Groups of Projectivities of Generalized Quadrangles

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Abstract

We characterize some classical quadrangles by means of properties of their groups of projectivities. In particular, we characterize all finite classical quadrangles with regular lines, and all symplectic quadrangles over quadratically closed fields.

1 Introduction and statement of the Main Result

1.1 Definitions and notation

A weak generalized quadrangle $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a point-line incidence geometry satisfying the following axioms.

(GQ1) Every point is incident with at least two, but not all lines.

(GQ2) Every line is incident with at least two, but not all points.

(GQ3) For every point x and every line L not incident with x, there is a unique incident point-line pair (y, M) with $L \,\mathbf{I}\, y \,\mathbf{I}\, M \,\mathbf{I}\, x$.

A (thick) generalized quadrangle is a weak generalized quadrangle satisfying the additional axiom (GQ4).

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(GQ4) Every point is incident with at least three lines and every line is incident with at least three points.

Generalized quadrangles were introduced by TITS [1959]. The above definition is taken from VAN MALDEGHEM [19**]. A standard reference on finite generalized quadrangles is the monograph by PAYNE & THAS [1984].

We now introduce some more terminology. A *flag* is a set consisting of a point and a line which are incident. A sequence $(u_0, u_1, \ldots, u_{d-1}, u_d)$ with $u_0 \mathbf{I} u_1 \mathbf{I} \ldots \mathbf{I} u_{d-1} \mathbf{I} u_d$, is called a *path (of length d)*, $d \geq 0$. The *distance* $\delta(v, w)$ between two elements v, w of a generalized quadrangle Γ is the length of a minimal path joining v and w. Two elements at distance 4 are called *opposite*. By Axiom (GQ3), no two elements can be at distance ≥ 5 from each other. The set of elements at distance i from a certain element u of Γ is denoted by $\Gamma_i(u)$. For $i = 1$, we also write $\Gamma(u) = \Gamma_1(u)$. Two lines at distance 0 or 2 are called *concurrent* (notation $L \perp M$) and we write $L^\perp$ for the set of lines concurrent with L, i.e. $L^\perp = \Gamma_2(L) \cup \{L\}$. For a set of lines $\{L_1, \ldots, L_m\}$ the set of lines concurrent with all of them is denoted by $\{L_1, \ldots, L_m\}^\perp = \bigcap_{i=1}^m L_i^\perp$. Two points at distance 0 or 2 are called *collinear* (similar notation).

If u and v are two distinct elements which are not opposite, then there is a unique element $\mathrm{proj}_u v$ incident with u nearest to v; we call it *the projection of v onto u*.

An element u of a generalized quadrangle is called *regular* if for w opposite u, the set $\Gamma_2(u) \cap \Gamma_2(w)$ is determined by any two of its elements. In other words, for w_1 and w_2 opposite u, one has

$$|\Gamma_2(u) \cap \Gamma_2(w_1) \cap \Gamma_2(w_2)| \geq 2 \Longrightarrow \Gamma_2(u) \cap \Gamma_2(w_1) = \Gamma_2(u) \cap \Gamma_2(w_2).$$

The set $\Gamma_2(u) \cap \Gamma_2(w)$, u opposite w, is sometimes called a *trace* if u and w are points, and a *dual trace* if u and w are lines. We abbreviate $\Gamma_2(u) \cap \Gamma_2(w)$ often by u^w or w^u. A *projective point* is a regular point x for which $x^y \cap x^z$ is never empty, for all points y and z opposite x. Clearly, it is enough to require that $x^y \cap x^z$ is never empty for all points y, z opposite x, with y opposite z. The motivation for the name "projective" is that the set of points $\{x\} \cup \Gamma_2(x)$ together with the traces x^y, y opposite x, and the lines $\Gamma(x)$ form a projective plane precisely when x is a projective point, see VAN MALDEGHEM [19**].

It is well known that in any generalized quadrangle Γ, the number $s + 1$ of points on a line is a constant (possibly infinite), and, dually, the number $t + 1$ of lines through a point is a constant. The pair (s, t) is usually called *the order of* Γ.

There is the principle of *duality* for generalized quadrangles. Indeed, interchanging the names "point" and "line" of a given generalized quadrangle Γ produces another generalized quadrangle Γ^D, the *dual* of Γ.

Now let v and u be two opposite elements of a generalized quadrangle Γ. We define the map $[v; u] : \Gamma(v) \rightarrow \Gamma(u)$ as follows:

$$w^{[v;u]} = w' \Longleftrightarrow \delta(w, w') = 2.$$

By Axiom (GQ3), this is well-defined and bijective. We call the map $[v; u]$ a *perspectivity* from v to u. Now let $\{w_i : i \in \{0, 1, 2, \ldots, k\}\}$, $k \in \mathbb{N} \setminus \{0\}$, be a set of elements with w_{i-1} opposite w_i, $i = 1, 2, \ldots, k$. We put

$$[w_0; w_k] := [w_0; w_1][w_1; w_2] \ldots [w_{k-1}; w_k],$$

and we call the map $[w_0; w_k] : \Gamma(w_0) \rightarrow \Gamma(w_k)$ a *projectivity* from w_0 to w_k. If $w_0 = w_k$, then we have a *self-projectivity.* The set of all self-projectivities of an element u of a generalized quadrangle Γ clearly forms a group under composition, and we call it the *group of projectivities of* u. For x and y points, the groups of projectivities of x and y, viewed as permutation groups acting on $\Gamma(x)$ and $\Gamma(y)$, respectively, are isomorphic. Similarly for groups of projectivities of lines. We denote by $\Pi(\Gamma)$ the permutation group corresponding to the group of projectivities of any line of Γ, and call it *the general group of projectivities of* Γ. Dually, we denote by $\Pi^*(\Gamma)$ the permutation group corresponding to the group of projectivities of a point and call it the *general dual group of projectivities of* Γ. It turns out that for an element u of Γ, the set of self-projectivities which can be written as a composition of an even number of perspectivities forms a subgroup of index at most 2 of the full group of projectivities of u. Again, this is independent of u (but depending on the type of u, i.e., point or line) and we denote by $\Pi_+(\Gamma)$ the corresponding subgroup of $\Pi(\Gamma)$ (*the special group of projectivities of* Γ), and by $\Pi^*_+(\Gamma)$ the corresponding subgroup of $\Pi^*(\Gamma)$ (*the special dual group of projectivities of* Γ). Note that the special (dual) group of projectivities is doubly transitive. All these facts are well known, see e.g. Knarr [1988].

Now we turn to some examples.

1.2 Examples

Let $\mathbb{K}$ be any (commutative) field and let θ be a symplectic polarity of the projective space $\mathbf{PG}(3, \mathbb{K})$. The points of $\mathbf{PG}(3, \mathbb{K})$ together with the totally isotropic lines of $\mathbf{PG}(3, \mathbb{K})$ with respect to θ form a generalized quadrangle, which we call *the symplectic quadrangle over* $\mathbb{K}$, notation $\mathsf{W}(\mathbb{K})$. We write $\mathsf{W}(q)$ for $\mathsf{W}(\mathbf{GF}(q))$, where $\mathbf{GF}(q)$ is the Galois field of q elements. The permutation group $\Pi_+(\mathsf{W}(\mathbb{K}))$ coincides with $\Pi(\mathsf{W}(\mathbb{K}))$ and is equivalent to $\mathbf{PSL}_2(\mathbb{K})$ acting naturally on $\mathbf{PG}(1, \mathbb{K})$; the permutation group $\Pi^*_+(\mathsf{W}(\mathbb{K}))$ coincides with $\Pi^*(\mathsf{W}(\mathbb{K}))$ and is equivalent to $\mathbf{PGL}_2(\mathbb{K})$ acting naturally on the projective line $\mathbf{PG}(1, \mathbb{K})$, see Van Maldeghem [19**].

The points and lines of any non-degenerate quadric of Witt index 2 in some projective space form a weak generalized quadrangle, called an *orthogonal quadrangle*. In the finite case, two classes of (thick) generalized quadrangles arise this way: in $\mathbf{PG}(4,q)$ and $\mathbf{PG}(5,q)$, and we denote them respectively by $\mathsf{Q}(4,q)$ and $\mathsf{Q}(5,q)$. Their order is, respectively, (q,q) and (q,q^2). Notice that $\mathsf{W}(q)$ is the dual of $\mathsf{Q}(4,q)$, and $\mathsf{W}(q)$ is isomorphic to $\mathsf{Q}(4,q)$ if and only if q is even, see e.g. Payne & Thas [1984], where one can also find that all lines of both $\mathsf{Q}(4,q)$ and $\mathsf{Q}(5,q)$ are regular (and hence all points of $\mathsf{W}(q)$ are regular; more generally, all points of $\mathsf{W}(\mathbb{K})$ are regular, for any field $\mathbb{K}$).

An important open problem is the question: is every finite generalized quadrangle all lines of which are regular necessarily isomorphic to $\mathsf{Q}(4,q)$ or to $\mathsf{Q}(5,q)$? A fair amount of characterizations of $\mathsf{Q}(5,q)$ exists using the assumption of regularity of all the lines, plus an extra condition. Few results though characterize $\mathsf{Q}(4,q)$ and $\mathsf{Q}(5,q)$ at the same time using the regularity of the lines. With a condition on the groups of projectivities we provide such a characterization in this paper.

Also, we provide a characterization of $\mathsf{Q}(5,q)$ only using an assumption on the special dual group of projectivities, together with the condition that $t=s^2$ for the order (s,t). We also characterize $\mathsf{W}(\mathbb{K})$ for some particular fields $\mathbb{K}$.

1.3 Main Results

For our first result, we recall that a field in which every quadratic equation has at least one solution is a *quadratically closed* field. A *separably quadratically closed field* is a field which has no separably quadratic extension. Every quadratically closed field is separably quadratically closed. The converse is true whenever the characteristic of the field is not equal to 2. If the characteristic is equal to 2, then there are separably quadratically closed fields which are not quadratically closed (e.g. the separable quadratic closure of a non-perfect field).

Theorem 1 *Let Γ be a generalized quadrangle all points of which are regular. If every element of $\Pi^*(\Gamma)$ has a fixed element, then $\Gamma \cong \mathsf{W}(\mathbb{K})$, for some separably quadratically closed field $\mathbb{K}$.*

For our second main result, we introduce the following terminolgy. A *Zassenhaus (permutation) group* is a permutation group acting 2-transitively such that only the identity stabilizes at least 3 points.

Theorem 2 *Let Γ be a generalized quadrangle all lines of which are regular. Suppose $\Pi(\Gamma)$ is a Zassenhaus group which satisfies the following additional properties:*

(i) *the set of all elements of* $\Pi(\Gamma)$ *fixing only a point* p *forms, together with the identity, a commutative subgroup of* $\Pi(\Gamma)_p$ *(the stabilizer of the point* p *in* $\Pi(\Gamma)$*);*

(ii) *every non-identity element of* $\Pi(\Gamma)$ *with an involutory couple has exactly two fixed elements.*

Then Γ *is either an orthogonal quadrangle or the dual of a quadrangle arising from a* σ*-hermitian form in a vector space of dimension* 4 *over a skew field; in particular,* Γ *is a Moufang quadrangle.*

If moreover

(iii) $\Pi(\Gamma)_{p,q}$ *(the stabilizer in* $\Pi(\Gamma)$ *of two distinct points* p, q*) is abelian,*

then there is a field $\mathbb{K}$ *of characteristic* $\neq 2$ *and with* -1 *a square in* $\mathbb{K}$ *such that* Γ *is isomorphic to the dual of* $\mathsf{W}(\mathbb{K})$*.*

Corollary 1 *Let* Γ *be a generalized quadrangle all points of which are regular, and let* $\mathbb{K}$ *be some (commutative) field. If* $\Pi^*(\Gamma)$ *is permutation equivalent to* $\mathbf{PSL}_2(\mathbb{K})$ *acting on the projective line* $\mathbf{PG}(1,\mathbb{K})$*, with either* $\mathbb{K}$ *separably quadratically closed, or* $\operatorname{char}\mathbb{K} \neq 2$ *and* -1 *is a square in* $\mathbb{K}$*, then* $\Gamma \cong \mathsf{W}(\mathbb{K})$*.*

For the next result, we need some notation. Namely, for any prime power q, we denote by $\mathbf{PGL}_2^{(\sqrt{q})}(q)$ the group of all projective transformations of $\mathbf{PG}(1,q)$ generated by $\mathbf{PGL}_2(q)$ and the transformation induced by the semi-linear mapping with identity matrix and corresponding field automorphism $x \mapsto x^{\sqrt{q}}$, if q is a square. If q is not a square, then we read $\sqrt{q}$ as the identity in this definition.

Theorem 3 *Let* Γ *be a finite generalized quadrangle of order* (s,t) *all points of which are regular. Then* Γ *is dual to* $\mathsf{Q}(4,s)$ *or to* $\mathsf{Q}(5,s)$ *if and only if* $\Pi^*(\Gamma)$ *is permutation equivalent to a subgroup of* $\mathbf{PGL}_2(t)$ *acting naturally on* $\mathbf{PG}(1,t)$ *and* $\Pi(\Gamma)$ *is permutation equivalent to a subgroup of* $\mathbf{PGL}_2^{(\sqrt{s})}(s)$ *acting naturally on* $\mathbf{PG}(1,s)$*.*

Theorem 4 *Let* Γ *be a finite generalized quadrangle of order* $(s,\sqrt{s})$*. Then* Γ *is dual to* $\mathsf{Q}(5,q)$ *if and only if* $\Pi_+(\Gamma)$ *is a Zassenhaus group.*

Notice that the finite classical projective planes are characterized by a very simple property: a finite projective plane of order $s \neq 23$, $s > 4$, is classical if and only if its group of projectivities does not contain the alternating group in its natural action, see Grundhöfer [1988]. Theorem 4 is the best approximation of that result for quadrangles that we are aware of.

We are still far away from the analogue of the classical results for projective planes: *a projective plane is Moufang if and only if the stabilizer of a point in the group of projectivities has a regular abelian normal subgroup, and it is pappian if and only if its group of projectivities is sharply* 3*-transitive.* Nevertheless, the theorems in this paper are a first step towards such a characterization for Moufang quadrangles with regular points or lines.

2 Coordinatization

2.1 Introduction of the coordinates

In this section, we recall some facts about coordinatization of generalized quadrangles that we will need in the proof of Theorem 2. Everything is due to Hanssens & Van Maldeghem [1988, 1989].

Let Γ be a generalized quadrangles of order (s,t). Choose any point and label it (∞); choose any line through (∞) and label it $[\infty]$. Let R_1 and R_2 be sets of cardinalities s and t, respectively, containing the distinguished elements 0 and 1, but not containing ∞. As a general rule, we denote the coordinates of lines with square brackets; those of points by parentheses.

We complete the flag $\{(\infty),[\infty]\}$ to an ordinary quadrangle

$$(\infty)\,\mathbf{I}\,[\infty]\,\mathbf{I}\,(0)\,\mathbf{I}\,[0,0]\,\mathbf{I}\,(0,0,0)\,\mathbf{I}\,[0,0,0]\,\mathbf{I}\,(0,0)\,\mathbf{I}\,[0]\,\mathbf{I}\,(\infty).$$

We choose bijectively a coordinate $(a,0,0)$, $a\in R_1$, for the points of $[0,0,0]$ distinct from $(0,0)$ (in conformity with the already defined coordinate $(0,0,0)$), and we do the same dually for the lines through $(0,0,0)$ (replacing R_1 by R_2). We label the projection of $(a,0,0)$ onto $[\infty]$ by (a) (similarly dually); we also label the projection onto $[0]$ of $\mathrm{proj}_{[1,0,0]}(a')$ by $(0,a')$ (similarly dually), and we label the projection of $(0,a')$ onto $[0,0]$ by $(0,0,a')$ (similarly dually). Furthermore, we label the projection of $(0,0,b)$ onto any line $[k]$, $k\in R_2$, by (k,b) (and dually), and we label the projection of $(0,a')$ onto the line $[a,l]$ by (a,l,a') (and dually). This way, every point and line has been given unique coordinates. We define the quaternary operations Q_1 and Q_2,

$$\begin{array}{rcrcl} Q_1 & : & R_1\times R_2\times R_1\times R_2 & \rightarrow & R_1, \\ Q_2 & : & R_1\times R_2\times R_1\times R_2 & \rightarrow & R_2, \end{array}$$

as follows:

$$Q_1(a,k,b,k')=a' \Leftrightarrow \delta(\mathrm{proj}_{[k,b,k']}(a),(0,a'))=2$$

and

$$Q_2(a,k,b,k')=l \Leftrightarrow \delta([a,l],[k,b,k'])=2.$$

Then clearly we have

$$(a,l,a')\,\mathbf{I}\,[a,l]\,\mathbf{I}\,(a)\,\mathbf{I}\,[\infty]\,\mathbf{I}\,(\infty)\,\mathbf{I}\,[k]\,\mathbf{I}\,(k,b)\,\mathbf{I}\,[k,b,k']$$

and

$$(a,l,a')\,\mathbf{I}\,[k,b,k']$$

$$\Updownarrow$$

$$\begin{cases} Q_1(a,k,b,k') &= a', \\ Q_2(a,k,b,k') &= l. \end{cases}$$

We define the following binary operation $\oplus$ in R_1:

$$a \oplus b := Q_1(a,1,b,0).$$

We have the following properties (which are easy to verify):

$$\begin{aligned} &Q_1(a,0,b,k') = b = Q_1(0,k,b,0), \\ &Q_2(a,0,0,k') = k' = Q_2(0,k,b,k'), \\ &0 \oplus a = a = a \oplus 0. \end{aligned}$$

We call the quadruple (R_1, R_2, Q_1, Q_2) a *coordinatizing ring for* Γ. We note that $[\infty]$ is a regular line if and only if Q_2 is independent of its third argument.

2.2 Root elations and Moufang quadrangles

Let Γ be any generalized quadrangle. Let $u\,\mathbf{I}\,v\,\mathbf{I}\,w$, with $u \neq w$. Let u_1 and u_2 be two elements incident with a common element which is also incident with u. Then there is at most one automorphism $\theta(u_1,u_2)$ of Γ which fixes all elements incident with u, v and w, and which maps u_1 to u_2. We call $\theta(u_1,u_2)$ an (u,v,w)-*elation*, or a *root elation.* If there is at least one such root elation for every choice of u_1, u_2, with $u_1 \neq u \neq u_2$, then we say that Γ is (u,v,w)-transitive (this is a generalization of a similar notion for projective planes). If Γ is (x,L,y)-transitive for all points x,y and all lines L with $x\,\mathbf{I}\,L\,\mathbf{I}\,y$, $x \neq y$, then we call Γ *half Moufang.* If moreover the dual of Γ is also half Moufang, then we say that Γ is *a Moufang quadrangle.* This Moufang condition was introduced by Tits [1974]. All Moufang quadrangles are classified by Tits & Weiss [1997]. If there is at least one regular element, and if there are no involutory root elations, then the classification follows from a result of Faulkner [1977].

¿From Hanssens & Van Maldeghem [1989], we recall that a coordinatized generalized quadrangle Γ is $((\infty),[\infty],(0))$-transitive if and only if

$$Q_1(a,k,b \oplus B,k') = Q_1(a,k,b,k') \oplus B$$

and

$$Q_2(a,k,b\oplus B,k') = Q_2(a,0,B,Q_2(a,k,b,k')),$$

for all $a,b,B\in R_1$ and all $k,k'\in R_2$. In that case, the action of an $((\infty),[\infty],(0))$-elation on the points of the line $[0]$ is given by $(0,a)\mapsto(0,a\oplus B)$, for some (fixed) $B\in R_1$.

3 Proofs of the theorems

3.1 Proof of Theorem 1

Let p be any point. Let L and M be two distinct lines through p. Let x_i and y_i, $i=1,2$ be two points incident with L and M respectively, but different from p. Suppose also $x_1\neq x_2$ and $y_1\neq y_2$. Let p_i, $i=1,2$, be any point opposite p but collinear with both x_i and y_i. We want to show that $p^{p_1}\cap p^{p_2}$ is non-empty, and we may assume that p_1 and p_2 are not collinear. By assumption, the projectivity $[p;p_1;p_2;p]$ has at least one fixed element. Let N be a fixed element, then the projections of p_1 and p_2 onto N coincide unless Γ contains a triangle. Hence the traces p^{p_1} and p^{p_2} have either at least 2 elements in common, in which case they coincide (in view of the regularity of p), a contradiction, or they have exactly one element in common. This shows that p is a projective point. Hence every point is projective and by SCHROTH [1992], we conclude that $\Gamma\cong\mathsf{W}(\mathbb{K})$, for some field $\mathbb{K}$.

Suppose now first that $\mathbb{K}$ has characteristic 2 and let $Ax^2+Bx+C=0$ be an arbitrary quadratic equation over $\mathbb{K}$, with $B\neq 0$. We may assume that $A=1$. If $B+C=1$, then $x=1$ is a solution of that equation. So suppose that $B+C\neq 1$. If $B\neq 0$, then we put $B_1=B^{-1}(1+C)$ and $B_2=B+B_1$. It is now easy to see that the element $x\mapsto\frac{B_1x+C}{x+B_2}$ of $\mathbf{PGL}_2(\mathbb{K})$ belongs to $\mathbf{PSL}_2(\mathbb{K})$ and a fixed element x_0 satisfies $x_0^2+Bx_0+C=0$. This implies that $\mathbb{K}$ does not have a proper separable quadratic extension.

Now suppose that $\mathbb{K}$ has characteristic different from 2. The following argument is deduced from one we learned from Norbert Knarr. Let $a\in\mathbb{K}$; we must show that a is a square in $\mathbb{K}$. Put $r=\frac{a-1}{2}$ and $s=\frac{a+1}{2}$. The element $x\mapsto\frac{2sx-r}{rx}$ of $\mathbf{PGL}_2(\mathbb{K})$ clearly belongs to $\mathbf{PSL}_2(\mathbb{K})$, hence it has a fixed point x_0 which satisfies the quadratic equation $rx_0^2-2sx_0+r=0$. Consequently the discriminant $4s^2-4r^2=4a$ is a square.

This completes the proof of Theorem 1.

3.2 Proof of Theorem 2

In order to prove Theorem 2, we note that a line L is regular if whenever L,L_1,L_2 are pairwise skew lines with the property that there exist lines M_1,M_2 meeting all three of them, then any other line meeting two of $\{L,L_1,L_2\}$ also has to meet the third one.

We put $G = \Pi(\Gamma)$ and we denote by H the (abstract) stabilizer in G of a point. Furthermore, we let $N \subset H$ be the set of elements which fix exactly one point, together with the identity. Then by assumption, N is an abelian (normal) subgroup of H acting on a line minus one point.

1. *N is transitive.*

   Let (R_1, R_2, Q_1, Q_2) be an arbitrary coordinatizing ring. We may assume that H is the stabilizer of the point labeled (∞) acting on the line $[\infty]$. Let $b \in R_1$ be an arbitrary element. The projectivity

   $$\theta_b = [[\infty], [1, b, 0], [0], [1, 0, 0], [\infty]]$$

   maps 0 to b, and, by definition of $\oplus$, maps a to $a \oplus b$. Suppose θ_b has a fixed point (x), $x \neq \infty$. Since $[\infty]$ is a regular line, there is a line L_x through (x) meeting both $[1, b, 0]$ and $[1, 0, 0]$. Since $(x)^{\theta_b} = (x)$, the projections onto $[0]$ of the intersections of L_x with $[1, b, 0]$ and $[1, 0, 0]$ coincide, hence $b = 0$ or there arises a triangle. So for $b \neq 0$ there are no fixed elements besides (∞), i.e. $\theta_b \in N$ and in fact $N = \{\theta_b; b \in R_1\}$ because a transitive abelian subgroup is sharply transitive.

2. *The set-up.*

   Considering N as an additive group with operation law $+$ and identity denoted by 0, we can now identify $\theta_b \in N$ and $b \in R_1$. Then the action of $N \leq G$ on $R_1 \cup \{\infty\}$ is given by right translation (fixing ∞) and for every projectivity ρ of the point row $[\infty]$, the mapping $\theta : N \cup \{\infty\} \to N \cup \{\infty\} : x \mapsto x^\theta$ defined by $(x)^\rho = (x^\theta)$ is an element of G. By projecting successively onto $[1, 0, 0]$ and $[0]$, it follows that, identifying (∞) with $(0, \infty)$, for every projectivity ρ of the point row $[0]$, the mapping $\theta : N \cup \{\infty\} \to N \cup \{\infty\} : x \mapsto x^\theta$ defined by $(0, x)^\rho = (0, x^\theta)$ is an element of G. Our first aim is to show that Γ is $((\infty), [\infty], (0))$-transitive. To that end, we have to show that $Q_1(a, k, b \oplus B, k') = Q_1(a, k, b, k') \oplus B$ and $Q_2(a, k, b \oplus B, k') = Q_2(a, 0, B, Q_2(a, k, b, k'))$. It will follow that Γ is a half Moufang quadrangle. Our second aim is then to construct all other root elations and restrict the possibilities for Γ. Finally, we have to show that, if the stabilizer in H of some point is abelian, then Γ is dual to the symplectic quadrangle $\mathsf{W}(\mathbb{K})$ for some field $\mathbb{K}$.

3. $Q_2(a, k, b \oplus B, k') = Q_2(a, 0, B, Q_2(a, k, b, k'))$.

   This identity is automatically satisfied since $[\infty]$ is a regular line (see above: Q_2 is independent of its third argument in that case, and we always have $Q_2(a, 0, 0, x) = x$).

4. *$a \oplus b$ in R_1 is the same as $a + b$ in N.*

   Just note that $\theta_a + \theta_b = \theta_{a \oplus b}$ since they agree at 0 and N acts sharply transitively.

5. *For any* $k, k' \in R_2 \setminus \{0\}$*, the projectivity* $[[0]; [0,0]; [k]; [0,k']; [0]]$ *has no fixed points besides* (∞).

   Let x be a fixed point of the projectivity mentioned. Let x', x'' and x''' be the successive projections onto $[0,0]$, $[k]$ and $[0,k']$. Then the projectivity $[[\infty]; xx'; x''x'''; [\infty]]$ interchanges the points (∞) and (0). By assumption, there are two fixed points (a_i), $i = 1,2$.

   This means that (a_i) and its projections onto xx' and $x''x'''$ are collinear (since otherwise there would be a triangle), say they are incident with M_i, $i = 1,2$. Since $[\infty]$ is a regular line, every line meeting two elements of $\{[\infty], xx', x''x'''\}$ meets also the third, contradicting the fact that $[k]$ meets $[\infty]$ and $x''x'''$, but not xx'.

6. $Q_1(a,k,b \oplus B,k') = Q_1(a,k,b,k') \oplus B$.

   Put $a_1 = Q_1(a,k,b \oplus B,k')$ and $a_2 = Q_1(a,k,b,k')$. We have to show that $a_1 = a_2 + B$. Let L be the line that joins the point (a) to its projection x_2 onto $[k,b,k']$. Then L also meets $[k,b+B,k']$, say in the point x_1 (by regularity of the line $[\infty]$ and the fact that both $[k]$ and $[0,k']$ meet all three of $[\infty], [k,b,k'], [k,b+B,k']$). By definition, the projection of x_i onto $[0]$ is the point $(0,a_i)$, $i = 1,2$. Now consider the projectivity $\theta = [[0]; L; [k]; [0,0]; [0]]$. If $a = 0$, then $L = [0,k']$ and θ has no fixed points except for (∞) by the previous paragraph. Suppose now $a \neq 0$ and assume that θ has a fixed point $(0,x)$, $x \in N$. By the previous paragraph, the projectivity $\theta' = [[0]; [0,0]; [k]; [0,k']; [0]]$ has no fixed points except (∞). Since N is a subgroup, the projectivity $\theta\theta'$ has some fixed element $(0,x')$, $x' \in N$. But $\theta\theta' = [[0]; L; [k]; [0,k']; [0]]$, and since any line joining a point of $[k]$ to its projection onto $[0,k']$ has to intersect L by the regularity of $[\infty]$, this is equal to $\theta\theta' = [[0]; L; [0,k']; [0]]$. Now if $(0,x')$ is a fixed point of $\theta\theta'$, then $[k]$ has to intersect the line defined by $(0,x')$ and its projection to L, yielding a triangle unless $k = 0$, in which case $a_1 = b + B$ and $a_2 = b$ by the identities in the introduction, so the result follows trivially. Hence we can assume that θ has no fixed points distinct from (∞) and so we can write $\theta : (0,x) \mapsto (0,x+C)$ for some $C \in K$. But $a_1^\theta = b+B$ and $a_2^\theta = b$. Hence $a_1 + C = b + B$ and $a_2 + C = b$, consequently $a_1 = a_2 + B$.

7. Γ *is a Moufang quadrangle.*

   We have already shown that Γ is a half Moufang quadrangle. Now let p be any point, let L_1 and L_2 be distinct lines through p, let x_i be incident with L_i, $i = 1, 2$, with $x_i \neq p$. Let (x_1, M_1, y, M_2, x_2) and $(x_1, M_1', y', M_2', x_2)$ be two paths with $y \neq p \neq y'$. We have to show that there exists an (L_1, p, L_2)-elation mapping M_1 onto M_1'. Let z be any point on M_2, $y \neq z \neq x_2$. Let (M_1', z', P, z) and (P, z'', P', p) be two paths (which uniquely define the elements z', z'', P, P'). Let θ be the (p, L_2, x_2)-elation mapping y to z. Let θ' be the (p, P', z'')-elation mapping z to z'. Then θ' induces on the point row of L_2 an element η belonging to N (considering N with respect to L_2 and p, i.e., considering N as a subgroup of the group of self-projectivities of L_1 and each element of N fixes p). Let θ'' be the (x_1, L_1, p)-elation wich maps x_2^η onto x_2 (or equivalently, z' to y'), then θ'' induces on L_2 the mapping η^{-1} because x_2 is fixed by $\theta'\theta''$. Also, if θ induces the mapping η' on the point row of L_1, then θ' similarly induces ${\eta'}^{-1}$ on L_1 since $\theta\theta'$ fixes x_1. Hence the collineation $\theta\theta'\theta''$ is an (L_1, p, L_2)-elation which maps y to $y^{\theta\theta'\theta'} = z^{\theta'\theta''} = {z'}^{\theta''} = y'$.

8. *Which Moufang quadrangles have regular lines?*

   By the classification of Moufang quadrangles mentioned above, the only Moufang quadrangles with regular lines are the orthogonal quadrangles, the duals of the quadrangles arising from σ-hermitian forms over skew fields in vector spaces of dimension 4 (for precise definitions, see e.g. TITS [1974], Chapter 8), the duals of some quadrangles arising from σ-hermitian forms over skew fields of characteristic 2 in vector spaces of dimension > 4, and so-called mixed quadrangles (which are subquadrangles of symplectic quadrangles in characteristic 2). This follows from VAN MALDEGHEM [19**], Table 5.1. However, no Moufang quadrangle satisfying the hypotheses of Theorem 2 has root elations of even order (because these root elations have involutory couples without having two fixed points). This rules out all Moufang quadrangles defined over a field in characteristic 2, in particular the last two classes mentioned above. This proves the first statement of Theorem 2.

   We now assume that the stabilizer in G of two points is abelian and we take a closer look at orthogonal quadrangles and dual hermitian quadrangles in vector spaces of dimension 4.

9. *Moufang quadrangles arising from σ-hermitian forms in vector spaces of dimension* 4.

   Let Γ be a Moufang quadrangle arising from a σ-hermitian form in a vector space of dimension 4 over the skew field $\mathbb{L}$. It is shown in TITS [1974] (10.10, page 213), that the dual group of projectivities contains all maps of the form

$$\mathbb{L}_{\sigma,-1} \to \mathbb{L}_{\sigma,-1} : x \mapsto a^\sigma x a,$$

where $a \in \mathbb{L} \setminus \{0\}$, and $\mathbb{L}_{\sigma,-1} = \{t + t^\sigma : t \in \mathbb{L}\}$. These projectivities all fix two elements and hence they must commute with each other (since they are a subgroup of the stabilizer of two elements in $\mathbf{PSL}_2(\mathbb{K})$). By TITS [1974] (10.5) and (10.9), Γ is the dual of an orthogonal quadrangle (remember that $\mathbb{K}$ does not have characteristic 2).

10. *Orthogonal quadrangles.*

First, we note:

Suppose Γ' is a subquadrangle of the generalized quadrangle Γ. Let L be a line of Γ' (note that L is also a line of Γ). Then the group of self-projectivities of L in Γ' is, as a permutation group acting on $\Gamma'(L)$, a subgroup of the stabilizer of $\Gamma'(L)$ of the group of self-projectivities of L in Γ, viewed as a permutation group acting on $\Gamma(L)$.

To prove this, it is enough to remark that every perspectivity in Γ' extends uniquely to a perspectivity in Γ in the obvious way.

Now we can show the following result:

If Γ is an orthogonal quadrangle arising from a non-degenerate quadric in $\mathbf{PG}(d, \mathbb{K})$, with $\mathbb{K}$ a field of characteristic $\neq 2$, then $\Pi(\Gamma)$ is permutation equivalent to $\mathbf{PSL}_2(\mathbb{K})$ if and only if Γ is dual to $\mathsf{W}(\mathbb{K})$.

If $\mathbb{K}$ is quadratically closed, then $d = 4$ and the result follows. So we may assume that $\mathbb{K}$ is not quadratically closed.

By our observation above, it clearly suffices to show that for $d = 5$, the group $\Pi(\Gamma)$ contains an element of the form $x \mapsto mx$, with m a non-square in $\mathbb{K}$.

By VAN MALDEGHEM [19**], Γ is coordinatized by $R_1 = \mathbb{K}$ and $R_2 = \mathbb{K} \times \mathbb{K}$, and there exists a non-square $m \in \mathbb{K}$ such that

$$\begin{aligned} Q_1(a, (k_1, k_2), b, (k_1', k_2')) &= b + a(k_1^2 + mk_2^2) + 2(k_1k_1' + mk_2k_2'), \\ Q_2(a, (k_1, k_2), b, (k_1', k_2')) &= (k_1', k_2') - (ak_1, ak_2). \end{aligned}$$

Now the map determined by

$$\begin{cases} (a, (l_1, l_2), a') &\mapsto (a, (ml_2, l_1), ma') \\ [(k_1, k_2), b, (k_1', k_2')] &\mapsto [(mk_2, k_1), mb, (mk_2', k_1')] \end{cases}$$

clearly preserves incidence, it fixes every point on $[\infty]$, and it induces on $[0]$ the map $(0, a) \mapsto (0, ma)$. By VAN MALDEGHEM [19**], Lemma(8.3.1), $\Pi(\Gamma)$ contains the element of $\mathbf{PGL}_2(\mathbb{K})$ determined by $x \mapsto mx$. Since m is not a square in $\mathbb{K}$, the result follows.

This result now implies that every orthogonal quadrangle which is not dual to a symplectic quadrangle has fixed point free involutory self-projectivities (of a line).

Indeed, by the 2-transitivity of $\Pi(\Gamma)$, there exists at least one involution $\theta \in \Pi(\Gamma) \geq \mathbf{PSL}_2(\mathbb{K})$ of the form $x \mapsto -a^2/x$. If -1 is not a square in $\mathbb{K}$, this has no fixed points. If -1 is a square in $\mathbb{K}$, then we compose with $x \mapsto mx$, m a non-square as above, to obtain the mapping $x \mapsto -ma^2/x$. The latter mapping has no fixed points.

The theorem is proved.

3.3 Proof of Corollary 1

Let $\mathbb{K}$ be a field of characteristic $\neq 2$ in which -1 is a square. We show that $\mathbf{PSL}_2(\mathbb{K})$ acting on $\mathbf{PG}(1, \mathbb{K})$ satisfies the assumptions of Theorem 2. Indeed, this action can be identified with the action of the rational transformations $x \mapsto \frac{ax+b}{cx+d}$, with $ad - bc$ a non-zero square in $\mathbb{K}$. The stabilizer of ∞ is $\mathbf{AGL}_1^+(\mathbb{K})$, its elements being the maps of the form $x \mapsto a^2x + b$. If such a function has no fixed point apart from ∞, then clearly $a = 1$, and the elements with $a = 1$ form a subgroup of $\mathbf{PSL}_2(\mathbb{K})$. Moreover, the stabilizer of ∞ and 0 is commutative (and consists of the elements of the form $x \mapsto a^2x$, $a \neq 0$). Also, any element with an involutory couple, say $(\infty, 0)$, has the form $x \mapsto -a^2/x$, $a \neq 0$. Since -1 is a square in $\mathbb{K}$, this has two distinct fixed points (ia) and $(-ia)$ where $i^2 = -1$.

Also, if $\mathbb{K}$ is a separably quadratically closed field, then obviously the conditions of Theorem 1 are satisfied, and the result follows.

The corollary is proved.

We should point out that the conditions on the permutation group $\Pi(\Gamma)$ in Theorem 2 are a special case of a characterization of subgroups of $\mathbf{PGL}_2(\mathbb{K})$ due to MÄURER[1983]. By his result, $\Pi(\Gamma)$ satisfies assumptions $(i) - (iii)$ if and only if it is permutation equivalent to $\mathbf{PSL}_2(\mathbb{K})$ for a field of characteristic $\neq 2$ with -1 a square in $\mathbb{K}$ acting on the projective line $\mathbf{PG}(1, \mathbb{K})$.

3.4 Proof of Theorem 3

As in the proof of Theorem 2, the result follows if Γ does not contain any 3×3-grid (a $k \times \ell$-grid being a weak generalized quadrangle with lines containing k points, with lines containing ℓ points, and such that every point is incident with 2 lines). Indeed, Condition (ii) of Theorem 2 is only used in the proof of that theorem in Paragraph 5. With the notation of that paragraph, the points (∞), (0), x, x', x'' and x''' form the dual of a 3×3-grid, if x is a fixed point of the projectivity under consideration. So Condition (ii) may be replaced by the condition that no such dual grid exists. Of course, Condition (i) is satisfied by every 2-transitive subgroup of $\mathbf{PGL}_2(q)$ (acting naturally on $\mathbf{PG}(1, q)$).

Hence we assume that Γ does contain a 3×3-grid. If $\Pi(\Gamma)$ is contained in $\mathbf{PGL}_2(s)$, then Γ contains a regular pair of lines, hence $s = t$ and Γ is dual to $\mathsf{W}(s)$ with s even. Hence we may assume that s is a perfect square, say $s = q^2$, and that $\Pi(\Gamma)$ contains a non-linear semi-linear transformation. Note that, since Γ contains regular points, $s \geq t$.

1. *Every* 3×3*-grid is contained in a maximal* $(q+1) \times (q+1)$*-grid.*
   Let $\{L_1, L_2, L_3\} \subseteq \{M_1, M_2, M_3\}^\perp$ with M_i opposite M_j, $i \neq j$, $i, j \in \{1, 2, 3\}$, and with $L_1 \neq L_2 \neq L_3 \neq L_1$. The projectivity $\sigma = [L_1; L_2; L_3; L_1]$ has at least three fixed points. Identifying in $\mathbf{GF}(q^2) \cup \{\infty\}$ these points with $0, 1, \infty$, we readily see that σ is either the identity or the map $x \mapsto x^q$. If a is fixed under σ, then $\{a, \mathrm{proj}_{L_2} a, \mathrm{proj}_{L_3} a\}$ forms a triangle, hence $\mathrm{proj}_a L_2 = \mathrm{proj}_a L_3$. It follows that $|\{L_1, L_2, L_3\}^\perp| \geq q + 1$. Suppose $\{M_0, M_1, \ldots, M_q\} \subseteq \{L_1, L_2, L_3\}^\perp$, with $M_i \neq M_j$ for $i \neq j$, $i, j \in \{0, 1, \ldots, q\}$. The projectivity $\sigma_i = [M_0; M_i; M_q; M_0]$, $0 < i < q$, has at least three fixed points which are independent of i. Identifying these points with $0, 1, \infty$ again, we deduce similarly as before that there are at least $q + 1$ lines L (namely, one line through each point on M_0 corresponding to an element of $\mathbf{GF}(q) \cup \{\infty\}$) in $\{M_0, M_1, \ldots, M_q\}^\perp$. Hence we already have a $(q+1) \times (q+1)$-grid $\mathcal{G}$ containing $L_1, L_2, L_3, M_1, M_2, M_3$. We now show that this grid is maximal. In fact, we will show that whenever a line L meets three of the lines $M_0, M_1, \ldots, M_q$, then it must meet all of them and it belongs to $\mathcal{G}$. We may assume $L_1 \neq L \neq L_2$. By considering the projectivity $[L; L_1; L_2; L]$, we see as above that L must belong to $\{M_0, M_1, \ldots, M_q\}^\perp$. Now, if $L \notin \mathcal{G}$, then the projectivity $[M_0; M_i; M_q; M_0]$, $0 < i < q$, has at least $q + 2$ fixed points, hence it is the identity and we easily deduce (as before) that $|\{M_0, M_1, \ldots, M_q\}^\perp| = q^2 + 1$. Now consider a point w on L_1 not incident with any M_i, $0 \leq i \leq q$. Let $\{M_0, M_1 \ldots, M_q\}^\perp = \{L_0, L_1, \ldots, L_{q^2}\}$. Let $N_i = \mathrm{proj}_w L_i$, $0 \leq i \leq q^2$, $i \neq 1$. If $N_i = N_j$ for some $i \neq j$, then N_i meets three of the lines $L_0, L_1, \ldots, L_{q^2}$ and as before, we deduce that it must meet every such line, and again as before this implies that $\mathcal{G}$ is contained in an $(s+1) \times (s+1)$-grid. Hence $s = t$, a contradiction. So the lines N_i, $0 \leq i \leq q^2$, $i \neq 1$, are pairwise distinct and we obtain $s \leq t$. Consequently $s = t$, a contradiction. We conclude that $\mathcal{G}$ is maximal.

2. *We have* $t = q$*.*
   Let $\mathcal{G}$ be a $(q+1) \times (q+1)$-grid with line set $\{L_0, L_1, \ldots, L_q, M_0, M_1, \ldots, M_q\}$, and with $L_i \perp M_j$, for all $i, j \in \{0, 1, \ldots, q\}$. Let w_0 be a point on L_0 not belonging to the grid. By the previous paragraph, the q lines N_i, $1 \leq i \leq q$, incident with w_0 and concurrent with L_i are mutually distinct. Now assume that $t \neq q$, i.e., $t > q$. Then there is some further line N through w_0, $N \neq N_i$, $i = 1, 2, \ldots, q$. Consider the projectivity $\sigma_i = [L_0; L_i; L_q; L_0]$. It has exactly $q + 1$ fixed points, hence it is an involution σ (in fact, independent of i). Let w_0' be the image of w_0 under σ. Let w_i be the projection of w_0' onto L_i, and let w_i' be the projection of w_0 onto L_i, $1 \leq i \leq q$.

Using the fact that σ_i is involutory, we easily see that $w'_0, w'_1, \ldots, w'_q \in \{w_0, w_q\}^\perp$ and $w_0, w_1, \ldots, w_q \in \{w'_0, w'_q\}^\perp$. By the regularity of points in Γ, we deduce that w_i and w'_j are collinear for all $i, j \in \{0, 1, \ldots, q\}$. Now let w' be the point of N collinear with w_q, and hence with w_i, for all $i \in \{0, 1, \ldots, q\}$. Let N' be the line through w' concurrent with M_0. Let w be the projection of w'_0 onto N'. Then w is collinear with w'_i, for all $i \in \{0, 1, \ldots, q\}$. Denote by x_{ij} the intersection of L_i with M_j. Then the projectivity $\theta = [L_0; L_q; N'; L_0]$ has a fixed point x_{00} and an involutory couple (w_0, w'_0). Hence it is an involution. If a semi-linear involution fixes ∞ and has an involutory couple $(0, b)$, it is of the form $x \mapsto -(b/b^q)x^q + b$. Hence if θ is semi-linear but not linear, then it coincides with the involution $\sigma = [L_0; L_1; L_q; L_0]$ since they agree on x_{00} and the involutory couple (w_0, w'_0). Hence θ has in particular the same set of fixed points as σ. As before, this implies that N' meets all M_i, $i \in \{0, 1, \ldots, q\}$, and so $\mathcal{G}$ is not maximal, a contradiction. We conclude that θ is linear. Similarly the projectivity $\theta' = [L_0; L_1; N'; L_0]$ is a linear involution. We deduce $\theta = \theta'$. Hence $\theta'\theta^{-1} = [L_0; L_1; N'; L_q; L_0]$ is the identity. It readily follows that the line M_i, $i \in \{0, 1, \ldots, q\}$, is concurrent with N'. Hence again, $\mathcal{G}$ is not maximal, a contradiction. We conclude that $t = q$.

3. Γ *is a Moufang quadrangle.*
   We already have that Γ has order (q^2, q). By Payne & Thas [1984](1.2.4), every three pairwise opposite lines are contained in a $(q+1) \times 3$-grid, in particular in a 3×3-grid. Hence, every three pairwise opposite lines are contained in a maximal $(q+1) \times (q+1)$-grid. The result now follows directly from the dual of Payne & Thas [1984](5.3.2(i)).

3.5 Proof of Theorem 4

Let L_0 be any line of Γ. Let L_1 be any line opposite L_0, and let M_0, M_1, M_2 be three different lines concurrent with both L_0 and L_1. As above, we know that L_0, L_1, M_0, M_1, M_2 are contained in a $(\sqrt{s}+1) \times 3$-grid containing $\sqrt{s}+1$ lines $L_0, , L_1, L_2, \ldots, L_{\sqrt{s}}$ which are all concurrent with M_0, M_1, M_2. Similarly, there are $\sqrt{s}+1$ lines $M_0, M_1, M_2, \ldots, M_{\sqrt{s}}$ concurrent with L_0, L_1, L_2. If we show that L_j meets M_i, for $i, j \in \{3, 4, \ldots, \sqrt{s}\}$, then as above, we are done (again using Payne & Thas [1984](5.4.2(i))). Therefore, consider the even projectivity $\theta = [L_0; L_1; L_2; L_j; L_0]$. Clearly the intersection points of L_0 with M_0, M_1, M_2, respectively, are fixed by θ. By assumption, also the intersection point x of L_0 and M_i is fixed. This yields a triangle with vertices $x, \mathrm{proj}_{L_2}x, \mathrm{proj}_{L_j}x$ if L_j does not meet M_i.

The theorem is proved.

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