Generalized Quadrangles Weakly Embedded in Finite Projective Space

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Abstract

We show that every weak embedding of any finite thick generalized quadrangle of order (s, t) in a projective space $\mathbf{PG}(d, q)$, q a prime power, is a full embedding in some subspace $\mathbf{PG}(d, s)$, where $\mathbf{GF}(s)$ is a subfield of $\mathbf{GF}(q)$, except in some well-known cases where we classify these exceptions. This generalizes a result of Lefèvre-Percsy [4], who considered the case $d = 3$.

1 Introduction and Statement of the Main Result

A *weak embedding* of a point-line geometry Γ with point set $\mathcal{S}$ in a projective space $\mathbf{PG}(d, \mathbb{K})$ is a monomorphism θ of Γ into the geometry of points and lines of $\mathbf{PG}(d, \mathbb{K})$ such that

(WE1) the set $\mathcal{S}^\theta$ generates $\mathbf{PG}(d, \mathbb{K})$,

(WE2) for any point x of Γ, the subspace generated by the set $X = \{y^\theta \parallel y \in \mathcal{S}$ is collinear with $x\}$ meets $\mathcal{S}^\theta$ precisely in X.

In such a case we say that the image Γ^θ of Γ is weakly embedded in $\mathbf{PG}(d, \mathbb{K})$.

A *full embedding* in $\mathbf{PG}(d, \mathbb{K})$ is a weak embedding with the additional property that for every line L, all points of $\mathbf{PG}(d, \mathbb{K})$ on the line L^θ have an inverse image under θ.

Weak embeddings were introduced by Lefevre-Percsy [2, 3], who required a third condition, namely

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(WE3) if for two lines L_1 and L_2 of Γ the images L_1^θ and L_2^θ meet in some point x, then x belongs to $\mathcal{S}^\theta$.

But it was shown by THAS & VAN MALDEGHEM [8] that (WE3) is a consequence of (WE1) and (WE2) if Γ is a non-degenerate polar space of rank at least 2, that is, they proved that "sub-weak" is equivalent to "weak" where "sub-weak" was used for a θ satisfying (WE1), (WE2) and "weak" was used for a θ satisfying (WE1), (WE2), (WE3). LEFEVRE-PERCSY [4] classified all finite thick generalized quadrangles weakly embedded in a finite projective 3-space (in fact she erroneously thought that she had proved a stronger result, namely she only assumed conditions (WE1) and (WE3) mentioning without proof that (WE2) follows from these; this is unfortunately not true and we will meet counterexamples in the proof of Lemma 8). In the present paper we will classify all weak embeddings of all finite thick generalized quadrangles in any finite projective space.

Let Γ be a finite thick generalized quadrangle with point set $\mathcal{S}$ and line set $\mathcal{L}$. Let θ be a monomorphism from Γ into the point-line geometry of a projective space $\mathbf{PG}(d,q)$. Usually, we simply say that Γ is weakly embedded in $\mathbf{PG}(d,q)$ without referring to θ, that is, we identify the points and lines of Γ with their images in $\mathbf{PG}(d,q)$.

For finite polar spaces of rank at least 3, it follows from THAS & VAN MALDEGHEM [8] that any weak embedding is a full embedding in some subspace over some subfield. This is certainly not true for generalized quadrangles as the following counterexample shows. Let x_1, x_2, x_3, x_4, x_5 be the consecutive vertices of a proper pentagon in $W(2)$, with $W(2)$ the generalized quadrangle arising from a symplectic polarity in $\mathbf{PG}(3,2)$ (note that the automorphism group of $W(2)$ acts regularly on the set of all such pentagons). Let $\mathbb{K}$ be any field and identify x_i, $i \in \{1,2,3,4,5\}$, with the point $(0,\ldots,0,1,0,\ldots,0)$ of $\mathbf{PG}(4,\mathbb{K})$, where the 1 is in the ith position. Identify the unique point y_{i+3} of $W(2)$ on the line $x_i x_{i+1}$ and different from both x_i and x_{i+1}, with the point $(0,\ldots,0,1,1,0,\ldots,0)$ of $\mathbf{PG}(4,\mathbb{K})$, where the 1's are in the ith and the $(i+1)$th position (subscripts are taken modulo 5). Finally, identify the unique point z_i of the line $x_i y_i$ (it is easy to see that this is indeed a line of $W(2)$) different from both x_i and y_i, with the point whose coordinates are all 0 except in the ith position, where the coordinate is -1, and in the positions $i-2$ and $i+2$, where it takes the value 1 (again subscripts are taken modulo 5). It is an elementary excercise to check that this defines a weak embedding of $W(2)$ in $\mathbf{PG}(4,\mathbb{K})$. We call this the *universal weak embedding* of $W(2)$ in $\mathbf{PG}(4,\mathbb{K})$.

Our Main Result reads as follows.

Theorem 1 *Let Γ be a finite thick generalized quadrangle of order (s,t) weakly embedded in the projective space $\mathbf{PG}(d,q)$. Then either s is a prime power, $\mathbf{GF}(s)$ is a subfield of $\mathbf{GF}(q)$ and Γ is fully embedded in some subspace $\mathbf{PG}(d,s)$ of $\mathbf{PG}(d,q)$, or Γ is isomorphic to $W(2)$, the unique generalized quadrangle of order 2, and the weak embedding is the universal one in a projective 4-space over an odd characteristic finite field.*

Note that all full embeddings of finite thick generalized quadrangles are known (see BUEKENHOUT & LEFEVRE [1] and PAYNE & THAS [6]) and only the natural modules of the groups of the classical quadrangles turn up. So the above theorem gives a complete classification of all weakly embedded thick generalized quadrangles in finite projective space.

Finally, we introduce one more notion. Let Γ be a polar space weakly embedded in some projective space $\mathbf{PG}(d,\mathbb{K})$ for some field $\mathbb{K}$. Let L be any line of $\mathbf{PG}(d,\mathbb{K})$. If L contains at least two points of Γ which are not collinear in Γ, then we call L a *secant* . It is easy to show that no secant line contains two collinear points of Γ (see Lemma 1 of THAS & VAN MALDEGHEM [8]). LEFEVRE-PERCSY [2] shows that the number of points of Γ on a secant line is a constant δ. We call δ the *degree* of the embedding.

For notation about generalized quadrangles not explained here, we refer to the monograph by PAYNE & THAS [6].

2 Proof of the Main Result

In this section, we prove our Main Result in a series of lemmas. So from now on we suppose that Γ is a thick generalized quadrangle of finite order (s,t) with point set $\mathcal{S}$ and line set $\mathcal{L}$. We also assume that Γ is weakly embedded of degree δ in $\mathbf{PG}(d,q)$ for some prime power q and some positive integers d and δ (necessarily $d \geq 3$).

First we recall the result of LEFEVRE-PERCSY [4], which we will use in the course of our proof.

Lemma 1 *If $d=3$, then there is a subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$ and a subspace $\mathbf{PG}(3,s)$ of $\mathbf{PG}(3,q)$ such that Γ is fully embedded in $\mathbf{PG}(3,s)$.*

In this case either $\Gamma \cong W(s)$ or $\Gamma \cong H(3,s)$ and $\delta = s+1$ respectively $\delta = \sqrt{s}+1$. Note also that $t+1=\delta$ in these cases.

And we also recall from LEFEVRE-PERCSY [3], Proposition 4, the following result.

Lemma 2 *If L and M are two lines of Γ meeting in a point x of Γ, then the plane LM of $\mathbf{PG}(d,q)$ contains exactly δ lines of Γ through x.*

We now consider the case $d = 4$.

Lemma 3 *If $d = 4$ and $\delta = 2$, then $\Gamma \cong Q(4,s)$.*

PROOF. Consider any two opposite lines L and M of Γ (i.e. two lines which do not meet in Γ). By (WE3), L and M span a 3-dimensional subspace U (over $\mathbf{GF}(q)$) of $\mathbf{PG}(4,q)$. By 2.3.1 of Payne & Thas [6] the points of $\mathcal{S}$ in U together with the lines of $\mathcal{L}$ in U form a subquadrangle Γ' of Γ. By (WE2) and Lemma 2, the order of Γ' is $(s,1)$. Hence (L,M) is a pair of regular lines of Γ, and consequently every line of Γ is regular. By 1.3.6(i) of Payne & Thas [6] this implies $s \leq t$. Now consider two opposite points x and y of Γ. By (WE2) the points of Γ collinear with x span at most a 3-dimensional subspace of $\mathbf{PG}(4,q)$; by Lemma 2 these points span at least a 3-dimensional subspace of $\mathbf{PG}(4,q)$. Hence the points of Γ collinear with x span a 3-dimensional subspace U_x of $\mathbf{PG}(4,q)$; similarly one defines U_y. By (WE2) the spaces U_x and U_y are distinct and hence they meet in a plane π. Suppose there exists a point z of Γ collinear (in Γ) with three different points u,v and w of Γ in π (since $\delta = 2$, u,v,w are not on a common line of $\mathbf{PG}(4,q)$). Then, by (WE2), z is collinear with the $t+1$ points of Γ in π. If z', with $z' \notin \{x,y,z\}$, is collinear (in Γ) with u and v, then, as w is collinear with x,y,z, the point w is collinear with all points collinear with u,v, hence w is collinear with z'. It follows that z' is collinear with all points of Γ in π. Consequently we obtain a subquadrangle of order $(1,t)$. Hence in this case the pair (x,y) is a regular pair of points and so by 1.3.6(i) of Payne & Thas [6] we have $s \geq t$. Now by Payne & Thas [6], 5.2.1 (dual statement) and 3.3.1(i), $\Gamma \cong Q(4,s)$ and s is even. So we may assume that no point of Γ, distinct from x and y, is collinear with at least three points of π. But then the pair (x,y) is antiregular and by 1.3.6(i) of Payne & Thas [6] we again have $s \geq t$. Hence, again by the same theorem as in the even case, $\Gamma \cong Q(4,s)$ and s is odd. □

Lemma 4 *If $d = 4$, $\delta = 2$ and, if for q odd we assume $s \neq 2$, then the points of every line L of Γ form a subline of L over the subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$. Also, every subquadrangle of order $(s,1)$ is fully embedded in some 3-dimensional subspace over $\mathbf{GF}(s)$ of $\mathbf{PG}(4,q)$.*

PROOF. By Lemma 3, $\Gamma \cong Q(4,s)$. Let L and M be two opposite lines of Γ. Let π be a plane of $\mathbf{PG}(4,q)$ containing M and disjoint from L. Clearly, any grid of Γ is defined by two such lines and therefore lies in a unique 3-dimensional subspace $\mathbf{PG}(3,q)$ of $\mathbf{PG}(4,q)$; by the first part of the proof of Lemma 3, this grid is formed by all points and lines of Γ in $\mathbf{PG}(3,q)$. Consider now all grids of Γ containing L, and intersect the corresponding 3-spaces with π. We obtain s^2 lines of π which are said to be of type (I). Also, consider the $s+1$ 3-spaces of $\mathbf{PG}(4,q)$ spanned by the points of Γ collinear with any point of Γ on L. Since these 3-spaces contain L, each of them meets π in a line. Hence we obtain $s+1$ lines (not of type (I)) in π which are said to be of type (II). Suppose K_1 and K_2 are two lines in π of type (I) or (II), but not both of type (II). Since $\Gamma \cong Q(4,s)$, the corresponding grids, or the corresponding grid and the set of all lines of Γ incident with a fixed point of Γ on L share exactly two lines one of which is L. These two lines span a plane meeting π in a point of π lying on both K_1 and K_2. Also, any two distinct lines of type (II) do not meet. As there are $s(s+1)$ lines of Γ concurrent with L, but different from L, and as each of these lines defines a common point of two lines in π of type (I) or (II), but not both of type (II), there are exactly s^2+s points in π on at least two lines in π of type (I) or (II), but not both of type (II). It is also clear that any two distinct of these s^2+s points in π are contained in exactly one line of type (I) or (II) (this follows directly from the construction of the points and from the definition of the lines of type (I) and (II)). Hence we obtain a linear space with s^2+s points and s^2+s+1 blocks; each block of type (I) contains $s+1$ points, each block of type (II) contains s points and each point is contained in $s+1$ blocks. Deleting one block K of type (II) with all points on it gives us an affine plane of order s, which is a substructure of the projective plane π. By LIMBOS [5], either this affine plane uniquely embeds in a projective subplane π' of π, or $s \in \{2,3\}$. If $s=3$, then again by LIMBOS [5], no three distinct lines of any parallel class of that affine plane meet in a common point, contradicting the fact that for three of the four parallel classes (those which have as point at infinity a point of the deleted block K) all lines meet in a unique point. We conclude that the $s+1$ points of Γ on M form a subline $\mathbf{PG}(1,s)$ over $\mathbf{GF}(s)$, since for $s=2$ and q even this is trivial.

Now let Γ' be any subquadrangle of order $(s,1)$ of Γ. Consider two opposite lines L and M of Γ' and let N be a line of Γ' meeting both L and M. The 3-space U in $\mathbf{PG}(4,q)$ defined by L and M meets $\mathcal{S}$ in the point set of a subquadrangle which is clearly Γ'. Since the points of Γ' on each of the lines L, M, N form a subline over $\mathbf{GF}(s)$, the points of Γ on the lines L and N generate a unique subplane over $\mathbf{GF}(s)$. Also, that subplane together with the points of Γ' on M generate a unique sub-3-space V over $\mathbf{GF}(s)$ of U. Let x be any point of Γ'. Since Γ' is a grid, there is a unique line N' of Γ' containing x and intersecting L and M. So N', as a line of U, is also a line of V since it contains at least two points of V, namely its intersections with L and M. Similarly, all lines of Γ' opposite N' are lines of V. Let N'' be such a line, $N \neq N'' \neq N'$ (N'' exists since $s > 1$). Also, there is a unique line L' of Γ' containing x and intersecting N in, say, the point y of Γ'. The line L' is the unique line in U containing y and meeting the two skew lines N' and N''. But as y and both N', N'' lie in V, we see that also L' lies in V. Hence $x \in V$. So Γ' is fully embedded in V. $\square$

Lemma 5 *If $d = 4$, then every line L of Γ meets every subspace of $\mathbf{PG}(4,q)$ (not containing L) generated by two arbitrary opposite lines of Γ in a point of Γ.*

PROOF. If $\delta = 2$, then this follows from Lemma 3. Suppose now $\delta > 2$. By 2.3.1 of Payne & Thas [6] the points and lines of Γ in a 3-space of $\mathbf{PG}(4,q)$ containing two opposite lines of Γ, form a subquadrangle Γ' of Γ of order (s,t'); by (WE2) and Lemma 2 we have $t'+1 = \delta$. Also, Γ' is weakly embedded in that 3-space, hence by Lemma 1 we have $t' = s$ or $t' = \sqrt{s}$. If $t' = s$, then by 2.2.2 and 2.2.1 of Payne & Thas [6] $t = s^2$ and every line of Γ not in Γ' meets Γ' in a unique point. So suppose that $t' = \sqrt{s}$. Let L be a line of Γ not belonging to Γ' and let U be the 3-space over $\mathbf{GF}(q)$ generated by the points of Γ'. Suppose L meets U in a point not belonging to $\mathcal{S}$. Let x be any point of Γ on L. Then $x \notin U$. The set of all points of Γ' collinear in Γ with x forms an ovoid $\mathcal{O}$ of Γ' (see 2.2.1 of Payne & Thas [6]). Hence there are $1+s\sqrt{s}$ such points. Let y be any element of $\mathcal{O}$. The plane π generated by L and xy contains by Lemma 2 exactly $t'+1$ lines of Γ (through x). If M is one of these lines and $M \neq xy$, then we show that M does not meet U in a point of Γ'. Assume, by way of contradiction, that z is the (unique) point of M in U. Then the line yz of $\mathbf{PG}(4,q)$ is a secant and contains $t'+1$ points of Γ', hence, by (WE2) and Lemma 2, it also contains a point of L, a contradiction. So with every line of Γ through x which meets Γ' in an element of $\mathcal{O}$, one associates $t'-1 = \sqrt{s}-1$ other lines of Γ through x different from L and meeting U in points not belonging to Γ'. If $z, z' \in \mathcal{O}$, $z \neq z'$, then clearly the plane generated by L and xz does not contain xz', otherwise by the same argument as above L would meet U in an element of $\mathcal{O}$. Hence we obtain $1 + (1+s\sqrt{s})\sqrt{s} > s^2+1 \geq t+1$ lines of Γ through x, a contradiction. Hence the result. $\square$

Lemma 6 *If $d > 4$, then $d = 5$, $\delta = 2$, $\Gamma \cong Q(5, s)$ and every line L of Γ meets every subspace of $\mathbf{PG}(d, q)$ (not containing L) generated by three lines L_1, L_2, L_3 of Γ, such that L_1 and L_2 are opposite and L_3 meets L_2 but does not lie in the space (over $\mathbf{GF}(q)$) generated by L_1 and L_2, in a point of Γ.*

PROOF. Consider two opposite lines L_1 and L_2 of Γ. By (WE3), they generate a 3-space U_3 of $\mathbf{PG}(d, q)$. The points and lines of Γ in U_3 form a subquadrangle Γ' of Γ of order (s, t') with $t' + 1 = \delta$. If $t = t'$, then $d = 3$; so $t' < t$ and hence there exists a line L of Γ containing any point of Γ' but not contained in Γ'. Let U_4 be the 4-space of $\mathbf{PG}(d, q)$ generated by U_3 and L. Then the points and lines of Γ in U_4 form a subquadrangle Γ'' of Γ of order (s, t'') and again $t' < t'' < t$. But by 2.2.2 of Payne & Thas [6], $t' = 1$, $t'' = s$ and $t = s^2$; so $\delta = t' + 1 = 2$. Hence there exists a line L' of Γ containing any point of Γ'' but not contained in Γ''. Let U_5 be the 5-space of $\mathbf{PG}(d, q)$ generated by U_4 and L'. Then the points and lines of Γ in U_5 form a subquadrangle Γ''' of Γ of order (s, t'''). We have $t' < t'' < t'''$ and hence again $t''' = s^2$. So $\Gamma''' = \Gamma$ and $d = 5$. The other assertions immediately follow from 5.3.5(i) and 2.2.1 of Payne & Thas [6]. □

Lemma 7 *If $s > 2$ or $d = 3$ or q is even, then s is a prime power, $\mathbf{GF}(s)$ is a subfield of $\mathbf{GF}(q)$ and Γ is fully embedded in a projective subspace $\mathbf{PG}(d, s)$ of $\mathbf{PG}(d, q)$.*

PROOF. We prove this by (a very short) induction on d. For $d = 3$, this is Lemma 1. Suppose the lemma is true for $d = 3$ or $d = 4$. We show that it is true for $d + 1$.

Let U' be a d-dimensional subspace (over $\mathbf{GF}(q)$) of $\mathbf{PG}(d + 1, q)$, q even or $s > 2$, such that the points and lines of Γ in U' generate U' and form a subquadrangle Γ' of Γ of order (s, t') with $t = st'$ (this is possible by Lemmas 5 and 6, and by 2.2.1 of Payne & Thas [6]). Clearly Γ' is weakly embedded in U' and if $t' > 1$, we can apply induction. If $t' = 1$, then $s = t$ and $d = 3$ (by the proof of Lemma 6); as $\delta = t' + 1 = 2$ (by (WE2) and Lemma 2) we can apply Lemma 4. In either case we conclude that Γ' is fully embedded in a projective subspace V' of U' of dimension d and defined over the subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$. Also, every line of Γ not contained in Γ' meets Γ' in a unique point. Let L be such a line and let x be the point of Γ' on L. Since also L belongs to a subquadrangle with similar properties as Γ', we deduce that the points of Γ on L form a projective subline over $\mathbf{GF}(s)$ of L. Hence the points of V' together with the points of Γ on L generate a unique $(d + 1)$-dimensional subspace V over $\mathbf{GF}(s)$ of $\mathbf{PG}(d + 1, q)$. Let z be any point of Γ. We show that z belongs to V and this will imply the assertion. We may suppose that z is not incident with L.

Let M be the unique line of Γ containing z and intersecting L in, say, the point y of Γ. First suppose $y \neq x$. Let u be the intersection point of M with U'. As above, u belongs to Γ'. Let N be any line of Γ' containing u. Clearly the lines N and L are skew and the 3-dimensional subspace U'' (over $\mathbf{GF}(q)$) of $\mathbf{PG}(d+1, q)$ generated by L and N meets the point set of Γ in the point set of a subquadrangle Γ'' which is, by Lemma 1, fully embedded in some subspace V'' over $\mathbf{GF}(s)$ of U'' and which contains z. But V'' contains the points of Γ on L, N and on the line of Γ' meeting N and containing x. Hence V'' is a subspace of V. We conclude that z belongs to V.

Now suppose $y = x$. Let x' be any point of Γ' collinear with x, $x \neq x'$, and let L' be any line of Γ not belonging to Γ' and incident with x'. By the preceding paragraph all points of Γ on L' belong to V. Clearly z is not collinear with x' and substituting L' for L in the preceding paragraph shows that $z \in V$. The lemma is proved. $\square$

We now handle the case $s = 2$ and q odd.

Lemma 8 *If $s = 2$ and q is odd, then $t = 2$, $d = 4$ and the weak embedding is isomorphic to the universal one defined in the introduction.*

PROOF. If $s = 2$, then $t = 2$ or $t = 4$.

First let $t = 2$. By Lemma 1, $d > 3$ and by Lemma 6, $d < 5$. Hence $d = 4$. Note that $\Gamma \cong W(2)$. The generalized quadrangle $W(2)$ can be defined as the geometry of pairs of a set of six elements where triples of mutually disjoint pairs are the lines (see 6.1.1 of PAYNE & THAS [6]). If we consider the set $\{1, 2, 3, 4, 5, 6\}$, then, without loss of generality, we can coordinatize $\mathbf{PG}(4, q)$ according to the following table.

$\{1,2\}$	$(1,0,0,0,0)$
$\{1,4\}$	$(0,1,0,0,0)$
$\{4,5\}$	$(0,0,1,0,0)$
$\{5,6\}$	$(0,0,0,1,0)$
$\{2,6\}$	$(0,0,0,0,1)$

This follows from the fact that $W(2)$ is generated by any pentagon and that Γ generates $\mathbf{PG}(4, q)$.

The five points above are the vertices of a pentagon Ω of Γ, and for each of the five sides of Ω we know the coordinates of two points of Γ on it. Choosing the intersection of the hyperplanes $\{3,6\}\{1,4\}\{5,6\}\{2,6\}$, $\{2,3\}\{1,2\}\{4,5\}\{2,6\}$, $\{1,3\}\{1,2\}\{1,4\}\{5,6\}$, $\{3,4\}\{1,4\}\{4,5\}\{2,6\}$ as point $(1,1,1,1,1)$ we obtain

$\{3,6\}$	$(1,0,1,0,0)$
$\{2,3\}$	$(0,1,0,1,0)$
$\{1,3\}$	$(0,0,1,0,1)$
$\{3,4\}$	$(1,0,0,1,0)$
$\{3,5\}$	$(0,1,0,0,a)$

where $a \in \mathbf{GF}(q) \setminus \{0\}$. Each other point of Γ, e.g. $\{1,6\}$, lies on exactly one line through one of the vertices of Ω; we know already the coordinates of two points of such a line. For example, $\{1,6\}$ lies on the line $\{2,3\}\{4,5\}$. Hence there exist $b_1, b_2, \ldots, b_5 \in \mathbf{GF}(q) \setminus \{0\}$ such that we can make the following identification:

$\{1,5\}$	$(1,0,0,1,b_1)$
$\{2,4\}$	$(0,0,1,b_2,1)$
$\{1,6\}$	$(0,1,b_3,1,0)$
$\{2,5\}$	$(1,b_4,1,0,0)$
$\{4,6\}$	$(b_5,1,0,0,a)$

Expressing that the three points of every line of Γ are also collinear in $\mathbf{PG}(4,q)$, we obtain

$b_1 = b_2 = \cdots = b_5 = -1 = -a$. Hence we have the universal weak embedding. Note that, for q large enough, we can always find a point x in $\mathbf{PG}(4,q)$ not lying in any of the 10 3-spaces generated by opposite pairs of lines of Γ, and not lying in any of the 15 3-spaces generated by the points of Γ collinear to some point of Γ. Hence the projection of Γ from x onto a hyperplane not containing x will satisfy (WE1) and (WE3), but not (WE2). This shows that (WE2) is not a consequence of (WE1) and (WE3).

Next suppose $t = 4$. By Lemma 1 we have $d > 3$. Assume that $\delta > 2$. Then, by Lemmas 1 and 2, in any 3-space generated by opposite lines L, M of Γ a subquadrangle Γ' of Γ of order (s,t'), with $1 < \delta - 1 = t' < 4$, is induced. Hence, again by Lemma 1, $t' = 2$ and $s = 2$ divides q, a contradiction. Hence $\delta = 2$. By Lemma 3, $d = 4$ implies $s = t$, contradicting $(s,t) = (2,4)$. Hence $d = 5$ by Lemma 6. It follows that every subquadrangle Γ' of Γ of order $(2,2)$ is weakly embedded in a 4-dimensional subspace U (over $\mathbf{GF}(q)$) of $\mathbf{PG}(5,q)$. By the first part of the proof, it must be isomorphic to the universal weak embedding of $W(2)$ in U. Now consider any point x of Γ not lying in Γ'.

The points of Γ' collinear with x form an ovoid in Γ'. It is well-known that each ovoid of Γ' can be obtained in this way. So without loss of generality and with the above description of Γ', we can take as points of the ovoid the points $\{1,2\}, \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}$. By (WE2) these points should be linearly dependent in U, hence the determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 2$$

should be equal to 0. This implies that q is even, a contradiction.

Now the lemma is completely proved. $\square$

Putting all previous lemmas together, our Main Result follows.

REMARK. In the infinite case, a different approach is needed. The situation is also totally different. Actually, the theorem is not true in the infinite case, e.g. all Moufang quadrangles of so-called mixed type are weakly embedded in projective 3-space over a field of characteristic 2 and some of them are not fully embedded in a projective subspace.

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