On certain twin buildings over tree diagrams

Bernhard Mühlherr * Hendrik Van Maldeghem†

Abstract

We classify twin buildings over tree diagrams such that all rank 2 residues are either finite Moufang polygons but no octagons, or Moufang polygons associated to the groups PGL_3, PSp_4 or G_2. As a byproduct, we obtain some results on Moufang sets.

1 Introduction

Twin buildings generalize spherical buildings in a natural way and it is presumed in [12] that the methods used in the classification of the spherical buildings can be applied to classify the 2-spherical (all m_{ij} are finite) twin buildings as well. In *loc. cit.* it is shown that the foundation of a 2-spherical twin building (i.e. its local structure) is Moufang (see Section 6.1. in *loc. cit.*) and it is conjectured that almost all Moufang foundations can be realized as local structures of twin buildings (see Conjecture 2 in *loc. cit.*).

The purpose of this note is to classify all finite Moufang foundations over 'tree diagrams' and to verify the validity of the conjecture mentioned above for those foundations which have no octagons as rank 2 residues.

Using [10], [12] and [5] one obtains the following theorem as a consequence of our results:

*The first author is supported by the Deutsche Forschungsgemeinschaft

†The second author is a Research Director of the Fund for Scientific Research FWO – Flanders (Belgium)

Received by the editors September 1997.

Communicated by Francis Buekenhout.

1991 *Mathematics Subject Classification.* 51E24.

Key words and phrases. Moufang set, Moufang foundation, Steinberg twist, Kac-Moody twin buildings, 2-spherical twin buildings.

Main Theorem. *Let M be a Coxeter diagram over a finite set I satisfying $m_{ij} < 8$ whose underlying graph is a connected tree and let $\mathcal{B}$ be a twin building of type M such that $\mathcal{B}$ has no rank 2 residue associated to* $\mathrm{PSp}_4(2)$*,* $\mathrm{G}_2(2)$ *or* $\mathrm{G}_2(3)$*.*

Suppose $\mathcal{B}$ satisfies one of the following:

(i) *There exists a field k such that each rank 2 residue is associated to* $\mathrm{PGL}_2(k) \times \mathrm{PGL}_2(k)$*,* $\mathrm{PGL}_3(k)$*,* $\mathrm{PSp}_4(k)$ *or* $\mathrm{G}_2(k)$*.*

(ii) *All rank 1 residues are finite.*

Then $\mathcal{B}$ is known.

In order to establish the uniqueness in parts (i) and (ii), one uses [12] and [5].

The twin buildings arising in part (i) are precisely those which correspond to Kac-Moody groups of adjoint type over k, which are associated to the appropriate generalized Cartan matrices. The existence of these twin buildings follows from [10] and [12].

The buildings arising in part (ii) can be obtained by twisting twin buildings which are associated to Kac-Moody groups over finite fields. The existence of these twin buildings will be established in Section 5.

The classification of the finite Moufang foundations over 'tree diagrams' relies on the fact that each finite Moufang polygon satisfies a certain property (Ind) (see Proposition 2); the fact that each finite polygon and each polygon associated to a split algebraic group over any field has the property (Ind) is proved in Section 2. In order to construct buildings over tree diagrams whose local structure is isomorphic to a given finite Moufang foundation we extend the theory of Steinberg twists to certain Kac-Moody groups over fields (see Section 4); starting with a Moufang foundation $\mathcal{F}$ over a tree diagram we will construct a twin building $\mathcal{B}$ and an automorphism γ of $\mathcal{B}$ such that the fixed point structure of γ in $\mathcal{B}$ is a twin building whose local structure is isomorphic to $\mathcal{F}$.

Remark. If the Coxeter diagram M is simply laced, then our main result is already an immediate consequence of [12], Section 6.5 and [5].

As described above, the present paper can be seen as a contribution to the classification program which is outlined in [12]. Most of the concepts occuring in this note are already contained in *loc. cit.*, to which we refer the reader for the notation and definitions used here and for further information about twin buildings.

2 Moufang sets induced from Moufang polygons

We adopt the definitions from [12] 4.4:

A *Moufang set* is a system $(X, (U_x)_{x \in X})$ consisting of a set X and a family of groups of permutations of X indexed by X itself and satisfying the following conditions:

(MoS1) U_x fixes x and is simply transitive on $X \setminus \{x\}$.

(MoS2) In the full permutation group of X, each U_x normalizes the set of subgroups $\{U_y \mid y \in X\}$.

The point x in (MoS1) is called the *center* of U_x. The group U_x shall be called a *root group.* And the elements of U_x are often called *root elations*.

Let $(X, (U_x)_{x \in X}), (Y, (V_y)_{y \in Y})$ be Moufang sets. A mapping $\alpha : X \to Y$ is an isomorphism between the Moufang sets $(X, (U_x)_{x \in X})$ and $(Y, (V_y)_{y \in Y})$, if the following conditions are satisfied:

(i) α is a bijection from X onto Y,

(ii) For each $x \in X$ the mapping $\pi \to \alpha \circ \pi \circ \alpha^{-1}$ defines an isomorphism from U_x onto $U_{\alpha(x)}$.

Let $n \geq 3$, let Γ be a generalized Moufang n-gon and let $\mathcal{B}$ be the associated building of rank 2. Let P be a panel of $\mathcal{B}$, let x be a chamber in P and let ϕ be a root containing x such that P is on the boundary of ϕ. Then the root group U_ϕ fixes x and it acts regularly on $P \setminus \{x\}$; it is a fact that this action does not depend on the choice of the root ϕ. This shows in particular that the panels of a Moufang polygon carry an induced structure of a Moufang set. We denote that Moufang set by $\mathcal{U}_P$.

Throughout this note we will be interested in Moufang polygons Γ which have the following property:

(Ind) The full automorphism group of Γ induces on each panel the full group of automorphisms of $\mathcal{U}_P$.

It seems that there are very few Moufang polygons which satisfy (Ind). This will be made clear in the remainder of this section. Our main goal is to prove the following result.

Proposition 1 (i) *Let k be a field and let Γ be the polygon associated to the group* $\mathrm{PGL}_3(k)$, $\mathrm{PSp}_4(k)$ *or* $\mathrm{G}_2(k)$. *Then Γ has the property* (Ind).

(ii) *Let Γ be a finite Moufang polygon. Then Γ has the property* (Ind).

This will follow from lemmas 1 and 3 below. However, in order to provide a more solid reference for these things, we will prove more results than we will use in this paper. For instance, we note that Moufang polygons which are not projective planes have two kinds of panels. For some Moufang polygons, (Ind) is satisfied only for one kind of panels (see Lemma 4).

Since the remainder of this section deals with generalized polygons, we adopt some of the notation of the theory of these objects. Let Γ be a generalized n-gon, $n \geq 3$. For a point or a line P, we denote by $\Gamma(P)$ the set of elements of Γ incident with P. An *axial* collineation of Γ is an automorphism of Γ fixing all points and lines at distance $\leq n/2$ from some fixed line L, which is called the *axis* of the collineation.

Lemma 1 *Let Γ be a finite Moufang polygon. Then Γ has property* (Ind).

Let P be a panel (a point or a line) of any finite Moufang polygon Γ. Let α be an automorphism of $\mathcal{U}_P$. Then, by definition, α induces by conjugation in the full symmetric group of $\Gamma(P)$ an automorphism of the group generated by the root elations. All these groups are well-known: they are $\mathrm{PSL}_2(q)$, $\mathrm{PSU}_3(q)$ or $\mathrm{Sz}(q)$, for appropriate prime powers q. The automorphism groups are respectively the groups $\mathrm{PGL}_2(q)$, $\mathrm{PGU}_3(q)$ and $\mathrm{Sz}(q)$ extended by the field automorphisms of respectively $\mathrm{GF}(q)$, $\mathrm{GF}(q^2)$ and $\mathrm{GF}(q)$. But these are also the stabilizers, modulo the kernel, of $\Gamma(P)$ in the full group of automorphisms of Γ. The lemma is proved. ∎

Lemma 2 *A Moufang projective plane* Γ *satisfies* (Ind) *if and only if it is a desarguesian plane over a skew field* k *admitting no anti-automorphism. In particular, all pappian planes satisfy* (Ind)*; no non-desarguesian Moufang plane satisfies* (Ind).

Proof. First let Γ be a desarguesian projective plane coordinatized by the skew field k, i.e., $\Gamma = \mathrm{PG}(2,k)$ can be defined — up to duality — as follows. The set of points is the set of classes of triples $(x,y,z)k^\times = \{(xr,yr,zr) : r \in k^\times\}$, where $k^\times = k \setminus \{0\}$; the set of lines likewise consists of the classes of triples $k^\times[A,B,C]$; a point $(x,y,z)k^\times$ is incident with a line $k^\times[A,B,C]$ if and only if $Ax+By+Cz=0$. The points of the line $L := k^\times[0,0,1]$ form a Moufang set $\mathcal{U}_L$ with respect to the translations (elations) induced on L. It is well known (and, in fact, easily calculated) that a generic element of the root group with center $(1,0,0)k^\times$ can be written as

$$\phi_X : \Gamma(L) \to \Gamma(L) : \left\{ \begin{array}{lcl} (x,1,0)k^\times & \mapsto & (x+X,1,0)k^\times, \\ (1,0,0)k^\times & \mapsto & (1,0,0)k^\times, \end{array} \right.$$

with $X \in k$. Similarly, one can write a generic element of the root group with center $(0,1,0)k^\times$ as

$$\theta_Y : \Gamma(L) \to \Gamma(L) : \left\{ \begin{array}{rcll} (x,1,0)k^\times & \mapsto & ((x^{-1}+Y^{-1})^{-1},1,0)k^\times, & x \neq -Y \\ (-Y,1,0)k^\times & \mapsto & (1,0,0)k^\times & \\ (1,0,0)k^\times & \mapsto & (Y,1,0)k^\times, & \end{array} \right.$$

with $Y \in k^\times$ (the identity is obtained by symbolically putting $Y = \infty$ with $\infty^{-1} = 0$ and $(\infty,1,0) = (1,0,0)$). Note that the same root groups are obtained by the opposite skew field k^{opp}. The group $\mathrm{PGL}_2(k)$, which acts on $\Gamma(L)$ as a subgroup of $\mathrm{PGL}_3(k)$ (the group of *projective linear transformations* of Γ), is of course an automorphism group of $\mathcal{U}_L$. Note that here the group $\mathrm{PGL}_2(k)$ is the group of all similarities of the projective line $\Gamma(L)$; a general element is of the form

$$\Gamma(L) \to \Gamma(L) : (x,y,0)k^\times \mapsto (ax+b, cy+d, 0)k^\times,$$

with $a,b,c,d \in k$, $ac^{-1} - bd^{-1} \neq 0$. This group clearly acts triply transitively on $\Gamma(L)$. Hence, if σ is an arbitrary automorphism of $\mathcal{U}_L$, we may assume that it fixes the points $(1,0,0)k^\times$, $(0,1,0)k^\times$ and $(1,1,0)k^\times$. Consequently σ stabilizes the root groups with center $(1,0,0)k^\times$ and $(0,1,0)k^\times$, respectively, and we can view σ as a permutation of the skew field k via the identification $x \mapsto (x,1,0)$ (by abuse of language, we denote that permutation also by σ). This readily implies that ϕ_X is mapped by conjugation onto $\phi_{\sigma(X)}$; hence $\sigma(a+X) = \sigma(a)+\sigma(X)$ and σ is additive. Also, θ_1 is mapped by conjugation onto itself, hence $\sigma((1+x^{-1})^{-1}) = (1+(\sigma(x))^{-1})^{-1}$. By a result of Hua [1], this implies that σ is either an automorphism or an anti-automorphism. Conversely, every automorphism or anti-automorphism of k clearly gives rise to an automorphism of $\mathcal{U}_L$. Hence we have determined all automorphisms of $\mathcal{U}_L$. In fact, this result is well-known, see Tits [8], 8.10.

Now we show that the automorphisms of $\mathcal{U}_L$ which correspond to proper anti-automorphisms of k (i.e., anti-automorphisms which are not automorphisms) are not induced by Γ. Indeed, let Ψ be a collineation of Γ stabilizing L, $(1,0,0)k^\times$, $(0,1,0)k^\times$ and $(1,1,0)k^\times$ and such that $\Psi((x,y,0)k^\times) = (\sigma(x),\sigma(y),0)k^\times$, with σ a proper anti-automorphism of k. By composing with a suitable axial collineation of Γ (with axis L), we may assume that Ψ fixes the points $(0,0,1)k^\times$ and $(1,1,1)k^\times$. Now, Ψ takes the point $(1,y,0)k^\times$, $y \in k^\times$, to the point $(y^{-\sigma},1,0)k^\times$. Using the projection from the point $(0,-1,1)k^\times$ onto the line $k^\times[0,1,0]$, we deduce that Ψ maps $(1,0,z)k^\times$ to $(\sigma(z^{-1}),0,1)k^\times$, $z \in k^\times$. The intersection of the lines through $(0,0,1)k^\times$ and $(1,y,0)k^\times$, and through $(0,1,0)k^\times$ and $(1,0,z)k^\times$, respectively, is the point $(1,y,z)k^\times$. Applying Ψ, we see that $(1,y,z)k^\times$ is mapped onto $(1,\sigma(y),\sigma(z))k^\times$. Similarly, $(y^{-1},1,zy^{-1})k^\times$ is mapped onto

$$(\sigma(y^{-1}),1,\sigma(zy^{-1}))k^\times = (\sigma(y)^{-1},1,\sigma(y^{-1}\sigma(z))k^\times = (1,\sigma(y),\sigma(yzy^{-1}))k^\times,$$

which implies that $zy = yz$, for all $z, y \in k^\times$, a contradiction (since this means that σ is an automorphism). This shows the lemma for desarguesian planes.

Similarly, one shows that, if an alternative division ring contains an anti-automorphism, then the corresponding projective plane does not satisfy property (Ind). But every alternative division ring admits an anti-automorphism (its "standard involution"), hence the lemma is completely proved. ■

Any Moufang set isomorphic to $\mathcal{U}_L$ in the above proof will be called a *projective line over* k. In the remainder, we will only use this notion for k a commutative field.

Lemma 3 *Let* Γ *be a Moufang polygon associated to the symplectic group* $\mathrm{PSp}_4(k)$*, Dickson's group* $\mathrm{G}_2(k)$*, or* $\mathrm{PGL}_3(k)$*, for some commutative field* k*, then* Γ *has property* (Ind)*.*

Proof. The assertion has been proved for $\mathrm{PGL}_3(k)$ in the previous lemma. So we may assume that Γ is the symplectic quadrangle (case $\mathrm{PSp}_4(k)$), or the split Cayley hexagon (case $\mathrm{G}_2(k)$). Let P be a panel of Γ. Then $\mathcal{U}_P$ is isomorphic to a projective line over a (commutative) field k. Let G be the automorphism group of $\mathcal{U}_P$ inherited from $\mathrm{Aut}\,\Gamma$. First we claim that G acts triply transitively on $\Gamma(p)$. Indeed, this follows immediately from [14], 4.5.10(*ii*) ($\mathrm{PSp}_4(k)$) and 4.5.11 ($\mathrm{G}_2(k)$). Now we remark that every field automorphism of k induces a collineation in Γ via its standard embedding in $\mathrm{PG}(3,k)$ (symplectic quadrangle) or in $\mathrm{PG}(6,k)$ (split Cayley hexagon). Indeed, for instance the split Cayley hexagon has a representation in $\mathrm{PG}(6,k)$ such that its points are all points of the quadric with equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$ and its lines are all lines whose grassmannian coordinates satisfy six linear equations with coefficients 0, 1 or -1 (see [6]). Hence applying a field automorphism to the coordinates in $\mathrm{PG}(6,k)$ induces a collineation of the hexagon. Such a collineation can be chosen to stabilize P and at least three chambers containing p. It follows that the full automorphism group of $\mathcal{U}_P$ is induced by Γ.

The lemma is proved. ■

Lemma 4 *Let Γ be the Moufang octagon associated to the Ree group ${}^2\mathrm{F}_4(k,\tau)$, with k a perfect field of characteristic 2 in which the Frobenius automorphism has the square root τ. Let P be a panel for which $\mathcal{U}_P$ is induced by multiple roots (or equivalently, for which the group generated by the root elations is a Suzuki group over k). Then the automorphism group of $\mathcal{U}_P$ is induced by* $\mathrm{Aut}\,\Gamma$.

Proof. The root groups of Moufang set $\mathcal{U}_P$ correspond to the normal regular subgroups of the parabolic subgroups of the Suzuki group $\mathrm{Sz}(k,\tau)$ over k and with respect to τ, and we may identify $\Gamma(P)$ with a Suzuki-Tits ovoid in $\mathrm{PG}(3,k)$. Let Ψ be an automorphism of $\mathcal{U}_P$. By the double transitivity, we may assume that Ψ fixes two given points x and y of the Suzuki-Tits ovoid. By [9], we may identify the elements of $\Gamma(P)\setminus\{y\}$ with the pairs $(a,b)\in k^2$ (identifying x with $(0,0)$) and the root group with center y defines an operation $(a,b)(c,d)=(a+c,b+d+c\tau(a))$ (this is the image of (c,d) under the root elation with center y mapping $x=(0,0)$ to (a,b)). Now Ψ acts by conjugation as an automorphism of the root group with center y. Hence it stabilizes the set of elements of order 2. This set clearly consists of all root elations mapping x to $(0,b)$, $b\in k$. This implies that Ψ, as a permutation of the Suzuki-Tits ovoid (viewed as an inversive plane) $\Gamma(P)$, stabilizes the block through x and y with *corner* x (with the terminology of [13]; this corner is called *le "nœud"* in Tits [7]). Consequently, every automorphism of $\mathcal{U}_p$ stabilizes the corresponding Suzuki-Tits ovoid as an STI-plane in the sense of *loc. cit.*). Since by *loc. cit.*, the symplectic quadrangle over k, and consequently the projective space $\mathrm{PG}(3,k)$, can be defined only in terms of the elements, the geometric structure implied by the corners, and the incidence relation of the corresponding Suzuki-Tits ovoid, Ψ induces an automorphism of $\mathrm{PG}(3,k)$. Hence Ψ acts as a projective semi-linear transformation of $\mathrm{PG}(3,k)$. If Ψ acts as a projective linear transformation, then it is generated by root elements (it belongs to $\mathrm{Sz}(k,\tau)$) and consequently induced by Γ. If it is not a projective linear transformation, then, it is induced by a field automorphism σ (which centralizes τ). Considering a suitabe coordinate respresentation of Γ, see e.g. [2], we see that Ψ is induced by letting σ act on the coordinates of the elements, hence Ψ is induced by Γ. ∎

The previous lemmas are the most general things that can be proved on condition (Ind). For other Moufang polygons, the specific properties of the underlying skew field will be important, in particular, the structure of the automorphism and duality groups of this skew field. For instance, if Γ is the Moufang octagon associated to the Ree group ${}^2\mathrm{F}_4(k,\tau)$, and P is a panel such that $\mathcal{U}_P$ is the projective line over k, then every field automorphism of k, viewed as an automorphism of $\mathcal{U}_P$, and which is induced by Γ, must commute with τ. Hence if there is some field automorphism which does not commute with τ, then Γ does not satisfy (Ind); if every field automorphism commutes with τ, then Γ does satisfy (Ind).

3 Moufang foundations

Let I be a set and let M be a Coxeter diagram over I; for each subset J of I we let M_J denote the restriction of M onto the set J and I' denotes the set of all subsets of I which have cardinality 2.

A *foundation of type* M is a triple $((\mathcal{B}_J)_{J\in I'},(c_J)_{J\in I'},(\theta_{jik})_{j\neq i\neq k\in I})$ such that

(i) $\mathcal{B}_J$ is a building of type M_J and c_J is a chamber of $\mathcal{B}_J$ for each $J \in I'$.

(ii) θ_{jik} is a bijection from the i-panel of $c_{\{i,j\}}$ onto the i-panel of $c_{\{i,k\}}$ which maps $c_{\{i,j\}}$ on $c_{\{i,k\}}$ and $\theta_{kil} \circ \theta_{jik} = \theta_{jil}$ for all $i, j, k, l \in I$ satisfying $i \notin \{j, k, l\}$.

Suppose now that M is 2-spherical (i.e. all m_{ij} are finite) and that M has no isolated nodes. A foundation $\mathcal{F} = ((\mathcal{B}_J)_{J \in I'}, (c_J)_{J \in I'}, (\theta_{jik})_{j \neq i \neq k \in I})$ will be called a *Moufang foundation* if the following holds:

(MoF1) If $J \in I'$ is such that M_J is irreducible, then $\mathcal{B}_J$ is a Moufang building.

(MoF2) If $j \neq i \neq k$ are elements of I such that $M_{\{i,j\}}$ and $M_{\{i,k\}}$ are irreducible, then θ_{jik} is an isomorphism between the Moufang sets which are induced from $\mathcal{B}_{\{i,j\}}$ and $\mathcal{B}_{\{i,k\}}$ respectively.

The following proposition is obtained by an easy induction on the cardinality of I:

Proposition 2 *Suppose that M is a 2-spherical Coxeter diagram over a set I whose graph is a tree and let $\mathcal{F} = ((\mathcal{B}_J)_{J \in I'}, (c_J)_{J \in I'}, (\theta_{ijk})_{j \neq i \neq k \in I})$ be a Moufang foundation of type M. If $\mathcal{B}_J$ satisfies the condition* (Ind) *for all $J \in I'$ for which M_J is irreducible, then $\mathcal{F}$ is uniquely determined by $(\mathcal{B}_J)_{J \in I'}$.*

Let $\mathcal{B}$ be a building of type M and let c be a chamber of $\mathcal{B}$. Then the union of all rank 2 residues of $\mathcal{B}$ which contain c provides a foundation $\mathcal{F}_c$ in a canonical way. A foundation $\mathcal{F}$ of type M will be called *integrated in the building* $\mathcal{B}$ if there is a chamber c in $\mathcal{B}$ such that $\mathcal{F}$ is isomorphic to $\mathcal{F}_c$. A foundation $\mathcal{F}$ of type M will be called *integrated* if there exists a twin building $\mathcal{B}$ such that $\mathcal{F}$ is integrated in a 'half' of $\mathcal{B}$; in this case we will also say that $\mathcal{F}$ is integrated in the twin building $\mathcal{B}$.

4 Steinberg twists of Kac-Moody twin buildings

We start with a general remark about Moufang twin buildings: Let I be a finite set of cardinality at least 2 and let M be a Coxeter diagram over I having no isolated nodes. Suppose that $\mathcal{B}$ is a twin building of type M which satisfies the Moufang property. Then the root groups are uniquely determined in $\mathrm{Aut}(\mathcal{B})$ by the combinatorial structure of $\mathcal{B}$; in particular: the group $G(\mathcal{B})$ generated by all root groups is a normal subgroup of $\mathrm{Aut}(\mathcal{B})$. Hence each automorphism γ of $\mathcal{B}$ induces an automorphism γ^* on $G(\mathcal{B})$.

In the following we let I be a set and $A = (A_{ij})_{i,j \in I}$ be a generalized Cartan matrix over I such that $A_{ij}A_{ji} \leq 3$ for any two distinct $i, j \in I$ and such that the associated Coxeter diagram M over I has no irreducible component of rank smaller than 3. Given a field k, then we let $\mathcal{B}_A(k)$ denote the twin building associated to Kac-Moody group of adjoint type over k which corresponds to A (cf. [10], [12]). The building $\mathcal{B}_A(k)$ is Moufang and the root groups are all isomorphic to the additive group of k (which will be denoted by k^+ in the following).

Proposition 3 *Let k be a field of cardinality at least 4 and let $\mathcal{B} = \mathcal{B}_A(k)$. Let Σ be a twin apartment of $\mathcal{B}$, let c be a chamber contained in Σ; for $i \in I$ let α_i denote the root of Σ which contains c and which has the i-panel of c on its boundary and let U_i denote the corresponding root group.*

(*i*) *There exists a system of isomorphisms* $(x_i : k^+ \to U_i)_{i\in I}$ *with the following property: If* π *is a permutation of* I *such that* $A_{i^\pi j^\pi} = A_{ij}$ *for all* $i, j \in I$ *and if* σ *is an automorphism of* k, *then there exists a unique automorphism* γ *of* $\mathcal{B}$ *fixing* c *and stabilizing* Σ *such that* $(x_i(t))^{\gamma^*} = x_{i^\pi}(t^\sigma)$ *for all* $i \in I, t \in k$.

(*ii*) *Let* $(x_i)_{i\in I}, \pi, \sigma$ *and* γ *be as above, let* $\tilde{I}$ *be the set of orbits of* $\langle\pi\rangle$ *in* I *and suppose that the restriction of* M *onto each orbit is spherical. Then* $\tilde{\mathcal{B}} := C_{\mathcal{B}}(\gamma)$ *is a thick Moufang twin building.*

Note about the proof. The local structure of the twin building $\mathcal{B}_A(k)$ is obtained from the commutation relations in the group $\mathbf{St}_A(k)$ (as defined in [10]). This provides the isomorphisms $(x_i)_{i\in I}$. Now one applies the main result of [5] (here the condition $|k| \geq 4$ comes in!) in order to construct the automorphism γ with the required properties; its uniqueness follows by the Rigidity Theorem for twin buildings (Théorème 1 in [11]).

Part (*ii*) of the previous proposition is well-known in the spherical case and a proof of it can be extracted — for instance — from [4]. The arguments generalize to the situation of twin buildings without much change.

5 Finite Moufang foundations over tree diagrams

In the following Z_l denotes the cyclic group of order l for each natural number l.

Let M be a Coxeter diagram over a set I satisfying $m_{ij} < 8$ and such that its graph is a connected tree. Let $\mathcal{F} = ((\mathcal{B}_J)_{J\in I'}, (c_J)_{J\in I'}, (\theta_{jik})_{j\neq i\neq k\in I})$ be a finite Moufang foundation of type M. For each $i \in I$ we have a Moufang set $\mathcal{M}_i = (X_i, (U_x)_{x\in X_i})$ and the condition $m_{ij} \neq 8$ implies that there is a prime p_i and a natural number k_i such that the Moufang set $\mathcal{M}_i$ is associated to a group $\mathrm{PSL}_2(p_i^{k_i})$ or $\mathrm{PSU}_3(p_i^{k_i/2})$. As the graph of M is connected it follows that $p_i = p_j$ for all $i, j \in I$; in the following we denote this prime by p.

From basic facts about finite Moufang polygons and Proposition 2 it follows that the foundation $\mathcal{F}$ is uniquely determined by the following data:

(*i*) the Moufang sets $\mathcal{M}_i$;

(*ii*) the tree underlying M (i.e., all $(i, j) \in I \times I$ such that $m_{ij} > 2$);

(*ii*) the Dynkin diagram (in other words: the Cartan matrix) of $\mathcal{B}_{\{ij\}}$ for all distinct $i, j \in I$ which satisfy $|X_i| = |X_j|$.

Given these data, we put $K = \mathrm{lcm}\{k_i \mid i \in I\}$ and $G = Z_K$. For $i \in I$ we set $l_i = K/k_i$ and $G_i = G/Z_{l_i}$. Let $\bar{I}$ denote the set of all pairs (i, C) consisting of an element $i \in I$ and a coset $C \in G_i$. We are going to define now a generalized Cartan matrix over the set $\bar{I}$:
We put $A_{\bar{i},\bar{i}} = 2$ for all $\bar{i} \in \bar{I}$.
If $i, j \in I$ are such that $m_{ij} = 2$, then we put $A_{(i,C)(j,D)} = A_{(j,D)(i,C)} = 0$ for all $C \in G_i, D \in G_j$.
If $i, j \in I$ are such that $m_{ij} > 2$ and $|X_i| = |X_j|$ then we set $A_{(i,C)(j,D)} = A_{ij}$ if $C = D$ and $A_{(i,C)(j,D)} = 0$ in the remaining cases.

If $i, j \in I$ are such that $m_{ij} > 2$ and $|X_i| > |X_j|$ then we set $A_{(i,C)(j,D)} = A_{(j,D)(i,C)} = -1$ if C is contained in D and we set $A_{(i,C)(j,D)} = A_{(j,D)(i,C)} = 0$ in the remaining cases.
If $\mathcal{M}_i$ is associated to $L_2(p^{k_i})$ then we put $A_{(i,C)(i,D)} = A_{(j,D)(i,C)} = 0$ if $C \neq D$.
If $\mathcal{M}_i$ is associated to $U_3(p^{k_i/2})$ then we put $A_{(i,C)(i,D)} = A_{(j,D)(i,C)} = -1$ if $C^{-1}D$ has order 2 in G_i; we put $A_{(i,C)(i,D)} = A_{(j,D)(i,C)} = 0$ if the order of $C^{-1}D$ is bigger than 2.

The group Z_K has a permutation representation $\Pi : Z_K \to \mathrm{Sym}(\bar{I})$ via $(i, C)^{\Pi(z)} := (i, Cz)$ for all $z \in Z_K$; it is easily verified that $A_{\bar{i}^{\Pi(z)}\bar{j}^{\Pi(z)}} = A_{\bar{i}\bar{j}}$ for all $\bar{i}, \bar{j} \in \bar{I}, z \in Z_K$. In the following let z_o be a generator of Z_K and set $\pi = \Pi(z_0)$.

Let k be the Galois field $\mathrm{GF}(p^K)$ of order p^K, let $\sigma \in \mathrm{Aut}(k)$ be the Frobenius automorphism of k and let $\bar{\mathcal{B}} = \mathcal{B}_A(k)$. Suppose first that $|k| \leq 3$, then it is readily verified that the foundation $\mathcal{F}$ is integrated in $\bar{\mathcal{B}}$.

Suppose now that $|k| \geq 4$: Let $\Sigma, c, (\alpha_{\bar{i}})_{\bar{i}\in\bar{I}}$ be as in Proposition 3 and let $(x_{\bar{i}} : k^+ \to U_{\alpha_{\bar{i}}})_{\bar{i}\in\bar{I}}$ be as asserted in part (i) of Proposition 3; By part (i) of Proposition 3 we obtain an automorphism γ of $\bar{\mathcal{B}}$ which is associated to the pair (π, σ); We put $\mathcal{B} := C_{\bar{\mathcal{B}}}(\gamma)$. The set of orbits of π in $\bar{I}$ can be identified with the set I. Let $\bar{M}$ be the Coxeter diagram over $\bar{I}$ which is associated to the generalized Cartan matrix A; by considering the action of γ on $\bar{M}$ it follows from [3] that the Coxeter diagram associated to $\mathcal{B}$ is precisely M.

Let $i \neq j \in I$. In order to determine the $\{i, j\}$-residue of c in $\mathcal{B}$ we have to determine the set of fixed points in the $i \cup j$-residue of c in $\bar{\mathcal{B}}$. It is straightforward to check, that this fixed point set is isomorphic to $\mathcal{B}_{\{i,j\}}$.

We illustrate this at an example: Let $i \neq j \in I$ be such that $\mathcal{B}_{\{i,j\}}$ is a generalized hexagon associated to a group ${}^3\mathrm{D}_4$ and let us assume that $k_j = 3k_i$. Let $\bar{\mathcal{B}}_{ij}$ be the $i \cup j$-residue of $\bar{\mathcal{B}}$ which contains c. Then the Coxeter diagram of $\bar{\mathcal{B}}_{ij}$ is the product of k_i copies of the diagram D_4 and $\bar{\mathcal{B}}_{ij}$ is associated to the direct product of k_i copies of the Chevalley group $\mathrm{D}_4(p^K)$. The fixed point structure of γ^{k_j} in $\bar{\mathcal{B}}_{ij}$ — call it $\mathcal{B}'_{ij}$ — is the building associated to the direct product of k_i copies of the Chevalley group $\mathrm{D}_4(p^{k_j})$. Now γ^{k_i} acts on $\mathcal{B}'_{ij}$ as an automorphism of order 3 and its fixed point structure — call it $\mathcal{B}''_{ij}$ — is isomorphic to the building associated to the product of k_i copies of the group ${}^3\mathrm{D}_4(p^{k_i})$. Finally, γ acts on $\mathcal{B}''_{ij}$ as an automorphism of order k_i which permutes the components of its Coxeter diagram regularly. Thus the fixed point structure of γ in the building $\bar{\mathcal{B}}_{ij}$ is isomorphic to $\mathcal{B}_{\{i,j\}}$.

We obtain the following proposition:

Proposition 4 *Let M be a Coxeter matrix satisfying $m_{ij} < 8$ for all $i, j \in I$ and whose underlying graph is a tree. Then each finite Moufang foundation of type M is integrated in a twin building.*

References

[1] L.-K. Hua. On the automorphisms of a sfield. *Proc. Nat. Acad. Sci. USA*, 35:386 – 389, 1949.

[2] M. Joswig and H. Van Maldeghem. An essay on the Ree octagons. *J. Alg. Combin.*, 4:145 – 164, 1995.

[3] B. Mühlherr. Coxeter groups in Coxeter groups. In F. De Clerck et al., editors, *Finite Geometries and Combinatorics*, volume 191 of *London Math. Soc. Lecture Note Ser.*, pages 277 – 287. Deinze 1992, Cambridge University Press, 1993.

[4] B. Mühlherr. *Some Contributions to the Theory of Buildings Based on the Gate Property.* PhD thesis, University of Tübingen, 1994.

[5] B. Mühlherr and M. Ronan. Local to global structure in twin buildings. *Invent. Math.*, 122:71 – 81, 1995.

[6] J. Tits. Sur la trialité et certains groupes qui s'en déduisent. *Inst. Hautes Etudes Sci. Publ. Math.*, 2:13 – 60, 1959.

[7] J. Tits. Ovoïdes et groupes de Suzuki. *Arch. Math.*, 13:187 – 198, 1962.

[8] J. Tits. *Buildings of Spherical Type and Finite BN-pairs*, volume 386 of *Lecture Notes in Math.* Springer, Berlin – Heidelberg – New York, 1974.

[9] J. Tits. Moufang octagons and the Ree groups of type ${}^2F_4$. *Amer. J. Math.*, 105:539 – 594, 1983.

[10] J. Tits. Uniqueness and presentations of Kac-Moody groups over fields. *J. Algebra*, 105:542 – 573, 1987.

[11] J. Tits. Résumé de cours. *Annuaire du Collège de France*, 89e année:81 – 95, 1988-89.

[12] J. Tits. Twin buildings and groups of Kac-Moody type. In M. W. Liebeck and J. Saxl, editors, *Groups, Combinatorics and Geometry*, volume 165 of *London Math. Soc. Lecture Note Ser.*, pages 249 – 286. Durham 1990, Cambridge University Press, 1992.

[13] H. Van Maldeghem. A geometric characterization of the perfect Suzuki-Tits ovoids. *J. Geom.*, 58:192 – 202, 1997.

[14] H. Van Maldeghem. *Generalized Polygons.* Birkhäuser, Basel, to appear.

Bernhard Mühlherr
Fachbereich Mathematik, Lehrstuhl II
Universität Dortmund
D–44221 Dortmund
e-mail: bernhard.muehlherr@mathematik.uni-dortmund.de

Hendrik Van Maldeghem
Vakgroep voor Zuivere Wiskunde en Computeralgebra
Universiteit Gent
Galglaan 2,
B–9000 Gent
e-mail: hvm@cage.rug.ac.be