w and c with $v \neq z$. Set $\{u\} = \Gamma_2(w) \cap \Gamma_2(v)$. Let M' be the focus line of any (other) Suzuki trace of γ and set $\{w'\} = \Gamma_3(M') \cap \Gamma_4(v)$ and $\{u'\} = \Gamma_2(w') \cap \Gamma_2(v)$. It is clear that $c^w = c^{w'}$ and hence that w' is a post of τ. Also, $x^u = x^{u'}$ and so from the definition of Suzuki cycles it follows that there is a unique Suzuki cycle γ' with origin x and center z such that the focus lines of the elements of γ' are the same as for γ and such that for every focus line M' the point w' defined above is incident with an element of the corresponding Suzuki trace. By Lemma 31, γ and γ' have exactly one Suzuki trace in common. This Suzuki trace is collinear in $\mathcal{M}(\Gamma)$ with τ because they are both contained in a unique track — which is a line of type (IB2). This completes the proof of this case.

Now assume that p has type (I) and λ contains no point of type (O). We take the same notation as in the previous paragraph for p. If λ has type (IB1) then the proof is easy but non-trivial. Indeed, let μ be a bundle mark with center y and base circle C. We leave the case that x and y are collinear to the reader. So let x and y be not collinear. Suppose first that p is collinear — in $\mathcal{M}(\Gamma)$ — with at least two points of h incident with μ. Hence τ has two points u_1 and u_2 of two different elements of μ as a post, this implies by Lemma 5 that u_1 and u_2 lie on lines through y contained in the same circle of ${}^y\Gamma$ whose corner coincides with the corner of C. So we can assume that u_1 and u_2 are collinear, hence they have to be equal and so every element of μ is collinear with τ and lies in a fixed plane of type [IB2] containing p. It also follows that the unique point of type (B) on λ lies in that plane. Similarly, one shows that, if the unique point of type (B) on λ is collinear with p, then either each other or no other point of λ is collinear with p and in the former case, there is again a plane containing p and all points of λ. Now we show that there is at least one point of λ collinear with p. By definition of our hyperline of type [B], there exists a post u of τ collinear with y. If u lies on an element of C, then p is collinear with the unique element of type (B) on λ. If u does not lie on any element of C, then it is contained in a unique element τ' of μ. By definition, there is a curtain containing τ and τ', hence the result.

Now suppose that λ has type (IB2). So let λ be a track with focus z and Suzuki direction

ς'. The case $x = z$ is easy to handle and so we consider the general case $x \neq z$. Suppose first that p is collinear with the point ς' of $\mathcal{M}(\Gamma)$ and with some other point τ' of λ (where we again adopt the notation τ for p; τ has center c and meets L in x). Let N be the focus line of ς' and put $\{y\} = \Gamma_1(N) \cap \Gamma_1(L)$. The case $y = x$ is again easy, and so we leave that to the reader. We consider the general case $y \neq x$. Let M be a hem of the curtain defined by τ and τ'. Let M_y be the projection of M onto y. The collinearities mentioned imply by Lemma 45 that $N{=}M_y$. Moreover if $\{u_x\} = \Gamma_5(M) \cap \Gamma_2(x)$, $\{u_z\} = \Gamma_5(M) \cap \Gamma_2(z)$ and $\{w\} = \Gamma_3(M) \cap \Gamma_3(N)$, then, by Lemma 4, $w^{u_x} = w^{u_z}$ and there is a line λ' of $\mathcal{M}(\Gamma)$ of type (IB2) through the trace mark $w^{[u_x]} = w^{[u_z]}$ and the Suzuki trace ς (viewed as points of $\mathcal{M}(\Gamma)$). The points ς', τ and τ' all lie in the plane $(\lambda')^\sigma$ of type [IB2] and so the line λ is contained in $(\lambda')^\sigma$ (because it is uniquely determined by ς' and τ'). Hence (BSM) holds in this case.

So we may assume that p is not collinear with ς'. Now note that there is a unique plane β of type [IB2] containing λ (indeed, β must contain the point y and hence it is uniquely determined) and contained in the hyperline ς^σ. The plane β of $\mathcal{M}(\Gamma)$ containes lines of type (OI) (and let λ'' be one of those) and one line of type (OB2), say λ'''. Since these lines contain a point of type (O) (namely y), p is collinear with a point p'' on λ'' (not with all of them; that follows from inspection of the types of planes in ς^σ: none of them can contain all points of the line λ'' (including z !) and the point p) and with a point p''' on λ''' (again not with all of them since otherwise p is also collinear with ς'). As above, p, p'', p''' are inside a unique plane β' of type [IB2]. This plane contains the line $p''p'''$ of $\mathcal{M}(\Gamma)$, but so does β. And in β the line $p''p'''$ meets the line λ in a unique point which is collinear with p (since it lies in β'). Hence the result.

Now let p again be a point of type (O) in h. By Proposition 51 the residue of $\{p^\sigma, h^\sigma\}$ in $\mathcal{M}(\Gamma)$ is a generalized quadrangle, hence also the residue of $\{p, h\}$ is.

Now let p be again a point of type (I). Let p' be any point of type (O) not collinear with p. For any line λ through p in h, there is a unique line λ' of h through p' meeting λ; this follows easily from the above cases in which (BSM) holds. Moreover, the mapping $\Theta : \lambda \mapsto \lambda'$ is a bijection from the set of lines through p to the set of lines through p' (considering only lines in h). This follows also from the above cases for which we checked (BMS) noting that no line of h through p' can be of type (II). Let β be a plane in h through p. For every line in β through p, there is a concurrent line of h through p'; let λ_1 and λ_2 be two distinct lines through p in β and let λ_i' be the unique line through p' meeting λ_i, $i = 1, 2$, in the point, say, q_i. Then by (BSM) there is a plane β' of $\mathcal{M}(\Gamma)$ containing p', q_1, q_2 meeting the plane β in the line q_1q_2. Clearly the plane β' is unique with the property that it meets β in a line and that it contains p'. Conversely, every plane β' through p' can be obtained in such a way (by interchanging the roles of p and p' in the previous argument and noting that no line in β' can have type (II)). Hence Θ defines an isomorphism from the residue of (p, h) to the residue of (p', h). Consequently the residue of (p, h) is a generalized quadrangle.

Finally let p be a point of type (B). Axiom (BSM) is easily checked when the type of λ is not (IB2) or (II). Suppose first that λ has type (II). Let p be a Suzuki trace ς' with focus line M concurrent with L. Let x be the intersection of L and M. The line λ is a curtain with some hem H and rail L. We leave the easy case where the projection M' of H onto x is M to the reader. Suppose now that $M' \neq M$. Let $\{N'\} = \Gamma_2(M') \cap \Gamma_4(H)$ and let ς'' be the Suzuki trace with focus line M' and fringes L and N'. Let z be the unique point of type (O) in $\mathcal{M}(\Gamma)$ incident with the unique line of type (OB2) determined by ς' and ς''. Let z_H be the projection of z onto H, then Lemma 22 implies that the line N of Γ concurrent with M and at distance 5 from z_H is a fringe of ς'. If y is the projection of z_H onto N and $\{x_0\} = \Gamma_2(z) \cap \Gamma_4(z_H)$, then $x_0^{[y]}$ is a trace mark incident in $\mathcal{M}(\Gamma)$ with λ. Hence p is in $\mathcal{M}(\Gamma)$ collinear with at least one point of λ. Collinearity of p with a second point of λ would be in conflict with Corollary 33. Indeed, suppose that p is collinear with a trace mark whose corresponding trace contains the point z' of Γ incident with L, $z \neq z'$. Clearly $z' \neq x$. Let z'_H be the projection of z' onto H and let L'_H be the projection of z' onto z'_H. By assumption $\varsigma' = M_{\bigcirc}^{L'_H}$, hence by Corollary 33 the point z'_H is at distance 6 from z, which implies $z = z'$.

Finally let λ be a line of type (IB2). It is easily seen that there is a unique plane β of type [IB1] incident with λ in $\mathcal{M}(\Gamma)$. Let u be the unique point of type (O) in β, then u is collinear with p in h. We now assume that p is not incident with β, otherwise (BSM) follows. Let λ' be the line of h incident with u and p. Since the residue in h of u is a generalized quadrangle, there is a unique plane β' incident with λ' and meeting β in a line λ''. Clearly $\lambda \neq \lambda''$ (they have different types) and so these two lines meet in a point q. The point q is collinear with p since they both lie in the plane β'. We now show uniqueness of q. Suppose the point q' on λ is collinear with p. By Axiom (BSM) the point u is collinear with all points of the line of $\mathcal{M}(\Gamma)$ joining p and q' and there is a plane β'' containing p, q', u. Clearly β'' meets β in a line; by the uniqueness of β' we have $\beta'' = \beta'$. Hence $q = q'$. This completes the proof of (BSM) in this situation.

Under the assumption that p is a point of type (B) of $\mathcal{M}(\Gamma)$, let p' be any point of type (I) not collinear in h with p. Similarly as above there is a bijection Θ from the set of lines (in h) through p to the set of lines in h through p'. Now let β be a plane of $\mathcal{M}(\Gamma)$ in h through p. We show that there is at least one plane β' through p' meeting β in a line of $\mathcal{M}(\Gamma)$. The plane β contains a line of type (OO) (if β has type [OB1]) or a line of type (IB1) incident with p (if β has type [OB2] or [IB2]), or a line of type (IB2) incident with p (if β has type [IB1] or [IB2]). In all these case one deduces readily (using (BSM)) that p' is collinear with a point q of β with q of type (O) or (I) (since p' is not collinear with p). Let λ be the line of $\mathcal{M}(\Gamma)$ incident with q and p'. Since the residue of q in h is a generalized quadrangle, there is a unique plane β' containing λ and meeting β in line. The claim is proved.

Hence if two lines λ_i, $i = 1, 2$, are incident with p in h and if they are contained in a plane

β, then the two unique corresponding lines λ_i^Θ, $i = 1, 2$, meeting λ_i and incident with p' lie in a plane β' (because of the previous paragraph).

Similarly one shows that, if λ'_i, $i = 1, 2$, are two lines in h through p' lying in a plane β', then the corresponding lines $\lambda_i'^{\Theta^{-1}}$, $i = 1, 2$, meeting λ'_i and incident with p lie in a common plane β. Indeed, this follows from the fact that every plane through p' contains a line of type (OI) or (II) and hence p is collinear with at least one point q of type (O) or (I) in β'. The result is obtained by looking at the residue in h of q, as before.

So Θ is an isomorphism and hence the residue of p in h is isomorphic to the residue of p' in h, which was shown to be a generalized quadrangle.

This completes the proof of the proposition.

Proposition 53 *The residue in $\mathcal{M}(\Gamma)$ of an incident point-hyperline pair, where the hyperline has type* [I]*, is a generalized quadrangle.*

PROOF. Let τ be a trace mark in Γ. Let x be the center of the corresponding trace X and let $\mathcal{C}$ be the post set of τ. Then the point x — as a point of type (O) in $\mathcal{M}(\Gamma)$ — is incident with the hyperline $h = \tau^\sigma$. Let Y be any crown trace of τ. Then Y supplied with posts collinear to elements of $\mathcal{C}$ defines a unique trace mark which is incident with h. Let $z \in X$ and let L be any line in the gate set of X, $L \neq xz$. Then the posts of τ define a unique Suzuki trace ς with focus line L by considering the lines at distance 2 from L and 3 from posts which are at distance 5 from L. The definition of h implies that ς is — as a point of type (B) — incident with h. Note that, if ς' is the Suzuki trace incident with h with focus line L', where L' is also incident with z in Γ, then ς, ς' and x are three points of $\mathcal{M}(\Gamma)$ lying on the same line of type (OB2). Finally let u be any post of τ. Then the trace $u^{[x]}$ is also — as a point of type (I) — incident with h.

Let p be a point of $\mathcal{M}(\Gamma)$ in h. We have to show again that the residue of the pair $\{p, h\}$ is a generalized quadrangle. If p has type (O) or (B), then by Propositions 51 and 52 the residue of the pair $\{h^\sigma, p^\sigma\}$ is a generalized quadrangle. Since σ is an incidence preserving involution, the result follows.

Hence we may assume that p has type (I). First let p be not collinear with x in h, i.e. the trace belonging to p does not contain x, which is equivalent with saying that x is a post of p. It is very easy and straight forward to check (BSM) for the point x and a line of any type. Also, (BSM) — which is equivalent with (BS) in this case — is readily verified for the point p and any line in h incident with the point x in $\mathcal{M}(\Gamma)$. This already implies that there is a bijective correspondence Θ from the set of lines in h through x to the set of lines in h through x which takes collinear elements in the residues of p to collinear elements in the residue of x in h (and two lines correspond if and only if they meet in a unique point). Hence Θ extends to an injective map from the set of planes in h through p

to the set of planes in h through x. If we show that it is also surjective onto the latter set of planes, then Θ is an isomorphism and the result for p follows. To prove surjectivity, let β be any plane of h through x. The type of β is either [OI2] or [IB1]. In both cases, there is a unique line λ of type (OB2) in β incident with x. Let λ' be the unique line through p in h for which ${\lambda'}^{\Theta} = \lambda$. Then the common point q of λ and λ' has type (B). Since the residue in h of q is a generalized quadrangle, there is a plane β' through λ' meeting β in a line through q. Clearly ${\beta'}^{\Theta} = \beta$, whence the result.

Now let p be collinear with x in h. Then p is a trace mark with trace Y which has center $y \in X$. Let L be an element of the gate set of Y distinct from xy through x. Let M be any line of Γ through x contained in the circle of ${}^{x}\Gamma$ with corner L containing xy, $L \neq M \neq xy$. Let F, $F \neq M$, be any element of the gate set of X through z, where z is the unique point on M contained in X. Let ς be the unique Suzuki trace with focus line F incident in $\mathcal{M}(\Gamma)$ with h. As a point of h, we denote ς by p'. By our conditions on L, M and xy, we see that p and p' are not collinear in h. Our first aim is to show that for every line λ' in h through p', there exists a unique line λ of h through p meeting λ' (in a point of h). This is obvious if λ' has type (OB2) because in this case x is the unique point on λ' collinear with p in h. Also, if λ' is contained in a plane of h containing x, then the assertion follows easily. Indeed, let β' be the plane generated by λ' and x, then there is a unique plane β containing xp (a line of type (OI) in h) and meeting β' in a line λ'' through x (because the residue in h of x is a generalized quadrangle). Now λ' and λ'' meet in a point q and since q is incident with β, q is collinear with p. If q' is some other point on λ' collinear with p, then since (BSM) holds for x, there is a plane β'' containing x, p, q'. This plane meets β' in the line xq' and contains the line xp, contradicting the fact that the residue in h of x is a generalized quadrangle.

Of course, a similar argument shows that, whenever a line λ through p in h has all its points collinear with x (in $\mathcal{M}(\Gamma)$), then there exists a unique line λ' in h through p' meeting λ.

Now suppose that λ' is a line through p' for which only p' is collinear with x. Since only the trace marks with post x are not collinear with x in h, λ' has necessarily type (IB2). Let u be the point at distance 3 from F contained in every trace belonging to some trace mark incident with λ' in $\mathcal{M}(\Gamma)$. We are looking for a line L^* containing a point of Y and containing a point of a trace belonging to some trace mark incident with λ'. Clearly, if L^* exists, then $\delta(L^*, u) = 5$ and $\delta(L^*, y) = 3$. Hence $L^* \in y^{u}_{[3]}$. Let B be the circle of ${}^{u}\Gamma$ containing the back up of the set of all centers of the track defined by λ'. Then L^* should also be at distance 4 from an element of B. By Lemma 6, this is equivalent with L^* meeting an element of the gate set A of X through y in a point of Y. Since xy is not the corner of the circle of ${}^{x}\Gamma$ containing L, M, xy, the traces Y and y^u meet in a set of points the back up of which onto y is a circle A' of ${}^{y}\Gamma$ meeting A in at least one line, namely xy. But xy is the corner of A and not of A', so A and A' have exactly one other line L_y in common. Put $\{L^*\} = \Gamma_2(L_y) \cap \Gamma_5(u)$ and it now follows easily that L^* is the unique line meeting our requirements. Let τ' be the trace mark of λ' whose center c lies at distance 3 from L^*. Since x is a post of τ', the projection of u onto L^* is a point of τ'. Moreover, c^x is the trace belonging to τ'. By Lemma 46 (putting $w_1 = c$, $w_2 = y$, $L = L^*$, $y = x$ and $\{x\} = \Gamma_2(c) \cap \Gamma_5(L)$ where the left hand sides are the notations used in the statement of the lemma, and the right hand sides the corresponding notation in the present proof), p and τ' (as points of type (I)) are collinear.

Now we show that, conversely, for every line λ of h incident with p, there exists a unique line λ' of h incident with p' meeting λ (in h). By the foregoing, we only have to deal with the case that λ is incident with points not collinear with x, i.e. λ has type (II). Let R be the rail of the curtain η corresponding with λ. Clearly we can choose a hem H concurrent with L. Since L is the corner of the circle of ${}^x\Gamma$ through L, M, xy, there exists a point v by Axiom [RT4] at distance 3 from R and at distance 5 from both H and F and by Lemma 13, v is unique with that property. But clearly these conditions on v are necessary and sufficient for $v^{[x]}$ to be a trace mark of the curtain η (expressed by $\delta(v, R) = 3$ and $\delta(v, H) = 5$) and for $v^{[x]}$ to be collinear with p' (expressed by $\delta(v, F) = 5$ and hence some element of v^x is at distance 3 from F). This shows our assertion.

So we have shown that "meeting" is a bijective correspondence between the set of lines of h through p and those through p'. Call this correspondence Θ. Suppose now β is a plane in h through p. Let $\mathcal{A}$ be the set of lines in β through p. Either all points of β are collinear with x in h, or $\mathcal{A}$ contains an element of type (II). In the former case, β contains an element of type (OI). In both cases, there is a point q in β of type (I) or (O) collinear with p' and if q has type (I), then it is not collinear with x in h. Hence the residue in h of q is a generalized quadrangle and there exists a unique plane β' through $p'q$ meeting β in a line of h. By the uniqueness of the line through p' meeting a given line through p, all lines through p' meeting the elements of $\mathcal{A}$ lie in β. By interchanging the roles of p and p' in this argument, we see that Θ extends bijectively to the set of planes through p and p' in h. Hence we conclude that the residue in h of p is isomorphic to the one of p', which is a generalized quadrangle.

This concludes the proof of the proposition.

Now we can prove:

Proposition 54 *The space $\mathcal{M}(\Gamma)$, furnished with points, lines, planes and hyperlines as above, is the flag complex of a building $\Delta(\mathcal{M}(\Gamma))$ of type F_4. The polarity σ is a type permuting automorphism of order 2 of $\Delta(\mathcal{M}(\Gamma))$. The chambers of $\Delta(\mathcal{M}(\Gamma))$ fixed under σ form a building of type $I_2^{(8)}$, i.e. the building associated with a generalized octagon, and this generalized octagon is isomorphic to Γ provided incident point-hyperline pairs are called points, and incident line-plane pairs are called lines.*

PROOF. In order to show the first part of the statement, it suffices to prove by TITS [22], Proposition 9, the following properties.

(LL) If two lines are both incident with two distinct points, they coincide;

(LH) if a line and a hyperline are both incident to two distinct points, they are incident;

(HH) if two distinct hyperlines are both incident with two distinct points, the latter are collinear;

(OT) if two lines are incident with the same set of points, they coincide.

Of course, (OT) follows from (LL) which in turn follows from the definition of lines and from the various properties we have shown in Section 3. Also (LH) is easily checked. That leaves (HH). From the explicit description of the hyperlines in terms of the points they contain (in the proofs of Propositions 51, 52 and 53), it is tedious but straightforward and easy to verify that the respective sets of points incident with two respective hyperlines can only meet in the set of points incident with a plane, in a single point, or they are disjoint (one must bear in mind that in fact hyperlines are defined as unions of certain planes). Alternatively, one can determine the possible pairs of opposite points in the respective hyperlines and notice that no pair appears twice. In view of the description of hyperlines in terms of the points it is incident with, this is again an easy but tiresome exercise. So we may conclude that $\mathcal{M}(\Gamma)$, furnished with points, lines, planes and hyperlines is the flag complex of a building $\Delta(\mathcal{M}(\Gamma))$ of type F_4.

It is obvious that if σ fixes a chamber, then the point of the chamber has to be a point of type (O) (since these are the only points p for which p^σ contains p) and the line of that chamber must be a line of type (OO) (similar reason). The proposition now follows easily.

6 End of the Proof of the "if"-part of the Main Result

We can now end the proof of the fact that Γ is a perfect Ree-Tits octagon.

Lemma 55 *The residue of an incident point-hyperline pair in $\mathcal{M}(\Gamma)$ is a symplectic generalized quadrangle over some perfect field of characteristic* 2.

PROOF. It suffices to show the result for one specific residue. We therefore consider a point x of Γ — as a point of $\mathcal{M}(\Gamma)$ — and the hyperline x^σ. The residue of $\{x, x^\sigma\}$ is a generalized quadrangle Δ. Let $\Omega = {}^x\Gamma$ be the block geometry in x of Γ. Let us call points, respectively lines, of Δ the lines, respectively planes, of $\mathcal{M}(\Gamma)$ through x in x^σ. Then points of Δ are either lines of Γ through x, or Suzuki bundles with focus flag (x, L) for some line L of Γ incident with x. We can identify the former with the points of Ω and the latter with the circles of Ω (identifying a Suzuki bundle with its foundation). The lines of Δ are either planes of type [OO] of the form L^σ, where L is a line of type (OO) and hence also a line of Γ through x; or planes of type [OB1] of the form ζ^σ, where ζ is a Suzuki bundle with focus flag (x, L), for some line L of Γ through x. In the former case, we can identify these lines of Δ with the points of Ω and in the latter case we can identify these lines of Δ with circles (the foundation of ζ again) of Ω. So if a is a point Ω we can either view a as a point of Δ — and we denote the corresponding point as a_p — or as a line of Δ — in which case we write a_ℓ. Similarly one defines the notations C_p and C_ℓ for any circle C of Ω. It is easy to verify that, for a, b points of Ω and C, D circles of Ω, and denoting incidence in Δ by $\mathbb{I}$, we have

$$a_p \mathbb{I} b_\ell \Longleftrightarrow a = b,$$

$$a_p \mathbb{I} D_\ell \Longleftrightarrow a = \partial D,$$

$$C_p \mathbb{I} b_\ell \Longleftrightarrow b = \partial C,$$

$$C_p \mathbb{I} D_\ell \Longleftrightarrow \partial C \in D \text{ and } \partial D \in C.$$

From this, it follows immediately that, denoting collinearity in Δ by $\perp$,

$$a_p \perp b_p \Longleftrightarrow a = b,$$

$$a_p \perp C_p \Longleftrightarrow a \in C,$$

$$C_p \perp D_p \Longleftrightarrow C \text{ touches } D$$

(using Axiom [ST2] or, equivalently, result [MC3]). We now show that any point a_p, a a point of Ω, is distance-2-regular. Let C, D be circles of Ω and suppose that C_p and D_p are collinear to x_p, but not mutually collinear. Then they do not touch and hence they have a unique other point b in common, $b \neq a$. Suppose E is a circle of Ω such that $C_p, D_p \in a_p^{E_p}$. Then there are two circles C, D containing a and b and touching E, hence all circles through a, b touch E. Axiom [MP2] now readily implies that $a_p^{E_p} = a_p^{b_p}$. Since Δ has the Moufang property — as a residue of a spherical rank 4 building — all points of Δ are distance-2-regular. Clearly the sets $a_p^{b_p}$ and $a_p^{c_p}$, for distinct points a, b, c of Ω, share the unique point C_p, where C is the unique circle of Ω through a, b, c. Hence by Schroth [14], Δ is a symplectic quadrangle over some commutative field $\mathbb{K}$. Clearly σ induces a polarity in Δ, hence $\mathbb{K}$ is a perfect field of characteristic 2 (if $\mathbb{K}$ were not perfect, then the symplectic quadrangle Δ would not be isomorphic to its dual, see Van Maldeghem [29](7.3.2), which restates an unpublished result of Tits).

This completes the proof of the lemma.

Theorem 56 *The generalized octagon Γ is a perfect Ree-Tits octagon.*

PROOF. Let $(x_0, L_0, x_1, L_1, \ldots, x_7, L_7)$ be a circuit of length 8 in Γ, where L_i is incident with x_i and x_{i+1} (subscipts to be taken modulo 8). Let τ_i, $0 \leq i < 8$ be the trace mark with center x_i and post x_{i+4} (again with subscripts taken modulo 8). Finally let ς_i, $o \leq i < 8$, be the Suzuki trace $(L_i)_{\bigcirc}^{L_{i+4}}$ (subscripts modulo 8). The twenty-four points x_i, τ_i, ς_i, $i = 0, \ldots, 7$, are the points of an apartment $\mathcal{A}$ of the building $\mathcal{M}(\Gamma)$.

The polarity σ induces in the residue of $\{x_0, x_0^\sigma\}$ a polarity of a symplectic generalized quadrangle. By TITS [17], this polarity defines a unique field automorphism θ whose square is the Frobenius. Let σ' be a polarity of $\mathcal{M}(\Gamma)$ for which the set of absolute points and lines constitutes a Ree-Tits octagon Γ', such that x_i, $i = 1, \ldots, 7$ are points of Γ', and L_i, $i = 0, \ldots, 7$ are lines of Γ', such that $x_i^{\sigma'}$ and $L_i^{\sigma'}$ belong to $\mathcal{A}$, for all $i \in \{0, \ldots, 7\}$ and such that the underlying field automorphism is precisely θ (this is possible by TITS [23]). It then follows from BOREL & TITS [1] that σ can be written as $t \cdot \sigma'$, where t belongs to the subgroup of the full automorphism group of $\mathcal{M}(\Gamma)$ generated by the torus T corresponding to $\mathcal{A}$ and the automorphisms fixing $\mathcal{A}$ pointwise and being defined by field automorphisms. Looking at the residue of $\{x_0, x_0^\sigma\}$, one sees that the restrictions of σ and σ' to that residue are conjugate (in the full automorphism group of the residue) by TITS [17] (because they define the same field automorphism). Hence we may assume that no field automorphism is involved in t.

Now, the torus T is 4-dimensional and we can write every element in the form $(k_1, k_2, k_3, k_4) \in (\mathbb{K}^\times)^4$. Since the tori related to Γ' are two-dimensional, we may assume that $(k_1, k_2, 1, 1)^{\sigma'} = (k_1, k_2, 1, 1)$, for all $k_1, k_2 \in \mathbb{K}^\times$. We can also put $(1, 1, k_3, 1)^{\sigma'} = (1, 1, 1, k_3)$, for all $k_3 \in \mathbb{K}^\times$. Put $t = (k_1, k_2, k_3, k_4)$. Since $\sigma = t \cdot \sigma'$ is an involution, we see that $t \cdot t^{\sigma'} = 1$, hence $t = (1, 1, k_3, k_3^{-1})$. Putting $t' = (1, 1, 1, k_3)$, we have $(t')^{-1} \cdot (t')^{\sigma'} = t$ and so we obtain $\sigma = t \cdot \sigma' = ((t')^{-1} \cdot \sigma' \cdot t' \cdot \sigma) \cdot \sigma = (\sigma')^{t'}$. Hence σ and σ' are conjugate in the full automorphism group of $\mathcal{M}(\Gamma)$. This now easily implies that Γ and Γ' are isomorphic. The theorem is proved.

7 Proof of the "only if"-part of the Main Result

7.1 A model of the Ree-Tits octagons

By TITS [23], every Ree-Tits octagon is determined by a field $\mathbb{K}$ of characteristic 2 and an endomorphism σ in $\mathbb{K}$ whose square is the Frobenius endomorphism $x \mapsto x^2$. This generalized octagon will be denoted by $O(\mathbb{K}, \sigma)$.

The following description of $O(\mathbb{K}, \sigma)$, $\mathbb{K}$ and σ as above, is due to JOSWIG & VAN MALDEGHEM [8]. We write $x^{\sigma+j} := x^\sigma x^j$, $j = 1, 2$.

Let $\mathbb{K}_\sigma^{(2)}$ be the group on the set of all pairs $(k_0, k_1) \in \mathbb{K} \times \mathbb{K}$ with operation law $(k_0, k_1) \oplus (l_0, l_1) = (k_0 + l_0, k_1 + l_1 + l_0 k_0^\sigma)$. For $k = (k_0, k_1)$, set $tr(k) = k_0^{\sigma+1} + k_1$ (the *trace* of

k) and set $N(k) = k_0^{\sigma+2} + k_0 k_1 + k_1^\sigma$ (the *norm* of k). Define a multiplication $a \otimes k = a \otimes (k_0, k_1) = (ak_0, a^{\sigma+1} k_1)$. Also write $(k_0, k_1)^\sigma$ for (k_0^σ, k_1^σ). Then the points of $O(\mathbb{K}, \sigma)$ are the elements of

$$
\begin{array}{l}
\{(\infty)\} \\
\cup \mathbb{K} \\
\cup \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \\
\cup \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \\
\cup \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \\
\cup \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \\
\cup \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \\
\cup \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \times \mathbb{K}
\end{array}
$$

(and these are all denoted by round parentheses); the lines of $O(\mathbb{K}, \sigma)$ are the elements of

$$
\begin{array}{l}
\{[\infty]\} \\
\cup \mathbb{K}_\sigma^{(2)} \\
\cup \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \\
\cup \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \\
\cup \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \\
\cup \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \\
\cup \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \\
\cup \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \times \mathbb{K}_\sigma^{(2)} \times \mathbb{K} \times \mathbb{K}_\sigma^{(2)}
\end{array}
$$

(and denoted by square brackets); incidence is given by the sequence

$$
\begin{gathered}
(a, l, a', l', a'', l'', a''')\ I\ [a, l, a', l', a'', l'']\ I\ (a, l, a', l', a'')\ I\ \ldots \\
\ldots (a)\ I\ [\infty]\ I\ (\infty)\ I\ [k]\ I\ (k, b)\ I\ \ldots \\
\ldots [k, b, k', b', k'']\ I\ (k, b, k', b', k'', b'')\ I\ [k, b, k', b', k'', b'', k'''],
\end{gathered}
$$

and the rule : $(a, l, a', l', a'', l'', a''')$ is incident with $[k, b, k', b', k'', b'', k''']$ if and only if the following six equations hold:

$$(k_0''', k_1''') = (l_0, l_1) \oplus a \otimes (k_0, k_1) \oplus (0, al_0' + a^\sigma l_0'') \qquad (I1)$$

$$\begin{aligned} b'' &= a' + a^{\sigma+1}N(k) + k_0(al_0' + a^\sigma l_0'' + tr(l)) \\ &\quad + a^\sigma(a''' + l_0k_1) + al_0''^\sigma + l_0l_0' \end{aligned} \qquad (I2)$$

$$\begin{aligned} (k_0'', k_1'') &= a^\sigma \otimes (k_1, tr(k)N(k)) \oplus k_0 \otimes (l_0, l_1)^\sigma \\ &\quad \oplus (0, tr(k)N(l) + a^{\sigma+1}l_0N(k)^\sigma + tr(k)(aa' + a^\sigma l_0 l_0'' + a^{\sigma+1}a''') \\ &\quad + tr(l)(k_1^\sigma a + a''') + k_1^\sigma a^{\sigma+1}l_0'' + k_0^{\sigma+1}a^2 l_0''^\sigma \\ &\quad + k_0(a' + al_0''^\sigma + k_1 a^\sigma l_0 + a^\sigma a''')^\sigma \\ &\quad + k_0^\sigma l_0(a' + al_0''^\sigma + k_1a^\sigma l_0 + a^\sigma a''') \\ &\quad + a(l_1'' + a'''^\sigma l_0 + a'''l_0') + l_0''(a' + a^\sigma a''') + a''l_0 + l_0l_0'l_0'') \\ &\quad \oplus (l_0', l_1') \end{aligned} \qquad (I3)$$

$$\begin{aligned} b' &= a'' + a^{\sigma+1}N(k)^\sigma + a(k_0l_0'' + l_0k_1 + a''')^\sigma + tr(k)(l_1 + a^\sigma l_0'') \\ &\quad + k_0^\sigma(a' + a^\sigma a''') + l_0'l_0'' + l_0^\sigma a''' \end{aligned} \qquad (I4)$$

$$\begin{aligned} (k_0', k_1') &= (l_0'', l_1'') \oplus a \otimes (tr(k), k_0N(k)^\sigma) \oplus l_0 \otimes (k_0, k_1)^\sigma \\ &\quad \oplus (0, N(k)(a^\sigma l_0'' + l_1) + k_0(a'' + l_0'l_0'' + aa'''^\sigma + l_0^\sigma a''') \\ &\quad + k_1(k_1l_0a^\sigma + a' + al_0''^\sigma + a^\sigma a''') + k_0k_1^\sigma al_0^\sigma + a'''^\sigma l_0 + a'''l_0') \end{aligned} \qquad (I5)$$

$$b = a''' + aN(k) + l_0k_1 + k_0l_0'' \qquad (I6)$$

where $a, a', a'', a''', b, b', b'' \in \mathbb{K}$ and $k, k', k'', k''', l, l', l'' \in \mathbb{K}_\sigma^{(2)}$ and $k = (k_0, k_1)$, etc. These elements are called *the coordinates*.

This description is valid for σ bijective or not. But we will only use it in the perfect case (σ a bijection).

7.2 The proof

In VAN MALDEGHEM [26] it is shown that perfect Ree-Tits octagons satisfy Axioms [RT1], [RT1′] and the part of [RT5] not between parentheses, see *loc. cit.*, Lemma 4.4 (and the discussion preceding it). Also, by *loc. cit.*, the discussion preceding Lemma 4.6, and Lemma 4.7., one easily deduces Axiom [RT2]

Let Γ be a perfect Ree-Tits octagon described using coordinates as in the previous section. From *loc. cit.*, we recall the following result. Let o be the point with coordinates $(0,0,0,0,0,0,0)$. Then the set of points v of Γ such that the back up of $(\infty)^o \cap (\infty)^v$ contains the circle of ${}^{(\infty)}\Gamma$ through $[0]$ with corner $[\infty]$ is equal to

$$\mathcal{V} = \{(0, (0, l_1), a', l', a'', l'', 0) : l_1, a', a'' \in \mathbb{K}; l', l'' \in \mathbb{K}_\sigma^{(2)}\}.$$

Without loss of generality, the point y of Axiom [RT3] can be chosen to be equal to $(0,0)$, granted we put $x = (\infty)$, $u = o$ and $z = (0)$. By *loc. cit.*, Section 4.1, the circle of ${}^{(0,b)}\Gamma$ with corner $[0]$ and containing $[0,b,0]$ consists of $\{[0,b,(0,k_1')] : b,k_1' \in \mathbb{K}\} \cup \{[0]\}$. Putting $b=0$, [RT3] is proved if we show that, whenever $v \in \mathcal{V}$ and $\Gamma_1((0,0)) \cap \Gamma_5(v) = \{[0,0,(0,k_1'0]\}$, for some $k_1' \in \mathbb{K}$, then necessarily $(\infty)^o = (\infty)^v$. So we let $v \in \mathcal{V}$ and $\Gamma_1((0,0)) \cap \Gamma_5(v) = \{[0,0,(0,k_1'0]\}$, for some $k_1' \in \mathbb{K}$. Equation $(I5)$ implies that $v = (0,(0,l_1),a',l',a'',(0,k_1'),0)$. The result now follows directly from Equation $(I6)$, noting that $(\infty)^o$ consists of those points with two coordinates the second coordinate of which is 0.

Next we consider Axiom [RT4]. Put $x = (\infty)$, $y = o$, $L = [0,0,0,0]$, $z = (0)$, $z' = (0,0)$ and $z'' = (k,0)$, with $k \in \mathbb{K}_\sigma^{(2)}$ and $k_0 \neq 0$. We can do all this without loss of generality by the transitivity properties of the automorphism group of Γ, see Tits [23]. Also, we may assume that $L' = [0,0,0]$ and $L'' = [k,0,(0,k_1')]$ (similarly as above). If y' exists, then it has coordinates of the form $(0,0,0,0,a'',l'',a''')$, $a'',a''' \in \mathbb{K}$, $l'' \in \mathbb{K}_\sigma^{(2)}$. Distance 5 from L' readily implies $l'' = 0$ and $a''' = 0$. From Equation $(I5)$, we deduce that the line M'' at distance 5 from $(0,0,0,0,a'',0,0)$ and incident with z'' has coordinates $[k,0,(0,k_0a'')]$. Putting $a'' = k_1'k_0^{-1}$, we obtain the result. Now let y'' be any point at distance 3 from L, opposite (∞) and at distance 6 from o. Then y'' has coordinates of the form $(0,0,0,0,a'',l'',a''')$, $a'',a''' \in \mathbb{K}$, $a'' \neq 0$, $l'' \in \mathbb{K}_\sigma^{(2)}$. Expressing that $(\infty)^o = (\infty)^{y''}$, one obtains easily, using Equation $(I6)$, that $a''' = l_0'' = 0$. Since the line at distance 2 from the line $[k]$, $k \in \mathbb{K}_\sigma^{(2)}$ and at distance 5 from o has coordinates $[k,0,0]$, we are looking for solutions of the equation

$$0 = l_1'' + k_0 a'',$$

for given $l_1'', a'' \in \mathbb{K}$ with $a'' \neq 0$, and unknown k_0 (this follows from Equation $(I5)$). This has clearly a solution for $k = (k_0,k_1)$ and hence Axiom [RT4] is shown.

8 Proof of the Main Result — Finite Case

In this section we suppose that Γ is a finite generalized octagon of order (s,t) satisfying the Axioms [RT1f], [RT1′] and [RT3].

For the first three lemmas, we suppose that $t = s^2$.

Let Ω be an inversive plane, i.e. a geometry satisfying Axioms [MP1] and [MP2]. If a is a point of Ω, then the geometry Ω_a with point set the set of points of Ω except for a, and line set all circles of Ω containing a, is an affine plane. Usually this is called the *internal affine plane in* a of Ω. If we take for Ω the block geometry ${}^x\Gamma$ for some point x of Γ, then Ω contains s^2+1 points — since there are s^2+1 lines of Γ incident with x. Each circle of Ω contains $s+1$ points.

Lemma 57 *The generalized octagon* Γ *satisfies Axiom* [RT1]. *Also, s is an odd power of 2.*

PROOF. By Feit & Higman [6], the number $2ss^2$ is a perfect square, hence s is even. By Axiom [RT1f], ${}^x\Gamma$ is a finite inversive plane of even order, for all points x of Γ. Hence s is a power of 2 by Dembowski [5]. Since $2s^3$ is a perfect square, s must be an odd power of 2. A circle of ${}^x\Gamma$ not containing some point L of ${}^x\Gamma$ is an oval in the internal affine plane in L of ${}^x\Gamma$ and its nucleus is an affine point (otherwise the line "at infinity" is a tangent and hence the oval has only s affine points). This implies Axioms [CH1] and [CH2]. The lemma is proved.

Lemma 58 *Let x be a point of Γ, let X be a trace with center x and let z (respectively z') be a point of X (respectively of* $(\Gamma_2(x) \setminus X) \setminus \Gamma_2(z)$*). Then there exists a unique trace with center x containing z' and meeting X trivially in z.*

PROOF. Let X, x, z, z' be as in the statement of the lemma. Let y be such that $X = x^y$. Let X' be a trace with center x containing both z and z' and suppose that $X \cap X' = \{z\}$. Let $\{L\} = \Gamma_1(y) \cap \Gamma_5(z)$ and let $\{y'\} = \Gamma_1(L) \cap \Gamma_6(z')$. We must show that $X' = x^{y'}$. Suppose by way of contradiction that $X' \neq x^{y'}$. Then $|X' \cap x^{y'}| = s+1$ by Axiom [RT1f]. Consider the set $W = \{x^w : w \in \Gamma_2(y) \cap \Gamma_6(z)\}$ of s traces with center x containing z and pairwise meeting trivially in z. For $u \in X'$, $u \neq z$, the trace x^v, with $\{v\} = \Gamma_1(L) \cap \Gamma_6(u)$, belongs to W, hence the set $\{(Y \cap X') \setminus \{z\} : Y \in W\}$ is a partition of $X' \setminus \{z\}$, with possibly some empty classes. By Axiom [RT1f], each non-empty class contains s elements (since s^2 is impossible because there is at least one class with s elements, namely $(X' \cap x^{y'}) \setminus \{z\}$). It follows that s classes contain s elements, hence also $(X \cap X') \setminus \{z\}$ contains s elements, a contradiction to our assumption. The lemma is proved.

A set of traces $\{x^y : y \in \Gamma_1(L) \setminus \Gamma_6(x)\}$, for any line L at distance 7 from the point x, is called a *pencil of traces with origin u*, where $\{u\} = \Gamma_2(x) \cap \Gamma_5(L)$. It is clear that every trace is contained in a pencil of traces with origin any of its elements.

Lemma 59 *The generalized octagon* Γ *satisfies Axiom* [RT2].

PROOF. Let x be a point of Γ, let X be a trace with center x, let z_1, z_2 be two distinct points of X and let Z_i be a trace with center x meeting X trivially in z_i, $i = 1, 2$. By the discussion preceding this lemma, X and Z_1 belong to a pencil $\mathcal{T}$ of traces with base point z_1. Each element of $\mathcal{T}$ meets Z_2 either in a point or in a set of $s+1$ points. Suppose k elements meet in a point, then, since $|\mathcal{T}| = s$, we have $k + (s+1)(s-k) = s^2$. Hence $k = 1$ and so X is the unique element of $\mathcal{T}$ meeting Z_2 trivially. So we obtain a *flock* of the inversive plane ${}^x\Gamma$ by considering back ups onto x of these intersections. By Thas

[15], this flock is linear, which means that it is unique with respect to the two elements xz_1 and xz_2 which are not covered. So this flock is a transversal partition with extremities xz_1 and xz_2. Hence the lemma.

Now we suppose that Γ has order (s,t), with s and t arbitrary, and that Γ satisfies Axioms [RT1f], [RT1$'$], [RT2f] and [RT3]. We show that $t = s^2$, where (s,t) is the order of Γ.

Lemma 60 *If the finite generalized octagon* Γ *of order* (s,t) *satisfies Axioms* [RT1f], [RT1$'$], [RT2f] *and* [RT3], *then* $t = s^2$ *and consequently* Γ *satisfies Axioms* [RT1] *and* [RT2].

PROOF. Let x be a point of Γ, let X be a trace with center x, let z_1, z_2 be two distinct points of X and let Z_1 be any trace through z_1 meeting X trivially. Let $\mathcal{T}$ be a pencil of traces with origin z_2. Then $|\mathcal{T}| = s$ and every element of $\mathcal{T}$ meets Z_1 in a set of $\sqrt{t}+1$ points except for X, by Axiom [RT2f]. Hence the set of $t-1$ points of $Z_1 \setminus \{z_1, z_2\}$ is partitioned into $s-1$ sets of $\sqrt{t}+1$ elements. So $s = \sqrt{t}$.

The second assertion follows from Lemma 57 and Lemma 59. The lemma is completely proved.

Finally we show that Axiom [RT4] is a consequence of the others (still in the finite case).

Lemma 61 *If a finite generalized octagon* Γ *of order* (s,s^2) *satisfies Axioms* [RT1], [RT1$'$], [RT2] *and* [RT3], *then it also satisfies Axiom* [RT4].

PROOF. Note that Section 3 of this paper does not use Axiom [RT4], hence we can use some of the properties shown there.

Let x and y be two opposite points of Γ and $L \in \Gamma_5(x) \cap \Gamma_3(y)$. Let $\{z\} = \Gamma_2(x) \cap \Gamma_3(L)$. Let $z', z'' \in x^y$ be such that the lines xz, xz', xz'' do not lie in a circle of ${}^x\Gamma$ with corner xz. Let L' and L'' be any lines of the gate set of x^y through z' and z'' respectively. Let u be a variable point on L, $u \notin \Gamma_4(x)$, and let y_u be the point at distance 2 from u and 5 from L'. We have $x^{y_u} = x^y$ by Lemma 4 of *loc. cit.* Let $\{L''_u\} = \Gamma_1(z'') \cap \Gamma_5(y_u)$. By Lemma 13 of *loc. cit.*, the mapping $u \mapsto L''_u$ is injective. But u runs over a set of s elements and L''_u runs over a set of $\sqrt{t} = s$ elements (the gate set in z'' of x^y is a circle of ${}^{z''}\Gamma$). Hence this mapping is bijective and L'' is an image. This shows the first part of Axiom [RT4].

Now, let y'' be opposite x, such that $x^{y''} = x^y$, $\delta(y, y'') = 6$ and $\delta(y'', L) = 3$. Let y'' be collinear with u'' on L. Then $y''u''$ is a line which lies at distance 4 from some element of the gate set of x^y through z' (except for xz'), and y'' is unique on $y''u''$ with respect to the property $x^y = x^{y''}$. Hence there are exactly $s^2 - s$ choices for y''. But $s-1$ of these choices lead to $L_0 \in x^y_{[3]} \cap x^{y''}_{[3]}$ for $L_0 \in \Gamma_3(x) \cap \Gamma_5(y)$. Let z_0 be collinear with x and incident with L_0. By Lemma 13 of *loc. cit.*, the $s-1$ points y'' obtained by considering L_0 differ from the $s-1$ obtained by considering L_0' if and only if xz_0 and xz_0' (self-explanatory notation) are contained in a circle of ${}^x\Gamma$ with corner xz. Since there are s such circles, we have in total $s(s-1)$ points y'' for which $|x_{[3]}^y \cap x_{[3]}^{y''}| > 1$. This proves the assertion completely.

Puttting all lemmas of this section together, the Main Result – Finite Case readily follows.

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