Translation ovoids of generalized quadrangles and hexagons.

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Abstract

We define the notion of a translation ovoid in the classical generalized quadrangles and hexagons of order q, and we enumerate all known examples; translation spreads are defined dually. A modification of the known ovoids in the generalized hexagon $H(q)$, $q = 3^{2h+1}$, yields new ovoids of that hexagon. Dualizing and projecting along reguli, we obtain an alternative construction of the Roman ovoids due to Thas & Payne [21]. Also, we construct a new translation spread in $H(q)$ for any $q \equiv 1 \bmod 3$, q odd, with the property that any projection along reguli yields the classical ovoid in the generalized quadrangle $Q(4, q)$; finally, we prove that for q odd, the new example is the only non-hermitian translation spread in $H(q)$ with the property that any projection along reguli yields the classical ovoid in $Q(4, q)$.

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1 Introduction

A finite generalized n-gon is a point-line geometry whose incidence graph has diameter n and girth $2n$. A generalized 3-gon is a (generalized) projective plane; generalized 4-gons

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and 6-gons are usually called *(generalized) quadrangles and hexagons* respectively. These notions were introduced by TITS [22]. If for a generalized n-gon Γ there exist positive integers s, t such that all lines are incident with $s+1$ points and all points are incident with $t+1$ lines, then we say that Γ *has order* (s,t). In this paper, we will only consider finite generalized n-gons of order (s,t) with $s, t > 1$; finiteness means that both s and t are finite. In fact, we will only consider two specific examples of generalized quadrangles and hexagons, i.e., the quadrangle $Q(4,q)$ and the hexagon $H(q)$ (for the definitions, see below), together with their duals (obtained by switching the terms 'points' and 'lines'). Note that finite generalized n-gons of order (s,t) with $s,t > 1$ only exist for $n = 2,3,4,6,8$ (by the result of FEIT & HIGMAN [3]).

Let $n = 2m$ be equal to 4 or 6 and suppose that Γ is a finite generalized n-gon of order (s,t) with $s,t > 1$. We call two elements *opposite* if they are at distance n in the incidence graph. An *ovoid* of Γ is a set of $s^{m-1}t+1$ mutually opposite points (that is the maximal number of mutually opposite points); dually one defines a *spread*. The aim of the paper is to construct new examples of ovoids in the polygons $Q(4,q)$, $H(q)$ and their duals, and to give new constructions of some known ovoids in these polygons. Along with these constructions, we prove some classification results. The basic idea is to look for examples with a relatively large automorphism group. This is expressed by introducing the notion of a translation ovoid or spread (with respect to a point, a line or a flag).

The paper is organized as follows. In section 2 we introduce the various notions we need: the quadrangle $Q(4,q)$ and its dual $W(q)$, the generalized hexagon $H(q)$ and its dual, coordinatization, the polar space $Q(6,q)$, translation ovoids and spreads, the projection of an ovoid of $H(q)$ into $Q(4,q)$, the projection along reguli of a translation spread of $H(q)$ into $Q(4,q)$. In section 3, we briefly survey the known examples of ovoids and spreads in $Q(4,q)$ and $H(q)$, and we prove some characterization results for them. In section 4, we construct new ovoids of $H(q)$ for $q = 3^h$ and we show that in the dual of $H(q)$ the projection along reguli of the dual of these ovoids yields a geometric construction of the Roman ovoids due to THAS & PAYNE [21]. In section 5, we construct a new translation ovoid in the dual of $H(q)$ for all q odd with $q \equiv 1 \bmod 3$, or equivalently, a new translation spread in $H(q)$ for any such q. We also classify for all odd q the translation spreads of $H(q)$ (or more generally, the locally hermitian spreads of $H(q)$) for which any projection along reguli yields a classical ovoid in $Q(4,q)$.

Part of this paper was included in the Ph.D. thesis of the first author.

2 Definitions and notation

2.1 The quadrangle $Q(4,q)$

Let Γ be the point-line geometry obtained from a non-singular quadric $Q(4,q)$ in $\mathbf{PG}(4,q)$ by taking for points the points of the quadric, for lines the lines of $\mathbf{PG}(4,q)$ which lie entirely on $Q(4,q)$ together with the natural incidence relation. Then Γ is a generalized quadrangle, also denoted by $Q(4,q)$; see PAYNE & THAS [12]. If $Q(4,q)$ has equation $X_0^2 - X_1X_2 - X_3X_4 = 0$, then we may relabel the points and lines according to Table 1 and call this a *coordinatization*. The incidence relation in terms of these coordinates is as follows:

$$[k,b,k']\,\mathbf{I}\,(k,b)\,\mathbf{I}\,[k]\,\mathbf{I}\,(\infty)\,\mathbf{I}\,[\infty]\,\mathbf{I}\,(a)\,\mathbf{I}\,[a,l]\,\mathbf{I}\,(a,l,a')$$

for all $a,a',b,k,k',l \in \mathbf{GF}(q)$, and

$$(a,l,a')\,\mathbf{I}\,[k,b,k'] \quad\Leftrightarrow\quad \left\{\begin{array}{lcl} b &=& ak^2+a'-2kl, \\ l &=& ak+k'. \end{array}\right. \tag{1}$$

POINTS	
Coordinates in $Q(4,q)$	Coordinates in $\mathbf{PG}(4,q)$
(∞)	$(0,1,0,0,0)$
(a)	$(0,-a,0,1,0)$
(k,b)	$(k,-b,0,k^2,1)$
(a,l,a')	$(l,l^2-aa',1,a',a)$
LINES	
Coordinates in $Q(4,q)$	Representation in $\mathbf{PG}(4,q)$
$[\infty]$	$\langle(0,1,0,0,0),(0,0,0,1,0)\rangle$
$[k]$	$\langle(0,1,0,0,0),(k,0,0,k^2,1)\rangle$
$[a,l]$	$\langle(0,-a,0,1,0),(l,l^2,1,0,a)\rangle$
$[k,b,k']$	$\langle(k,-b,0,k^2,1),(k',k'^2,1,b+2kk',0)\rangle$

Table 1: Coordinatization of $Q(4,q)$.

This coordinatization is a special case of a more general theory; see HANSSENS & VAN MALDEGHEM [5, 6].

2.2 The hexagon $H(q)$ and the polar space $Q(6,q)$

Let $Q(6,q)$ be a non-singular quadric in $\mathbf{PG}(6,q)$. An *i-system* of $Q(6,q)$, $i \in \{0,1,2\}$, is a set $\mathcal{S}$ of q^3+1 i-dimensional projective subspaces of $Q(6,q)$ such that any plane of

$Q(6,q)$ containing a member of $\mathcal{S}$ has no point in common with any other member of $\mathcal{S}$ (see SHULT & THAS [15]). A 0-system is called an *ovoid* of $Q(6,q)$; a 2-system is called a *spread* of $Q(6,q)$.

Now let $Q(6,q)$ be defined by the equation $X_3^2 = X_0X_4 + X_1X_5 + X_2X_6$. Then TITS [22] defines the generalized hexagon $H(q)$ as follows. The points are all points of $Q(6,q)$. The lines are those lines of $Q(6,q)$ whose Grassmann (Plücker) coordinates satisfy

$$\begin{array}{lll} p_{34} = p_{12}, & p_{35} = p_{20}, & p_{36} = p_{01}, \\ p_{03} = p_{56}, & p_{13} = p_{64}, & p_{23} = p_{45} \end{array}$$

(for the definition of Grassmann coordinates, see e.g. HIRSCHFELD & THAS [7]). Just as for $Q(4,q)$, there exists a coordinatization of $H(q)$; it is given in Table 2. Incidence is given by

$$[k,b,k',b',k'']\,\mathbf{I}\,(k,b,k',b')\,\mathbf{I}\,[k,b,k']\,\mathbf{I}\,(k,b)\,\mathbf{I}\,[k]\,\mathbf{I}\,(\infty)\,\mathbf{I}$$
$$[\infty]\,\mathbf{I}\,(a)\,\mathbf{I}\,[a,l]\,\mathbf{I}\,(a,l,a')\,\mathbf{I}\,[a,l,a',l']\,\mathbf{I}\,(a,l,a',l',a''),$$

for all $a,a',a'',b,b',k,k',k'',l,l' \in \mathbf{GF}(q)$, and by

$$(a,l,a',l',a'')\,\mathbf{I}\,[k,b,k',b',k''] \;\Leftrightarrow\; \left\{ \begin{array}{lll} b & = & a'' - ak, \\ a' & = & a^2k + b' + 2ab, \\ l & = & k'' - ka^3 - 3ba^2 - 3ab', \\ k' & = & k^2a^3 + l' - kl - 3a^2a''k - 3a'a'' + 3aa''^2 \end{array} \right. \quad (2)$$

(see DE SMET & VAN MALDEGHEM [2]). A *point regulus* on $H(q)$ is a set of $q+1$ points at distance 3 (in the incidence graph) from two opposite lines, and hence at distance 3 from $q+1$ opposite lines by RONAN [14]. Dually, one defines a *line regulus*. The point regulus on $H(q)$ defined by the lines $[0,0]$ and $[0,0,0]$ is the set $\{(\infty)\} \cup \{(0,0,a',0,0) | a' \in \mathbf{GF}(q)\}$. In $\mathbf{PG}(6,q)$, this is readily seen to be the set of points with coordinates $(a'^2,0,0,a'd,d^2,0,0)$, with $(0,0) \neq (a',d) \in \mathbf{GF}(q)^2$. Hence this represents a conic on $Q(6,q)$. Dually, the line regulus $\mathcal{R}$ on $H(q)$ defined by the points $(0,0)$ and $(0,0,0)$ is the set

$$\{[\infty]\} \cup \{[0,0,k',0,0] | k' \in \mathbf{GF}(q)\}.$$

In $\mathbf{PG}(6,q)$, all these lines form one set of generators of the hyperbolic quadric obtained by intersecting $Q(6,q)$ with the subspace with equations $X_1 = X_3 = X_5 = 0$. The hyperbolic quadric itself has equations $X_0X_4 + X_2X_6 = X_1 = X_3 = X_5 = 0$. Hence, for every point x on any member of the line regulus $\mathcal{R}$, there is a unique line $L_x^{\mathcal{R}}$ in $Q(6,q)$ through x and meeting every member of $\mathcal{R}$ non-trivially. We call $L_x^{\mathcal{R}}$ the *transversal* of $\mathcal{R}$ in x.

POINTS	
Coordinates in $H(q)$	Coordinates in $\mathbf{PG}(6,q)$
(∞)	$(1,0,0,0,0,0,0)$
(a)	$(a,0,0,0,0,0,1)$
(k,b)	$(b,0,0,0,0,1,-k)$
(a,l,a')	$(-l-aa',1,0,-a,0,a^2,-a')$
(k,b,k',b')	$(k'+bb',k,1,b,0,b',b^2-b'k)$
(a,l,a',l',a'')	$(-al'+a'^2+a''l+aa'a'',-a'',-a,-a'+aa'',$ $1,l+2aa'-a^2a'',-l'+a'a'')$

LINES	
Coordinates in $H(q)$	Representation in $\mathbf{PG}(6,q)$
$[\infty]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,0,1)\rangle$
$[k]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,1,-k)\rangle$
$[a,l]$	$\langle(a,0,0,0,0,0,1),(-l,1,0,-a,0,a^2,0)\rangle$
$[k,b,k']$	$\langle(b,0,0,0,0,1,-k),(k',k,1,b,0,0,b^2)\rangle$
$[a,l,a',l']$	$\langle(-l-aa',1,0,-a,0,a^2,-a'),$ $(-al'+a'^2,0,-a,-a',1,l+2aa',-l')\rangle$
$[k,b,k',b',k'']$	$\langle(k'+bb',k,1,b,0,b',b^2-b'k),$ $(b'^2+k''b,-b,0,-b',1,k'',-kk''-k'-2bb')\rangle$

Table 2: Coordinatization of $H(q)$.

Note that every line regulus $\mathcal{R}$ is determined by any two of its elements. Therefore, we denote sometimes the line regulus containing two lines L, M by $\mathcal{R}(L, M)$. Similarly for point reguli. Also, the set of points at distance 3 from all elements of a line regulus forms a point regulus and dually, all lines at distance 3 from all points of a point regulus forms a line regulus. We call such reguli *associated*.

2.3 Translation ovoids

The definition of translation ovoid is inspired by the definition of a translation oval in the projective plane $PG(2, q)$, for q even. We introduce translation ovoids in general for quadrangles and hexagons of order (s, t), but we will only deal with the case $s = t$. Note that also the infinite case can be included in the following definitions, but we do not insist on this because almost all interesting features of ovoids and spreads happen in finite geometry.

Let $\Gamma = (\mathcal{P}, \mathcal{B}, I)$ be a generalized $2m$-gon, $m = 2, 3$, of order (s, t) and let O be an ovoid of Γ which contains the point $x \in \mathcal{P}$. Let us denote by $\Gamma_i(y)$ the set of points of Γ at distance i from the point y (measured in the incidence graph of Γ). Let L be a line of Γ

incident with x. It is easily seen that for each point $y \neq x$ on L, there are exactly $s^{m-2}t$ elements in $V_y := (O \setminus \{x\}) \cup \Gamma_{2m-2}(y)$. We call the ovoid O a *translation ovoid with respect to the flag* $\{x, L\}$ if there exists a group $G_{\{x,L\}} < \mathrm{Aut}(\Gamma)$, which fixes the ovoid O, which fixes x linewise and L pointwise, and which acts transitively on the points of each set V_y, with $y \in L \setminus \{x\}$ (viewing a line as the set of points incident with it). Also, the ovoid O is a *translation ovoid with respect to the point* x if O is a translation ovoid with respect to the flag $\{x, M\}$, for each line M through x. Note that, if O is a translation ovoid with respect to the flag $\{x, L\}$, then in the above necessarily $|G_{\{x,L\}}| = s^{m-2}t$ and the action mentioned is regular. Indeed, any collineation of $H(q)$ fixing all lines through x, fixing all points on L and fixing at least one point of $H(q)$ opposite x is readily seen to be the identity. If O is a translation ovoid with respect to the flag $\{x, L\}$, respectively the point x, the group $G_{\{x,L\}}$, respectively $G_{\{x\}} = \langle G_{\{x,M\}} | M \,\mathbf{I}\, x\rangle$, is referred to as the group associated with the translation ovoid O with respect to $\{x, L\}$, respectively x.

Of course, the dual of a translation ovoid w. r. t. a point (resp. a flag) is called a *translation spread w. r. t. a line (resp. a flag).*

Now let O be any ovoid of $H(q)$. Then by Thas [17], O is an ovoid of $Q(6, q)$ and, conversely, any ovoid of $Q(6, q)$ is an ovoid of $H(q)$. Also, a spread of $H(q)$ is a 1-system of $Q(6, q)$; see Shult & Thas [15]. Finally, if O is an ovoid of $H(q)$, then the set of planes of $Q(6, q)$ whose points are the points collinear in $H(q)$ with a point of O forms a spread of $Q(6, q)$.

Let O be any ovoid of $Q(6, q)$ and consider a point x of $Q(6, q)$ not contained in O. The geometry Γ_x whose point set is the set of lines of $Q(6, q)$ through x and whose line set is the set of planes of $Q(6, q)$ through x (with natural incidence) is isomorphic to the quadrangle $Q(4, q)$. The set of lines xy of $Q(6, q)$, where $y \in O$, defines an ovoid O' of $Q(4, q)$. We call O' the projection of O from x. Hence every ovoid of $H(q)$ gives rise to ovoids of $Q(4, q)$.

Now let $\mathcal{S}$ be a spread of $H(q)$ with the following property: there is a line L of $\mathcal{S}$ such that for every $M \in \mathcal{S} \setminus \{L\}$ we have $\mathcal{R}(L, M) \subseteq \mathcal{S}$. Then we say that $\mathcal{S}$ *is locally hermitian (in L)* and we call L a *hermitian line of* $\mathcal{S}$. Let x be any point on L. In Γ_x (see above) we consider the set of points O formed by the transversals in x of the line reguli in $\mathcal{S}$ containing L. We show that O is an ovoid of Γ_x, and we call that ovoid the *projection of* $\mathcal{S}$ *from* x *along reguli.* To that end, we first mention the following theorem.

Theorem 1 (Shult & Thas [15], Theorems 5 and 6) *Let* $\mathcal{S}$ *be a* 1*-system of* $Q(6, q)$*.*
(a) If $\mathbf{PG}(5, q)$ *is a hyperplane of* $\mathbf{PG}(6, q)$ *which intersects* $Q(6, q)$ *in a hyperbolic quadric* $Q^+(5, q)$*, then* $\mathbf{PG}(5, q)$ *contains exactly* $q + 1$ *elements of* $\mathcal{S}$*.*
(b) If the hyperplane $\mathbf{PG}(5, q)$ *of* $\mathbf{PG}(6, q)$ *is tangent to* $Q(6, q)$ *at the point* y*, where* y *is not on an element of* $\mathcal{S}$*, then* $\mathbf{PG}(5, q)$ *contains exactly* $q + 1$ *elements of* $\mathcal{S}$*.*

We will usually apply this theorem to spreads of $H(q)$, in view of the previously mentioned fact that every spread of $H(q)$ is a 1-system of $Q(6,q)$.

Theorem 2 *Let $\mathcal{S}$ be a spread of $H(q)$ which is locally hermitian in some line L. Let x be any point incident with L. Then the projection along reguli of $\mathcal{S}$ from x is an ovoid of $\Gamma_x \cong Q(4,q)$.*

PROOF. Consider the set O of $\Gamma_x \cong Q(4,q)$ which is the projection along reguli of the locally hermitian spread $\mathcal{S}$ from a point x on a hermitian line of $\mathcal{S}$. Suppose two points of O are collinear in Γ_x. Then the two corresponding reguli $\mathcal{R}_1$ and $\mathcal{R}_2$ of $\mathcal{S}$ are contained in a hyperplane π of $\mathbf{PG}(6,q)$ which meets $Q(6,q)$ in a quadric Q containing planes. By the previous theorem, Q is not of hyperbolic non-singular type. Hence π is a tangent hyperplane of $Q(6,q)$ and all points of Q are, in $H(q)$, at distance ≤ 4 from some fixed point y of $H(q)$. It is easily seen that y must belong to the point reguli associated with respectively $\mathcal{R}_1$ and $\mathcal{R}_2$. But then clearly $\mathcal{R}_1 \cup \mathcal{R}_2$ contains pairs of elements at distance 4 from each other, a contradiction. $\square$

3 Examples and properties

3.1 Translation ovoids and translation spreads in $H(q)$

Some general results

Lemma 3 *Let $\mathcal{S}$ be a spread of $H(q)$ containing the line $[\infty]$. Then for every triple $(k,b,k') \in \mathbf{GF}(q)^3$, there exist unique elements $f_1(k,b,k')$ and $f_2(k,b,k')$ in $\mathbf{GF}(q)$ such that the line $[k,b,k',f_1(k,b,k'),f_2(k,b,k')]$ belongs to $\mathcal{S}$.*

PROOF. There are q^2+q+1 lines of $H(q)$ concurrent with a fixed line of $\mathcal{S}$. In total, there are $(q^3+1)(q^2+q+1)$ lines of the hexagon concurrent with some line of $\mathcal{S}$, since no line of $H(q)$ can meet at least two members of $\mathcal{S}$. But $H(q)$ contains exactly $(q^3+1)(q^2+q+1)$ lines, hence every line of $H(q)$ not belonging to $\mathcal{S}$ is concurrent with exactly one member of $\mathcal{S}$. The mappings f_1 and f_2 are now defined by saying that the line $[k,b,k',f_1(k,b,k'),f_2(k,b,k')]$ of $\mathcal{S}$ is concurrent with the line $[k,b,k']$ (the latter does not belong to $\mathcal{S}$ and is not concurrent with $[\infty]$). $\square$

Lemma 4 *The group $G^{[\infty]}$ of automorphisms of $H(q)$ generated by all collineations fixing all points incident with $[\infty]$ and stabilizing all lines through some point of $[\infty]$ has order q^5 and acts regularly on the set of lines of $H(q)$ opposite $[\infty]$ (or equivalently, the set of lines with 5 coordinates).*

PROOF. This follows from the fact that $H(q)$ satisfies the so-called Moufang condition. In Tits' notation, our group $G^{[\infty]}$ is contained in U_+ (see TITS [24]) and is in fact equal to the product of 5 so-called root groups $U_1.U_2.\ldots.U_5$ (all of which fix $[\infty]$ pointwise). The result follows (see also VAN MALDEGHEM [25]). $\square$

In coordinates, a general element $\Theta[K, B, K', B', K'']$, $K, B, K', B', K'' \in \mathbf{GF}(q)$, of $G^{[\infty]}$ can be written as (and we give the action on the elements with five coordinates; the action on the other elements is obtained by restricting coordinates):

$$\begin{cases} (a,l,a',l',a'')^{\Theta[K,B,K',B',K'']} &= (a, l + K'' - 3aB' - 3a^2B - a^3K, a' + B' + 2aB + a^2K, \\ & \quad l' + K' + KK'' + 3aB^2 + 3a'B + 3BB' + a^3K^2 + lK + \\ & \quad +3aa'K + 3a^2BK, a'' + B + aK), \\ [k,b,k',b',k'']^{\Theta[K,B,K',B',K'']} &= [k + K, b + B, k' + K' - kK'' - 3bB', b' + B', k'' + K'']. \end{cases}$$

Using the incidence relation (2) of Section 2, it can be easily checked that the above mapping preserves incidence, that it fixes all points on $[\infty]$, that it is equal to the product $\Theta[0, B, K', B', K'']\Theta[K, 0, 0, 0, 0]$ of two elements fixing all points of $[\infty]$ and stabilizing all lines through respectively (∞) and (0). Moreover, it maps $[0, 0, 0, 0, 0]$ to $[K, B, K', B', K'']$, hence the group of all such collineations acts regularly on the lines opposite $[\infty]$. So we have exhibited the explicit expression of an arbitrary element of $G^{[\infty]}$.

Lemma 5 *If $\mathcal{S}$ is a spread of $H(q)$ containing $[\infty]$ such that the subgroup $G_{\mathcal{S}}$ of $G^{[\infty]}$ stabilizing $\mathcal{S}$ has order q^3, then $\mathcal{S}$ is a translation spread with respect to $[\infty]$.*

PROOF. Let x be a point incident with $[\infty]$ and L a line distinct from $[\infty]$ incident with x. First we notice that by the regular action of $G^{[\infty]}$ on the lines opposite $[\infty]$, the group $G_{\mathcal{S}}$ acts transitively on $\mathcal{S} \setminus \{[\infty]\}$. Let M_1 and M_2 be two elements of $\mathcal{S}$ at distance 4 from L (in the incidence graph). The element of $G_{\mathcal{S}}$ mapping M_1 to M_2 stabilizes every line incident with x (this is because $G^{[\infty]}$ fixes either one line through x, that is, the line $[\infty]$, or all lines through x, see WEISS [26], Lemma 1). The assertion is proved. $\square$

Theorem 6 *If $\mathcal{S}$ is a translation spread of $H(q)$ with respect to the flags $\{[\infty], (\infty)\}$ and $\{[\infty], (0)\}$, then $\mathcal{S}$ is locally hermitian in $[\infty]$. If moreover $q \not\equiv -1 \mod 3$, then $\mathcal{S}$ is a translation spread of $H(q)$ with respect to the line $[\infty]$ and the group $G_{\{[\infty]\}}$ acts regularly on $\mathcal{S} \setminus \{[\infty]\}$.*

PROOF. First suppose that the characteristic of $\mathbf{GF}(q)$ is different from 3. Let us assume without loss of generality that $[0, 0, 0, 0, 0]$ belongs to $\mathcal{S}$. If $[0, B, K', B', K''] \in \mathcal{S}$, then by the regular action of $G^{[\infty]}$, the group element $\Theta[0,B,K',B',K'']$ belongs to $G_{\{[\infty],(\infty)\}}$. It follows that for any $[k,b,k',b',k''] \in \mathcal{S}$, the line $[k,b,k',b',k'']^{\Theta[0,B,K',B',K'']}$ also belongs to $\mathcal{S}$. Hence for arbitrary $k,b,k',B,K' \in \mathbf{GF}(q)$, the line

$$[k, b+B, k'-3bf_1(0,B,K')-kf_2(0,B,K')+K', f_1(k,b,k')+f_1(0,B,K'), f_2(k,b,k')+f_2(0,B,K')]$$

belongs to $\mathcal{S}$. Hence, for $i=1,2$,

$$f_i(k, b+B, k'-3bf_1(0,B,K')-kf_2(0,B,K')+K') = f_i(k,b,k')+f_i(0,B,K'). \tag{3}$$

If we put $k=b=0$ in Equation (3), then we obtain

$$f_i(0,B,k'+K') = f_i(0,0,k')+f_i(0,B,K'). \tag{4}$$

Similarly, we can also put $B=0$ in Equation (3) to obtain

$$f_i(k,b,k'-3bf_1(0,0,K')-kf_2(0,0,K')+K') = f_i(k,b,k')+f_i(0,0,K'). \tag{5}$$

If we put $k=0$ in Equation (5), then we can use Equation (4) to obtain

$$f_1(0,0,-3bf_1(0,0,K')) = 0. \tag{6}$$

If there exists $K' \in \mathbf{GF}(q)$ such that $f_1(0,0,K') \neq 0$, then putting $b = \frac{-K'}{3f_1(0,0,K')}$ in Equation (6), we obtain the contradiction $f_1(0,0,K')=0$. So $f_1(0,0,K')=0$ for all $K' \in \mathbf{GF}(q)$. Now let $K' \in \mathbf{GF}(q)$ be arbitrary. Let $M_{K'}$ be the line of $H(q)$ concurrent with both $[0,0,K']$ and $[0,0,0,K']$; $M_{K'}$ has coordinates $[0,0,K',0,0]$ and belongs to the regulus $\mathcal{R}([\infty],[0,0,0,0,0])$. Then clearly the unique element of $\mathcal{S}$ concurrent with $[0,0,K']$ is also concurrent with $M_{K'}$, as it has coordinates $[0,0,K',0,f_2(0,0,K')]$. Since $\mathcal{S}$ is a translation spread with respect to the flag $\{[\infty],(0)\}$, we may similarly conclude that the unique element of $\mathcal{S}$ concurrent with $[0,0,0,K']$ is also concurrent with $M_{K'}$. Since two elements of $\mathcal{S}$ cannot meet the same line, we conclude that $M_{K'}$ belongs to $\mathcal{S}$. But this means that the regulus $\mathcal{R}([\infty],[0,0,0,0,0])$ is contained in $\mathcal{S}$. Since the line $[0,0,0,0,0]$ was chosen arbitrarily in $\mathcal{S}$, the spread $\mathcal{S}$ is locally hermitian in $[\infty]$.

Now suppose that q is odd and that -3 is a non-zero square in $\mathbf{GF}(q)$, that is, assume that q is odd with $q \equiv 1 \bmod 3$. Suppose that some line $[0,b,k',b',0]$ belongs to $\mathcal{S}$, with $b \neq 0$. By the previous paragraph, also $[0,b,K',b',0]$ belongs to $\mathcal{S}$, for all $K' \in \mathbf{GF}(q)$. Then one can check by direct computation that the line $[0,b,k'_1,b',0]$, with

$$k'_1 = \frac{-3bb'+\sqrt{-3}bb'}{2},$$

lies at distance 3 from the point $(\frac{k'_1+bb'}{b^2},0,0,0,0)$, which is incident with $[0,0,0,0,0]$. Hence the two lines $[0,0,0,0,0]$ and $[0,b,k'_1,b',0]$ of $\mathcal{S}$ are not opposite, a contradiction. So $f_2(0,b,k') \neq 0$ for all $b,k' \in \mathbf{GF}(q)$, $b \neq 0$; as $\mathcal{R}([\infty],[0,0,0,0,0])$ belongs to $\mathcal{S}$, we have $f_2(0,0,k')=0$. Now choose $b_1 \in \mathbf{GF}(q)\setminus\{0\}$. Then by applying $\Theta[0,-b_1,0,-f_1(0,b_1,0),-f_2(0,b_1,0)]$ to $\mathcal{S}$, we obtain a spread $\mathcal{S}'$ again containing $[\infty]$ and $[0,0,0,0,0]$; also, $\mathcal{S}'$ is a translation spread with respect to the flag $\{[\infty],(\infty)\}$. It is clear that, if we denote the analogue to f_2 for $\mathcal{S}'$ by f_2', we have the relation $f_2(0,x,k')=f_2'(0,x-b_1,k')+f_2(0,b_1,k')$, for all $x,k'\in\mathbf{GF}(q)$. Applying our previous result to f_2', we have $f_2'(0,b-b_1,k')\neq 0$, for all $b,k'\in\mathbf{GF}(q)$, $b\neq b_1$. Noting that $f_2(0,b_1,y)$ is independent of $y\in\mathbf{GF}(q)$, we deduce that, for all $b,k',k_1'\in\mathbf{GF}(q)$, $f_2(0,b,k')\neq f_2(0,b_1,k_1')$ whenever $b_1\neq b$. This means that $x\mapsto f_2(0,x,0)$ is surjective onto $\mathbf{GF}(q)$. We claim that $G_{\{[\infty],(\infty)\}}$ acts transitively on the set of lines of $H(q)$ through (0) different from $[\infty]$. Indeed, the element of $G_{\{[\infty],(\infty)\}}$ mapping $[0,0,0,0,0]$ onto $[0,x,0,f_1(0,x,0),f_2(0,x,0)]$ has the form $\Theta[0,x,z,f_1(0,x,0),f_2(0,x,0)]$, for some suitable $z\in\mathbf{GF}(q)$; it is now clear that this element maps the line $[0,0]$ to the line $[0,f_2(0,x,0)]$ and the claim follows. Put $G=\langle G_{\{[\infty],(\infty)\}},G_{\{[\infty],(0)\}}\rangle$. Since (0) is fixed by G, and since G is a subgroup of $G^{[\infty]}$ (which acts regularly on the set of lines opposite $[\infty]$), we deduce that G has order q^3. Hence by Lemma 5, $\mathcal{S}$ is a translation spread with respect to the line $[\infty]$.

If q is an even square, and ω is a solution of the equation $X^2+X+1=0$, then the same argument as in the previous paragraph can be performed with $k_1'=\omega bb'$.

Now suppose that q is power of 3. Consider two lines $[k,b_0,k',b_0',k'']$ and $[k,b_1,k',b_1',k'']$ and suppose these lines belong to $\mathcal{S}$. Applying a suitable element of $G^{[\infty]}$ on $\mathcal{S}$, we may assume that $k=k'=k''=b_0=b_0'=0$ (use the fact that the characteristic is equal to 3). Now the line $[0,0]$ is at distance 4 (in the incidence graph) from the lines $[0,b_1,0,b_1',0]$, $[0]$ and $[0,b_1,0,0,0]$, all of which are concurrent with $[0,b_1,0]$. But $[0,0,0,0,0]$ is also at distance 4 from $[0]$ and $[0,b_1,0,0,0]$, hence by the distance-2-regularity of the lines in $H(q)$, for q a power of 3, we have that $[0,0,0,0,0]$ and $[0,b_1,0,b_1',0]$ are not opposite, a contradiction. This means that $x\mapsto f_2(k,x,k')$ is injective for all $k,k'\in\mathbf{GF}(q)$, hence surjective, and as before this implies that $G=\langle G_{\{[\infty],(\infty)\}},G_{\{[\infty],(0)\}}\rangle$ has order q^3 and acts regularly on $\mathcal{S}$.

We may now modify slightly our argument of the first paragraph of this proof as follows.

Let us assume without loss of generality that $[0,0,0,0,0]$ belongs to $\mathcal{S}$. If $[K,B,K',B',K'']\in\mathcal{S}$, then by the regular action of $G^{[\infty]}$, the group element $\Theta[K,B,K',B',K'']$ belongs to G. It follows that for any $(k,b,k',b',k'')\in\mathcal{S}$, the line $(k,b,k',b',k'')^{\Theta[K,B,K',B',K'']}$ also belongs to $\mathcal{S}$. Hence for arbitrary $k,b,k',K,B,K'\in\mathbf{GF}(q)$, the line

$$[k+K,b+B,k'-kf_2(K,B,K')+K',f_1(k,b,k')+f_1(K,B,K'),f_2(k,b,k')+f_2(K,B,K')]$$

belongs to $\mathcal{S}$. So, for $i=1,2$,

$$f_i(k+K,b+B,k'-kf_2(K,B,K')+K')=f_i(k,b,k')+f_i(K,B,K'). \tag{7}$$

If we put $k = b = 0$ in Equation (7), then we obtain

$$f_i(K, B, k' + K') = f_i(0, 0, k') + f_i(K, B, K'). \tag{8}$$

Similarly, we can also put $K = B = 0$ in Equation (7) to obtain

$$f_i(k, b, k' - kf_2(0, 0, K') + K') = f_i(k, b, k') + f_i(0, 0, K'). \tag{9}$$

Now we apply Equation (8) to Equation (9), and obtain

$$f_2(0, 0, -kf_2(0, 0, K')) = 0. \tag{10}$$

Similarly as in the first part of the proof, this implies that $f_2(0, 0, K') = 0$ for all $K' \in \mathbf{GF}(q)$. If we interchange lower-case letters and upper-case letters in Equation (7), then we obtain, for $i = 1$,

$$f_1(k + K, b + B, k' + K' - kf_2(K, B, K')) = f_1(k + K, b + B, k' + K' - Kf_2(k, b, k'))$$

which we can simplify in view of Equation (8) to

$$f_1(0, 0, Kf_2(k, b, k') - kf_2(K, B, K')) = 0, \tag{11}$$

for all values of $k, b, k', K, B, K' \in \mathbf{GF}(q)$. So we put $k = k' = 0$ and $K = 1$. since $b \mapsto f_2(0, b, 0)$ is surjective, we have $f_1(0, 0, K') = 0$, for all $K' \in \mathbf{GF}(q)$. As above, this implies that the line regulus $\mathcal{R}([\infty], [0, 0, 0, 0, 0])$ is contained in $\mathcal{S}$. So we conclude that $\mathcal{S}$ is locally hermitian in $[\infty]$.

The theorem is completely proved. $\square$

Remark 7 From the previous proof follows readily that, if q is odd and not congruent to -1 modulo 3, and if $\mathcal{S}$ is a locally hermitian spread of $H(q)$ in $[\infty]$, then the map $f_2(0, x, 0)$ is a permutation of $\mathbf{GF}(q)$. Similarly one shows that $f_2(a, x, b)$ is a permutation of $\mathbf{GF}(q)$, for fixed $a, b \in \mathbf{GF}(q)$. Hence there are maps $g_i : \mathbf{GF}(q)^3 \to \mathbf{GF}(q)$, $i = 1, 2$, such that $\mathcal{S}$ can be written as $\{[\infty]\} \cup \{[k, g_1(k, k', k''), k', g_2(k, k', k''), k''] : k, k', k'' \in \mathbf{GF}(q)\}$.

We will see below that there exist translation spreads with respect to a flag which are not locally hermitian in any line.

We can also prove the converse of Lemma 5 for odd characteristic. However, since we will not use this result, we only very briefly sketch a proof.

Theorem 8 *If $\mathcal{R}$ is a translation spread of $H(q)$, q odd, with respect to a line L, then the collineation group G generated by all groups $G_{\{x,L\}}$, with x incident with L, acts regularly on $\mathcal{S} \setminus \{L\}$ and hence has order q^3.*

PROOF. Similarly as in the proof of Theorem 12 below (now calculating that two distinct lines M_1 and M_2 of $\mathcal{S} \setminus \{L\}$ can be at distance 4 from 0, 1, 2, 3 or $q+1$ lines concurrent with L, and that the case $q+1$ only occurs for at most $q-1$ choices of M_2 if M_1 is fixed), we obtain that the orbit of any line M of $\mathcal{S}$ under G has at least length $q^3/3$. Since the characteristic of $\mathbf{GF}(q)$ may be assumed to be different from 2 (by assumption) and 3 (by Theorem 6), it follows that the length of the orbit is equal to q^3. $\square$

Examples of translation spreads and ovoids in $H(q)$

The hermitian spreads If we intersect the non-singular quadric $Q(6,q) : X_3^2 = X_0X_4 + X_1X_5 + X_2X_6$ with the hyperplane $\Pi : X_5 = \gamma X_1$, with γ a non-square of $GF(q)$, then we obtain an elliptic quadric $Q^-(5,q)$. The lines of $H(q)$ which are contained in Π, hence also in $Q^-(5,q)$, constitute a spread $\mathcal{S}$ of $H(q)$. This spread is called the *hermitian spread* (see Thas [17]). Now consider the coordinatization of $H(q)$ (see Table 2 above). The line $[\infty]$ of $H(q)$ corresponds with the line on $Q(6,q)$ through the points $(1,0,0,0,0,0,0)$ and $(0,0,0,0,0,0,1)$. Hence $[\infty] \in \mathcal{S}$. The line $[k,b,k',b',k'']$ of $H(q)$ corresponds with the line $\langle (k'+bb', k, 1, b, 0, b', b^2 - b'k), (b'^2 + k''b, -b, 0, -b', 1, k'', -kk'' - k' - 2bb') \rangle$ on $Q(6,q)$. So the line $[k,b,k',b',k'']$ is an element of $\mathcal{S}$ if and only if $k'' = -\gamma b$ and $b' = \gamma k$. Hence

$$\mathcal{S}_H = \{[\infty]\} \cup \{[k,b,k',\gamma k,-\gamma b] : (k,b,k') \in GF(q)^3\}$$

is a hermitian spread $\mathcal{S}_H$ of $H(q)$. Here the functions f_1 and f_2 are given by $f_1(k,b,k') = \gamma k$ and $f_2(k,b,k') = -\gamma b$. Since both f_1 and f_2 are independent of k', we easily deduce that $\mathcal{S}_H$ is locally hermitian in $[\infty]$. But the line $[\infty]$ plays the same role here as any other line in $\mathcal{S}_H$, hence $\mathcal{S}_H$ is locally hermitian in any of its elements. This motivates the earlier notion of "locally hermitian". In fact, the property of being locally hermitian in every element characterizes the hermitian spread. But we can do even better:

Theorem 9 *If $\mathcal{S}$ is a spread of $H(q)$ which is locally hermitian in at least two of its elements, then $\mathcal{S}$ is a hermitian spread.*

PROOF. Suppose that $\mathcal{S}$ is locally hermitian in the lines L_1 and L_2, $L_1 \neq L_2$. Let $M_1 \in \mathcal{S} \setminus \{L_1, L_2\}$ be such that $M_1 \notin \mathcal{R}(L_1, L_2)$. By Theorem 1 and the fact that every spread of $H(q)$ is a 1-system of $Q(6,q)$, the lines L_1, L_2, M_1 define a $\mathbf{PG}(5,q)$ which intersects $Q(6,q)$ in a $Q^-(5,q)$. Put $\mathcal{R}(L_2, M_1)$ (all elements of which lie in $\mathbf{PG}(5,q)$) equal to $\{L_2, M_1, M_2, \ldots, M_q\}$. Also, we put, for all $i \in \{1, 2, \ldots, q\}$, the regulus $\mathcal{R}(L_1, M_i)$ equal to $\{L_1, M_i, M_{i,1}, M_{i,2}, \ldots, M_{i,q-1}\}$. Every line $M_{i,j}$, $1 \le i \le q$, $1 \le j < q$, belongs to $\mathbf{PG}(5,q)$, hence to $Q^-(5,q)$. Also all the elements of $\mathcal{R}(L_2, M_{i,j})$, for $1 \le i \le q$, $1 \le j < q$, belong to $Q^-(5,q)$. Since a hermitian spread in $H(q)$ provided with its reguli defines a linear space which is isomorphic to the linear space obtained from a hermitian curve in

$\mathbf{PG}(2, q^2)$ by considering secant lines (see THAS [17] or DE SMET & VAN MALDEGHEM [2]), we may interpret L_1, L_2, M_i and $M_{i,j}$, $1 \le i \le q$, $1 \le j < q$, as respective points l_1, l_2, m_i and $m_{i,j}$ of a hermitian curve $\mathcal{H}$ in $\mathbf{PG}(2, q^2)$. Since by O'Nan's property, see O'NAN [11], no line $l_2 m_{i,j}$ coincides with $l_2 m_{i',j'}$, for $(i, j) \neq (i', j')$, we obtain a set $\mathfrak{R}$ of $2 + (q-1)q$ different reguli $\mathcal{R}(L_2, M_1)$, $\mathcal{R}(L_2, L_1)$, $\mathcal{R}(L_2, M_{i,j})$, $1 \le i \le q$, $1 \le j < q$, all contained in the hermitian spread defined by $Q^-(5, q)$. Now we apply the same argument switching the roles of M_1 and $M_{1,1}$. Let $\mathcal{R}(L_2, M_{1,1}) = \{L_2, M_{1,1}, N_1, \ldots, N_{q-1}\}$. Let n_i, $1 \le i < q$, be the point on $\mathcal{H}$ corresponding with N_i. Suppose that we can show that, up to a renumbering, the lines $l_1 n_i$, $1 < i < q$, have the following property (*): if $\{n_{i,j} | 1 \le j < q\} \cup \{l_1, n_i\}$ is the intersection of $l_1 n_i$ with $\mathcal{H}$, then there exists a unique $j \in \{1, 2, \ldots, q-1\}$ such that the regulus in $Q^-(5, q)$ corresponding with the secant line $l_2 n_{i,j}$ does not belong to $\mathfrak{R}$. Then it follows immediately that the $q-2$ reguli through L_2 which do not belong to $\mathfrak{R}$ also belong to $\mathcal{S}$. Hence $\mathcal{S}$ is the hermitian spread obtained from $Q^-(5, q)$.

So it remains to show that indices can be chosen in such a way that the lines $l_1 n_i$, $1 < i < q$, satisfy property (*). Let $\mathbf{PG}(1, q^2)$ be a line of $\mathbf{PG}(2, q^2)$ not containing l_2. It is well known that the points of $\mathbf{PG}(1, q^2)$ together with the projective sublines over $\mathbf{GF}(q)$ form the classical inversive plane $\mathfrak{I}$ of order q. The $q^2 - q + 2$ secants of $\mathcal{H}$ corresponding to the elements of $\mathcal{R}$ intersect $\mathbf{PG}(1, q^2)$ in $q^2 - q + 2$ points of $\mathfrak{I}$; let R be the set of these points. Let p be the point of $\mathfrak{I}$ corresponding with the tangent line of $\mathcal{H}$ at l_2. Clearly $p \notin R$. The sets $l_1 m_i \cap \mathcal{H}$, $1 \le i \le q$, are projected from p onto q circles of $\mathfrak{I}$ all containing the projection l_1^* and m_1^* of respectively l_1 and m_1. Similarly, the secants $l_1 n_i$, $1 \le i < q$, give rise to $q - 1$ circles of $\mathfrak{I}$ all containing l_1^* and the projection $m_{1,1}^*$ of $m_{1,1}$. Translated to the internal (or derived) affine plane $\mathcal{A}$ of $\mathfrak{I}$ at l_1^*, this means that, except for the line $m_{1,1}^* m_1^*$, we have all lines through m_1^* except for the line $m_1^* p$, and all lines through $m_{1,1}^*$ except for $m_{1,1}^* p$. Now let the projection of $l_1 n_1 \cap \mathcal{H}$ correspond to the unique line N in $\mathcal{A}$ through $m_{1,1}^*$ parallel to $m_1^* p$. Then the circle N of $\mathfrak{I}$ is incident with $q + 1$ elements of R. On the other hand, each other line in $\mathcal{A}$ through $m_{1,1}^*$ and not through m_1^* has a unique intersection point p' with $m_1^* p$, hence the line $l_2 p'$ in $\mathbf{PG}(2, q^2)$ does not contain a point of R, which shows property (*).

This completes the proof of the proposition. $\square$

We also have the following theorem.

Theorem 10 *A hermitian spread in $H(q)$ is a translation spread with respect to every line.*

PROOF. It suffices to show that $\mathcal{S}_H$, defined as above, is a translation spread with respect to $[\infty]$. Therefore, we notice that the group

$$\{\Theta[K, B, K', \gamma K, -\gamma B] | K, B, K' \in \mathbf{GF}(q)\}$$

stabilizes $\mathcal{S}_H$. The result now follows from Lemma 5. $\square$

As a corollary we obtain:

Corollary 11 *If a spread $\mathcal{S}$ of $H(q)$ is a translation spread with respect to two different lines, then it is a hermitian spread.*

PROOF. By Theorem 6, $\mathcal{S}$ is locally hermitian in two different elements. The result follows from Theorem 9. $\square$

If $q = 3^h$, then the generalized hexagon $H(q)$ is self-dual. Dualize $H(q)$ as follows. Consider the map τ which acts as follows on the points and lines of $H(q)$:

$$\begin{array}{lcl} [\infty] & \mapsto & (\infty) \\ [k] & \mapsto & (k) \\ [k,b,k'] & \mapsto & (k,b^3,k') \\ [k,b,k',b',k''] & \mapsto & (k,b^3,k',b'^3,k'') \\ (a) & \mapsto & [a^3] \\ (k,b) & \mapsto & [k,b^3] \\ (a,l,a') & \mapsto & [a^3,l,a'^3] \\ (k,b,k',b') & \mapsto & [k,b^3,k',b'^3] \\ (a,l,a',l',a'') & \mapsto & [a^3,l,a'^3,l',a''^3]. \end{array} \tag{12}$$

Using the incidence relation (2), it is easy to check that τ preserves the incidence and so it dualizes $H(q)$. It follows from the proof of Theorem 6 that in characteristic 3 the map $x \mapsto f_2(k,x,k')$ (for f_2 associated with a given spread $\mathcal{S}$ of $H(q)$ containing $[\infty]$) is bijective for all $k,k' \in \mathbf{GF}(3^h)$. Hence there are maps

$$g_i : \mathbf{GF}(q)^3 \rightarrow \mathbf{GF}(q) : (k,k,k'') \mapsto g_i(k,k',k''), \qquad i = 1,2,$$

such that

$$\mathcal{S} = \{[\infty]\} \cup \{[k, g_1(k,k',k''), k', g_2(k,k',k''), k'']|k,k',k'' \in \mathbf{GF}(q)\}.$$

It is now convenient to write the hermitian spread as (substituting γ for γ^{-1})

$$S_H = \{[\infty]\} \cup \{[k, -\gamma k'', k', \gamma^{-1}k, k'']|(k,k',k'') \in GF(q)^3\}.$$

The duality τ maps a hermitian spread onto an ovoid $\mathcal{U}_H$ of $H(q)$. The ovoid $\mathcal{U}_H$ is called a *hermitian ovoid of* $H(q)$, $q = 3^h$, and is given by the set of points (use the formulae (12) above)

$$\mathcal{U}_H = \{(\infty)\} \cup \{(a, -\gamma' a''^3, a', \gamma'^{-1} a^3, a'')|(a,a',a'') \in GF(q)^3\},$$

with $\gamma' = \gamma^3$ a non-square of $GF(q)$. It has all dual properties of the hermitian spreads in $H(q)$. So $\mathcal{U}_H$ is a translation ovoid with respect to any of its points, and it is locally hermitian in every point. Note that the dual of Theorem 9 and Corollary 11 characterize hermitian ovoids in $H(3^h)$.

The Ree-Tits ovoids in $H(3^{2e+1})$ A Ree-Tits ovoid is an ovoid that arises as the set of absolute points of a polarity in $H(q)$. Note that $H(q)$ is self-polar if and only if $q = 3^{2e+1}$ for some non-negative integer e. The standard form of the Ree-Tits ovoid $\mathcal{U}_R$ is given by (see De Smet & Van Maldeghem [2])

$$\mathcal{U}_R = \{(\infty)\} \cup \{(a, a''^s - a^{3+s}, a', a^{3+2s} + a'^s + a^s a''^s, a'')|(a, a', a'') \in \mathbf{GF}(q)\},$$

with $q = 3^{2e+1}$ and $s = 3^{e+1}$.

Suppose the ovoid $\mathcal{U}_R$ described above is the set of absolute points with respect to the polarity σ (for an explicit form of σ, see De Smet & Van Maldeghem [2]). The group

$$\{\Theta[0, B, B'^s, B', B^s]|B, B' \in \mathbf{GF}(3^{2e+1})\}$$

stabilizes $\mathcal{U}_R$. This implies that $\mathcal{U}_R$ is a translation ovoid with respect to the flag $\{(\infty), [\infty]\}$. By the transitivity properties of the automorphism group of $\mathcal{U}_R$, we may conclude that the Ree-Tits ovoid $\mathcal{U}_R$ is a translation ovoid with respect to any flag $\{x, x^\sigma\}$, for all $x \in \mathcal{U}_R$.

Suppose that $\mathcal{U}_R$ is a translation ovoid with respect to the flag $\{(\infty), M\}$, with $M \,\mathbf{I}\, (\infty)$ and $M \neq [\infty]$. Then by the dual of Theorem 6 there follows that $\mathcal{U}_R$ is a translation ovoid with respect to (∞) and that $\mathcal{U}_R$ is locally hermitian. By the transitivity, $\mathcal{U}_R$ would be hermitian by Theorem 9. But this would force the functions $g_1(a, a', a'') = a''^s - a^{3+s}$ and $g_2(a, a', a'') = a^{3+2s} + a'^s + a^s a''^s$ to be independent from a', a contradiction. Hence $\mathcal{U}_R$ is no translation ovoid with respect to the flag $\{(\infty), M\}$, for each $M \,\mathbf{I}\, (\infty)$ with $M \neq [\infty]$.

3.2 Translation ovoids in $Q(4, q)$

Some general facts

Let O be an ovoid of $Q(4, q)$. Since the incidence relation between a point of type (a, l, a') and a line of type $[k, b, k']$ is given by the formula (1) in Section 2, two points (a, l, a') and (x, m, x') are collinear if and only if $(l - m)^2 = (a - x)(a' - x')$. This means that the set $O = \{(\infty)\} \cup \{(a, l, f(a, l))|a, l \in \mathbf{GF}(q)\}$, with f a mapping from $\mathbf{GF}(q)^2$ to $\mathbf{GF}(q)$, is an ovoid of $Q(4, q)$ if and only if

$$(l - m)^2 \neq (a - x)(f(a, l) - f(x, m)),$$

for all $a, l, x, m \in \mathbf{GF}(q)$, with $a \neq x$. Without loss of generality we may suppose that $(0, 0, 0) \in O$, i. e. $f(0, 0) = 0$.

Now we want to determine the conditions the function f, which obviously defines the ovoid O, has to satisfy in order that O is a translation ovoid w. r. t. the point (∞), resp. the flag $\{(\infty), M\}$, $M \,\mathbf{I}\, (\infty)$.

Suppose that O is a translation ovoid of $Q(4,q)$ w. r. t. the flag $\{(\infty), M\}$ and let $G_{\{(\infty),M\}}$ be the corresponding group. Similarly as for $H(q)$, the group $G_{\{(\infty),M\}}$ is a subgroup of the group $G^{(\infty)}$ of all collineations $\Phi(A,L,A')$, where $\Phi(A,L,A')$ is defined as

$$\begin{cases} (a,l,a')^{\Phi(A,L,A')} &= (a+A, l+L, a'+A'), \\ [k,b,k']^{\Phi(A,L,A')} &= [k, b+A'+k^2A-2kL, k'+L-kA] \end{cases}$$

(this can be checked using the incidence relation (1)). Hence completely similar to the case of $H(q)$, one has that O is a translation ovoid with respect to $\{(\infty), [\infty]\}$ if and only if the corresponding mapping f has the property

$$f(a,l) + f(0,L) = f(a, l+L),$$

for all $a, l, L \in \mathbf{GF}(q)$.

Now we note that being a translation ovoid with respect to two different flags $\{(\infty), L\}$ and $\{(\infty), M\}$ is not enough to guarantee that O is a translation ovoid with respect to the point (∞). Counterexamples are provided by the ovoids O_{K_1} of Kantor, see below.

Now we consider the case where O is a translation ovoid with respect to (∞). Let f be the corresponding mapping. Since we assume that $(0,0,0) \in O$, the corresponding group $G_{\{(\infty)\}}$ must be contained in the translation group

$$\{\Phi(A,L,f(A,L)) | A, L \in \mathbf{GF}(q)\},$$

hence $|G_{\{(\infty)\}}| \leq q^2$. In the next theorem we will prove that the group $G_{\{(\infty)\}}$ acts regularly on the q^2 points of the set $O \setminus \{(\infty)\}$, which implies that

$$G_{\{(\infty)\}} = \{\Phi(A,L,f(A,L)) | A, L \in \mathbf{GF}(q)\}.$$

Theorem 12 *If O is a translation ovoid of $Q(4,q)$ with respect to a point x, then the associated group $G_{\{x\}}$ acts regularly on the set of points $O \setminus \{x\}$.*

PROOF. We coordinatize $Q(4,q)$ in such a way that x is the point (∞) of $Q(4,q)$. We already have that $|G_{\{(\infty)\}}| \leq q^2$ and

$$G_{\{(\infty)\}} \leq \{\Phi(A,L,f(A,L)) | A, L \in \mathbf{GF}(q)\}.$$

We will show that the group $G_{\{(\infty)\}}$ acts transitively on the set of points $O \setminus \{(\infty)\}$. Let y and z be two points of the set $O \setminus \{(\infty)\}$. If the triad $\{(\infty), y, z\}$ is centric and if L is the line through (∞) and one of the centers of this triad, then there exists an element $\theta \in G_{\{(\infty),L\}}$ which maps y onto z. If q is even, then each triad is centric (see Payne & Thas [12]). So suppose that q is odd and that the triad $\{(\infty), y, z\}$ is not centric. Let

W_y be the set of points w of $O \setminus \{(\infty)\}$ for which the triad $\{(\infty), y, w\}$ is centric and let W_z be the set of points w of $O \setminus \{(\infty)\}$ for which the triad $\{(\infty), z, w\}$ is centric. Using the fact that a centric triad of $Q(4,q)$ has two centers, we have that

$$|W_y| = \frac{|\{(\infty), y\}^{\perp}| \cdot (q-1)}{2} = \frac{q^2-1}{2}.$$

Similarly $|W_z| = \frac{q^2-1}{2}$. If the intersection $W_y \cap W_z$ is empty, then $O \setminus \{(\infty)\}$ contains at least $|W_y| + |W_z| + |\{y,z\}| = q^2+1$ points, a contradiction. Hence let w be an element of $W_y \cap W_z$. Since in $G_{\{(\infty)\}}$ there exists a collineation which maps y onto w and a collineation which maps w onto z, there follows that the group $G_{\{(\infty)\}}$ contains an element which maps y onto z. Hence $G_{\{(\infty)\}}$ acts transitively on the points of $O \setminus \{(\infty)\}$. $\square$

Notice that, just like in the dual of $H(q)$, we have the following property:

Lemma 13 *An ovoid of $Q(4,q)$ containing (∞) and admitting a subgroup of $G^{(\infty)}$ of order q^2, is necessarily a translation ovoid with respect to (∞).*

PROOF.

The proof is similar to the proof of Lemma 5 (relying on the fact that a translation of $Q(4,q)$ with base point (∞) fixing a point $y \neq (\infty)$ collinear with (∞) also fixes every point of the line $(\infty)y$). $\square$

The next corollary and definition prepare some characterization results.

Corollary 14 *If O is an ovoid of $Q(4,q)$, with $q = p^e$, then O is a translation ovoid with respect to (∞) if and only if*

$$O = \{(\infty)\} \cup \{(a, l, \sum_{i=0}^{e-1}(\lambda_i a^{p^i} + \beta_i l^{p^i}))|a, l \in \mathbf{GF}(q)\},$$

with $\lambda_i, \beta_i \in \mathbf{GF}(q)$.

PROOF. Let O be an ovoid of $Q(4,q)$ which contains (∞), so

$$O = \{(\infty)\} \cup \{(a, l, f(a,l))|a, l \in \mathbf{GF}(q)\},$$

with f a mapping from $\mathbf{GF}(q)^2$ to $\mathbf{GF}(q)$. If O is a translation ovoid with respect to (∞), then each translation $\Phi(A, L, f(A,L))$, with $A, L \in GF(q)$, fixes the ovoid O. This means that $(a, l, f(a,l))^{\Phi(A,L,f(A,L))} \in O$, for all $a, l, A, L \in \mathbf{GF}(q)$, or equivalently that $f(a,l) + f(A,L) = f(a+A, l+L)$, for all $a, l, A, L \in GF(q)$. Hence we have that $f(a,0) + f(0,L) = f(a,L)$, $f(a,0) + f(A,0) = f(a+A,0)$ and $f(0,l) + f(0,L) = f(0,l+L)$.

This implies that $f(a,l) = \sum_{i=0}^{e-1}(\lambda_i a^{p^i} + \beta_i l^{p^i})$, with $\lambda_i, \beta_i \in \mathbf{GF}(q)$, see e. g. LIDL & NIEDERREITER [9].

Conversely, if the ovoid O is given by

$$O = \{(\infty)\} \cup \{(a, l, \sum_{i=0}^{e-1}(\lambda_i a^{p^i} + \beta_i l^{p^i}))|a, l \in \mathbf{GF}(q)\},$$

then each translation $\Phi(A, L, f(A, L))$, with $A, L \in \mathbf{GF}(q)$, fixes O. The result now follows from Lemma 13. $\square$

Let O be a translation ovoid with respect to the point (∞) of $Q(4,q)$, $q = p^e$, i. e. $O = \{(\infty)\} \cup \{(a, l, f(a, l))|a, l \in \mathbf{GF}(q)\}$, with $f(a,l) = \sum_{i=0}^{e-1}(\lambda_i a^{p^i} + \beta_i l^{p^i})$. Then the *kernel of* O is the subfield $K = GF(q')$, q' maximal, of $\mathbf{GF}(q)$ for which the following holds:

$$\forall x \in K, \forall a, l \in \mathbf{GF}(q) : f(xa, xl) = xf(a, l).$$

Also, we call any polynomial $f(a,l)$ of the form

$$f(a,l) = \sum_{i=0}^{e-1}(\lambda_i a^{p^i} + \beta_i l^{p^i})$$

automorphic.

Remark. Let g_x, $x \in \mathbf{GF}(q)^*$, be the $((\infty), (0,0,0))$-generalized homology which maps (a, l, a') onto (xa, xl, xa') (see e. g. DE SMET & VAN MALDEGHEM [2]). Then g_x fixes the translation ovoid O if and only if x is an element of the kernel of O.

Some examples

The classical ovoid O_E. Let $\mathbf{PG}(3,q)$ meet $Q(4,q)$ in an elliptic quadric. Then the intersection is an ovoid O_E of $Q(4,q)$. Without loss of generality we may take, in the case that q is odd, as an equation for $\mathbf{PG}(3,q)$ the equation $X_3 = \gamma X_4$, with γ a non-square in $\mathbf{GF}(q)$. It is easily computated (using Table 1) that

$$O_E = \{(\infty)\} \cup \{(a, l, \gamma a)|a, l \in \mathbf{GF}(q)\}.$$

Obviously this is a translation ovoid of $Q(4,q)$ with respect to the point (∞). The kernel K of O_E is the field $GF(q)$.

The ovoid O_{K_1} of Kantor. The standard form of the ovoid O_{K_1} of Kantor is

$$O_{K_1} = \{(\infty)\} \cup \{(a, l, \gamma a^\sigma) | a, l \in \mathbf{GF}(q)\},$$

with γ a non-square and σ an automorphism of $GF(q)$, $\sigma \neq 1$ and q odd (see KANTOR [8]). Since the polynomial $f(a, l) = \gamma a^\sigma$ is automorphic, the ovoid O_{K_1} is a translation ovoid with respect to (∞). The kernel of O_{K_1} is the subfield $\{x | x^\sigma = x, x \in \mathbf{GF}(q)\}$ of $\mathbf{GF}(q)$.

The Roman ovoid O_{TP} of Thas and Payne. The standard form of the ovoid O_{TP} of Thas and Payne (see [21]) is

$$O_{TP} = \{(\infty)\} \cup \{(a, l, \gamma^{-1} a + (\gamma a)^{\frac{1}{9}} + l^{\frac{1}{3}}) | a, l \in \mathbf{GF}(q)\},$$

with $q = 3^h$, $h > 2$, and γ a non-square in $GF(q)$. Since the function $f(a, l) = \gamma^{-1} a + (\gamma a)^{\frac{1}{9}} + l^{\frac{1}{3}}$ is automorphic, there follows that O_{TP} is a translation ovoid of $Q(4, q)$ with respect to (∞). One easily sees that the subfield $\mathbf{GF}(3)$ is the kernel of O_{TP}.

The ovoid O_{K_2} of Kantor. The standard form of the ovoid O_{K_2} of Kantor is

$$O_{K_2} = \{(\infty)\} \cup \{(a, l, a^{2s+3} + l^s) | a, l \in \mathbf{GF}(q)\},$$

with $q = 3^{2e-1}$, $e > 2$, and $s = 3^e$; see Kantor [8]. Since $f(a, 0) + f(a', 0) \neq f(a + a', 0)$ for some $a, a' \in \mathbf{GF}(q)$, the ovoid O_{K_2} is not a translation ovoid with respect to (∞). But since clearly the collineations $\Phi(0, L, L^s)$ leave the ovoid invariant, O_{K_2} is a translation ovoid with respect to the flag $\{(\infty), [\infty]\}$.

The ovoid O_{K_2} is obtained from the Ree-Tits ovoid in $H(q)$ by projection.

The Suzuki-Tits ovoid O_S. The standard form of the Suzuki-Tits ovoid O_S is

$$O_S = \{(\infty)\} \cup \{(a, l, a^{2^{e+1}+1} + l^{2^{e+1}}) | a, l \in \mathbf{GF}(q)\},$$

with $q = 2^{2e+1}$ and $e \geq 1$; see TITS [23]. Since $f(a, 0) + f(a', 0) \neq f(a + a', 0)$ for some $a, a' \in \mathbf{GF}(q)$, the ovoid O_S is not a translation ovoid with respect to (∞). But since clearly the collineations $\Phi(0, L, L^{2^{e+1}})$ leave the ovoid invariant, O_S is a translation ovoid with respect to the flag $\{(\infty), [\infty]\}$.

The ovoid O_S is obtained as the set of absolute points of a polarity of $Q(4, q)$.

A sporadic ovoid in $Q(4,3^5)$**.** It was recently proved by PENTTILA & WILLIAMS [13] that the set

$$O_{PW} = \{(\infty)\} \cup \{(a, l, a^9 + l^{81}) | a, l \in \mathbf{GF}(3^5)\},$$

is a translation ovoid of $Q(4,3^5)$ with respect to (∞).

The ovoids mentioned above are the only known translation ovoids with respect to a point or a flag of the generalized quadrangle $Q(4,q)$. In fact, they are the only known ovoids in $Q(4,q)$ at all.

Now we prove some useful characterization results.

The following result is due to THAS [18] and GEVAERT, JOHNSON & THAS [4].

Lemma 15 *If O is a non-classical ovoid of $Q(4,q)$, then $O \cong O_{K_1}$ if and only if O is the union of q conics on $Q(4,q)$ (viewed as a quadric in $\mathbf{PG}(4,q)$) all containing a common point x (O is a translation ovoid with respect to the point x).*

We have the following corollaries.

Corollary 16 *Let O be a translation ovoid with respect to the point (∞) of $Q(4,q)$. If O contains a conic C of $Q(4,q)$, then either $O \cong O_E$ or $O \cong O_{K_1}$.*

PROOF. This follows from the previous lemma by noting that the group $G_{\{(\infty)\}}$ maps C to q mutually tangent conics at (∞). □

If the kernel K is "large" enough with respect to the field $GF(q)$, then one can prove something more.

Corollary 17 *Let O be a translation ovoid with respect to the point ∞ of $Q(4,q)$ and let $K = \mathbf{GF}(q')$ be the kernel of O. If $q = q'$, then O is isomorphic to the classical ovoid O_E and if $q = {q'}^2$ then O is isomorphic to the ovoid O_{K_1} of Kantor.*

PROOF. Let O be a translation ovoid with respect to the point (∞), i. e. $O = \{(\infty)\} \cup \{(a, l, f(a, l)) | a, l \in \mathbf{GF}(q)\}$ with $f(a, l) = \sum_{i=0}^{e-1} (\lambda_i a^{p^i} + \beta_i l^{p^i})$ and $q = p^e$. Then O has kernel $\mathbf{GF}(q')$, $q = {q'}^h$, if and only if $\mathbf{GF}(q')$ is the largest subfield of $\mathbf{GF}(q)$ consisting of elements x for which $f(xa, xl) = xf(a, l)$, for each $(a, l) \in \mathbf{GF}(q)^2$. This means that

$$\sum_{i=0}^{e-1} (\lambda_i (xa)^{p^i} + \beta_i (xl)^{p^i}) = x \sum_{i=0}^{e-1} (\lambda_i a^{p^i} + \beta_i l^{p^i}),$$

for each $(a,l) \in \mathbf{GF}(q)^2$ and $x \in \mathbf{GF}(q')$, or, equivalently,

$$\sum_{i=0}^{e-1}(\lambda_i(x - x^{p^i})a^{p^i} + \beta_i(x - x^{p^i})l^{p^i}) = 0, \ \ \forall (a,l) \in \mathbf{GF}(q)^2, \forall x \in \mathbf{GF}(q').$$

This implies that $\lambda_i(x - x^{p^i}) = \beta_i(x - x^{p^i}) = 0$ for each $i \in \{0,\ldots,e-1\}$ and each $x \in \mathbf{GF}(q')$. Hence $\lambda_i = \beta_i = 0$, for each $i \in \{0,\ldots,e-1\}$ for which $p^i \notin \{1, q', \ldots, {q'}^{h-1}\}$.

If $q = q'$, then $f(a,l) = \lambda a + \beta l$, so (referring to Table 1) O is contained in the hyperplane with equation $X_3 = \lambda X_4 + \beta X_0$, hence $O \cong O_E$.

If $q = {q'}^2$ then $f(a,l) = \lambda_0 a + \lambda_1 a^{q'} + \beta_0 l + \beta_1 l^{q'}$. If $\beta_1 = 0$, then the q points $(0, l, \beta_0 l)$, together with (∞), lie in the plane with equations $X_4 = X_3 - \beta_0 X_0 = 0$ (use Table 1 again), hence on a conic and the result follows from Corollary 16. If $\beta_1 \neq 0$, then the q points

$$(a, -\frac{\lambda_1^{q'} a}{\beta_1^{q'}}, \lambda_0 a - \frac{\beta_0 \lambda_1^{q'} a}{\beta_1^{q'}}),$$

together with (∞), lie in the plane with equations

$$X_3 - \lambda_0 X_4 - \beta_0 X_0 = \lambda_1^{q'} X_4 + \beta_1^{q'} X_0 = 0.$$

The result again follows from Corollary 16. $\square$

If the kernel is smaller, but still large enough, then we have the following computer result.

Theorem 18 *Each translation ovoid w. r. t. a point of $Q(4, {q'}^3)$, whose kernel contains $GF(q')$, $3 \le q' \le 31$ and q' odd, is either isomorphic to the the classical ovoid O_E or to the Kantor ovoid O_{K_1} or to the Roman ovoid O_{TP}.*

PROOF. By computer using GAP (see below). $\square$

Remark. There is a connection between semifield flocks of quadratic cones and translation ovoids with respect to a point of $Q(4,q)$. In fact, they are equivalent objects. This is due to Thas [19]; see also Bloemen [1]. So Corollary 17 and Theorem 18 can also be formulated in terms of semifield flocks. In fact, the proof of Theorem 18 makes extensive use of that connection, as we will explain now.

More about the proof of Theorem 18. By the above remark, we must classify all semifield flocks in $\mathbf{PG}(3, {q'}^3)$ whose kernel contains $\mathbf{GF}(q')$. Dualizing the situation, and assuming we do not have a classical ovoid O_E or a Kantor ovoid O_{K_1}, it is readily seen that this implies that we must classify all subplanes Π isomorphic to $\mathbf{PG}(2,q')$ in a projective plane $\mathbf{PG}(2, {q'}^3)$ containing only internal points of a given conic C. First, one classifies all external lines of C containing sublines isomorphic to $\mathbf{PG}(1,q')$ only consisting of internal points of C. For example, if $9 \leq q' \leq 31$, the computer does not find such lines! It follows that for $9 \leq q' \leq 31$ only bisecants are possible in Π. But then considering any line pencil in Π and applying the polarity associated with C, we obtain an external line L containing a subline isomorphic to $\mathbf{PG}(1,q')$ consisting only of external points. As the group $\mathbf{PGL}(2,{q'}^3)$ of L contains an element which interchanges the internal and external points of C on L, this implies that on L there is also a subline isomorphic to $\mathbf{PG}(1,q')$ consisting only of internal points, a contradiction. This takes care of $9 \leq q' \leq 31$. If $3 \leq q' \leq 7$, then we do find such lines. We also find all sublines isomorphic to $\mathbf{PG}(1,q')$ which consist only of internal points and which lie on a bisecant of C. It is then easy to write a programme to find all subplanes Π. The result is that only for $q' = 3$ one finds such subplanes, and they are all elements of one orbit with respect to the projective group fixing C. The theorem easily follows.

For a complete detailed proof, including the programmes used, we refer to Bloemen [1].

A connection between locally hermitian ovoids of $H(3^h)$ and Kantor ovoids O_{K_1} of $Q(4,3^h)$.

Theorem 19 *Let O be a locally hermitian ovoid in x of $H(q)$, $q = 3^e$, and let $H(q)$ be embedded in the quadric $Q(6,q)$. Then each ovoid O_v of the generalized quadrangle $Q(4,q)$ which is obtained by projecting O from a point v collinear with x in $H(q)$, $v \neq x$, is isomorphic to an ovoid O_{K_1} of Kantor.*

PROOF. This follows from the fact that the point reguli of $H(q)$ through x are conics. So O_v is the union of conics having two by two just the projection of x in common, and the result follows from Lemma 15. $\square$

Remark. Applying the definition of translation ovoid to the generalized quadrangle $H(3,q^2)$, one can show that in this case translation ovoids are the union of q^2 projective sublines over $\mathbf{GF}(q)$ of the space $\mathbf{PG}(3,q^2)$, all containing a common point. Since we will not need that result, we omit the proof.

4 New classes of ovoids of $H(q)$, $q = 3^h$

Embed the generalized hexagon $H(q)$ in the non-singular quadric $Q(6,q)$. In this section we will determine all ovoids of $H(q)$ which contain a point-regulus $\mathcal{R}$ and are isomorphic to the Hermitian ovoid $\mathcal{U}_H$ under the group $\mathbf{PGO}(7,q)$ of $Q(6,q)$ (but which are not necessarily isomorphic to $\mathcal{U}_H$ under the group $G_2(q)$ of the generalized hexagon $H(q)$).

Let $Q(6,q)$ be represented by the equation $X_0X_4+X_1X_5+X_2X_6=X_3^2$, as before. Again we coordinatize the generalized hexagon $H(q)$ as in Section 2, see Table 2. Let

$$\mathcal{U}_H = \{(\infty)\} \cup \{(a, -\gamma a''^3, a', \gamma^{-1}a^3, a'')|(a,a',a'') \in GF(q)^3\},$$

with γ a given non-square of $GF(q)$ (see 3.1).

If $\mathcal{R}$ is the regulus through the points (∞) and $(0,0,0,0,0)$, then

$$\mathcal{R} = \{(\infty)\} \cup \{(0,0,a',0,0)|a' \in \mathbf{GF}(q)\},$$

or in projective coordinates

$$\mathcal{R} = \{(1,0,0,0,0,0,0)\} \cup \{(a'^2,0,0,a',1,0,0)|a' \in \mathbf{GF}(q)\}.$$

Now we determine all transformations $\psi \in \mathbf{PGO}(7,q)$, which fix the point $(1,0,0,0,0,0,0)$, which fix the regulus $\mathcal{R}$ and which do not necessarily fix the generalized hexagon $H(q)$. Then each ovoid $\mathcal{U}_H^\psi$ is an ovoid of $H(q)$ which is not necessarily isomorphic to $\mathcal{U}_H$ under the group $G_2(q)$ of $H(q)$.

Let Q be the matrix which defines the quadric $Q(6,q)$, i. e. Q is the permutation matrix associated to the permutation $(0\ 4)(1\ 5)(2\ 6)$ (fixing 3). If ψ is the transformation defined by the matrix T_ψ, then ψ fixes the quadric $Q(6,q)$ if and only if $T_\psi^t Q T_\psi = \lambda Q$, with $\lambda \in GF(q)$. The transformation ψ fixes $Q(6,q)$, $(1,0,0,0,0,0,0)$ and $\mathcal{R}$ only if T_ψ is a non-singular matrix of the following form:

$$T_\psi = \begin{pmatrix} \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2 & c_2 & 0 & 0 & d_2 & e_2 \\ 0 & b_3 & c_3 & 0 & 0 & d_3 & e_3 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & b_6 & c_6 & 0 & 0 & d_6 & e_6 \\ 0 & b_7 & c_7 & 0 & 0 & d_7 & e_7 \end{pmatrix}.$$

Since the automorphisms ψ and $\psi\phi$, with $\phi \in G_2(q)$, define mutually isomorphic ovoids $\mathcal{U}_H^\psi$ and $\mathcal{U}_H^{\psi\phi}$ with respect to $G_2(q)$, we only have to determine the different cosets $\psi G_2(q)$, with $\psi \in \mathbf{PGO}(7,q)$. We will use this to simplify the matrices T_ψ.

Let T_{xy} and U_k be the following matrices defining automorphisms (the first one being generated by generalized homologies, see De Smet & Van Maldeghem [2]) of $H(q)$:

$$T_{xy} = \mathrm{diag}(x^4y^2, xy, x, x^2y, 1, x^3y, x^3y^2),$$

$$U_k = I + kA_{1,2} - kA_{6,5},$$

where I is the 7×7 identity matrix, and $A_{i,j}$ is the matrix with all entries equal to 0 except for the one on row i and column j, which is equal to 1 (starting with row 0 and column 0, conform to our notation for coordinates in $\mathbf{PG}(6,q)$). If W_r is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -r & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -r \end{pmatrix},$$

then W_r defines an automorphism of $H(q)$ which fixes the point (∞) and maps the line $[\infty]$ onto the line $[r]$.

Considering the matrix $T_{xy}T_\psi$, with $y = (\lambda x^2)^{-1}$, we see that we may assume that $\lambda = 1$ (geometrically, this means that ψ fixes the regulus $\mathcal{R}$ pointwise).

If $b_2 \neq 0$ and $b_3 \neq 0$, then we consider the matrix $U_k T_\psi$, with $k = -b_2 b_3^{-1}$, and if $b_2 \neq 0$ and $b_3 = 0$, then we consider the matrix $W_r T_\psi$, with $r = 0$. In both cases, we see that the $(1,1)$-entry of the resulting matrix is zero. Hence without loss of generality we may suppose that T_ψ has $b_2 = 0$.

The automorphism ψ fixes the quadric $Q(6,q)$ if and only if $T_\psi^t Q T_\psi = \lambda Q$; for such a T_ψ with $\lambda = 1$ we necessarily have $\mathbb{T}_\psi \in \mathbf{PSO}(7,q)$. This condition is satisfied if and only if the following 10 equations hold:

$$\begin{aligned} b_3b_7 &= 0, && (13)\\ b_3c_7 + b_6c_2 + b_7c_3 &= 0, && (14)\\ b_3d_7 + b_6d_2 + b_7d_3 &= 1, && (15)\\ b_3e_7 + b_6e_2 + b_7e_3 &= 0, && (16)\\ c_2c_6 + c_3c_7 &= 0, && (17)\\ c_2d_6 + c_3d_7 + c_6d_2 + c_7d_3 &= 0, && (18)\\ c_2e_6 + c_3e_7 + c_6e_2 + c_7e_3 &= 1, && (19)\\ d_2d_6 + d_3d_7 &= 0, && (20)\\ d_2e_6 + d_3e_7 + d_6e_2 + d_7e_3 &= 0, && (21)\\ e_2e_6 + e_3e_7 &= 0. && (22) \end{aligned}$$

1. First we suppose that $b_3 \neq 0$, so let $b_3 = \lambda$, with $\lambda \in GF(q)^*$. Equations (13), (14), (15) and (16) yield

$$
\begin{aligned}
b_7 &= 0, \\
c_7 &= c_2\beta, && (23) \\
d_7 &= d_2\beta + \lambda^{-1}, && (24) \\
e_7 &= e_2\beta, && (25)
\end{aligned}
$$

with $\beta = -b_6\lambda^{-1}$, $\beta \in \mathbf{GF}(q)$.

(a) Suppose that $c_2 \neq 0$. So $c_2 = \alpha$, $\alpha \in \mathbf{GF}(q)^*$. Equations (23), (24), (25), (17), (18), (19), (20), (21) and (22) then yield

$$
\begin{aligned}
c_7 &= \alpha\beta, && (26) \\
c_6 &= -\beta c_3, && (27) \\
d_6 &= -\frac{\alpha\beta\lambda d_3 + c_3}{\alpha\lambda}, && (28) \\
e_6 &= \frac{1-\alpha\beta e_3}{\alpha}, && (29) \\
d_3 &= \frac{d_2 c_3}{\alpha}, && (30) \\
d_2 &= \frac{e_2 c_3 - \alpha e_3}{\lambda}, && (31) \\
e_2 &= 0. && (32)
\end{aligned}
$$

From $e_2 = 0$ there follows that equation (31) is equivalent with $d_2 = -\frac{\alpha e_3}{\lambda}$. If we substitute this in (30), then we have that $d_3 = -\frac{c_3 e_3}{\lambda}$. This, together with (28) yields $d_6 = c_3\frac{\alpha\beta e_3 - 1}{\alpha\lambda}$. If we substitute $d_2 = -\frac{\alpha e_3}{\lambda}$ in equation (24) there follows that $d_7 = \frac{1-\alpha\beta e_3}{\lambda}$. Let $c_3 = \mu$ and $e_3 = n$, with $\mu, n \in \mathbf{GF}(q)$. Then the matrix T_ψ is given by

$$
T_1(\lambda,\alpha,\beta,\mu,n) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha & 0 & 0 & -\frac{\alpha n}{\lambda} & 0 \\
0 & \lambda & \mu & 0 & 0 & -\frac{\mu n}{\lambda} & n \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -\lambda\beta & -\mu\beta & 0 & 0 & -\mu\frac{1-\alpha\beta n}{\alpha\lambda} & \frac{1-\alpha\beta n}{\alpha} \\
0 & 0 & \alpha\beta & 0 & 0 & \frac{1-\alpha\beta n}{\lambda} & 0
\end{pmatrix},
$$

with $\alpha, \lambda \in \mathbf{GF}(q)^*$ and $\mu, \beta, n \in \mathbf{GF}(q)$.

(b) Suppose that $c_2 = 0$. If $e_2 = 0$, then equation (23) implies $c_7 = 0$ and from (25) we have that $e_7 = 0$. This together with (19) gives a contradiction. Hence $e_2 \neq 0$. Let $I_{2,6} \in \mathbf{PSO}(7, q)$ be the permutation matrix associated to the permutation (2 6). Consider the matrix $T'_\psi = T_\psi I_{2,6}$. Then the elements c_2 and e_2 are interchanged. So T'_ψ is a matrix of type $T_1(\lambda, \alpha, \beta, \mu, n)$ and the matrix T_ψ can be written as $T_1(\lambda, \alpha, \beta, \mu, n)I_{2,6}$. Denote this matrix by $T_3(\lambda, \alpha, \beta, \mu, n)$.

2. Suppose that $b_3 = 0$. If $b_7 = 0$, then $b_6 \neq 0$, otherwise the matrix T_ψ is singular. So from (14) and (16) follows that $c_2 = e_2 = 0$. A few calculations now show that T_ψ is singular, a contradiction. Hence $b_7 \neq 0$. Now consider the matrix $T'_\psi = I_{2,6}T_\psi$. Since the elements b_3 and b_7 are interchanged, we are back in the first case, hence $T'_\psi = T_1(\lambda, \alpha, \beta, \mu, n)$ or $T'_\psi = T_3(\lambda, \alpha, \beta, \mu, n)$, which implies that $T_\psi = I_{2,6}T_1(\lambda, \alpha, \beta, \mu, n)$ or $T_\psi = I_{2,6}T_3(\lambda, \alpha, \beta, \mu, n)$. Denote these matrices respectively by $T_2(\lambda, \alpha, \beta, \mu, n)$ and $T_4(\lambda, \alpha, \beta, \mu, n)$.

Now we want to reduce the number of parameters $\lambda, \alpha, \beta, \mu, n$, by taking another representative of the respective cosets $\mathcal{A}T_i(\lambda, \alpha, \beta, \mu, n)$, with $\mathcal{A}$ the group of all non-singular matrices representing the elements of $G_2(q)$, $i = 1, 2, 3, 4$.

- In the matrix $T_{xy}T_1(\lambda, \alpha, \beta, \mu, n)$, with $x = \lambda^{-1}$ and $y = \lambda^2$, we replace α by $\alpha\lambda^{-1}$, n by λn and μ by $\lambda\mu$. We then see that the parameter λ disappears. We denote the resulting matrix by $T'_1(\alpha, \beta, \mu, n)$, with $\alpha \in \mathbf{GF}(q)^*$ and $\mu, \beta, n \in \mathbf{GF}(q)$. Since $I_{2,6}T_{xy} = T_{xy}I_{2,6}$, it is clear that we may replace $T_2(\lambda, \alpha, \beta, \mu, n)$ by $T'_2(\alpha, \beta, \mu, n) = I_{2,6}T'_1(\alpha, \beta, \mu, n)$, $T_3(\lambda, \alpha, \beta, \mu, n)$ by $T'_3(\alpha, \beta, \mu, n) = T'_1(\alpha, \beta, \mu, n)I_{2,6}$ and $T_4(\lambda, \alpha, \beta, \mu, n)$ by $T'_4(\alpha, \beta, \mu, n) = I_{2,6}T'_1(\alpha, \beta, \mu, n)I_{2,6}$.

- The matrix $W_rT'_1(\alpha, \beta, \mu, n)$, with $r = \frac{\mu}{\alpha}$, is independent of μ, hence we may denote it by $T''_1(\alpha, \beta, n)$, with $\alpha \in \mathbf{GF}(q)^*$ and $\beta, n \in \mathbf{GF}(q)$.

  Since $T'_3(\alpha, \beta, \mu, n) = T'_1(\alpha, \beta, \mu, n)I_{2,6}$, we also have that $T''_3(\alpha, \beta, n) = W_rT'_3(\alpha, \beta, \mu, n) = T''_1(\alpha, \beta, n)I_{2,6}$.

- Similarly $W_rT'_2(\alpha, \beta, \mu, n)$, with $r = \beta$, does not depend on $\beta = r$, hence we may denote it by $T''_2(\alpha, \mu, n)$, $\alpha \in \mathbf{GF}(q)^*$ and $\mu, n \in \mathbf{GF}(q)$. The matrix $T''_4(\alpha, \mu, n) = W_\beta T'_4(\alpha, \beta, \mu, n)$ is equal to $T''_2(\alpha, \mu, n)I_{2,6}$.

From now on, since there is no confusion possible, we will write $T_i(\alpha, \beta, n)$ instead of $T''_i(\alpha, \beta, n)$. Let $\theta_i(\alpha, \beta, n)$ be the automorphism which corresponds with $T_i(\alpha, \beta, n)$, $i = 1, 2, 3, 4$, and let $K = \{\theta_i(\alpha, \beta, n) | i \in \{1, 2, 3, 4\}, (\alpha, \beta, n) \in \mathbf{GF}(q)^3, \alpha \neq 0\}$. In this way we obtain the following result.

Lemma 20 *If O is an ovoid of $H(q)$, $q = 3^e$, which contains a regulus and which is isomorphic to $\mathcal{U}_H$ under the group $\mathbf{PGO}(7, q)$, then there exists an automorphism ϕ of $H(q)$ and an automorphism $\theta \in K$ for which $O^\phi = \mathcal{U}_H^\theta$.*

We are now ready to prove the main result of this section.

Theorem 21 *If O is a translation ovoid of $H(q)$, $q = 3^h$, $h > 1$, with respect to (∞), which is isomorphic to $\mathcal{U}_H$ under the group $\mathbf{PGO}(7, q)$, then O is isomorphic under $G_2(q)$, either to the Hermitian ovoid $\mathcal{U}_H$ or to an ovoid of type*

$$O_\beta = \{(\infty)\} \cup \{(a, -\gamma a''^3 + \beta a, a', \gamma^{-1}a^3 + \beta a'', a'')|(a, a', a'') \in \mathbf{GF}(q)^3\},$$

with $\beta \in \mathbf{GF}(q)^*$. Also, for $\beta' \in \mathbf{GF}(q)^*$, the ovoids O_β and $O_{\beta'}$ are isomorphic with respect to $\mathrm{Aut}(H(q))$.

PROOF. The Hermitian ovoid $\mathcal{U}_H$ of $H(q)$ is given by the set of points

$$\mathcal{U}_H = \{(\infty)\} \cup \{(a, -\gamma a''^3, a', \gamma^{-1}a^3, a'')|(a, a', a'') \in GF(q)^3\},$$

or in projective coordinates

$$\begin{aligned}\mathcal{U}_H = \{(1,0,0,0,0,0,0)\} \cup \{&(x_3^2 - \gamma^{-1}(\gamma x_1^2 - x_2^2)^2, x_1, x_2, x_3, 1, f_1(x_1, x_2, x_3),\\ &f_2(x_1, x_2, x_3))|(x_1, x_2, x_3) \in GF(q)^3\},\end{aligned}$$

where $f_1(x_1, x_2, x_3) = \gamma x_1^3 - x_1x_2^2 - x_2x_3$ and $f_2(x_1, x_2, x_3) = x_2^3\gamma^{-1} - x_1^2x_2 + x_1x_3$, with γ a non-square of $GF(q)$. This means that

$$(x_3^2 - (x_1x_5 + x_2x_6), x_1, x_2, x_3, 1, x_5, x_6) \in \mathcal{U}_H \Leftrightarrow \left\{\begin{array}{lcl} x_5 & = & f_1(x_1, x_2, x_3),\\ x_6 & = & f_2(x_1, x_2, x_3).\end{array}\right.$$

Let $O_i(\alpha, \beta, n) = \mathcal{U}_H^{\theta_i(\alpha,\beta,n)}$, for each $i \in \{1, 2, 3, 4\}$, $\alpha \in \mathbf{GF}(q)^*$ and $\beta, n \in \mathbf{GF}(q)$. We will examine which of these ovoids are translation ovoids with respect to (∞). Since the automorphism group of $H(q)$ acts transitively on the incident pairs (x, point regulus containing x), x any point of $H(q)$, all translation ovoids of $H(q)$ which are isomorphic to $\mathcal{U}_H$ under the group $\mathbf{PGO}(7, q)$ are obtained in this way. From Theorem 6 we have that if $O_i(\alpha, \beta, n)$ is a translation ovoid with respect to (∞), then $O_i(\alpha, \beta, n)$ is the union of q^2 point reguli through (∞). The point regulus of $H(q)$ through (∞) and a point (a, l, a', l', a'') is given by the set

$$\{(\infty)\} \cup \{(a, l, a' + \lambda, l', a'')|\lambda \in \mathbf{GF}(q)\}.$$

If σ_λ is the group element $\Theta[0,0,0,\lambda,0] \in T$, then this set can also be written as

$$\{(\infty)\} \cup \{(a,l,a',l',a'')^{\sigma_\lambda} | \lambda \in \mathbf{GF}(q)\}.$$

From this follows that an ovoid $O_i(\alpha,\beta,n)$ is the union of q^2 point reguli through (∞) if and only if for each element p of $O_i(\alpha,\beta,n)$ and for each $\lambda \in \mathbf{GF}(q)$, there holds that $p^{\sigma_\lambda} \in O_i(\alpha,\beta,n)$. Since $O_i(\alpha,\beta,n) = \mathcal{U}_H^{\theta_i(\alpha,\beta,n)}$, this is equivalent with the condition that for each $p \in \mathcal{U}_H$, there must hold that $p^{\theta_i(\alpha,\beta,n)\sigma_\lambda\theta_i^{-1}(\alpha,\beta,n)} \in \mathcal{U}_H$.

Let T_λ be the matrix which is associated with σ_λ, i. e.

$$T_\lambda = \begin{pmatrix} 1 & 0 & 0 & \lambda & \lambda^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the ovoid $O_i(\alpha,\beta,n)$ is the union of q^2 point reguli through (∞) if and only if

$$T_i(\alpha,\beta,n)^{-1}T_\lambda T_i(\alpha,\beta,n) \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ 1 \\ f_1(x_1,x_2,x_3) \\ f_2(x_1,x_2,x_3) \end{pmatrix} \in \mathcal{U}_H,$$

with $x_0 = x_3^2 - \gamma^{-1}(\gamma x_1^2 - x_2^2)^2$, for each $(\lambda,x_1,x_2,x_3) \in \mathbf{GF}(q)^4$, $i \in \{1,2,3,4\}$, $\alpha \in \mathbf{GF}(q)^*$ and $\beta, n \in \mathbf{GF}(q)$. Denote this condition by $(*)$.

1. First we will determine which ovoids of type $O_1(\alpha,\beta,n)$ can be written as the union of q^2 point reguli through (∞).

   The ovoid $O_1(\alpha,\beta,n)$ is the union of q^2 point reguli through (∞) if and only if condition $(*)$ holds for $i=1$. This is equivalent with

   $$\begin{aligned}&(x_0+\lambda(x_3+\lambda), x_1 - \alpha\lambda n(nf_2(x_1,x_2,x_3)+x_1), \alpha\lambda(n^2 f_1(x_1,x_2,x_3)\\ &\quad -nx_2)+x_2, x_3-\lambda, 1, \alpha\lambda(nf_1(x_1,x_2,x_3)-x_2)+f_1(x_1,x_2,x_3),\\ &\quad \alpha\lambda(nf_2(x_1,x_2,x_3)+x_1)+f_2(x_1,x_2,x_3)) \in \mathcal{U}_H,\end{aligned}$$

for each $(\lambda, x_1, x_2, x_3) \in \mathbf{GF}(q)^4$. If $x_2 = x_3 = 0$, then $f_1(x_1, 0, 0) = \gamma x_1^3$ and $f_2(x_1, 0, 0) = 0$. If condition $(*)$ is satisfied there must hold that

$$\begin{cases} (\alpha\lambda n + 1)\gamma x_1^3 = f_1((1 - \alpha\lambda n)x_1, \alpha\lambda n^2\gamma x_1^3, -\lambda), \\ \alpha\lambda x_1 = f_2((1 - \alpha\lambda n)x_1, \alpha\lambda n^2\gamma x_1^3, -\lambda), \end{cases}$$

for each $(\lambda, x_1) \in \mathbf{GF}(q)^2$. The coefficient of λ in the first equation is given by $\alpha n x_1^3$ and must be zero for each $x_1 \in \mathbf{GF}(q)$. This implies that $n = 0$. If we use this in the second equation we find that $\alpha\lambda x_1 = f_2(x_1, 0, -\lambda) = -\lambda x_1$, for each $(\lambda, x_1) \in \mathbf{GF}(q)^2$; hence $\alpha = -1$.

Each ovoid $O_1(-1, \beta, 0)$ is given by the set of points

$$\{(1,0,0,0,0,0,0)\} \cup \{(x_3^2 - \gamma^{-1}(\gamma x_1^2 - x_2^2)^2, x_1, x_2, x_3, 1, \\ f_1(x_1, x_2, x_3) - \beta x_2, f_2(x_1, x_2, x_3) + \beta x_1) | (x_1, x_2, x_3) \in \mathbf{GF}(q)^3\},$$

or in coordinates of $H(q)$

$$\{(\infty)\} \cup \{(a, -\gamma a''^3 + \beta a, a', \gamma^{-1}a^3 + \beta a'', a'') | (a, a', a'') \in GF(q)^3\}.$$

If $\Theta(K, B, K', B', K'')$ corresponds to $\Theta[\sqrt[3]{K}, B, \sqrt[3]{K'}, B', \sqrt[3]{K''}]$ under the duality τ of $H(q)$ (see the formulae 12 in Section 3.1), then the group

$$\{\Theta(A, -\gamma A''^3 + \beta A, A', \gamma^{-1}A^3 + \beta A'', A'') | (A, A', A'') \in \mathbf{GF}(q)^3\}$$

acts transitively on the points of $O_1(-1, \beta, 0) \setminus \{(\infty)\}$. So by Lemma 5, we have that $O_1(-1, \beta, 0)$ is a translation ovoid with respect to (∞).

This means that an ovoid of type $O_1(\alpha, \beta, n)$ is a translation ovoid with respect to (∞) if and only if $n = 0$ and $\alpha = -1$.

2. The ovoid $O_2(\alpha, \beta, n)$ is the union of q^2 point-reguli through (∞) if and only if condition $(*)$, with $i = 2$, is satisfied, or equivalently, if and only if

$$(x_0 + \lambda(x_3 + \lambda), x_1 - \alpha\lambda x_2, x_2, x_3 - \lambda, 1, f_1(x_1, x_2, x_3), \\ \alpha\lambda f_1(x_1, x_2, x_3) + f_2(x_1, x_2, x_3)) \in \mathcal{U}_H,$$

for each $(\lambda, x_1, x_2, x_3) \in \mathbf{GF}(q)^4$. This is equivalent with

$$\begin{cases} f_1(x_1, x_2, x_3) = f_1(x_1 - \alpha\lambda x_2, x_2, x_3 - \lambda), \\ \alpha\lambda f_1(x_1, x_2, x_3) + f_2(x_1, x_2, x_3) = f_2(x_1 - \alpha\lambda x_2, x_2, x_3 - \lambda), \end{cases} \quad (**)$$

for each $(\lambda, x_1, x_2, x_3) \in \mathbf{GF}(q)^4$. It is easy to see that the first equation does not hold for each $(\lambda, x_1, x_2, x_3) \in \mathbf{GF}(q)^4$. This means that an ovoid of type $O_2(\alpha, \beta, n)$, with $\alpha \in \mathbf{GF}(q)^*$ and $\beta, n \in \mathbf{GF}(q)^2$, never is a translation ovoid with respect to

(∞). We can say even more about these ovoids. The points p of $O_2(\alpha, \beta, n)$ for which the regulus through (∞) and p is contained in $O_2(\alpha, \beta, n)$, are determined by the values $x_1, x_2, x_3 \in \mathbf{GF}(q)^3$ which satisfy the equations of $(**)$ for each $\lambda \in \mathbf{GF}(q)$. It is not difficult to see that these equations are satisfied for each $\lambda \in \mathbf{GF}(q)$ if and only if $x_1 = x_2 = 0$. This implies that $p \in \mathcal{R}$, with $\mathcal{R}$ the regulus through (∞) and $(0, 0, 0, 0, 0)$. Hence an ovoid of type $O_2(\alpha, \beta, n)$ contains a unique regulus through (∞).

3. The ovoid $O_3(\alpha, \beta, n)$ is the union of q^2 point reguli through (∞) if and only if condition $(*)$ holds for $i = 3$. Notice that, since $T_3(\alpha, \beta, n) = T_1(\alpha, \beta, n)I_{2,6}$, there follows that

$$T_3^{-1}(\alpha, \beta, n)T_\lambda T_3(\alpha, \beta, n) = I_{2,6}T_1^{-1}(\alpha, \beta, n)T_\lambda T_1(\alpha, \beta, n)I_{2,6}.$$

This means that $(*)$ is satisfied if and only if

$$\begin{aligned}&(x_0 + \lambda(x_3 + \lambda), x_1 - \alpha\lambda n(nx_2 + x_1), \alpha\lambda(nx_2 + x_1) + x_2, x_3 - \lambda, 1,\\ &\quad \alpha\lambda(nf_1(x_1, x_2, x_3) - f_2(x_1, x_2, x_3)) + f_1(x_1, x_2, x_3),\\ &\quad \alpha\lambda n(nf_1(x_1, x_2, x_3) - f_2(x_1, x_2, x_3)) + f_2(x_1, x_2, x_3)) \in \mathcal{U}_H,\end{aligned}$$

for each $(\lambda, x_1, x_2, x_3) \in \mathbf{GF}(q)^4$. Let $x_2 = x_3 = 0$. If the condition above is satisfied, there must hold that

$$\begin{cases} (\alpha\lambda n + 1)\gamma x_1^3 = f_1(x_1(1 - \alpha\lambda n), \alpha\lambda x_1, -\lambda),\\ \alpha\lambda n^2\gamma x_1^3 = f_2(x_1(1 - \alpha\lambda n), \alpha\lambda x_1, -\lambda),\end{cases}$$

for each $(\lambda, x_1) \in \mathbf{GF}(q)^2$. The first equation can be written as

$$(\alpha\lambda n + 1)\gamma x_1^3 = \gamma x_1^3(1 - \alpha\lambda n)^3 - x_1^3\alpha^2\lambda^2(1 - \alpha\lambda n) + \alpha\lambda^2 x_1,$$

for each $(\lambda, x_1) \in \mathbf{GF}(q)^2$. The coefficient of λ^2 is $-x_1^3\alpha^2 + \alpha x_1$, which is zero for each $x_1 \in \mathbf{GF}(q)$ if and only if $\alpha = 0$, a contradiction. This means that an ovoid of type $O_3(\alpha, \beta, n)$, with $\alpha \in \mathbf{GF}(q)^*$ and $\beta, n \in \mathbf{GF}(q)^2$, never is a translation ovoid with respect to (∞).

4. The ovoid $O_4(\alpha, \beta, n)$ is the union of q^2 point reguli through (∞) if and only if condition $(*)$ holds for $i = 4$. We have that $T_4(\alpha, \beta, n) = T_2(\alpha, \beta, n)I_{2,6}$, hence

$$T_4^{-1}(\alpha, \beta, n)T_\lambda T_4(\alpha, \beta, n) = I_{2,6}T_2^{-1}(\alpha, \beta, n)T_\lambda T_2(\alpha, \beta, n)I_{2,6}.$$

Condition $(*)$ is satisfied if and only if

$$\begin{aligned}&(x_0 + \lambda(x_3 + \lambda), x_1 - \alpha\lambda f_2(x_1, x_2, x_3), x_2 + \lambda\alpha f_1(x_1, x_2, x_3), x_3 - \lambda,\\ &\quad 1, f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3)) \in \mathcal{U}_H,\end{aligned}$$

for each $(\lambda, x_1, x_2, x_3) \in \mathbf{GF}(q)^4$. This is equivalent with

$$\begin{cases} f_1(x_1, x_2, x_3) = \\ \qquad\qquad f_1(x_1 - \alpha\lambda f_2(x_1, x_2, x_3), x_2 + \alpha\lambda f_1(x_1, x_2, x_3), x_3 - \lambda), \\ f_2(x_1, x_2, x_3) = \\ \qquad\qquad f_2(x_1 - \alpha\lambda f_2(x_1, x_2, x_3), x_2 + \alpha\lambda f_1(x_1, x_2, x_3), x_3 - \lambda), \end{cases}$$

for each $(\lambda, x_1, x_2, x_3) \in \mathbf{GF}(q)^4$. If $x_1 = x_3 = 0$, then the first equation becomes

$$0 = f_1(-\alpha\lambda f_2(0, x_2, 0), x_2, -\lambda),$$

for each $(\lambda, x_2) \in \mathbf{GF}(q)^2$. So there must hold that

$$-\gamma\alpha^3\lambda^3 f_2(0, x_2, 0)^3 + \alpha\lambda f_2(0, x_2, 0)x_2^2 + \lambda x_2,$$

for each $(\lambda, x_2) \in \mathbf{GF}(q)^2$. The coefficient of λ^3 is $-\gamma^{-2}\alpha^3 x_2^9$, which should be zero for each $x_2 \in \mathbf{GF}(q)$, a contradiction.

This means that an ovoid of type $O_4(\alpha, \beta, n)$, with $\alpha \in \mathbf{GF}(q)^*$ and $\beta, n \in \mathbf{GF}(q)^2$, never is a translation ovoid with respect to (∞).

So the ovoid $O_i(\alpha, \beta, n)$ of $H(q)$, $q = 3^e > 3$, is a translation ovoid with respect to (∞) if and only if $i = 1$, $\alpha = -1$ and $n = 0$, and then $O_\beta = O_i(-1, \beta, 0)$ is given by

$$\{(\infty)\} \cup \{(a, -\gamma a''^3 + \beta a, a', \gamma^{-1}a^3 + \beta a'', a'') | (a, a', a'') \in GF(q)^3\}.$$

If $\beta = 0$, then $O_\beta \cong \mathcal{U}_H$. We show that also the converse is true, i. e. if $O_\beta \cong \mathcal{U}_H$, then $\beta = 0$. Each element g of $G_2(q)$ which fixes elementwise the apartment with elements

$$[\infty], (0), [0, 0], (0, 0, 0), [0, 0, 0, 0], (0, 0, 0, 0, 0),$$

$$[0, 0, 0, 0, 0], (0, 0, 0, 0), [0, 0, 0], (0, 0), [0], (\infty)\ ,$$

corresponds with a collineation g_{xy} with matrix T_{xy}, see above. The automorphism g_{xy} maps $(a, -\gamma a''^3 + \beta a, a', \gamma^{-1}a^3 + \beta a'', a'')$ onto the point

$$(xa, x^3y(-\gamma a''^3 + \beta a), x^2ya', x^3y^2(\gamma^{-1}a^3 + \beta a''), xya'').$$

So g_{xy} fixes O_β if and only if

- $y^2 = 1$ if $\beta = 0$,
- $y^2 = 1$ and $y = 1/x^2$ if $\beta \neq 0$.

So if $\beta = 0$, then this is a subgroup of order $2(q-1)$ and if $\beta \neq 0$, then this is a subgroup of order 2 if $q = -1 \bmod 4$ and of order 4 if $q = 1 \bmod 4$. Since the automorphism group of $\mathcal{U}_H$ acts 2-transitively on the points of $\mathcal{U}_H$, and since it is readily checked that the stabilizer in $G_2(q)$ of $\mathcal{U}_H$ fixing two given distinct points of $\mathcal{U}_H$ and fixing all the lines of $H(q)$ through these points has order $q-1$, there follows that if O_β, $\beta \in \mathbf{GF}(q)$, is isomorphic to $\mathcal{U}_H \cong O_\beta$, then either $q = 3$ (in contradiction with our assumption) or $\beta = 0$. So $O_\beta \cong \mathcal{U}_H \Leftrightarrow \beta = 0$.

To prove the theorem completely, we only have still to show that O_β and $O_{\beta'}$ are isomorphic if $\beta, \beta' \in \mathbf{GF}(q)^*$. If $\beta'/\beta = A^2$ is a square in $\mathbf{GF}(q)$, then it is readily checked that the transformation with matrix $\mathrm{diag}(A^4, A, A, A^2, 1, A^3, A^3)$ stabilizes $H(q)$ and maps O_β to $O_{\beta'}$. If β'/β is not a square in $\mathbf{GF}(q)$, then we may write it as $\beta'/\beta = A^2/\gamma$. If we first apply the transformation defined on the coordinates in $\mathbf{PG}(6,q)$ by $X_i \mapsto X_i$ $(i = 0, 4)$, $X_j \leftrightarrow X_{j+1}$ $(i = 1, 5)$, $X_3 \leftrightarrow -X_3$, followed by the transformation with matrix $\mathrm{diag}(\gamma^{-2}A^4, -\gamma^{-1}A, A, -\gamma^{-1}A^2, 1, -\gamma^{-1}A^3, \gamma^{-2}A^3)$, then $H(q)$ is stabilized and O_β is transformed into $O_{\beta'}$. $\square$

From the above proof immediately follows that the condition of being a translation ovoid in Theorem 21 can be weakened to being a locally hermitian ovoid without any change in the result.

It is also clear that one obtains a lot of new ovoids which are not translation ovoids. By dualizing $H(q)$ one obtains a lot of spreads of $H(q)$ which are not isomorphic to any previously known spread; these new spreads are also new 1-systems in the quadric $Q(6,q)$ (cf. Section 2.3).

Now we dualize the new ovoids found in Theorem 21 (using the formulae (12)) and obtain a spread of type

$$\mathcal{S}_\beta = \{[\infty]\} \cup \{[k, -\gamma k'' + \beta k^{1/3}, k', \gamma^{-1}k + \beta k''^{1/3}, k'']|(k, k', k'') \in \mathbf{GF}(q)^3\},$$

with $\beta \in \mathbf{GF}(q)^*$ and γ a non-square of $\mathbf{GF}(q)$. If we project $\mathcal{S}$ along reguli from the point (∞), then one can easily calculate the coordinates of the projection, and we obtain

$$\mathcal{O}_\beta = \{(\infty)\} \cup \{(a, -\gamma a'' + \beta a^{1/3}, \gamma^{-1}a + \beta a''^{1/3})|a, a'' \in \mathbf{GF}(q)\},$$

where we have put $a = k$ and $a'' = k''$. Hence

$$\mathcal{O}_\beta = \{(\infty)\} \cup \{(a, l, \gamma^{-1}a + \beta^{4/3}\gamma^{-1/3}a^{1/9} - \beta\gamma^{-1/3}l^{1/3})|a, l \in \mathbf{GF}(q)\}.$$

Without loss of generality, we may consider $\mathcal{O}_{-\gamma^{1/3}}$ and see that this is nothing else than the Roman ovoid of Thas & Payne [21].

Next, consider the point (x) on $[\infty]$. We now want to project $\mathcal{S}_\beta$ along reguli from (x). Reconsidering O_β, the point (x) corresponds to the line $[x]$. We now apply the collineation

$\Theta[-x,0,0,0,0]$ followed by the transformation defined on the coordinates in $\mathbf{PG}(6,q)$ by $X_i \mapsto X_{6-i}$ $(i=0,1,\ldots,6)$, and then we dualize again. The point (x) is thus mapped on (∞). Hence we can again easily calculate the coordinates of the projection along reguli from (∞), and after a tedious calculation, putting $\gamma\beta^3(1-\gamma x^2)$ equal to y^{-3}, for some $y \in \mathbf{GF}(q)$, we find

$$\mathcal{O}_{\beta,x} = \{(\infty)\} \cup \{(a, l, \gamma\beta^3 y^2(\gamma y a + \beta^{1/3} a^{1/9} + l^{1/3} + \gamma^{2/3}\beta x^{1/3} y a^{1/3} + \gamma x y l) | a, l \in \mathbf{GF}(q)\}.$$

Now we apply the collineation defined on the coordinates of $Q(4,q)$ as follows:

$$\begin{array}{rcl} (a,l,a') & \mapsto & (\gamma^3\beta^3 y^6 a, \gamma y^3 l + \gamma^3\beta^3 x y^6 a, \gamma^{-1}\beta^{-3} a' + \gamma^3\beta^3 x^2 y^6 a - \gamma x y^3 l), \\ {[k,l,k']} & \mapsto & [\gamma^{-2}\beta^{-3} y^{-3} k + x, \gamma^{-1}\beta^{-3} b, \gamma y^3 k'] \end{array}$$

(which can readily be checked to be indeed a collineation using the formulae (1) of Section 2). We obtain now the ovoid $\mathcal{O}_{-y}$, which is isomorphic to the Roman ovoid O_{TP} (see the previous paragraph).

Thus we have proved the following theorem:

Theorem 22 *Let $\mathcal{S}_H$ be a hermitian spread of $H(q)$, $q = 3^h$. Let τ_0 be any anti-automorphism from $H(q)$ onto its dual. Then there exists an automorphism θ of $Q(6,q)$ such that $\mathcal{S}_H^{\tau_0\theta}$ is a non-hermitian translation ovoid with respect to some point x. For any such τ_0 and θ, the projection along reguli of $\mathcal{S}_H^{\tau_0\theta\tau_0^{-1}}$ from any point on $x^{\tau_0^{-1}}$ is isomorphic to a Roman ovoid of Thas and Payne.*

Remark. It is clear that many non-isomorphic new ovoids of $H(q)$, $q = 3^h$, arise by applying an automorphism θ of $Q(6,q)$ to a hermitian ovoid of some fixed $H(q)$, where θ does not preserve $H(q)$. One can do the same trick with the Ree-Tits ovoids (cf. Section 3.1). This gives us amongst others new translation ovoids $\mathcal{U}_{R,\beta}$ with respect to $\{(\infty),[\infty]\}$. The explicit form of those is:

$$\mathcal{U}_{R,\beta} = \{(\infty)\} \cup \{(a, a''^s - a^{3+s} + \beta a, a', a^{3+2s} + a'^s + a^s a''^s + \beta a'', a'') | a, a', a'' \in \mathbf{GF}(q)\},$$

with $q = 3^{2e+1}$ and $s = 3^{e+1}$. It is clear that, if $\mathcal{U}_{R,\beta}$ is an ovoid, then it is a translation ovoid with respect to $\{(\infty),[\infty]\}$. The fact that it is indeed an ovoid follows from the following theorem:

Theorem 23 *Let $\mathcal{O}$ be an ovoid of $H(q)$ containing the point (∞). Then the set*

$$\mathcal{O}_\beta = \{(\infty)\} \cup \{(a, l + \beta a, a', l' + \beta a'', a'') | (a, l, a', l', a'') \in \mathcal{O}\}$$

is an ovoid of $H(q)$.

PROOF. It is an elementary exercise to calculate that two points (x, m, x', m', x'') and (y, n, y', n', y'') of $H(q)$ are opposite if and only if

$$\begin{aligned}&(m' - n')(x - y) - (m - n)(x'' - y'')\\&\qquad \neq x''y''(x - y)^2 + (x - y)(x'x'' - y'y'' - 2x'y'' + 2x''y') + (x' - y')^2\end{aligned}$$

(use the coordinates in $\mathbf{PG}(6, q)$, see Table 2). The theorem now easily follows. $\square$

5 New classes of locally hermitian spreads of $H(q)$

5.1 Preliminary results

Consider a non-singular elliptic quadric $Q^-(5, q)$ containing some fixed line L of $H(q)$ (for the calculations later on, we will assume $L = [\infty]$). Let $\mathcal{S}$ be the corresponding hermitian spread. In $\mathbf{PG}(5, q) \supseteq Q^-(5, q)$ we consider a $\mathbf{PG}(3, q)$ skew to L. Let $\mathcal{S} = \{L, M_1, \ldots, M_{q^3}\}$ and let τ be the 3-dimensional space which is tangent to $Q^-(5, q)$ at L. Then the lines $\langle L, M_i \rangle \cap \mathbf{PG}(3, q)$ together with $\tau \cap \mathbf{PG}(3, q) = L'$ form a regular spread S of $\mathbf{PG}(3, q)$ (for the definition of regular spread, see Thas [20]). Let $p \, \mathbf{I} \, L$. Then the lines of $Q^-(5, q)$ through p are denoted by L, N_1, N_2, $\ldots$, N_{q^2}. The planes $\langle L, N_i \rangle$, $i = 1, 2, \ldots, q^2$, intersect $\mathbf{PG}(3, q)$ in q^2 points which, together with L', form a plane π_p. In the tangent space τ_p of $Q^-(5, q)$ at p we now choose a 3-dimensional space γ not containing p, and in the tangent space τ'_p of $Q(6, q)$ at p we choose a 4-dimensional space $\gamma' \supseteq \gamma$ not containing p. Then the lines $L, N_1, N_2, \ldots, N_{q^2}$ intersect γ in the points of an elliptic quadric O_p on $Q(4, q) = Q(6, q) \cap \gamma'$. With the lines of π_p different from L' there correspond the conics of O_p through l, with $\{l\} = O_p \cap L$. Hence each regulus of S containing L' defines a conic of O_p through l, and conversely. The set of q reguli of $\mathcal{S}$ defined by a regulus of S containing L' will be called an *$\mathcal{R}$-conic* of $H(q)$.

Theorem 24 *Let $\mathcal{S}$ be a non-hermitian locally hermitian spread in L of $H(q)$. Suppose that for any point $x \in L$, the projection O_x from x along reguli is a classical ovoid of $Q(4, q)$. If $M_1, M_2 \in \mathcal{S} \setminus \{L\}$ and if the reguli $\mathcal{R}(L, M_1)$ and $\mathcal{R}(L, M_2)$ are distinct, then the 5-dimensional space containing $\mathcal{R}(L, M_1)$ and $\mathcal{R}(L, M_2)$ intersects $Q(6, q)$ in a non-singular elliptic quadric $Q^-(5, q)$ which contains exactly q reguli $\mathcal{R}(L, M)$, with $M \in \mathcal{S} \setminus \{L\}$. These q reguli are the elements of an $\mathcal{R}$-conic of $H(q)$.*

PROOF. If M, N are distinct lines on $Q(6, q)$ and $\langle M, L \rangle \cap Q(6, q)$ is hyperbolic, then we will denote the regulus of $\langle M, N \rangle \cap Q(6, q)$ containing M, N by $[M, N]$; the opposite regulus will be denoted by $\overline{[M, N]}$ (note that, if M and N are lines of $H(q)$, then $[M, N] = \mathcal{R}(M, N)$ and $\overline{[M, N]}$ does not contain any line of $H(q)$). As $\mathcal{S}$ is a translation spread of $H(q)$ with respect to the line L, we have $[L, M] \subseteq \mathcal{S}$ for any $M \in \mathcal{S} \setminus \{L\}$.

Let $M_1, M_2 \in \mathcal{S} \setminus \{L\}$ with $[L, M_1] \neq [L, M_2]$. If $\langle L, M_1, M_2 \rangle \cap Q(6, q) = Q'$ is singular or hyperbolic, then, as $\mathcal{S}$ is a 1-system of $Q(6, q)$, the space $\langle L, M_1, M_2 \rangle$ contains exactly $q + 1$ elements of $\mathcal{S}$ (see Section 2.3), a contradiction. So $\langle L, M_1, M_2 \rangle \cap Q(6, q)$ is an elliptic quadric $Q^-(5, q)$. Let $p \in L$ and let N_1, N_2 be the transversals through p for $[L, M_1], [L, M_2]$ respectively. The lines N_1, N_2 define respective points n_1, n_2 of the elliptic quadric O_p. Let $\mathcal{C}$ be the conic of O_p containing n_1, n_2, l, with $\{l\} = O_p \cap L$. If $\mathcal{C} = \{l, n_1, n_2, \ldots, n_q\}$ and $pn_i = N_i$, $i = 1, 2, \ldots q$, then $N_1, N_2, \ldots, N_q$ belong to the opposite reguli of respective reguli $[L, M_1]$, $[L, M_2]$, $\ldots$,$[L, M_q]$ in $\mathcal{S}$. Clearly the lines $L, N_1, N_2, \ldots, N_q$ belong to a 3-space. As L, N_1, N_2 belong to $Q^-(5, q)$, also the lines $N_3, N_4, \ldots, N_q$ of $\langle L, N_1, N_2 \rangle$ belong to $Q^-(5, q)$. If $p' \in L$, $p' \neq p$, then similarly M_1, M_2 and p' define lines $L, N'_1, N'_2, \ldots, N'_q$ of some 3-space; here we assume that N'_1, N'_2 are the respective lines of $\overline{[L, M_1]}$ and $\overline{[L, M_2]}$ through p'. As L, N'_1, N'_2 belong to $Q^-(5, q)$, also the lines $N'_3, N'_4, \ldots, N'_q$ of $\langle L, N'_1, N'_2 \rangle$ belong to $Q^-(5, q)$. The lines of $H(q)$ in $Q^-(5, q)$ are the elements of a hermitian spread $\mathcal{S}'$ of $H(q)$. Clearly $\mathcal{S}'$ contains $[L, M_1]$ and $[L, M_2]$. The reguli of $\mathcal{S}'$ through L for which $N_1, N_2, \ldots, N_q$ are the lines through p of the opposite reguli, are exactly the reguli of $\mathcal{S}'$ through L for which $N'_1, N'_2, \ldots, N'_q$ are the lines through p' of the opposite reguli (these q reguli through L are uniquely determined by $[L, M_1]$ and $[L, M_2]$). So $\mathcal{S}'$ contains $[L, M_1]$, $[L, M_2]$ and the opposites of $q - 2$ reguli $[N_i, N'_j]$, $i = 3, 4, \ldots, q$; indices can be chosen in such a way that $i = j$. If $R \in \mathcal{S}$, with R not in $Q^-(5, q)$, intersects N_i, then let R' be the line of $\mathcal{S}$ which intersects N'_i and for which $U = \langle R \cap N_i, R' \cap N'_i \rangle \in \mathcal{S}'$. So there is a line U of $H(q)$ which is concurrent with distinct lines of $\mathcal{S}$, a contradiction. Consequently, all lines of $\mathcal{S}$ which are concurrent with $N_1, N_2, \ldots, N_q$ belong to $Q^-(5, q)$. It follows that $Q^-(5, q)$ contains at least q distinct reguli $[L, M_1]$, $[L, M_2]$, $\ldots$, $[L, M_q]$, where $M_1, M_2, \ldots, M_q \in \mathcal{S} \setminus \{L\}$ are the elements of an $\mathcal{R}$-conic of $H(q)$.

Assume, by way of contradiction, that $Q^-(5, q)$ contains at least $q + 1$ reguli $[L, M_1]$, $[L, M_2], \ldots, [L, M_{q+1}], \ldots$. Let $N_1, N_2, \ldots, N_{q+1}, \ldots$ be the respective transversals through p. Then $L, N_1, \ldots, N_{q+1}, \ldots$ define points $l, n_1, \ldots, n_{q+1}, \ldots$ of the elliptic quadric O_p (where $\{l, n_1, \ldots, n_q\}$ is a conic). As the internal (or derived) affine plane $(O_p)_l$ of O_p at l (where O_p is viewed as an inversive plane) is generated by the points $n_1, n_2, \ldots, n_q, n_{q+1}$, it follows from the preceding paragraph that all lines of $\mathcal{S}$ are contained in $Q^-(5, q)$. Hence $\mathcal{S}$ is hermitian, a contradiction. $\square$

Theorem 25 *Let $\{\mathcal{R}(L, M_i) | i = 1, 2, \ldots, q\} = \mathcal{C}$ be an $\mathcal{R}$-conic of $H(q)$. Further we consider a regulus $\mathcal{R}(L, M)$ on $H(q)$ which is not contained in the elliptic quadric $Q^-(5, q)$ defined by $\mathcal{C}$, such that each 5-dimensional space $\langle L, M, M_i \rangle$ intersects $Q(6, q)$ in an elliptic quadric. Then $\mathcal{R}(L, M) \cup \mathcal{R}(L, M_1) \cup \ldots \cup \mathcal{R}(L, M_q)$ is contained in exactly one (necessarily non-hermitian) locally hermitian spread $\mathcal{S}$ of $H(q)$ with respect to the line L, such that for any point $x \in L$ the projection from x of $\mathcal{S}$ along reguli is a classical ovoid of $Q(4, q)$.*

PROOF. First, let $q = 2$. The line L is contained in exactly one 3-dimensional space τ for which $\tau \cap Q(6,q) = L$. It follows that any hyperplane containing $[L,M]$ and intersecting $Q(6,q)$ in a non-singular elliptic quadric, also contains τ, and consequently is uniquely defined. Hence $[L,M]$, $[L,M_1]$, $[L,M_2]$ belong to a common elliptic quadric, a contradiction. It follows that $q > 2$.

Each pair $\{[L,M],[L,M_i]\}$, $i = 1,2,\ldots,q$, defines a unique $\mathcal{R}$-conic $\mathcal{C}_i$ of $H(q)$. Let $x \in L$ and let $\mathbf{PG}(4,q)$ be a hyperplane, not through x, of the tangent space $\mathbf{PG}(5,q)$ of $Q(6,q)$ at x. The lines of $Q(6,q)$ through x intersect $\mathbf{PG}(4,q)$ in the points of a non-singular quadric $Q(4,q)$. Let $\mathbf{PG}(4,q) \cap L = \{l\}$. The transversals $N_1, N_2, \ldots, N_q$ through x of the respective reguli $[L,M_1],[L,M_2],\ldots,[L,M_q]$ intersect $Q(4,q)$ in the respective points $n_1,n_2,\ldots,n_q$. Then $\{l,n_1,n_2,\ldots,n_q\}$ is a non-singular conic C on $Q(4,q)$. The transversal N in x of $[L,M]$ intersects $Q(4,q)$ in the point n. By assumption, for any point $y \in C \setminus \{l\}$ the plane lny intersects $Q(4,q)$ in a non-singular conic C_y. It follows that the 3-dimensional space ζ_x containing C and n intersects $Q(4,q)$ in a non-singular elliptic quadric O_x which contains all conics C_y. Assume, by way of contradiction, that a line M', with $L \neq M'$, in a regulus of $\mathcal{C}_i \setminus \{[L,M]\}$ intersects a line M'', with $L \neq M''$, in a regulus of $\mathcal{C}_j \setminus \{[L,M]\}$, $i \neq j$. If $z' \in M' \cap M''$ and if z is the point of L collinear in $Q(6,q)$ with z', then on the elliptic quadric O_z the $\mathcal{R}$-conics $\mathcal{C}_i$ and $\mathcal{C}_j$ define distinct conics having three points in common, a contradiction. Hence the q^2-q+1 reguli of the q $\mathcal{R}$-conics $\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_q$ contain q^3-q^2+q+1 mutually disjoint lines of $H(q)$. Let $\mathcal{W} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots \cup \mathcal{C}_q$, and let $\mathcal{V}$ be the set of all lines in the elements of $\mathcal{W}$.

Let $\mathcal{R}_1, \mathcal{R}_2$ be any two distinct elements of $\mathcal{W}$. These two reguli are contained in a 5-dimensional space π_5. If $\pi_5 \cap Q(6,q)$ is a non-singular hyperbolic quadric, then there exist two planes π and π' in π_5 each of which contains a line of the regulus $\overline{\mathcal{R}}_i$, $i = 1,2$; if $\pi_5 \cap Q(6,q)$ is singular, then there exists one plane π in π_5 which contains a line of $\overline{\mathcal{R}}_i$, $i = 1,2$. If x is the common point of π and L, then on O_x there are two points which are collinear in $Q(6,q)$, a contradiction. Hence $\pi_5 \cap Q(6,q)$ is a non-singular elliptic quadric and so all lines in $\mathcal{R}_1 \cup \mathcal{R}_2$ are at mutually distance 6 in $H(q)$. It follows that the q^3-q^2+q+1 lines of $\mathcal{V}$ are at mutually distance 6 in $H(q)$.

Let $Q_1^-(5,q)$ be any non-singular elliptic quadric containing L, let $\mathcal{C}'$ be an $\mathcal{R}$-conic whose elements belong to $Q_1^-(5,q)$, let $\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3$ be three distinct elements of $\mathcal{C}'$ and let $x \in L$. Further, let $\mathbf{PG}_1(3,q)$ be a 3-dimensional space, not through x, in the tangent hyperplane of $Q_1^-(5,q)$ at x and let $\mathbf{PG}(3,q)$ be a 3-dimensional space, skew to L, in the $\mathbf{PG}_1(5,q)$ of $Q_1^-(5,q)$. Put $\{l\} = L \cap \mathbf{PG}_1(3,q)$ and let r_1,r_2,r_3 be the respective intersections of $\mathbf{PG}_1(3,q)$ with the transversals through x of $\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3$. The spaces generated by $\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3$ intersect $\mathbf{PG}(3,q)$ in the respective lines U_1,U_2,U_3 and the tangent space of $Q_1^-(5,q)$ at L intersects $\mathbf{PG}(3,q)$ in the line U. Then the lines xr_1, xr_2, xr_3 and the tangent line at l of the conic $C' = Q_1^-(5,q) \cap \langle r_1,r_2,r_3\rangle$ intersect the respective lines U_1,U_2,U_3,U in collinear points. Hence the cross-ratio $(l,r_1;r_2,r_3)$ is equal to the cross-ratio $(U,U_1;U_2,U_3)$. It follows that the cross-ratio $(l,r_1;r_2,r_3)$ is independent of the choice of x on L.

Consider points $x, x' \in L$. $x \neq x'$, the corresponding elliptic quadrics O_x, $O_{x'}$, the points $l, n, n_i \in O_x$ and $l', n', n'_i \in O_{x'}$, which correspond respectively to L, $[L, M]$, $[L, M_i]$, with $i = 1, 2, \ldots, q$. Then by the previous section there is a linear isomorphism θ of O_x onto $O_{x'}$ for which $l^\theta = l'$, $n^\theta = n'$, $n_i^\theta = n'_i$, with $i = 1, 2, \ldots, q$. If $W \in \mathcal{W}$ and if w, w' are the corresponding elements of respectively O_x, $O_{x'}$, then, again by the previous section $w^\theta = w'$. Now we consider any two distinct reguli $\mathcal{R}_1, \mathcal{R}_2$ of $\mathcal{W}$. Let z_1, z_2 be the respective points of O_x defined by $\mathcal{R}_1, \mathcal{R}_2$. Assume that $\mathcal{R}_3$ is an element of the $\mathcal{R}$-conic containing $\mathcal{R}_1, \mathcal{R}_2$, for which the corresponding point z_3 on O_x belongs to the set χ consisting of the $q^2 - q + 1$ points of O_x defined by $\mathcal{W}$. Then l, z_1, z_2, z_3 belong to a common conic on O_x. Let $\mathcal{R}'_3$ be the regulus of $\mathcal{W}$ defining z_3. The point z'_3 of $O_{x'}$ corresponding to $\mathcal{R}'_3$ is the point $z'_3 = z_3^\theta$; also, $(l, z_1; z_2, z_3) = (l', z'_1; z'_2, z'_3)$. For the point z''_3 of $O_{x'}$ defined by $\mathcal{R}_3$ we also have $(l, z_1; , z_2, z_3) = (l', z'_1; z'_2, z''_3)$. Hence $z'_3 = z''_3$. It immediately follows that $\mathcal{R}_3 = \mathcal{R}'_3$. Hence the $\mathcal{R}$-conic defined by $\mathcal{R}_1$ and $\mathcal{R}_2$ contains either q or $q-1$ elements of $\mathcal{W}$.

Let $\widetilde{Q}^-(5, q)$ be the non-singular elliptic quadric containing $[L, M]$ and the tangent space τ of $Q^-(5, q)$ (the elliptic quadric containing the elements of $\mathcal{C}$) at L. Let $x \in L$, and as before, let O_x be the elliptic quadric of $Q(4, q)$ defined by x. Further, let $m \in O_x \setminus \chi$, with χ the set of $q^2 - q + 1$ points of O_x defined by $\mathcal{W}$, as above. If C is the conic of O_x which corresponds to $\mathcal{C}$ and if $n \in O_x$ corresponds to $[L, M]$, then xn and xT, with T the tangent line of C at l, belong to the space of $\widetilde{Q}^-(5, q)$. Hence, as m belongs to the plane nT, the line xm belongs to $\widetilde{Q}^-(5, q)$. Let C' be a conic through l and m, with $C' \neq nT \cap O_x = C_x$, and let z_1 and z_2 be distinct points of $C' \setminus \{l, m\}$. Then the $\mathcal{R}$-conic defined by z_1 and z_2 contains a regulus $\mathcal{R}'$ for which $xm \in \overline{\mathcal{R}}'$. This regulus $\mathcal{R}'$ is the unique regulus of the $\mathcal{R}$-conic which does not belong to $\mathcal{W}$. As x is any point of L, the regulus $\overline{\mathcal{R}}'$ contains $q + 1$ lines of $\widetilde{Q}^-(5, q)$. Hence $\mathcal{R}'$ is a regulus of $\widetilde{Q}^-(5, q)$ having xm as transversal, and so $\mathcal{R}'$ is uniquely defined by m. Let $\mathcal{C}_0$ be the $\mathcal{R}$-conic of $\widetilde{Q}^-(5, q)$ defined by the conic C_x of O_x. Now it is clear that $\mathcal{C}_0$ is independent of the choice of x on L. An argument used before shows that the lines in the reguli of $\mathcal{C}_0 \cup \mathcal{W}$ are mutually at distance 6 in $H(q)$. As there are $q^3 + 1$ lines in the reguli of $\mathcal{C}_0 \cup \mathcal{W}$, these lines form a spread $\mathcal{S}$ of $H(q)$. Finally $\mathcal{S}$ is locally hermitian in L and for any point of L the ovoid of $Q(4, q)$ defined by $\mathcal{S}$ is an elliptic quadric. $\square$

A locally hermitian spread $\mathcal{S}$ of $H(q)$ with respect to the line L, such that for any point $x \in L$ the projection from x of $\mathcal{S}$ along reguli is a classical ovoid of $Q(4, q)$, will be called a *semi-classical spread.*

Lemma 26 *Let q be odd and suppose γ is a non-square in $\mathbf{GF}(q)$. For $r \in \mathbf{GF}(q)$, put $M_r = [0, -\gamma r, 0, 0, r]$. Then $\mathcal{C} = \{\mathcal{R}([\infty], M_r) | r \in \mathbf{GF}(q)\}$ is an $\mathcal{R}$-conic of $H(q)$. If $M = [k, b, k', b', k'']$ is an arbitrary line of $H(q)$ not lying in the projective 5-space* *determined by $\mathcal{C}$ and skew to the reguli of $\mathcal{C}$ (i.e., $b'(\gamma b'-k)^2 \neq (b+\gamma k'')^2 k$), then $\langle L, M, M_r\rangle$ intersects $Q(6,q)$ in an elliptic quadric for all $r \in \mathbf{GF}(q)$, if and only if for each $r \in \mathbf{GF}(q)$, the line in $\mathbf{PG}(2,q)$ with equation*

$$((r-k'')(b+\gamma r)-b'^2)X-(k(r-k'')-b'(\gamma r+b))Y+(kb'-(b+\gamma r)^2)Z=0 \qquad (33)$$

does not contain any point of the conic $Y^2 = XZ$.

PROOF. Let $\mathcal{S}$ be the hermitian spread of $H(q)$ contained in the space with equation $X_1 = \gamma X_5$ (with respect to Table 2). Then it is clear that $\mathcal{S}$ contains the line $[\infty]$. The tangent space $\tau'_{(\infty)}$ of $Q(6,q)$ at (∞) has equation $X_4 = 0$ and we consider the 4-dimensional space δ' with equations $X_0 = X_4 = 0$ in $\tau'_{(\infty)}$ (δ' does not contain (∞)). The quadric $Q(4,q) = \delta' \cap Q(6,q)$ has equations $X_0 = X_4 = X_1X_5 + X_2X_6 - X_3^2 = 0$. Since

$$\mathcal{S} = \{[\infty]\} \cup \{[\gamma b', -\gamma k'', k', b', k'']| k', b'', k'' \in \mathbf{GF}(q)\}$$

(as one easily computes), and since the unique point on the line $[\gamma b', -\gamma k'', k', b', k'']$ collinear with (∞) (in $Q(6,q)$) has coordinates $(\gamma b', -\gamma k'', k', b')$ in $H(q)$ (and hence coordinates $(k' - \gamma k'' b', \gamma b', 1, -\gamma k'', 0, b', \gamma^2 k''^2 - \gamma b'^2)$ in $\mathbf{PG}(6,q)$), it is clear that the transversals through (∞) of the reguli of $\mathcal{S}$ through $[\infty]$ meet $Q(4,q)$ in the points with coordinates $(0, \gamma b', 1, -\gamma k'', 0, b', \gamma^2 k''^2 - \gamma b'^2)$. Hence these points lie on the elliptic quadric $O_{(\infty)}$ with equations

$$X_0 = X_4 = X_1 - \gamma X_5 = X_1X_5 + X_2X_6 - X_3^2 = 0.$$

The intersection of $O_{(\infty)}$ with the space $X_1 = 0$ is the irreducible conic $C: X_0 = X_1 = X_4 = X_5 = X_2X_6 - X_3^2 = 0$ containing the point $(0,0,0,0,0,0,1)$ (which corresponds to the line $[\infty]$ of $\mathcal{S}$). The q^2 lines of $\mathcal{S} \setminus \{[\infty]\}$ defined by the points of $C \setminus \{(0,0,0,0,0,0,1)\}$, together with $[\infty]$, are the lines of the reguli of an $\mathcal{R}$-conic; they all have $b' = 0$, hence they have coordinates $[0, -\gamma k'', k', 0, k'']$. This proves the first assertion.

Now let $M = [k, b, k', b', k'']$ be any line of $H(q)$ at distance 6 from $[\infty]$. Fix $r \in \mathbf{GF}(q)$ and let $\mathbf{PG}(5,q)$ be the subspace of $\mathbf{PG}(6,q)$ generated by $[\infty], [0, -\gamma r, 0, 0, r], M$. Suppose that $\mathbf{PG}(5,q)$ intersects $Q(6,q)$ in a non-elliptic quadric Q. This means that the hyperplane with equation

$$\begin{vmatrix} X_0 & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ k'+bb' & k & 1 & b & 0 & b' & b^2 - b'k \\ b'^2 + k''b & -b & 0 & -b' & 1 & k'' & -kk'' - k' - 2bb' \\ 0 & 0 & 1 & -\gamma r & 0 & 0 & \gamma^2 r^2 \\ -\gamma r^2 & \gamma r & 0 & 0 & 1 & r & 0 \end{vmatrix} = 0 \qquad (34)$$

has at least one plane through $[\infty]$ in common with the quadric $Q(6, q)$. It is clear that the coefficients of X_0 and X_6 in Equation 34 are both equal to 0. Hence Equation 34 is equivalent with

$$\begin{vmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \\ k & 1 & b & 0 & b' \\ -b & 0 & -b' & 1 & k'' \\ 0 & 1 & -\gamma r & 0 & 0 \\ \gamma r & 0 & 0 & 1 & r \end{vmatrix} = 0 \tag{35}$$

Let ζ be the tangent space of $Q(6, q)$ at $[\infty]$. The intersection $\mathbf{PG}(5, q) \cap \mathbf{PG}(6, q)$ is non-elliptic if and only if $\mathbf{PG}(5, q) \cap \mathbf{PG}(6, q) \cap \zeta$ contains a plane. The space ζ has equations $X_2 = X_4 = 0$. Since the points of $\zeta \cap Q(6, q)$ satisfy $X_2 = X_4 = X_1X_5 - X_3^2 = 0$, it follows that the intersection of the surfaces with equations

$$X_2 = X_4 = X_1X_5 - X_3^2 = 0$$

and

$$\begin{vmatrix} X_1 & 0 & X_3 & 0 & X_5 \\ k & 1 & b & 0 & b' \\ -b & 0 & -b' & 1 & k'' \\ 0 & 1 & -\gamma r & 0 & 0 \\ \gamma r & 0 & 0 & 1 & r \end{vmatrix} = 0$$

must contain plane. Since X_0 and X_6 can be chosen freely, this is equivalent to saying that the system of equations

$$\begin{cases} X_1X_5 = X_3^2 \\ \begin{vmatrix} X_1 & 0 & X_3 & 0 & X_5 \\ k & 1 & b & 0 & b' \\ -b & 0 & -b' & 1 & k'' \\ 0 & 1 & -\gamma r & 0 & 0 \\ \gamma r & 0 & 0 & 1 & r \end{vmatrix} = 0 \end{cases}$$

must have some solution. Interpreting this in a projective plane with coordinates X_1, X_3, X_5, and varying r, the result easily follows. $\square$

If $q \equiv -1 \bmod 3$, then it will be more convenient to have the following lemma at our disposal.

Lemma 27 *Let q be odd with $q \equiv -1 \bmod 3$. For $r \in \mathbf{GF}(q)$, put $M_r = [0, r, 0, r, 0]$. Then $\mathcal{C} = \{\mathcal{R}([\infty], M_r) | r \in \mathbf{GF}(q)\}$ is an $\mathcal{R}$-conic of $H(q)$. If $M = [k, b, k', b', k'']$ is an arbitrary line of $H(q)$ not lying in the projective 5-space determined by $\mathcal{C}$ and skew to the reguli of $\mathcal{C}$ (i.e., $(b - b')^3 + kk''(k - 3b + 3b' + k'') \neq 0$), then $\langle L, M, M_r\rangle$ intersects $Q(6, q)$ in an elliptic quadric for all $r \in \mathbf{GF}(q)$, if and only if for each $r \in \mathbf{GF}(q)$, the line in $\mathbf{PG}(3, q)$ with equation*

$$(k''(b - r) + (b' - r)^2)X - ((b - r)(b' - r) + kk'')Y + (-k(b' - r) + (b - r)^2)Z = 0 \quad (36)$$

does not contain any point of the conic $Y^2 = XZ$.

PROOF. This is completely similar to the proof of Lemma 26. Note that the projective 5-space containing the $\mathcal{R}$-conic $\mathcal{C}$ has equation $X_3 = X_1 + X_5$. □

5.2 Examples

Let q be odd, but not a power of 3. If we put $b = k'' = 0$ and $k = 9\gamma b'$, $b' \neq 0$, in Equation (33), then we obtain the q lines

$$(\gamma r^2 - b'^2)X - 8r\gamma b'Y + \gamma(9b'^2 - \gamma r^2)Z = 0, \qquad r \in \mathbf{GF}(q). \quad (37)$$

Eliminating X from (37) and $Y^2 = XZ$, we obtain

$$(\gamma r^2 - b'^2)Y^2 - 8r\gamma b'YZ + \gamma(9b'^2 - \gamma r^2)Z^2 = 0. \quad (38)$$

The discriminant of that quadratic equation equals $\gamma(4\gamma^2 r^4 + 36b'^4 + 24\gamma r^2 b'^2)$, which is obviously equal to $\gamma(2\gamma r^2 + 6b'^2)^2$. Clearly the latter is a non-square in $\mathbf{GF}(q)$, for all $r \in \mathbf{GF}(q)$, if and only if -3 is a square in $\mathbf{GF}(q)$. It is an elementary exercise to calculate the other elements of a spread thus arising and we obtain the following theorem.

Theorem 28 *Let q be odd and equal to 1 mod 3 (so that -3 is a non-zero square in $\mathbf{GF}(q)$). Then the set*

$$\mathcal{S}_{[9]} = \{[\infty]\} \cup \{[9\gamma b', -\gamma k'', k', b', k''] | k', b', k'' \in \mathbf{GF}(q)\}$$

is a semi-classical non-hermitian translation spread in $H(q)$ with respect to $[\infty]$.

PROOF. Using the formulae just preceding Lemma 5, one easily verifies that the group

$$\{\Theta[9\gamma K, -\gamma B, K', K, B] | K, B, K' \in \mathbf{GF}(q)\}$$

stabilizes $\mathcal{S}_{[9]}$. Lemma 5 now implies that the spread is a translation spread with respect to $[\infty]$.

Next, we show that $\mathcal{S}_{[9]}$ is not a hermitian spread. If it were hermitian, then all lines would lie in the same 5-dimensional subspace U of $\mathbf{PG}(6,q)$. From the proof of Lemma 26 we deduce that U has equation $X_1 = \gamma X_5$ (it is the space of the $\mathcal{R}$-conic $\mathcal{C}$). But the element $[9\gamma, 0, 0, 1, 0]$ of $\mathcal{S}_{[9]}$ does not belong to U. Indeed, using Table 2, we see that this line is spanned by the points $(0, 9\gamma, 1, 0, 0, 1, -9\gamma)$ and $(1, 0, 0, -1, 1, 0, 0)$. The first point does not lie in U. $\square$

6 Some characterizations

Lemma 29 *Let $P(X)$ be a monic polynomial of degree 4 with the property that $P(r)$ is a non-zero square for all $r \in \mathbf{GF}(q)$, q odd. Then $P(X)$ can be written as the square $(X^2+AX+B)^2$ of an irreducible quadratic polynomial X^2+AX+B, with $A, B \in \mathbf{GF}(q)$.*

PROOF. If $Y^2 - P(X)$ were absolutely irreducible, then by SMIDT [16], page 32, we would have $q+1 \leq 2\sqrt{q}$, a contradiction. Hence $Y^2 - P(X)$ is not absolutely irreducible. Lemma 6.54 on page 309 of LIDL & NIEDERREITER [9] now implies that $P(X) = (X - \alpha_1)^2(X-\alpha_2)^2$, with α_1, α_2 possibly equal, and contained in an extension of $\mathbf{GF}(q)$. Since all coefficients of $P(x)$ lie in $\mathbf{GF}(q)$, the result readily follows. $\square$

Theorem 30 *If q is a power of 3, then every semi-classical spread $\mathcal{S}$ in $H(q)$ is a hermitian spread.*

PROOF. By definition, $\mathcal{S}$ contains an $\mathcal{R}$-conic. Without loss of generality, we may take $\mathcal{C} = \{\mathcal{R}([\infty], M_r) | r \in \mathbf{GF}(q)\}$, where $M_r = [0, -\gamma r, 0, 0, r]$, for all $r \in \mathbf{GF}(q)$, and some non-square $\gamma \in \mathbf{GF}(q)$. By Remark 7 we may assume that the line $M = [1, b, 0, b', 0]$ belongs to $\mathcal{S}$, for some $b, b' \in \mathbf{GF}(q)$. By Lemma 26 there arises a semi-classical non-hermitian spread if $b'(\gamma b' - 1)^2 \neq b^2$ and if for each $r \in \mathbf{GF}(q)$ the equation

$$(r(b+\gamma r) - b'^2)X^2 - (r - b'(\gamma r + b))XY + (b' - (b+\gamma r)^2)Y^2 = 0 \tag{39}$$

has no solution in $\mathbf{GF}(q)$. Hence, for every $r \in \mathbf{GF}(q)$, the element

$$(r - b'(\gamma r + b))^2 - (b' - (b+\gamma r)^2)(r(b+\gamma r) - b'^2) \tag{40}$$

must be a non-square in $\mathbf{GF}(q)$. Working out (40), we see that $\gamma^3 r^4 + r^2 + b^3 r + b'^3$ must be a non-square for all $r \in \mathbf{GF}(q)$. Multiplying with γ^{-3}, we see that, using Lemma 29, there exist $A, B \in \mathbf{GF}(q)$ such that $X^4 + \gamma^{-3}X^2 + \gamma^{-3}b^3X + \gamma^{-3}b'^3 = (X^2+AX+B)^2$.

This implies $A = b = 0$ and $b' = \gamma^{-1}$, and so $b'(\gamma b' - 1)^2 = b^2$. We conclude that there does not exist a semi-classical non-hermitian spread. $\square$

In the same way, we can classify all semi-classical spreads in $H(q)$, q odd, with $q \equiv 1 \bmod 3$.

Theorem 31 *If $q \equiv 1 \bmod 3$ and q is odd, then every semi-classical spread $\mathcal{S}$ in $H(q)$ is either hermitian or isomorphic to $\mathcal{S}_{[9]}$.*

PROOF. We can copy the proof of the previous theorem up to Equation (39). The discriminant of that equation (which must be a non-square), is equal to

$$(r - b'(\gamma r + b))^2 - 4(b' - (b + \gamma r)^2)(r(b + \gamma r) - b'^2) \tag{41}$$

Multiplying (41) with 4γ and putting $b = \gamma b''$, $t = 2\gamma(r + b'')$, we see that

$$t^4 - 2\gamma b'' t^3 + (\gamma^{-1} - 6b' - 3\gamma b'^2)t^2 + 2(6\gamma b' b'' - 2b'')t + 16\gamma b'^3 + 4\gamma b''^2 \tag{42}$$

must be a non-zero square in $\mathbf{GF}(q)$ for all $t \in \mathbf{GF}(q)$. Then, by Lemma 29, there exists $A, B \in \mathbf{GF}(q)$ such that $(X^2 + AX + B)^2 = X^4 + KX^3 + LX^2 + MX + N$, with $K = -2\gamma b''$, $L = \gamma^{-1} - 6b' - 3\gamma b'^2$, $M = 2(6\gamma b' b'' - 2b'')$ and $N = 16\gamma b'^3 + 4\gamma b''^2$. This implies that we must have $8M = 4KL - K^3$, and $64N = (4L - K^2)^2$, and gives us the following system of equations:

$$\left\{ \begin{array}{rcll} 64(16\gamma b'^3 + 4\gamma b''^2) & = & (4\gamma^{-1} - 24b' - 12\gamma b'^2 - 4\gamma^2 b''^2)^2, & (*) \\ 0 & = & 3b'' - 6\gamma b' b'' + 3\gamma^2 b'^2 b'' + \gamma^3 b''^3. & (**) \end{array} \right.$$

If $b'' \neq 0$, then by dividing $(**)$ by b'', we obtain b''^2 in function of γ and b'. Plugging this in into $(*)$, we get, after some elementary calculations, $(\gamma b' - 1)^3 = 0$, which implies by $(**)$ that $b'' = 0$, a contradiction. Hence we may assume that $b'' = 0$. In this case, $(*)$ is equivalent with (after multiplying with $16^{-1}\gamma^2$)

$$9(\gamma b')^4 - 28(\gamma b')^3 + 30(\gamma b')^2 - 12(\gamma b') + 1 = 0.$$

This factors as $(\gamma b' - 1)^3(9\gamma b' - 1) = 0$. As $b'(\gamma b' - 1)^2 \neq \gamma^2 b''^2$, so $b'(\gamma b' - 1) \neq 0$, we must have $9\gamma b' = 1$. This yields the spread $\mathcal{S}_{[9]}$ constructed in the previous section. $\square$

To conclude the odd case, we show:

Theorem 32 *If $q \equiv -1 \bmod 3$ and q is odd, then every semi-classical spread $\mathcal{S}$ in $H(q)$ is a hermitian spread.*

PROOF. The proof is similar as the previous one. This time we use Theorem 27. So we assume that $\mathcal{S}$ is a semi-classical non-hermitian spread of $H(q)$ which, without loss of generality, contains the $\mathcal{R}$-conic $\mathcal{C} = \{\mathcal{R}([\infty], M_r) | r \in \mathbf{GF}(q)\}$, with $M_r = [0, r, 0, r, 0]$. It is clear that $\mathcal{S}$ must contain a line with coordinates $[k, b, k', b', k'']$ with $k \neq 0$ and $k'' \neq 0$. Indeed, each line concurrent with $[k]$, $k \neq \infty$, is concurrent with an element of $\mathcal{S}$, hence since each such line has coordinates $[k, \ldots]$, there are $q^2(q-1)$ elements of $\mathcal{S}$ with coordinates $[k, \ldots]$, with $k \neq 0$. Similarly, there are $q^2(q-1)$ elements of $\mathcal{S}$ with coordinates $[\ldots, k'']$, with $k'' \neq 0$. Since $q > 2$, there exists a line with coordinates $[k, b, k', b', k'']$, with $k \neq 0 \neq k''$. For any such line we have $(b-b')^3 + kk''(k-3b+3b'+k'') \neq 0$ (see Lemma 27). By applying a suitable generalized homology (see De Smet & Van Maldeghem [2]), we may assume that $k'' = 1$. Hence, we deduce from Lemma 27 that, for all $r \in \mathbf{GF}(q)$, the element

$$((b-r)(b'-r)+k)^2 - 4(-k(b'-r)+(b-r)^2)((b-r)+(b'-r)^2) \tag{43}$$

is a non-square in $\mathbf{GF}(q)$. If we put $b - r = x$ and $b' - b = \ell$, then (43) becomes, after calculation,

$$-3x^4 + (4k - 6\ell - 4)x^3 + (6k + 12k\ell - 3\ell^2)x^2 + (6k\ell + 12k\ell^2)x + k^2 + 4k\ell^3.$$

If we multiply with -27 (which is a non-square in $\mathbf{GF}(q)$), and substitute $y = 3x$, then there results that

$$y^4 - (4k - 6\ell - 4)y^3 - (18k + 36k\ell - 9\ell^2)y^2 - (54k\ell + 108k\ell^2)y - 27k^2 - 108k\ell^3 \tag{44}$$

is a non-zero square in $\mathbf{GF}(q)$, for all $y \in \mathbf{GF}(q)$. Similarly as in the previous proof, this implies that, if we write (44) as $y^4 + Ky^3 + Ly^2 + My + N$, the equalities $8M = 4KL - K^3$ and $64N = (4L - K^2)^2$ must hold true. Writing K, L, M, N in terms of k and ℓ, one finds after an elementary calculation:

$$\left\{ \begin{array}{rcl} 0 &=& 27k\ell^3 + (6k+3)^2\ell^2 + (12k^3 + 36k^2 + 27k + 6)\ell + (k^4 + 5k^3 + 15k^2 + 5k + 1), \\ 0 &=& 18(k-1)\ell^2 + 18(k-1)(k+1)\ell + (k-1)(k+2)(4k+2). \end{array} \right.$$

Note that if $k = 1$, then the first equation implies that $\ell = -1$, and so $(b-b')^3 + kk''(k - 3b + 3b' + k'') = 0$, a contradiction. So we may assume that $k \neq 1$ and we can divide by $k - 1$. Making appropriate linear combinations, we subsequently obtain the following systems of equations in k, ℓ:

$$\left\{ \begin{array}{rcl} 0 &=& (9k^2 + 9k + 9)\ell^2 + (6k^3 + 21k^2 + 21k + 6)\ell + (k^4 + 5k^3 + 15k^2 + 5k + 1), \\ 0 &=& 9\ell^2 + 9(k+1)\ell + (k+2)(2k+1). \end{array} \right.$$

$$\Longleftrightarrow \left\{ \begin{array}{rcl} 0 &=& (3k^3 - 3k^2 - 3k + 3)\ell + (k^4 + 2k^3 - 6k^2 + 2k + 1), \\ 0 &=& 9\ell^2 + 9(k+1)\ell + (k+2)(2k+1). \end{array} \right.$$

$$\Longleftrightarrow \begin{cases} 0 &= 3(k+1)\ell + (k^2+4k+1), \\ 0 &= 9\ell^2 + 9(k+1)\ell + (k+2)(2k+1). \end{cases}$$

We can now easily eliminate ℓ and we obtain $-k^3+2k^2-k=0$, hence $k=0$ or $k=1$, a contradiction. $\square$

Final Remark. Semi-classical spreads of $H(q)$, q even, will be studied in a forthcoming paper.

References

[1] **I. Bloemen**, *Substructures and Characterizations of Finite Generalized Polygons*, Ph.-D. thesis, University Ghent, Gent 1995.

[2] **V. De Smet & H. Van Maldeghem**, The finite Moufang hexagons coordinatized, *Beiträge Algebra Geom.* **34** (1993), 217 – 232.

[3] **W. Feit & G. Higman**, The nonexistence of certain generalized polygons, *J. Algebra* **1** (1964), 114 – 131.

[4] **H. Gevaert, N. L. Johnson & J. A. Thas**, Spreads covered by reguli, *Simon Stevin* **62** (1988), 51 – 62.

[5] **G. Hanssens & H. Van Maldeghem**, A new look at the classical generalized quadrangles, *Ars Combin.* **24** (1987), 199 – 210.

[6] **G. Hanssens & H. Van Maldeghem**, Coordinatization of generalized quadrangles, *Ann. Discr. Math.* **37** (1988), 195 – 208.

[7] **J. W. P. Hirschfeld & J. A. Thas**, *General Galois Geometries*, Oxford University Press, Oxford.

[8] **W. M. Kantor**, Ovoids and translation planes, *Canad. J. Math.* **34** (1982), 1195 – 1207.

[9] **R. Lidl & H. Niederreiter**, *Finite Fields*, Addison-Wesley Publishing Company, London Amsterdam Ontario Sydney Tokyo, 1983.

[10] **G. Lunardon**, Flocks, ovoids of $Q(4,q)$ and designs, *preprint.*

[11] **M. O'Nan**, Automorphisms of unitary block designs, *J. Algebra* **20** (1972), 495 – 511.

[12] **S. E. Payne & J. A. Thas**, *Finite Generalized Quadrangles*, Pitman, Boston London Melbourne, 1984.

[13] **T. Penttila & B. Williams**, Ovoids of parabolic spaces, *preprint.*

[14] **M. A. Ronan**, A geometric characterization of Moufang hexagons, *Invent. Math.* **57** (1980), 227 – 262.

[15] **E. E. Shult & J. A. Thas**, m-systems of polar spaces, *J. Combin. Theory Ser.* A **68** (1994), 184 – 204.

[16] **W. M. Smidt**, *Equations over Finite Fields, an Elementary Approach*, Lect. Notes Math. **536**, Springer–Verlag, Berlin–Heidelberg–New-York, 1976.

[17] **J. A. Thas**, Polar spaces, generalized hexagons and perfect codes, *J. Combin. Theory Ser* A **29** (1980), 87 – 93.

[18] **J. A. Thas**, Generalized quadrangles and flocks of cones, *European J. Combin.* **8** (1987), 441 – 452.

[19] **J. A. Thas**, Symplectic spreads in $\mathbf{PG}(3, q)$, inversive planes and projective planes, *Proceedings Combinatorics '94* (Montesilvano, 1994), *Discrete Math.*, to appear.

[20] **J. A. Thas**, Projective geometry over a finite field, **in** F. Buekenhout, editor, *Handbook of Incidence Geometry, Buildings and Foundations*, chapter 7, North-Holland, Amsterdam (1995), 295 – 348.

[21] **J. A. Thas & S. E. Payne**, Spreads and ovoids in finite generalized quadrangles, *Geom. Dedicata* **52** (1994), 227 – 253.

[22] **J. Tits**, Sur la trialité et certains groupes qui s'en déduisent, *Inst. Hautes Études Sci. Publ. Math.* **2** (1959), 14 – 60.

[23] **J. Tits**, Ovoïdes et groupes de Suzuki, *Arch. Math.* **13** (1962), 187 – 198.

[24] **J. Tits**, Moufang polygons, I. Root data, *Bull. Belgian Math. Soc., Simon Stevin* **1** (1994), 455 – 468.

[25] **H. Van Maldeghem**, *Generalized Polygons, A Geometric Approach*, Birkhäuser, *to appear.*

[26] **R. Weiss**, The nonexistence of certain Moufang polygons, *Invent. Math.* **51** (1979), 261 – 266.