Fuzzy projective geometries from fuzzy groups

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Abstract

First we construct a fuzzy group from a fuzzy projective geometry, and then we construct a fuzzy projective geometry from a fuzzy group.

1 Introduction

Fuzzy groups were introduced by Rosenfeld in 1971, in [7]. Fuzzy vector spaces were introduced by Katsaras and Liu in 1977, in [4]. In [6], we introduced fuzzy projective geometries.
These fuzzy projective geometries were deduced from fuzzy vector spaces. In this article, we deduce a fuzzy group corresponding with such a fuzzy projective geometry (section 4), thus obtaining a relationship between fuzzy vector spaces and fuzzy groups by means of these fuzzy projective geometries.
Moreover, we will give a construction of fuzzy projective geometries from fuzzy groups that yields **the same** fuzzy projective geometries as defined in [6].
The fuzzy projective geometries we constructed are thus an important link in the connection between the theories of fuzzy vector spaces and fuzzy groups.

2 Preliminaries

In this paper, $(G, \cdot)$ or shortly G will always denote a group, and its neutral element will be denoted by e.

Definition 2.1 ([7]) A fuzzy set μ on a group G is a **fuzzy subgroup** of G if, $\forall x, y \in G$ the following holds:
(1) $\mu(x \cdot y) \geq \mu(x) \wedge \mu(y)$, and
(2) $\mu(x^{-1}) = \mu(x)$.

From (1) we immediately see that $\mu(e) \geq \mu(a)$, for all $a \in G$.

Remark that the conditions (1) and (2) are equivalent with:
(3) $\mu(x \cdot y^{-1}) \geq \mu(x) \wedge \mu(y)$

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which looks like the classical condition for a subset H of a group G to be a subgroup of G: if $a, b \in H$, then $a \cdot b^{-1} \in H$.

Definition 2.2 ([10]) Let μ be a fuzzy set of some set X. Then for $t \in [0,1]$, the set $\mu_t = \{x \in X | \mu(x) \geq t\}$ is called a **level subset** of the fuzzy set μ.

Proposition 2.1 (Propositions 2.1 and 2.2 in [1]) *A fuzzy set μ on the group G is a fuzzy subgroup of G if and only if μ_t is a subgroup of G for every $t \in [0, \mu(e)]$.*

Proposition 2.2 (Proposition 3.3 in [5]) *If G is a group having a chain*

$$\{e\} = G_0 \subset G_1 \subset G_2 \subset \ldots \subset G_n = G$$

of subgroups, with n maximal, then a fuzzy subgroup μ of G is a step function from $G \rightarrow [0,1]$ having at most $n+1$ steps.

If the subgroups G_i are as in the proposition above, the membership degree of e must be higher than the membership degrees of the elements of the group $G_1 \backslash \{e\}$, which in turn have to be higher than the membership degrees of the elements in the group $G_2 \backslash G_1$ and so on, leaving the elements of the group $G_n \backslash G_{n-1}$ with the lowest membership degree (this follows from the definition of fuzzy group). This means that the level subsets of μ are given by $G_i \backslash G_{i-1}$. Remark that the chain of subgroups of G don't need to be chosen maximal.
In fact, in [5] the proposition is stated for normal fuzzy subgroups, but it holds for fuzzy subgroups as well.

Definition 2.3 A **projective space** $PG(V) = (D(V), I)$ corresponding to a vector space V, is defined as the collection $D(V)$ of subspaces of V together with the following incidence relation I on the subspaces: U and U' are incident if $U \subseteq U'$ or $U' \subseteq U$, which we denote $U \text{ I } U'$ (for more details, see e.g. [3]). The **dimension** $d(U)$ of a subspace U is equal to the number of base vectors of U. The **projective dimension** $pd(U)$ **of** U is defined by $pd(U) = d(U) - 1$. The definition of projective points, lines, planes, ... is based on this projective dimension: subspaces having projective dimension 0 are called projective points, projective lines have projective dimension 1 and so on. Throughout this article, we will denote a projective space by $\mathcal{P}$. The **order of a projective space** is the number of points on a line minus one. If we work with projective spaces over a finite field of order q, the number of points on a line equals $q+1$, so the order of the projective space will be q.

Definition 2.4 Suppose $\mathcal{P}$ is an n-dimensional projective space. A **flag** in $\mathcal{P}$ is a sequence of distinct, non-trivial (i.e. different from $\emptyset$ and $\mathcal{P}$) subspaces $(U_0, U_1, \ldots, U_m)$ such that $U_j \subset U_i$ for all $j < i \leq m \leq n-1$. The **rank** of a flag is the number of subspaces it contains. A **maximal flag** in $\mathcal{P}$ is a flag of length n.
Suppose from now on that the dimension of U_i equals i. A **flag of type** $\{i, i+1, \ldots, m\}$, where $m > i$ is a sequence of distinct, non-trivial subspaces $(U_i, U_{i+1}, \ldots, U_m)$ such that $U_j \subset U_k$ for all $i \leq j < k \leq m \leq n-1$. We denote this shortly as an $[i, m]$**-flag**.

Suppose $\mathcal{P}$ is an n-dimensional projective space.

Definition 2.5 The fuzzy set λ is a **fuzzy n-dimensional space** on $\mathcal{P}$ if $\lambda(p) \geq \lambda(q) \wedge \lambda(r)$, for all collinear points p, q, r of $\mathcal{P}$. We denote $[\lambda, \mathcal{P}]$. The projective space $\mathcal{P}$ is called the **base projective space** of $[\mathcal{P}, \lambda]$. If $\mathcal{P}$ is a fuzzy point, line, plane, ..., we use base point, base line, base plane, ..., respectively.

Property 2.3 *A fuzzy n-dimensional space $[\lambda, \mathcal{P}]$ is of the following form (see [6]):*

$$\begin{array}{llll} \lambda: \mathcal{P} & \rightarrow & [0,1] & \\ \quad p & \mapsto & a_1 & \textit{if } p = q \\ \quad p & \mapsto & a_2 & \textit{for } p \in U_1 \backslash \{q\} \\ \quad p & \mapsto & a_3 & \textit{for } p \in U_2 \backslash U_1 \\ & \dots & & \\ \quad p & \mapsto & a_n & \textit{for } p \in U_{n-1} \backslash U_{n-2} \\ \quad p & \mapsto & a_{n+1} & \textit{for } p \in \mathcal{P} \backslash U_{n-1}, \end{array}$$

for a certain maximal flag $(q, U_1, \dots, U_{n-1})$ in $\mathcal{P}$, and for some reals $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_{n+1}$ in $[0,1]$.

One can prove this in the same way as for fuzzy vector spaces (see Theorem 3.2 in [6]).

3 The classical case

We will start this section with a short explanation of how in the classical case, a flag-transitive geometry can be constructed from its automorphism group. To keep things simple, we construct the smallest possible projective space: the Fano plane.

The Fano configuration $PG(2,2)$ is the projective plane over the finite field $GF(2)$. Throughout this article we will use the notation $\mathcal{F}$. It consists of 7 points and 7 lines and it is the smallest non-trivial projective plane. Every point of $\mathcal{F}$ is incident with exactly three lines of $\mathcal{F}$ and every line of $\mathcal{F}$ contains exactly three points of $\mathcal{F}$. Its automorphism group is $L_3(2)$, and consists of 168 elements. The subgroups of $L_3(2)$ that stabilize points are symmetric groups of order 24, and so are the subgroups of $L_3(2)$ that stabilize a line.

Now, suppose we are given the group $L_3(2)$, how do we recover the corresponding Fano plane? We will construct a geometry $\mathcal{F}'$ based on $L_3(2)$, and show that $\mathcal{F}' = \mathcal{F}$. For this, we select two subgroups S_4 and S_4' (symmetric groups of order 24) of $L_3(2)$, which meet in a group isomorphic to D_8. The latter is a group that stabilize a line and fixes a point on that line, thus a group stabilizing a flag so we see that D_8 is the maximal possible intersection of S_4 and S_4'. Both S_4 and S_4' have 7 left cosets. We will denote a left coset by gS_4, with $g \in L_3(2)$. Since we will only consider left cosets, we will from now on write 'coset' instead of left coset, since no confusion is possible.
We define the points and lines of $\mathcal{F}'$ as follows:
points: the (7) cosets of S_4.
lines: the (7) cosets of S_4'.
incidence: $gS_4 \text{ I } hS_4' \iff gS_4 \cap hS_4' \neq \emptyset$, so a point is incident with a line if the cosets that

define them have a nonzero intersection (in fact, if they intersect in a coset of the group D_8; the flags are then the (21) cosets of $S_4 \cap S_4' = D_8$).
One can show that this geometry $\mathcal{F}'$ is indeed the Fano configuration $\mathcal{F}$ (see [2], 1.2.17).

4 Fuzzy groups from fuzzy projective spaces

To keep things clear, we start with a construction of the fuzzy group corresponding with a fuzzy projective geometry on the Fano plane $\mathcal{F} = \mathcal{GF}(\in, \in)$, before we turn to the general case.

4.1 The fuzzy group corresponding with the fuzzy Fano plane

Suppose $[\mathcal{F}, \lambda]$ is a fuzzy projective space, thus for a certain flag (q, L) in $\mathcal{F}$ and reals $a_0 \geq a_1 \geq a_2 \in [0, 1]$ we have:

$$\begin{array}{llll} \lambda : \mathcal{F} & \rightarrow & [0,1] & \\ \quad p & \mapsto & \alpha_0 & \text{if } p = q \\ \quad p & \mapsto & \alpha_1 & \text{if } p \in L \backslash \{q\} \ (1) \\ \quad p & \mapsto & \alpha_2 & \text{if } p \in \mathcal{F} \backslash \{L\}. \end{array}$$

Since we have to construct a fuzzy group, we know by proposition 2.2 that we have to find a chain of subgroups of $L_3(2)$. We do this as follows.
The stabilizor of $\mathcal{F}$ is just its automorphism group $L_3(2)$. We can consider $L_3(2)$ to be the group stabilizing all points of $[\lambda, \mathcal{F}]$ with a membership degree that is at least a_2. We now search the subgroup of $L_3(2)$ that stabilizes the points with membership degree at least a_1. Since all these points are on the line L, this subgroup is just the stabilizor of L, hence a symmetric group of order 24: S_4'. At last, we search for the subgroup of S_4' that fixes the unique point p with membership degree a_0 on L. This is a diheder group of order 8: D_8. (Remark that this is not the group that stabilizes the point p, since this is a symmetric group of order 24: S_4; but we have $S_4 \cap S_4' = D_8$.)

This reasoning yields the chain $D_8 \leq S_4' \leq L_3(2)$. With this chain we construct in a natural way the following fuzzy set on $L_3(2)$:

$$\begin{array}{llll} \mu : L_3(2) & \rightarrow & [0,1] & \\ \quad x & \mapsto & a_0 & \text{if } x \in D_8 \\ \quad x & \mapsto & a_1 & \text{if } x \in S_4' \backslash D_8 \\ \quad x & \mapsto & a_2 & \text{if } x \in L_3(2) \backslash S_4'. \end{array}$$

We have to check that this fuzzy set is indeed a fuzzy group. This is straightforward by the multiplication in the group $L_3(2)$ and its subgroups and since $a_0 \geq a_1 \geq a_2$.

4.2 General case

Suppose $\mathcal{P}$ is an n-dimensional projective space, and $[\lambda, \mathcal{P}]$ is an n-dimensional fuzzy projective space on $\mathcal{P}$. Like in the previous section, it is possible to define a fuzzy group on the automorphism group of $\mathcal{P}$.

Suppose $[\lambda, \mathcal{P}]$ is of the following form:

$$\begin{array}{llll}
\lambda: & \mathcal{P} & \rightarrow & [0,1] \\
& p & \mapsto & a_1 & \text{if } p = q \\
& p & \mapsto & a_2 & \text{for } p \in U_1 \backslash \{q\} \\
& p & \mapsto & a_3 & \text{for } p \in U_2 \backslash U_1 \\
& & \dots & & \\
& p & \mapsto & a_n & \text{for } p \in U_{n-1} \backslash U_{n-2} \\
& p & \mapsto & a_{n+1} & \text{for } p \in \mathcal{P} \backslash U_{n-1},
\end{array}$$

for a certain maximal flag $(q, U_1, \dots, U_{n-1})$ in $\mathcal{P}$, and for some reals $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_{n+1}$ in [0,1].

We now search for a chain of subgroups of G, in the same way as we did in the previous section. So we first search the stabilizor group St_{n-1}, stabilising all points in $[\lambda, \mathcal{P}]$ with membership degrees at least a_{n-1}, thus the points in the hyperplane U_{n-1} in $\mathcal{P}$. Then we search for the subgroup of St_{n-2} of St_{n-1}, stabilizing the $(n-2)$-dimensional subspace U_{n-2}, and so on, creating a chain of subspaces $(St_0, St_1, \dots, U_{n-2}, U_{n-1}, G)$. With this chain we define the following fuzzy set on G:

$$\begin{array}{lllll}
\mu: & G & \longrightarrow & [0,1] & \\
& x & \mapsto & a_n & \text{if } x \in G \backslash St_{n-1} \\
& x & \mapsto & a_{n-1} & \text{if } x \in St_{n-1} \backslash St_{n-2} \\
& & \dots & & \\
& x & \mapsto & a_1 & \text{if } x \in St_1 \backslash St_0 \\
& x & \mapsto & a_0 & \text{if } x \in St_0.
\end{array}$$

One can easily prove that this is a fuzzy group on $\mathcal{P}$. We define this μ to be the fuzzy group corresponding with the fuzzy projective geometry $[\lambda, \mathcal{P}]$.

5 Fuzzy projective spaces from fuzzy groups

We start this section with a concrete case: the construction of the fuzzy Fano plane from a certain fuzzy group on the automorphism group of $\mathcal{F}$. Afterwards, we give the construction in the n-dimensional case.

5.1 A small fuzzy example

We want to define a fuzzy projective Fano plane starting from a certain fuzzy subgroup μ of $L_3(2)$. Since the base plane will be the geometry deduced from $L_3(2)$, the base plane of this fuzzy projective plane will be the Fano plane.
To agree with [6], the resulting fuzzy projective plane $[\mathcal{F}, \lambda]$ has to be of the following form:

$$
\begin{array}{lllll}
\lambda: & \mathcal{F} & \rightarrow & [0,1] & \\
 & p & \mapsto & \alpha_0 & \text{if } p=q \\
 & p & \mapsto & \alpha_1 & \text{for } p\in L\backslash\{q\} \\
(1) & p & \mapsto & \alpha_2 & \text{for } p\in \mathcal{F}\backslash\{L\}
\end{array} \qquad (1)
$$

for some $\alpha_0 \geq \alpha_1 \geq \alpha_2 \in [0,1]$ and (q, L) a flag in $\mathcal{F}$.
Of what form must μ be to obtain such a fuzzy projective geometry? From proposition 2.2 we know that with μ there corresponds a chain of subgroups of $L_3(2)$.
We proof that if we choose this chain of subgroups as follows:

$$D_8 \leq S_4' \leq L_3(2),$$

S_4' being the stabilizer of a line, and if we take the point stabilizer S_4 such that $S_4 \cap S_4' = D_8$ (this S_4 is indeed unique), that we can recover the fuzzy Fano plane from the fuzzy group μ. (Note that this chain is not maximal! A maximal chain could for example be: $\{e\} \leq C_2 \leq K_4 \leq D_8 \leq S_4' \leq L_3(2)$). This chain allows us to write μ as the following fuzzy group on $L_3(2)$:

$$
\begin{array}{lllll}
\mu: & L_3(2) & \rightarrow & [0,1] & \\
 & x & \mapsto & a_0 & \text{if } x\in D_8 \\
 & x & \mapsto & a_1 & \text{if } x\in S_4'\backslash D_8 \\
 & x & \mapsto & a_2 & \text{if } x\in L_3(2)\backslash S_4',
\end{array}
$$

with $a_0 \geq a_1 \geq a_2 \in [0,1]$. In the sequel however we will suppose that the real numbers a_0, a_1 and a_2 are different, because it clarifies the explanation. In the case some of these values are the same, an analogue reasoning can be made.
We want to obtain the fuzzy projective plane $[\mathcal{F}, \lambda]$ for some α's, so we restrict our attention to the fuzzy points, since the shape of the fuzzy lines is completely deduced from the fuzzy points.
We define the base points to be the classical cosets of S_4. Every classical point lies in exactly 3 flags, i.e. in 3 cosets of D_8 in $L_3(2)$. Now we explain how the membership degrees of the fuzzy points are given.
Look at the group $D_8 = S_4 \cap S_4'$. It has 21 cosets, one of them is just D_8 itself, 2 of them are disjoint subsets of $S_4'\backslash D_8$ and the 18 others are disjoint subsets of $L_3(2)\backslash S_4'$.
We define the following fuzzy set on $\mathcal{K}$, the set of all cosets of D_8 in $L_3(2)$:

$$
\begin{array}{llll}
\nu: & \mathcal{K} & \rightarrow & [0,1] \\
 & gD_8 & \mapsto & \mu(g)
\end{array}
$$

Is this well–defined, i.e. is the membership degree of a coset independent of the chosen representant $g \in L_3(2)$ of that coset? It is, since from elementary group theory we know that

$$
\begin{array}{lll}
gD_8 = D_8 & \Longleftrightarrow & g \in D_8 \\
gD_8 \subset S_4'\backslash D_8 & \Longleftrightarrow & g \in S_4'\backslash D_8 \\
gD_8 \subset L_3(2)\backslash S_4' & \Longleftrightarrow & g \in L_3(2)\backslash S_4'
\end{array}
$$

So the fuzzy set ν on $\mathcal{K}$ is of the following form:

$$\begin{array}{rccll} \nu: & \mathcal{K} & \rightarrow & [0,1] & \\ & X & \mapsto & a_0 & \text{if } X = D_8 \\ & X & \mapsto & a_1 & \text{if } X \subseteq S_4' \backslash D_8 \\ & X & \mapsto & a_2 & \text{if } X \subseteq L_3(2) \backslash S_4', \end{array}$$

with $a_0 \geq a_1 \geq a_2 \in [0,1]$. Note that this fuzzy set has exactly the same structure as μ. The only difference is the base set ($L_3(2)$ for μ and $\mathcal{K}$ for ν).
Each base point lies in 3 flags. Consider a base point p, and the 3 flags it is incident with: gD_8, hD_8 and jD_8, with g, h and j different elements of $L_3(2)$. How do we determine the membership degree $\lambda(p)$ of the fuzzy point $[p, \lambda(p)]$? We define the following fuzzy set λ on the set P of all base points of $[\mathcal{F}, \lambda]$:

$$\begin{array}{rccl} \lambda: & P & \rightarrow & [0,1] \\ & p & \mapsto & \max(\nu(gD_8), \nu(hD_8), \nu(jD_8)), \end{array} \tag{2}$$

where gD_8, hD_8 and jD_8 are the three flags through the point p.

What is the result of this definition? There are 7 base points, every point lies in 3 flags and every flag contains exactly one point. The membership degree of each flag in the fuzzy set ν will be used only once, since there are 21 flags, and three flags are needed for the determination of the membership degree of one point (see (2)).
This means that exactly one fuzzy point will have the membership degree a_0. There are two flags with membership degree a_1 in the fuzzy set ν, let us call them nD_8 and mD_8. There can only be two fuzzy points with membership degree a_1, if these two flags do not contain the same point, and if no flag of membership degree a_1 contains the base point of the fuzzy point with membership degree a_0.

This is not the case, since $D_8 \cup nD_8 \cup mD_8 = S_4'$, the group that stabilizes a line L. Since D_8, nD_8 and mD_8 are mutually disjoint, and since they all stabilize the same line L, they all have to fix another point on that line, because they have to be different. This means that it is impossible that two of these flags appear in the determination of the membership degree (by the maximum in (2)) of the same fuzzy point. Thus we find 3 points on a line with values a_0, a_1 and a_1. All the other points will have membership degree a_2, since only flags with value a_2 are left. So we find a fuzzy projective plane of the form:

$$\begin{array}{rccll} \lambda: & \mathcal{F} & \rightarrow & [0,1] & \\ & p & \mapsto & a_0 & \text{if } p = q \\ & p & \mapsto & a_1 & \text{if } p \in L \backslash \{q\} \\ & p & \mapsto & a_2 & \text{if } p \in \mathcal{F} \backslash \{L\}, \end{array}$$

for $a_0 \geq a_1 \geq a_2 \in [0,1]$. Thus λ is a fuzzy projective plane of the form (1). So we proved the following theorem:

Theorem 5.1 *The fuzzy projective plane $[\mathcal{F}, \lambda]$:*

$$\begin{array}{rclll} \lambda: & \mathcal{F} & \rightarrow & [0,1] & \\ & p & \mapsto & \alpha_0 & \textit{if } p = q \\ & p & \mapsto & \alpha_1 & \textit{for } p \in L \backslash \{q\} \\ & p & \mapsto & \alpha_2 & \textit{for } p \in \mathcal{F} \backslash \{L\}, \end{array}$$

where $\mathcal{F}$ is the Fano plane, (p, L) is a flag in $\mathcal{F}$ and $a_0 \geq a_1 \geq a_2$ are reals in $[0,1]$ can be constructed from the following fuzzy subgroup μ on the automorphism group $L_3(2)$ of $\mathcal{F}$:

$$\begin{array}{rclll} \mu: & L_3(2) & \rightarrow & [0,1] & \\ & x & \mapsto & a_0 & \textit{if } x \in D_8 \\ & x & \mapsto & a_1 & \textit{if } x \in S_4' \backslash D_8 \\ & x & \mapsto & a_2 & \textit{if } x \in L_3(2) \backslash S_4', \end{array}$$

where S_4' is the stabilizer group of the line L and D_8 is the stabilizer group of the point p on the line L.

5.2 General case

Suppose $PG(n,q) = \mathcal{P}$ is an n-dimensional projective space (over some finite field K), with automorphism group G. We choose a flag $F = (U_0, U_1, U_2, \ldots, U_{n-1})$ in $\mathcal{P}$, where U_i is an i-dimensional subspace of $\mathcal{P}$, so U_0 is a point and U_{n-1} is a hyperplane, and construct a fuzzy projective space $[\lambda, \mathcal{P}]$ on $\mathcal{P}$ (see definition 2.3), based on this flag. We will now construct a fuzzy group μ on G that allows us to recover $[\lambda, \mathcal{P}]$. For this, we need a chain of subgroups of G. We choose this chain in the following way:

We search for the group that stabilizes (U_{n-1}), we call it St_{n-1}. Next, we consider the $[n-2, n-1]$-flag (U_{n-2}, U_{n-1}). We call its stabilizer group St_{n-2}. In general, we call St_{n-i} the group that stabilizes the $[n-i, n-1]$-flag $(U_{n-i}, U_{n-i+1}, \ldots, U_{n-1})$, so the group stabilizing the $[1, n-1]$-flag $(U_1, U_2, \ldots, U_{n-1})$ is called St_1, and the stabilizer of the whole maximal flag is called St_0.

These stabilizors form the following chain of subgroups of G:

$$St_0 \subseteq St_1 \subseteq \ldots \subseteq St_{n-1} \subseteq G.$$

With this chain we construct the following fuzzy group μ on G:

$$\begin{array}{rclll} \mu: & G & \longrightarrow & [0,1] & \\ & x & \mapsto & a_n & \text{if } x \in G \backslash St_{n-1} \\ & x & \mapsto & a_{n-1} & \text{if } x \in St_{n-1} \backslash St_{n-2} \\ & & \ldots & & \\ & x & \mapsto & a_1 & \text{if } x \in St_1 \backslash St_0 \\ & x & \mapsto & a_0 & \text{if } x \in St_0, \end{array}$$

where the real numbers a_i are the same as in the definition of $[\lambda, \mathcal{P}]$. Again we suppose that all the values a_i are different, for the sake of clarity of this explanation. An analogue reasoning can be made if some values are the same. In fact, also the demand for $\mathcal{P}$ to be a projective space over a *finite* field is not necessary. However, the finite case is much easier to explain and to understand, so we focus on finite fields. The same construction can be made for base projective spaces over infinite fields.

We now take the subgroup St' of G stabilising the point U_0, such that $St_1 \cap St' = St_0$. Then we know that the base points of the fuzzy points will be given by the classical cosets of St'. We will now determine the membership degree of the fuzzy points with these base points.

Like in section 5.1, we define the following fuzzy set on $\mathcal{K}$, the set of all cosets of St_0 in G:

$$\begin{array}{lll} \nu : & \mathcal{K} & \rightarrow & [0,1] \\ & gSt_0 & \mapsto & \mu(g) \end{array}$$

This definition is well–defined, since:

$$\begin{array}{lcl} gSt_0 = St_0 & \iff & g \in St_0 \\ gSt_0 \subset St_1 \backslash St_0 & \iff & g \in St_1 \backslash St_0 \\ \ldots & & \\ gSt_0 \subset St_{n-1} \backslash St_{n-2} & \iff & g \in St_{n-1} \backslash St_{n-2} \\ gSt_0 \subset St_n \backslash St_{n-1} & \iff & g \in St_n \backslash St_{n-1} \end{array}$$

So we can write the fuzzy set ν on $\mathcal{K}$ as follows:

$$\begin{array}{lllll} \nu : & \mathcal{K} & \rightarrow & [0,1] & \\ & X & \mapsto & a_0 & \text{if } X = St_0 \\ & X & \mapsto & a_1 & \text{if } X \subseteq St_1 \backslash St_0 \\ & & \ldots & & \\ & X & \mapsto & a_{n-1} & \text{if } X \subseteq St_{n-1} \backslash St_{n-2} \\ & X & \mapsto & a_n & \text{if } X \subseteq G \backslash St_{n-1} \end{array}$$

Since $\mathcal{P} = PG(n, q)$ is a projective space of order q, there are $q+1$ points on a line. Furthermore, there are $q^i + q^{i-1} + q^{i-2} + \ldots + q^2 + q + 1$ points in every subspace U_i, and this for all $i \in \{1, \ldots n\}$ (of course there is only one point 'in' U_0, since it is a point itself). From now on, we denote the number $q^i + q^{i-1} + q^{i-2} + \ldots + q^2 + q + 1$ by N_i for $i \in \{1, 2, \ldots, n\}$. There are $q + 1 = N_1$ hyperplanes passing through a fixed $(n-2)$-dimensional space. In general, there are N_{n-i} i-dimensional subspaces of $\mathcal{P}$ passing through a fixed $(i-1)$-dimensional subspace of $\mathcal{P}$. The total number of maximal flags in $\mathcal{P}$ is thus $N_1 \cdot N_2 \cdots N_{n-1} \cdot N_n$. From this we conclude that there are $N_1 \cdot N_2 \cdots N_{n-1}$ flags through a fixed point of $\mathcal{P}$.

Since St_0 stabilizes $(U_0, \ldots, U_n)$ and St_1 stabilizes $(U_1, U_2, \ldots, U_n)$ and since there are $q+1 = N_1$ points on U_1, we have:

$$|St_1| = (q + 1)|St_0|,$$

which means that there are $q + 1$ cosets of St_0 that are contained in St_1 (including St_0 itself). In the same way, since St_2 stabilizes $(U_2, U_3, \ldots, U_n)$, and since in U_2 we can choose a pair (p, L)

such that $p \mathrm{~I~} L$, p a point and L a line in $(q+1)(q^2+q+1)$ different ways, we find that:

$$|St_2| = (q+1)(q^2+q+1)|St_0|,$$

which means that there are $(q+1)(q^2+q+1) = N_1 \cdot N_2$ cosets of St_0 in St_2.

In general, we will find that there are $N_1 \cdot N_2 \cdots N_i$ cosets of St_0 in St_i, thus there are equally many $[0,i]$-flags in U_i. This means, since there are N_i points in U_i, that there are $N_1 \cdot N_2 \cdots N_{i-1}$ flags through every point in U_i, since the number of flags through a point is a constant in every subspace of $\mathcal{P}$.

We define the fuzzy projective space defined from the fuzzy group μ as follows:

$$\begin{array}{rccl} \lambda : & P & \longrightarrow & [0,1] \\ & p & \mapsto & \max_{i=1}^{N_{n-1}} \nu(x_i St_0), \end{array} \tag{3}$$

where $x_i St_0$ are the N_{n-1} flags through the point p. Since every flag contains just one point, the membership degree of a flag in ν will appear in the determination of the membership degree of just one fuzzy point in λ.

Since there is one coset of St_0 (St_0 itself) that has the membership degree a_0 in ν, the point p this flag contains (this is U_0) will be given the membership degree a_0, and U_0 will be unique with this membership degree.

St_1 contains $q+1$ cosets of St_0, this means that there will be exactly q flags with membership degree a_1 in ν. Since all these flags and the flag with membership degree a_0 stabilize the line U_1 that contains $q+1$ points, these flags all contain a different point. Since afterwards we will only find other flags through these points with a lower membership degree, this means that all points on $U_1 \backslash U_0$ have the membership degree a_1 in λ.

St_2 contains $N_2 \cdot N_1$ cosets of St_0, thus there are equally many flags in U_2, of which $N_2 \cdot N_1 - N_1 = (q^2+q)(q+1)$ with membership degree a_2 (this is the number of flags in $U_2 \backslash U_1$). There are N_2 points in U_2, such that there are $N_1 = q+1$ flags through every point. The membership degree of the points of U_1 do not change, since the flags that are added have a smaller membership degree in the fuzzy set ν, so they vanish in the definition of membership degree in the fuzzy set λ, where the maximum operator is used. All points of $U_2 \backslash U_1$ get the membership degree a_2.

St_i contains $N_i \cdots N_1$ cosets of St_0, thus U_i contains $N_i \cdots N_1$ flags. This means that there are $N_{i-1} \cdots N_1$ flags through every point, of which $N_i \cdots N_1 - N_{i-1} \cdots N_1$ with membership degree a_i (this is the number of flags in $U_i \backslash U_{i-1}$). The new flags that go through points in U_{i-1} that are already given a membership degree before, do not change these membership degrees, since the membership degrees of the new flags are lower than these of the flags already used, thus they will not contribute in the determination of the membership degree of the points, since for this the maximum of the membership degrees of all flags is taken (see (3)). The points in $U_i \backslash U_{i-1}$

will all get the membership degree a_i in λ.

We end up with a situation that every point of $\mathcal{P}$ is given a membership degree in the fuzzy set λ, such that there is 1 point having the (highest) membership degree a_0, there are q points with membership degree a_1, q^2 points with membership degree a_2, ..., in general, there are q^i points of $\mathcal{P}$ with the membership degree a_i. Moreover, since the point with membership degree $a_0 = U_0$, the line defined by the points with membership degree a_1 and a_0 is U_1, and in general, since U_i is the space determined by the points of $\mathcal{P}$ with membership degree a_0 or a_1 or ... or a_i in λ, we have proved that λ is a fuzzy n-dimensional projective space of the form of definition 2.3.

Theorem 5.2 *Let $\mathcal{P}$ be an n-dimensional projective space, not necessary over a finite field. The fuzzy projective space $[\mathcal{P}, \lambda]$:*

$$\begin{array}{llll} \lambda: \mathcal{P} & \rightarrow & [0,1] & \\ p & \mapsto & a_1 & \textit{if } p = q \\ p & \mapsto & a_2 & \textit{for } p \in U_1 \backslash \{q\} \\ p & \mapsto & a_3 & \textit{for } p \in U_2 \backslash U_1 \\ & \dots & & \\ p & \mapsto & a_n & \textit{for } p \in U_{n-1} \backslash U_{n-2} \\ p & \mapsto & a_{n+1} & \textit{for } p \in \mathcal{P} \backslash U_{n-1}, \end{array}$$

with $(q = U_0, U_1, \dots, U_{n-1})$ a maximal flag in $\mathcal{P}$ and $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_{n+1}$ reals in $[0,1]$, can be constructed from the following fuzzy subgroup μ on the automorphism group G of $\mathcal{P}$:

$$\begin{array}{lllll} \mu: & G & \longrightarrow & [0,1] & \\ & x & \mapsto & a_n & \textit{if } x \in G \backslash St_{n-1} \\ & x & \mapsto & a_{n-1} & \textit{if } x \in St_{n-1} \backslash St_{n-2} \\ & & \dots & & \\ & x & \mapsto & a_1 & \textit{if } x \in St_1 \backslash St_0 \\ & x & \mapsto & a_0 & \textit{if } x \in St_0, \end{array}$$

where $St_0 \subseteq St_1 \subseteq \dots \subseteq St_{n-1}$ is a chain of subgroups of G such that St_i stabilizes the $[i,n]$-flag $(U_i, U_{i+1}, \dots, U_{n-1}, U_n)$, for all $i \in \{1, 2, \dots n\}$.

We remark that the stabilizor groups St_i we used, are in fact parabolic subgroups of G. Hence it is clear how to generalize the procudure explained in the present paper to geometries belonging to (almost) simple classical groups, or to exceptional groups of Lie type. Also, one can do exactly the same for semi-simple algebraic groups, or, more generally, for all groups with a (B, N)-pair (or Tits system). The parabolic subgroups are the subgroups containing a Borel subgroup. It follows that the base geometry is just the associated *building*, as defined by Tits [8]. The level subgroups form (in the general case) a maximal chain of subgroups between a Borel subgroup and the whole group. Basically, the arguments of the present paper can be used, but for non-linear diagrams there are some additional choices to make due to the fact that a flag is not linearly ordered by inclusion! This will be investigated in a forthcoming paper. Also, one is tempted to apply the same ideas to (simple) Lie algebras where a Borel subalgebra must replace the role of the Borel subgroup used above.

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