Characterizations by Automorphism Groups of some Rank 3 Buildings,

II. A Half Strongly-Transitive Locally finite Triangle building is a Bruhat-Tits Building.

H. Van Maldeghem* K. Van Steen

Abstract

We complete the proof of the fact that every locally finite triangle building Δ with a half strongly-transitive automorphism group G (e.g., this happens when Δ is defined via a (B, N)-pair in G) is a Bruhat-Tits building associated with a classical linear group over a locally finite local skewfield.

1 Introduction and Main Result

In order to show that every half strongly-transitive locally finite triangle building Δ is a Bruhat-Tits building (this is an affine building arising from an algebraic, classical or mixed type group over some local field as in Bruhat & Tits [4]; see Part I [11]), we prove that the projective Hjelmslev planes of level n attached to each vertex of Δ satisfy the Moufang condition, for all positive integers n. In Part I [11] of this paper, we proposed a machinery to do so. In particular, a method based on an induction hypothesis was developed and it was shown that only the first step of the induction hypothesis must be verified, along with the construction of a certain type of automorphism (called a ${}^{1}h$-collineation) in each Hjelmslev plane. We briefly summerize these results below, after recalling the main definitions.

Let us first write down the Main Result of this part of the paper:

Theorem I. *If Δ is a locally finite triangle building with a half strongly-transitive automorphism group G, and if O is an arbitrary vertex of Δ, then the projective Hjelmslev plane*

*The first author is a Research Director of the Fund for Scientific Research – Flanders (Belgium).

${}^{n}\!H(O)$ *of level* n*,* $n \geq 1$*, attached to* O*, as in* VAN MALDEGHEM [9]*, satisfies the Moufang condition, i.e., it admits every elation and hence it is a desarguesian Hjelmslev plane.*

From this theorem, we will derive the following result, which is stated in Part I [11] as the Main Result of Parts I and II:

Main Result. *If* Δ *is a locally finite triangle building with a half strongly transitive automorphism group* G*, then* Δ^∞ *is associated to a desarguesian projective plane, and hence* Δ *is a Bruhat-Tits building and arises from a classical group* $\mathbf{PSL}_3(\mathbb{K})$ *over a locally finite local skewfield* $\mathbb{K}$*.*

2 Preliminaries

2.1 Definitions

We briefly recall some definitions from Part I [11].

Let Δ be an affine building of type $\tilde{A}_2$. If each residue is finite, then Δ is called *locally finite*. If there is a type-preserving automorphism group G acting transitively on the set of pairs of chambers at fixed Weyl-distance from each other, for each such Weyl-distance, then we say that G acts half strongly-transitively on Δ.

Let O be some vertex of Δ. Then we denote by ${}^{n}\!H(O)$ (or simply ${}^{n}\!H$ if no confusion is possible) the Hjelmslev plane of level n attached to O (this is the geometry of vertices at distance n from O in Δ, see VAN MALDEGHEM [9], or Part I [11]). The point set of ${}^{n}\!H$ is denoted ${}^{n}\mathcal{P}$, the line set ${}^{n}\mathcal{L}$. The natural epimorphism from ${}^{n}\!H$ onto ${}^{k}\!H$, $1 \leq k \leq n$, is denoted by ${}^{k}\pi$. Points (respectively lines) of ${}^{n}\!H$ with the same image under ${}^{k}\pi$ are called *k-neighbouring*, 1-neighbouring being abbreviated by *neighbouring* (and denoted $\sim$). A point and a line whose images under ${}^{k}\pi$ are incident are called *k-near* (and again, 1-near is simplified to *near*). Every collineation α of ${}^{n}\!H$ preserves all neighbour relations and hence induces a collineation $(\alpha)^{\star_k}$ in ${}^{k}\!H$, which we call the $()^{\star_k}$-projection of α. To simplify notation, we denote $(\alpha)^{\star_k}$ sometimes by α when acting on elements of ${}^{k}\!H$ (if no confusion is possible).

An *elation* in ${}^{n}\!H$ with *axis* some line l and *center* some point P, where P is incident with l, is a collineation of ${}^{n}\!H$ fixing all points on l and fixing all lines through P. If the group of all elations with axis l and center P acts transitively on the points not near l incident with some line m (which is itself not neighbouring l, but which is incident with P), then we say that ${}^{n}\!H$ is *(P, l)-transitive*. If ${}^{n}\!H$ is (P, l)-transitive for all choices of such P and l, then we say that ${}^{n}\!H$ is a *Moufang Hjelmslev plane*, or that ${}^{n}\!H$ *satisfies the Moufang condition*.

We will use the word *axis* (of a collineation) to denote a line which is pointwise fixed by a collineation. Dually for center.

A collineation δ of ${}^{n}\!H$, $n \geq 2$, is a *quasi-elation* if a point P and a line l of ${}^{n}\!H$ exist such that

(i) $(\delta)^{\star_{n-1}}$ is an elation with axis ${}^{n-1}\pi(l)$ and center ${}^{n-1}\pi(P)$, ${}^{n-1}\pi(P)\ {}^{n-1}I\ {}^{n-1}\pi(l)$;

(ii) all lines $(n-1)$-neighbouring l are fixed;

(iii) all points $(n-1)$-neighbouring P are fixed.

Every line m, ${}^{n-1}\pi(m) = {}^{n-1}\pi(l)$, that is incident with at least 3 two by two non-neighbouring fixed points is called a *quasi-axis* for δ. Every point Q, ${}^{n-1}\pi(Q) = {}^{n-1}\pi(P)$, that is incident with at least 3 two by two non-neighbouring fixed lines is a *quasi-center* for δ. We have shown in Part I [11](Remark 9) that every quasi-elation has at least one center and, dually, at least one axis. Also, every elation is a quasi-elation (see Lemma 5 of Part I [11]).

In Part I [11], we have proved several elementary properties of quasi-elations. We will use these in the present paper. We now recall the definition of some other types of collineations.

For all k, $1 \leq k \leq n-1$, a ${}^{k}h^{l}_{P}$-collineation of ${}^{n}H$ is an elation with axis $l \in {}^{n}\mathcal{L}$ and center $P \in {}^{n}\mathcal{P}$, and $()^{\star_{n-k}}$-projection trivial.

A *generalized* 1-*homology of* ${}^{n}H$ is a non-trivial collineation of ${}^{n}H$ with $()^{\star_{n-1}}$-projection trivial, and with an axis $l \in {}^{n}\mathcal{L}$ and a center $P \in {}^{n}\mathcal{P}$, with l not near P.

2.2 Some known results

We remind the reader of three important results of Part I [11]. Let G be an automorphism group of a triangle building Δ, and let ${}^{n}\Psi(O)$ (or ${}^{n}\Psi$ if no confusion is possible) be the group of automorphisms of ${}^{n}H(O)$ induced by G.

Proposition 1 *Suppose that for every vertex v of Δ, ${}^{1}H(v)$ is a Moufang plane (with all elations inherited from G), that there exists at least one ${}^{1}h$-collineation in ${}^{2}H(v)$, and that there exists at least one quasi-elation in ${}^{2}H(v)$ with non-trivial $()^{\star_1}$-projection. Then ${}^{2}H$ is a Moufang projective Hjelmslev plane and all elations belong to ${}^{2}\Psi$.*

Proposition 2 *Let $n \geq 3$ and suppose that ${}^{k}H(v)$ is a Moufang Hjelmslev plane of level k, for every $k \leq n-1$ (and all elations are induced by G) and for all vertices v of Δ, and that for every vertex v of Δ, there exists some non-trivial ${}^{1}h$-collineation in ${}^{n}H(v)$ (and induced by G). Then there exists a quasi-elation of ${}^{n}H$ in ${}^{n}\Psi$ with non-trivial $()^{\star_1}$-projection and ${}^{n}H$ is a Moufang projective Hjelmslev plane with all elations belonging to ${}^{n}\Psi$.*

A *well-formed triangle* in the projective Hjelmslev plane ${}^{n}H$ is a set of three pairwise non-neighbouring points $\{P, Q, S\}$ such that ${}^{1}\pi(P), {}^{1}\pi(Q), {}^{1}\pi(S)$ are not collinear in ${}^{1}H$.

Property 3 (transitivity on the well-formed triangles of ${}^{n}H$) *Suppose that G acts half strongly-tranisitively on Δ. Let $\{P_1, P_2, P_3\}$ and $\{Q_1, Q_2, Q_3\}$ be well-formed triangles of ${}^{n}H$. Then a collineation α in ${}^{n}\Psi$ exists such that $\alpha(P_i) = Q_i$, for all $i \in \{1, 2, 3\}$.*

3 Fixed point sets in finite Hjelmslev planes of level n

In this section, which is independent of any hypothesis on the automorphism group, we will show that, if $\overset{n}{H}$ is finite, then every collineation has an equal number of fixed points and fixed lines, just as in the case of a finite projective plane, see for instance Hughes & Piper [5]. The method of proof will be a straightforward generalization of the case $n = 1$ (projective planes). However, for reasons of notation, we will give the proof only for the case of $n = 2$. The general case is proved in detail in the thesis of the second author, see Van Steen [12].

Also, in this section, we temporarily use another notation for points and lines of $\overset{2}{H}$. This will be convenient for the proofs of the next lemmas.

We assume that $\overset{1}{H}$ is a finite projective plane of order q. We label the points of $\overset{1}{H}$ arbitrarily by $\overset{1}{p}_i$, $1 \leq i \leq q^2 + q + 1$, and we label the q^2 points in every ${}^1\pi^{-1}(\overset{1}{p}_i)$ arbitrarily by $\overset{2}{p}_j$, $1 \leq j \leq q^2$. We then label a point P of $\overset{2}{H}$ by the sequence $\overset{1}{p}_i\,\overset{2}{p}_j$ with $1 \leq j \leq q^2$, $1 \leq i \leq q^2 + q + 1$, where $\overset{1}{p}_i$ refers to ${}^1\pi(P)$ and $\overset{2}{p}_j$ to P in the obvious way.
In the same way we label any line of $\overset{2}{H}$ by a sequence $\overset{1}{l}_i\,\overset{2}{l}_j$, $1 \leq j \leq q^2$, $1 \leq i \leq q^2 + q + 1$, with $\overset{1}{l}_i$ referring to ${}^1\pi(l)$ and $\overset{2}{l}_j$ to l.

Notice that with this labelling we have $\begin{cases} \overset{1}{p}_i\,\overset{2}{p}_j \sim \overset{1}{p}_g\,\overset{2}{p}_h & \Leftrightarrow \quad i = g \\ \overset{1}{l}_i\,\overset{2}{l}_j \sim \overset{1}{l}_g\,\overset{2}{l}_h & \Leftrightarrow \quad i = g. \end{cases}$

Definition 4 An incidence matrix A for $\overset{2}{H}$ is said to be *normal* if the point $\overset{1}{p}_i\,\overset{2}{p}_j$ refers to row $q^2(i-1) + j$ of A, and the line $\overset{1}{l}_i\,\overset{2}{l}_j$ refers to column $q^2(i-1) + j$ of A.

We can therefore write $A = (a_{q^2(i-1)+j,q^2(g-1)+h})$ with $a_{q^2(i-1)+j,q^2(g-1)+h} = 1$ if $\overset{1}{p}_i\,\overset{2}{p}_j\,\overset{2}{I}\,\overset{1}{l}_g\,\overset{2}{l}_h$, and with $a_{q^2(i-1)+j,q^2(g-1)+h} = 0$ otherwise.

Lemma 5 *If $\overset{2}{H}$ is finite, and if A is a normal incidence matrix for $\overset{2}{H}$ and α a collineation in ${}^2\Psi$, then α can be represented by 2 permutation matrices B and C satisfying*

$$BA = AC.$$

Proof. This is a standard exercise. $\square$

Lemma 6 *If $\overset{2}{H}$ is finite, and if A is a normal incidence matrix for $\overset{2}{H}$, then $det(A) \neq 0$ (over $\mathbb{Q}$, the field of rational numbers).*

Proof. Consider the matrix product $B = AA^T$. Then the diagonal elements b_{ii}, $1 \leq i \leq v = q^2(q^2 + q + 1)$ are given by

$$\begin{aligned} b_{ii} &= \text{the number of lines that are incident with a point} \\ &= q(q+1). \end{aligned}$$

The non-diagonal elements of B, namely b_{ij}, $i \neq j$, for $1 \leq i, j \leq v$, satisfy

$$b_{ij} = \text{the number of lines that are incident with the points } P \text{ and } Q \text{ respectively corresponding with the i'th and j'th row of } A.$$

If P and Q are neighbouring points, then $b_{ij} = q$. If P and Q are non-neighbouring points, then $b_{ij} = 1$. Hence, the determinant of the matrix AA^T is equal to

$$det(AA^T) = det \begin{pmatrix} q^2 I_{q^2} + qJ_{q^2} & J_{q^2} & \cdots & J_{q^2} \\ J_{q^2} & q^2 I_{q^2} + qJ_{q^2} & \cdots & J_{q^2} \\ \vdots & \vdots & \ddots & \vdots \\ J_{q^2} & J_{q^2} & \cdots & q^2 I_{q^2} + qJ_{q^2} \end{pmatrix},$$

where I_{q^2} denotes the $(q^2 \times q^2)$-identity matrix and J_{q^2} denotes the $(q^2 \times q^2)$-matrix with all entries equal to 1. If we denote the rows and columns of the blockmatrix above by respectively R_i, $1 \leq i \leq q^2+q+1$, and K_i, $1 \leq i \leq q^2+q+1$, then, after replacing the rows R_i, $i \neq 1$, by $R_i - R_1$, and afterwards replacing the first column by the sum of all columns, we obtain

$$det(AA^T) = det \begin{pmatrix} q^2 I_{q^2} + (2q+q^2)J_{q^2} & J_{q^2} & \cdots & J_{q^2} \\ 0 & q^2 I_{q^2} + (q-1)J_{q^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q^2 I_{q^2} + (q-1)J_{q^2} \end{pmatrix}.$$

Hence

$$det(AA^T) = det(q^2 I_{q^2} + (2q+q^2)J_{q^2})\,(det(q^2 I_{q^2} + (q-1)J_{q^2}))^{q^2+q}.$$

After an elementary calculation, we obtain

$$det(AA^T) = (q+1)^2 q^{(2q^2+1)(q^2+q)+2q^2}.$$

Hence $det(AA^T) = 0$ if and only if $q \in \{-1, 0\}$. Since $q \geq 2$ and since $det(A) = det(A^T)$, the lemma follows. □

Lemma 6 enables us to formulate the following useful result.

Lemma 7 *If $\overset{2}{H}$ is finite, then every collineation of $\overset{2}{H}$ has an equal number of fixed points and fixed lines.*

Proof. Suppose α is a collineation of $\overset{2}{H}$. Then, using Lemma 5, α can be represented by permutation matrices B and C with $BA = AC$ and A a normal incidence matrix for $\overset{2}{H}$.

By definition of B, the trace $tr(B)$ equals the number of fixed points of α. In the same way

$tr(C)$ gives the number of fixed lines of α.
Using Lemma 5 again together with Lemma 6, which guarantees the existence of A^{-1}, we obtain that $B = ACA^{-1}$. Thus $tr(B) = tr(C)$. Hence α has an equal number of fixed points and fixed lines. $\square$

Similarly, one shows:

Lemma 8 *Let $\overset{n}{H}$ be finite. Then every collineation acting on $\overset{n}{H}$ has an equal number of fixed points and fixed lines, $n \geq 1$.* $\square$

In fact, only the non-singularity of an incidence matrix of $\overset{n}{H}$ is somewhat harder to prove than in the case $n = 2$. But this boils down to some calculation which are uninteresting and uniformative for the rest of this paper. As mentioned before, a complete detailed proof can be found in Van Steen [12].

4 Proof of Theorem I

In this section we prove Theorem I of the Introduction. So we assume that Δ is a locally finite triangle building with a half strongly-transitive automorphism group G. After some definitions of affine planes and dual affine planes occurring in $\overset{n}{H}$, we prove first Theorem I for the cases $n = 1, 2$.

4.1 Affine planes in $\overset{n}{H}$

Suppose $\overset{i}{\pi}(Q)$ is a point of $\overset{i}{H}(O)$, $1 \leq i \leq n$, for some point $Q \in \overset{n}{\mathcal{P}}(O)$. Then the projective plane, viewed as a completed affine plane (and which allows us to speak about points at infinity, once we defined a line at infinity), associated with $\overset{i}{\pi}(Q)$ is denoted by $\overset{1}{H}(\overset{i}{\pi}(Q))$ and defined as follows.
For $i = 1$, the vertex O is viewed as the line at infinity.
For $i > 1$, the point $\overset{i-1}{\pi}(Q)$ of $\overset{i-1}{\mathcal{P}}(O)$ corresponds with the line at infinity of $\overset{1}{H}(\overset{i}{\pi}(Q))$.

The projective Hjelmslev plane of level j associated with $\overset{i}{\pi}(Q)$, $1 \leq j$, is denoted by $\overset{j}{H}(\overset{i}{\pi}(Q))$ and defined as the projective Hjelmslev plane of level j attached to the vertex $\overset{i}{\pi}(Q)$ of the triangle building Δ such that $\overset{1}{\pi}(\overset{j}{H}(\overset{i}{\pi}(Q))) = \overset{1}{H}(\overset{i}{\pi}(Q))$.

Suppose $\overset{i}{\pi}(m)$ is a line of $\overset{i}{H}(O)$, $1 \leq i \leq n$, for some line $m \in \overset{n}{\mathcal{L}}(O)$. Then the projective plane, viewed as a completed dual affine plane, associated with $\overset{i}{\pi}(m)$ is denoted by $\overset{1}{H}(\overset{i}{\pi}(m))$ and is defined in a similar way as $\overset{1}{H}(\overset{i}{\pi}(Q))$.
For $i = 1$, we view O as the point at infinity of $\overset{1}{H}(\overset{1}{\pi}(m))$.
For $i > 1$, the line $\overset{i-1}{\pi}(m)$ of $\overset{i-1}{\mathcal{L}}(O)$ corresponds with the point at infinity of the dual projective plane $\overset{1}{H}(\overset{i}{\pi}(m))$,

The projective Hjelmslev plane of level j associated with ${}^{i}\pi(m)$, $1 \leq j$, is denoted by ${}^{j}H({}^{i}\pi(m))$ and defined as the projective Hjelmslev plane attached to ${}^{i}\pi(m)$ such that ${}^{1}\pi({}^{j}H({}^{i}\pi(m))) = {}^{1}H({}^{i}\pi(m))$.

See also Part I [11] for these definitions.

4.2 The case of levels 1 and 2

In this subsection we show:

Theorem Ia *If Δ is a locally finite triangle building with a half strongly-transitive group G, then for all vertices O of δ, the projective plane ${}^{1}H(O)$ and the projective Hjelmslev plane ${}^{2}H(O)$ satisfy the Moufang condition and both ${}^{1}\Psi(O)$ and ${}^{2}\Psi(O)$ contain all elations.*

Lemma 9 *${}^{1}H(O)$ is a desarguesian projective plane of order $q = p^s$, where p is some prime and $s \geq 1$. Also, all elations belong to ${}^{1}\Psi(O)$.*

Proof. This is a consequence of Property 3, the Theorem of Ostrom-Wagner (see Hughes & Piper [5]) and the locally finiteness assumption. $\square$

Note that ${}^{1}\Psi$ contains the little projective group $\mathbf{PSL}(3, q)$. From now on we denote the order of a vertex-residue in Δ by $q = p^s$, where p is a fixed prime and s is a fixed positive integer.

Lemma 10 *For all lines l of ${}^{2}H$, $|{}^{2}\Psi_l| = kq^7(q + 1)$, for some positive integer k.*

Proof. Suppose $K \in {}^{2}\mathcal{P}$ and $L \in {}^{2}\mathcal{P}$ determine a unique line l (so $K \not\sim L$). Let M be some point of ${}^{2}H$ not near l, and let m be the line defined by M and K. Put $|{}^{2}\Psi_{M,K,L}| = k$, $k \geq 1$. Then by Property 3, $|{}^{2}\Psi_l|$ is equal to k multiplied with the number of possible choices for K, L, M defined as above. An elementary counting argument shows that there are exactly $q^7(q + 1)$ such choices. $\square$

Lemma 11 *Suppose $l \in {}^{2}\mathcal{L}$ and $P \in {}^{2}\mathcal{P}$ such that $P \,{}^{2}I\, l$. Then every Sylow p-subgroup Γ of ${}^{2}\Psi_{l,P}$ acts transitively on ${}^{2}\mathcal{P} \setminus \{Q \in {}^{2}\mathcal{P} \,|\, Q \text{ is near } l\}$.*

Proof. By Lemma 9, ${}^{1}H$ is a projective plane of order $q = p^s$. Suppose $p^t \,|\, k$ with k as in Lemma 10 and where $t \geq 0$. By Lemma 10 the order of ${}^{2}\Psi_{l,P}$ equals kq^6. Hence $p \,|\, |{}^{2}\Psi_{l,P}|$ and the Sylow p-subgroups of ${}^{2}\Psi_{l,P}$ are non-trivial. Let Γ be such a Sylow p-subgroup. Then $|\Gamma| = p^{6s+t}$. Suppose now R is some point of ${}^{2}H$ with ${}^{1}\pi(R) \not{I}\; {}^{1}\pi(l)$ and put $|R^\Gamma| = p^u$, the order of the orbit of R under the group Γ.

Notice that $|R^\Gamma|$ is indeed a power of p, since $|\Gamma| = |\Gamma_R||R^\Gamma|$ and since $|\Gamma| = p^{6s+t}$.
Using $|\Gamma| = |\Gamma_R||R^\Gamma|$,

$$\begin{aligned} |\Gamma_R| &= \text{the order of the subgroup of } \Gamma \text{ fixing } R \\ &= p^{6s+t-u}. \end{aligned}$$

Since $\Gamma_R \leq {}^2\Psi_{l,P,R}$ and, by using Lemma 10 again ($|{}^2\Psi_{l,P,R}| = kq^2$), we obtain that $p^{6s+t-u} \mid p^{2s+t}$. Hence $6s + t - u \leq 2s + t$ or

$$4s \leq u. \tag{1}$$

But there are only q^4 possibilities to pinpoint a point R of ${}^2H$ that is not near l. Thus $|R^\Gamma| \leq p^{4s}$, which implies that $p^u \leq p^{4s}$ or that

$$u \leq 4s. \tag{2}$$

From 1 and 2 we conclude that $u = 4s$.

Consequently $|R^\Gamma| = q^4$. The result is the transitivity of Γ on ${}^2\mathcal{P} \setminus \{Q \in {}^2\mathcal{P} \mid Q \text{ is near } l\}$. $\square$

Lemma 12 *Suppose $l, m \in {}^2\mathcal{L}$, $l \not\sim m$. Suppose P is the point of ${}^2H$ determined by l and m, and suppose Q is some point incident with l not neighbouring P. Then every Sylow p-subgroup Γ of ${}^2\Psi_{l,m,Q}$ acts transitively on the set $\{S \in {}^2\mathcal{P} \mid S \,{}^2\mathrm{I}\, m,\ {}^1\pi(S) \neq {}^1\pi(P)\}$.*

Proof. Noting that $|{}^2\Psi_{l,m,Q}| = kq^2$ (consequence of Lemma 10), that $|{}^2\Psi_{l,m,Q,R}| = k$, where R is some element of $\{S \in {}^2\mathcal{P} \mid S \,{}^2\mathrm{I}\, m,\ {}^1\pi(S) \neq {}^1\pi(P)\}$ (Lemma 10 and Property 3), and that there are q^2 points of ${}^2H$ incident with m that do not neighbour P, the proof of Lemma 11 is easily adapted. $\square$

Now we note (see e.g. Huppert [6], Hilfssatz 7.7.):

Lemma 13 *Suppose Υ is some group and θ an epimorphism*

$$\theta : \Upsilon \to \theta(\Upsilon).$$

If Γ is a Sylow p-subgroup of Υ, for some $p \geq 2$, then $\theta(\Gamma)$ is a Sylow p-subgroup of $\theta(\Upsilon)$.

In view of Proposition 1, we have to exibit at least one quasi-elation with non-trivial $()^{\star_1}$-projection. This will be done in the following lemma.

Lemma 14 *At least one quasi-elation exists in ${}^2\Psi$ (with a quasi-axis and a quasi-center) with $()^{\star_1}$-projection non-trivial.*

Proof. Consider some points P and Q of ${}^{2}H$, ${}^{1}\pi(P) \neq {}^{1}\pi(Q)$, and a line m not near Q with $P\,{}^{2}I\,m$. Let l be the line in ${}^{2}\mathcal{L}$ incident with P and Q.

By property 3, ${}^{2}\Psi_{P,Q,m}$ acts transitively on the points that are incident with m but which do not neighbour P. So $q^2 \,|\, |{}^{2}\Psi_{P,Q,m}|$. So it is possible to consider a non-trivial Sylow p-subgroup Γ of ${}^{2}\Psi_{P,Q,m}$. By Lemma 13, $(\Gamma)^{\star_1}$ is a Sylow p-subgroup of $({}^{2}\Psi_{P,Q,m})^{\star_1}$.

We claim that $(\Gamma)^{\star_1}$ contains at least one elation with axis ${}^{1}\pi(l)$. Indeed, if we coordinatize ${}^{1}H$ such that

$$\begin{aligned} {}^{1}\pi(P) &:= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T, \\ {}^{1}\pi(Q) &:= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T, \\ {}^{1}\pi(m) &:= Y = 0, \end{aligned}$$

then $(\Gamma)^{\star_1}$ is contained in the group of semi-linear matrices

$$\begin{pmatrix} 1 & 0 & d \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}^{\theta}, \quad b, c, d \in \mathbf{GF}(q),$$

a group of order $(q-1)^2 qs$ with $q = p^s$ as before. In fact, since $(\Gamma)^{\star_1}$ is a p-group, $(\Gamma)^{\star_1}$ is contained in the group of matrices

$$\delta_{d,\theta} := \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}^{\theta}, \quad d \in \mathbf{GF}(q).$$

Suppose there exists for every automorphism θ of $\mathbf{GF}(q)$ that occurs in $(\Gamma)^{\star_1}$ only one $d \in \mathbf{GF}(q) \setminus \{0\}$ to form a matrix $\delta_{d,\theta}$ of $(\Gamma)^{\star_1}$. Then $|(\Gamma)^{\star_1}| \leq s < q$. However, by Lemma 12, we have $q \,|\, |(\Gamma)^{\star_1}|$, a contradiction.
Hence, different elements d and $d' \in \mathbf{GF}(q) \setminus \{0\}$ exist, and an automorphism θ of $\mathbf{GF}(q)$ exists such that $\delta_{d,\theta}$ and $\delta_{d',\theta}$ are both elements of $(\Gamma)^{\star_1}$. Thus, since $(\Gamma)^{\star_1}$ is a group, $\delta_{d',\theta}(\delta_{d,\theta})^{-1} = \delta_{d',\theta}\delta_{-d^{\theta^{-1}},\theta^{-1}} = \delta_{d'-d,1} \in (\Gamma)^{\star_1}$ and $d' - d \neq 0$. Consequently, $\delta_{d'-d,1}$ is an elation of $(\Gamma)^{\star_1}$ with axis ${}^{1}\pi(l)$ and center ${}^{1}\pi(P)$. The claim follows.

Let α be some element of Γ with $(\alpha)^{\star_1}$ a non-trivial elation with axis ${}^{1}\pi(l)$ and center ${}^{1}\pi(P)$. The lines of ${}^{1}H$ that are incident with ${}^{1}\pi(P)$ correspond with the points at infinity of ${}^{1}H({}^{1}\pi(P))$. Hence, in ${}^{1}H({}^{1}\pi(P))$ α induces a collineation with axis the line at infinity of ${}^{1}H({}^{1}\pi(P))$ and fixes at least one affine point of ${}^{1}H({}^{1}\pi(P))$, namely P. Since the order of α is a power of p (α being an element of a Sylow p-group), α induces the identity collineation in ${}^{1}H({}^{1}\pi(P))$. Dually, α also induces the identity in ${}^{1}H({}^{1}\pi(l))$.

We now consider an arbitrary point ${}^{1}\pi(R) \neq {}^{1}\pi(P)$, $R \in {}^{2}\mathcal{P}$, that is incident with ${}^{1}\pi(l)$. Then the collineation induced by α in ${}^{1}H({}^{1}\pi(R))$ has a center at infinity, corresponding with the line ${}^{1}\pi(l)$ of ${}^{1}H$, becaue α induces the identity in ${}^{1}H({}^{1}\pi(l))$. Since $(\alpha)^{\star_1} \neq 1$ and points at infinity of ${}^{1}H({}^{1}\pi(R))$ are in one-one correspondence with the lines in ${}^{1}\mathcal{L}$ that are incident with ${}^{1}\pi(R)$, no other point at infinity of ${}^{1}H({}^{1}\pi(R))$ can be fixed by α.

Hence α induces in ${}^{1}H({}^{1}\pi(R))$ a non-trivial elation with some affine axis determined by an $l' \cap {}^{1}H({}^{1}\pi(R))$ $l' \in {}^{2}\mathcal{L}$, ${}^{1}\pi(l') = {}^{1}\pi(l)$, where $l' \cap {}^{1}H({}^{1}\pi(R))$ is the formal notation for the unique vertex in Δ that is adjacent to l', ${}^{1}\pi(l')$ and ${}^{1}\pi(R)$.

Dually, we consider an arbitrary line ${}^{1}\pi(u) \neq {}^{1}\pi(l)$, $u \in {}^{2}\mathcal{L}$, such that ${}^{1}\pi(u)\, {}^{1}I\, {}^{1}\pi(P)$. Following the reasoning of the previous paragraph, we obtain that α induces a non-trivial elation in ${}^{1}H({}^{1}\pi(u))$ with some affine center, determined by a $P' \cap {}^{1}H({}^{1}\pi(u))$, $P' \in {}^{2}\mathcal{P}$, $P' \sim P$, where again $P' \cap {}^{1}H({}^{1}\pi(u))$ is a formal notation and indicates the unique vertex in Δ that is adjacent to P', ${}^{1}\pi(P')$, and ${}^{1}\pi(u)$.

We conclude that α is a quasi-elation with quasi-axis (quasi-axes) neighbouring l and quasi-center (quasi-centers) neighbouring P, such that $(\alpha)^{\star_1} \neq 1$. $\square$

Our next aim is the construction of a non-trivial ${}^{1}h$-colineation in ${}^{2}H$. We recall some definitions from Part I [11].

A ${}^{1}h^{l}$*-collineation in* ${}^{2}\Psi$ is a ${}^{1}h$-collineation with axis l. Dually, a ${}^{1}h_{R}$*-collineation in* ${}^{2}\Psi$ is a ${}^{1}h$-collineation with center R. A ${}^{1}h^{l}_{R}$*-collineation* in ${}^{2}\Psi$ is a ${}^{1}h^{l}$-collineation which is also a ${}^{1}h_{R}$-collineation.

We denote the sets of ${}^{1}h$-collineations, ${}^{1}h$-collineations with axis l, ${}^{1}h$-collineations with center R, and ${}^{1}h$-collineations with axis l and center R in ${}^{2}\Psi$ respectively as ${}^{1}h\mathcal{C}$, ${}^{1}h\mathcal{C}^{l}$, ${}^{1}h\mathcal{C}_{(R)}$ and ${}^{1}h\mathcal{C}^{l}_{(R)}$.

Now note that, by Lemma 2 of Part I [11], for every point P of ${}^{2}H$, a generalized 1-homology induces in ${}^{1}H({}^{1}\pi(P))$ either the identity or a non-trivial elation with axis at infinity. Since ${}^{1}H({}^{1}\pi(P))$ is a finite projective plane of order q (by Lemma 9), the order of such an induced non-trivial elation is equal to p. So we may denote the order of the subgroup of ${}^{2}\Psi$ consisting of all generalized 1-homologies with an axis l and a center R by $p^{r}(l, R)$, for some specific $r \geq 0$ (or p^{r} if no confusion is possible).

The next theorem is a result about finite projective planes, independent of our hypotheses. We use the notation of Hughes & Piper [5]. In particular, for an automorphism group Υ of a projective plane, a point Q and a line l of that plane, we denote by $\Upsilon_{(Q,l)}$ the set of all collineations in Υ with center Q and axis l.

Theorem 15 *Let H be a finite projective plane of order $q = p^{s}$ (p prime, $s \geq 1$), and l some line of H. If Υ is a collineation group of H such that $\Upsilon_{(Q,l)}$ is non-trivial and $|\Upsilon_{(Q,l)}|$ is some fixed power of p, for all points Q of H that are incident with l, then $|\Upsilon_{(Q,l)}| = q$.*

Proof. Suppose $|\Upsilon_{(Q,l)}| = p^{h}$, $h > 0$. Then $|\Upsilon_{(l,l)}| = (q+1)(p^{h}-1)+1$, since there are exactly $q+1$ points of H incident with l.

Using Theorem 4.16 of Hughes & Piper [5], $|\Upsilon_{(l,l)}|\,|\,q^2 = p^{2s}$. Hence $(q+1)(p^h-1)+1 = p^{s+h}+p^h-p^s = p^s(p^h+\frac{p^h}{p^s}-1) > p^s$, and is a power of p, say p^r $(r>s)$.

Consequently, p is a divisor of $p^h+\frac{p^h}{p^s}-1$, which is only possible for $p^h=p^s=q$. $\square$

The following lemma is the crux of the proof of Theorem I.

Lemma 16 *At least one non-trivial* $\overset{1}{h}$*-collineation exists in* $\overset{2}{\Psi}$.

Proof. By Lemma 14, a quasi-elation α with some quasi-axis $l \in \overset{2}{\mathcal{L}}$ and quasi-center neighbouring some point $P \in \overset{2}{\mathcal{P}}$, $P \overset{2}{I} l$, and with non-trivial $()^{\star_1}$-projection exists. Let $m \overset{2}{I} P$ be some fixed line in $\overset{2}{\mathcal{L}}$ for α with $\overset{1}{\pi}(m) \neq \overset{1}{\pi}(l)$ (use Property 3 if necessary), and let Q be some point of $\overset{2}{H}$ incident with l, $\overset{1}{\pi}(Q) \neq \overset{1}{\pi}(P)$

Part 1:

In this part we prove the claim that there exists a group of collineations fixing all lines neighbouring l, fixing all lines of $\overset{1}{H}$ that are incident with $\overset{1}{\pi}(P)$, and acting transitively on the points that neighbour P and that are also incident with l.

For the moment, let us denote by Υ the subgroup of $\overset{2}{\Psi}$ generated by all collineations fixing every point in $\overset{2}{\mathcal{P}}$ that neighbours P, fixing all lines of $\overset{2}{H}$ that neighbour l (and by considering intersection of such lines, thus fixing all points of $\overset{1}{H}$ that are incident with $\overset{1}{\pi}(l)$).
Then $\alpha \in \Upsilon$ and $(\Upsilon)^{\star_1}$ is a set of elations with axis $\overset{1}{\pi}(l)$ and center $\overset{1}{\pi}(P)$. Hence

$$|(\Upsilon)^{\star_1}| \leq q. \tag{3}$$

If $\Upsilon' \leq \Upsilon$ is the set of collineations of Υ with $()^{\star_1}$-projection trivial, then

$$\frac{|\Upsilon|}{|\Upsilon'|} = |(\Upsilon)^{\star_1}|. \tag{4}$$

So by (3) and (4),

$$|\Upsilon'| \geq \frac{|\Upsilon|}{q}.$$

By the dual of Lemma 11, any Sylow p-group Γ of the subgroup of $\overset{2}{\Psi}$ consisting of collineations fixing l and Q, acts transitively on the lines that are not near Q and that are in particular incident with some point $R \overset{2}{I} m$, $\overset{1}{\pi}(R) \overset{1}{\not{I}} \overset{1}{\pi}(l)$. Hence, for every line $m' \overset{2}{I} R$, and with $\overset{1}{\pi}(m') = \overset{1}{\pi}(m)$ some collineation $\delta_{m'}$ of Γ mapping m' to m exists. The collineations $\delta_{m'}^{-1}\alpha\delta_{m'}$ are again elements of Υ because $\overset{1}{\pi}(P)$ is fixed by $\delta_{m'}$. Since $(\alpha)^{\star_1} \neq 1$, we also have $(\delta_{m'}^{-1}\alpha\delta_{m'})^{\star_1} \neq 1$.

Suppose m'' and m''' are distinct lines of $\overset{2}{H}$, satisfying $\overset{1}{\pi}(m'') = \overset{1}{\pi}(m''') = \overset{1}{\pi}(m)$ and $m'' \overset{2}{I} R \overset{2}{I} m'''$. Suppose $\delta_{m''}^{-1}\alpha\delta_{m''} = \delta_{m'''}^{-1}\alpha\delta_{m'''}$. Then $\delta_{m''}^{-1}\alpha\delta_{m''}$ fixes the line $\delta_{m''}^{-1}(m)$ and $\delta_{m'''}^{-1}(m)$.

Since both $\delta_{m''}^{-1}(m)$ and $\delta_{m'''}^{-1}(m)$ are incident with R and neighbouring, ${}^{1}\pi(R)$ is fixed by $\delta_{m''}^{-1}\alpha\delta_{m''}$. Hence $(\delta_{m''}^{-1}\alpha\delta_{m''})^{\star_1} = 1$, using ${}^{1}\pi(R) \not I\ {}^{1}\pi(l)$, a contradiction.

As a consequence of the previous paragraphs, the q choices for m' as a line of ${}^{2}H$ that neighbours m and that is incident with R, correspond with two by two different collineations $\delta_{m'}^{-1}\alpha\delta_{m'} \in \Upsilon$, with $(\delta_{m'}^{-1}\alpha\delta_{m'})^{\star_1}$ a non-trivial elation (of order p) acting on ${}^{1}H$.

It can now be seen that

$$|\Upsilon| \geq q(p-1)+1.$$

Consequently

$$|\Upsilon'| > 1,$$

which guarantees the existence of a collineation $\eta \in \Upsilon$ such that $(\eta)^{\star_1} = 1$ but $\eta \neq 1$.

Since $\eta \neq 1$, some point U of ${}^{2}H$, ${}^{1}\pi(U) \neq {}^{1}\pi(P)$, exists such that η maps U to some point U', $U' \sim U$, $U' \neq U$.

By Lemma 2 of Part I [11], η induces in ${}^{1}H({}^{1}\pi(U))$ a non-trivial elation. The center of the induced elation is determined by a line of ${}^{1}H$, say ${}^{1}\pi(v)$, $v \in {}^{2}\mathcal{L}$, such that every line of ${}^{2}H$ incident with both U and U' neighbours v. Notice that ${}^{1}\pi(v)$ might coincide with ${}^{1}\pi(l)$.

The question was whether a group of collineations fixing all lines neighbouring l exists, fixing all lines of ${}^{1}H$ that are incident with ${}^{1}\pi(P)$, and acting transitively on the points that neighbour P and that are also incident with l. Consider the induced collineations in the Hjelmslev plane ${}^{2}H({}^{1}\pi(l))$ of level 2 associated with the vertex ${}^{1}\pi(l)$. We remark that the vertex O is now a point of ${}^{1}H({}^{1}\pi(l))$, that ${}^{1}\pi(P)$ is a line of ${}^{1}H({}^{1}\pi(l))$ incident (in ${}^{1}H({}^{1}\pi(l))$) with O, and that l is a line of ${}^{1}H({}^{1}\pi(l))$ which is different from the line ${}^{1}\pi(P)$ of ${}^{1}H({}^{1}\pi(l))$ and not incident (in ${}^{1}H({}^{1}\pi(l))$) with O. The lines of ${}^{2}H(O)$ that neighbour l correspond with lines of ${}^{1}H({}^{1}\pi(l))$ that are not incident (incidence in ${}^{1}H({}^{1}\pi(l))$) with O.

So dually, and after shifting the problem to ${}^{2}H(O)$, we should show the existence of a set of collineations with $()^{\star_1}$-projection trivial, that induce in ${}^{1}H({}^{1}\pi(T))$, for some point T of ${}^{2}H(O)$, q elations.

To prove this existence, we remark that all 'directions', or points at infinity of ${}^{1}H({}^{1}\pi(T))$, play the same role, using the transitivity of ${}^{2}\Psi$ on the well-formed triangles of ${}^{2}H$. Hence the number of elations for some 'fixed direction' (the identity included) acting on ${}^{1}H({}^{1}\pi(T))$ equals p^h, $0 \leq h \leq s$.

Using earlier results in this proof (concerning η), we know that $1 < p^h$. Hence, applying Theorem 15, we conclude $p^h = q$ and our claim is proved.

Part 2: In this Part we prove the actual occurence of a non-trivial ${}^{1}h$-collineation in ${}^{2}\Psi$.

For this purpose we consider the subgroup Υ'' of ${}^{2}\Psi$ fixing all points in ${}^{2}\mathcal{P}$ fixed by α. Then

$|(\Upsilon'')^{\star_1}| \leq q$ and

$$|\Upsilon'''| \geq \frac{|\Upsilon''|}{q},$$

where Υ''' consists of all elements of Υ'' with trivial $()^{\star_1}$-projection.
As proven in Part 1, for every point $P' \overset{2}{I} l$, $P' \sim P$, a collineation $\theta_{P'}$ exists that fixes all lines that neighbour l, fixes all lines of $\overset{1}{H}$ that are incident with $\overset{1}{\pi}(P)$, and that maps P' to P. The collineations $\theta_{P'}^{-1}\alpha\theta_{P'}$ are again elements of Υ'' with $()^{\star_1}$-projection not trivial.

Notice that the set $\{\theta_{P'}^{-1}\alpha\theta_{P'} \,|\, P' \sim P, P' \overset{2}{I} l\}$ consists of two by two different elements. This can be shown similarly as above (see the argument concerning δ_m in Part 1). Consequently,

$$|\Upsilon''| \geq q(p-1)+1$$

and so $|\Upsilon'''| > 1$. In other words, some non-trivial collineation θ' in $\overset{2}{\Psi}$ exists with $(\theta')^{\star_1} = 1$ and fixing all points of $\overset{2}{H}$ that are fixed by α. Applying Lemma 2 of Part I [11] of this paper, all points of $\overset{2}{H}$ that are near l are fixed by θ' (recall that by Lemma 14, α fixes at least one point neigbouring any point near l).

Suppose non-trivial $\overset{1}{h}$-collineations do not exist. Then by Lemma 16(ii) of Part I [11], α is a generalized 1-homology. Hence $p^r > 1$. Consider the subgroup Υ^{iv} of $\overset{2}{\Psi}$ consisting of all collineations fixing every point of $\overset{2}{H}$ near l, and fixing some arbitrary line u not neighbouring l. Then every element of this group has a trivial $()^{\star_1}$-projection and the order of the group is $p^r p^z$, where p^z $(z \geq 0)$ is the orbit under Υ^{iv} of some point $V \overset{2}{I} u$, $\overset{1}{\pi}(V) \not\overset{1}{I} \overset{1}{\pi}(l)$. We note that the only collineations active on $\overset{1}{H}(\overset{1}{\pi}(V))$ are elations, by Lemma 2 or Lemma 16 of Part I [11].

On the other hand, $|\Upsilon^{iv}|$ equals $q(p^r-1)+1$. This can be seen as follows. If a collineation $\beta \in \Upsilon^{iv}$ exists such that the only points of $\overset{2}{H}$ that are incident with u and fixed by β neighbour P, then by Lemma 7, and since the number of points fixed by β is $q^2(q+1)$ in this case, there are $q^2(q+1)$ fixed lines for β. Moreover, all these lines are near P. Hence $\beta = 1$, a contradiction.

Consequently, every collineation in Υ^{iv} fixes some point $U \overset{2}{I} u$, with $U \not\sim P$. Thus Υ^{iv} consists of all possible generalized 1-homologies with axis l that fix u. Continuing, we obtain that

$$p^r p^z = q(p^r-1)+1.$$

Since $p^r > 1$, and thus $p \,|\, p^r p^z$, it follows that $p \,|\, q(p^r-1)+1$, a contradiction.

We conclude that there is at least one non-trivial $\overset{1}{h}$-collineation available in $\overset{2}{\Psi}$. $\square$

By Proposition 1, we conclude that $\overset{2}{H}$ is a Moufang Hjelmslev plane and that all elations belong to $\overset{2}{\Psi}$. Whence Theorem Ia. Now we show that in fact we have a Desarguesian Hjelmslev plane.

In 1977, Dugas proved (with corrections made by Bacon) that a finite Moufang (projective) Hjelmslev plane whose canonical image is not $\mathbf{PG}(2,2)$ is desarguesian. In 1979, this result was extended by Bacon. He showed that a finite punctally cohesive Moufang (projective) Klingenberg plane (and in particular a finite punctally cohesive Moufang (projective) Hjelmslev plane) whose canonical image is not $\mathbf{PG}(2,2)$ is a desarguesian plane. In Baker, Lane & Lorimer [1], theorems are formulated and proven in order to eliminate the $\mathbf{PG}(2,2)$ restriction, as indicated in the proof of Theorem 17. We refer to Baker, Lane & Lorimer [1], [2], and [3].

Theorem 17 *If Δ is a locally finite triangle building with a half strongly-transitive automorphism group, then for each vertex O, ${}^{2}H(O)$ is a desarguesian Hjelmslev plane.*

Proof. Since ${}^{2}H$ is a Moufang Hjelmslev plane, it can be coordinatized by a local alternative ring R. Moreover, using Baker, Lane & Lorimer [1], R must be a projective Hjelmslev ring. By the definition of a projective Hjelmslev ring, R is a right chain ring. Therefore, ${}^{2}H$ is punctally cohesive. Hence so far, ${}^{2}H$ is a finite punctally cohesive Moufang Hjelmslev plane. Using Baker, Lane & Lorimer [1] again, ${}^{2}H$ is desarguesian. $\square$

Recall that, by Theorem 35 of Part I [11], we have:

Theorem 18 *The set of elations in ${}^{2}\Psi$ with some fixed axis $l \in {}^{2}\mathcal{L}$ is an abelian group.*

4.3 The case $n \geq 3$

In this subsection, we show:

Theorem Ib. *If Δ is a locally finite triangle building with a half strongly-transitive group G, then for all vertices O of δ, the projective Hjelmslev plane ${}^{n}H(O)$, $n \geq 3$, satisfies the Moufang condition and ${}^{n}\Psi(O)$ contains all elations.*

We assume throughout, by induction, that ${}^{k}H(v)$ is a Moufang projective Hjelmslev plane with all elations in ${}^{k}\Psi(v)$, for $1 \leq k \leq n-1$, $n \geq 3$, and for all vertices v. As for the case $n = 2$, this implies (and also the proof is similar, see Theorem 17)

Theorem 19 *For all k, $2 \leq k < n$, and all vertices v of Δ, ${}^{k}H(v)$ is desarguesian.*

Theorem 35 of Part I [11] implies:

Theorem 20 *For every vertex v, the set of elations in ${}^{k}\Psi(v)$ with some chosen axis l of ${}^{k}H$ forms a commutative group acting transitively on the set of points of ${}^{k}\mathcal{P} \setminus \{Q \in {}^{k}\mathcal{P} \mid {}^{1}\pi(Q)\ {}^{1}I\ {}^{1}\pi(l)\}$.*

Also, note that the following lemmas have proofs which are completely similar to Lemma 10 and Lemma 11, respectively. Note that we still have our main assumption: the group G acts strongly-transitively on Δ.

Lemma 21 *For every line $l \in {}^n\mathcal{P}$, $|{}^n\Psi_l|$ is a multiple of $q^{4n-1}(q+1)$.*

Lemma 22 *Suppose $l \in {}^n\mathcal{L}$ and $P \in {}^n\mathcal{P}$ such that $P \, {}^n\mathrm{I} \, l$. Then every Sylow p-subgroup Γ of ${}^n\Psi_{l,P}$ acts transitively on ${}^n\mathcal{P} \setminus \{Q \in {}^n\mathcal{P} \,|\, Q \text{ is near } l\}$.* $\square$

In view of Proposition 2, we must show that there is a non-trivial ${}^1h$-collineation in ${}^n\Psi$. We need a few lemmas before we can show this. The first lemma slightly generalizes Lemma 16 of Part I [11].

Lemma 23 *Suppose l is some line of ${}^n\mathcal{H}$ and P some point of ${}^n\mathcal{H}$ with $P \, {}^n\mathrm{I} \, l$, $n \geq 2$. Suppose γ is a collineation in ${}^n\Psi$ with $(\gamma)^{\star_{n-1}} = 1$, fixing all lines incident with P except maybe for lines that neighbour l and such that all occurring fixed points are near l. Then γ is a ${}^1h$-collineation in ${}^n\Psi$ with axis l and center P.*

Proof. Suppose m is an arbitrary line of ${}^n\mathcal{H}$ that is incident with P and for which ${}^1\pi(m) \neq {}^1\pi(l)$.

We claim that γ induces the identity in ${}^1\mathcal{H}({}^{n-1}\pi(m))$. Indeed, the vertices in $cl({}^{n-1}\pi(m), {}^1\pi(T))$, for all ${}^1\pi(T) \; {}^1\mathrm{I} \; {}^1\pi(m)$, that are adjacent to both ${}^{n-1}\pi(m)$ and ${}^{n-2}\pi(m)$, correspond with the lines at infinity of ${}^1\mathcal{H}({}^{n-1}\pi(m))$, where, for $n = 2$, we set ${}^{n-2}\pi(m) = O$.

The lines m' that are incident with P and for which ${}^{n-1}\pi(m') = {}^{n-1}\pi(m)$, are fixed by γ, and give rise to an affine (affine in the dual projective plane ${}^1\mathcal{H}({}^{n-1}\pi(m))$) center for the by γ induced collineation in ${}^1\mathcal{H}({}^{n-1}\pi(m))$. Thus γ induces a collineation with two centers in ${}^1\mathcal{H}({}^{n-1}\pi(m))$. Necessarily, $\gamma_{{}^1\mathcal{H}({}^{n-1}\pi(m))} = 1$. Hence the claim.

In fact, all lines of ${}^n\mathcal{H}$ that are near P and do not neighbour l, are fixed for γ. Indeed, suppose that m is some line of ${}^n\mathcal{H}(O)$ such that ${}^1\pi(P) \, {}^1\mathrm{I} \, {}^1\pi(m)$, ${}^1\pi(m) \neq {}^1\pi(l)$, and $P \; {}^n\!\not\mathrm{I} \; m$. Let T and T' be two non-neighbouring points of ${}^n\mathcal{H}$ satisfying $T \, {}^n\mathrm{I} \, m \, {}^n\mathrm{I} \, T'$ and ${}^1\pi(T) \; {}^1\!\not\mathrm{I} \; {}^1\pi(l) \; {}^1\!\not\mathrm{I} \; {}^1\pi(T')$. Then the line m' of ${}^n\mathcal{H}(O)$ that is incident with P and T is a fixed line for γ. The line m'' determined by P and T' is fixed for γ as well. Additionally, $m' \cap {}^1\mathcal{H}({}^{n-1}\pi(T))$ and $m'' \cap {}^1\mathcal{H}({}^{n-1}\pi(T'))$ are both fixed by γ. Note again that $(\gamma)^{\star_{n-1}} = 1$. Since $m' \cap {}^1\mathcal{H}({}^{n-1}\pi(T)) = m \cap {}^1\mathcal{H}({}^{n-1}\pi(T))$ and $m'' \cap {}^1\mathcal{H}({}^{n-1}\pi(T')) = m \cap {}^1\mathcal{H}({}^{n-1}\pi(T'))$, we have $\gamma(m) = m$. Consequently, γ induces the identity collineation in ${}^{n-1}\mathcal{H}({}^1\pi(P))$.

Thus the number of lines in ${}^n\mathcal{L}$ fixed by γ is at least $qq^{2(n-1)} = q^{2n-1}$. Since γ fixes an equal number of lines and points, by Lemma 8, and since there are $q^{2(n-1)}$ points of ${}^n\mathcal{H}$ that neighbour P, some point R exists in ${}^{n}\mathcal{P}$, ${}^{1}\pi(R) \neq {}^{1}\pi(P)$, for which $\gamma(R) = R$. Since all occurring fixed points are near l, ${}^{1}\pi(R)\ {}^{1}I\ {}^{1}\pi(l)$.

Since all points of ${}^{n}H$ that neighbour P are fixed by γ, every line of ${}^{n}H(O)$ that is incident with R and neighbours l is fixed by γ. Using earlier arguments in the proof, it can be seen that γ induces the identity in ${}^{n-1}H({}^{1}\pi(l))$.

Under the assumption that all fixed points for γ are near l, and applying Lemma 8 again, there must be $(q+1)q^{2n-2}$ points near l that are fixed by γ. Since there are only $(q+1)q^{2n-2}$ points near l, γ is a ${}^{1}h$-collineation in ${}^{n}\Psi$ with axis l and center P. $\square$

Now we recall from Part I [11] (Lemma 18):

Lemma 24 *At least one quasi-elation γ in ${}^{n}\Psi$ exists with non-trivial $()^{\star_1}$-projection.*

Lemma 25 *Let k be some integer $1 \leq k \leq n-1$. If there is a collineation α in ${}^{n}\Psi$ fixing all points $(n-1)$-near some line $l \in {}^{n}\mathcal{L}$, with $(\alpha)^{\star_{n-1}}$ an elation with axis ${}^{n-1}\pi(l)$ and some center ${}^{n-1}\pi(P)$, $P\ {}^{n}I\ l$, and with $(\alpha)^{\star_k} = 1$,$(\alpha)^{\star_{k+1}} \neq 1$, then a non-trivial ${}^{1}h$-collineation exists in ${}^{n}\Psi$.*

Proof. The lemma is true for $k = n-1$ by Lemma 19 of Part I [11]. We proceed by induction as follows. Suppose the statement of the lemma is true for all k, $h \leq k \leq n-1$, with h such that $1 < h \leq n-1$. Then we prove the statement holds for $h-1$.

So suppose α is a collineation in ${}^{n}\Psi$ fixing all points that are $(n-1)$-near some line $l \in {}^{n}\mathcal{L}$, with $(\alpha)^{\star_{n-1}}$ an elation with axis ${}^{n-1}\pi(l)$ and some center ${}^{n-1}\pi(P)$, $P\ {}^{n}I\ l$, and for which $(\alpha)^{\star_{h-1}} = 1$ but $(\alpha)^{\star_h} \neq 1$. Suppose R is some point in ${}^{n}\mathcal{P}$, ${}^{1}\pi(R)\ {}^{1}\!\!\not{I}\ {}^{1}\pi(l)$. Then $\alpha(R)$ is some point S of ${}^{n}H$, with ${}^{h}\pi(R) \neq {}^{h}\pi(S)$, ${}^{h-1}\pi(R) = {}^{h-1}\pi(S)$. Any line incident with R and S intersects l in a unique point of ${}^{n}H$, a point which is fixed by α. Thus any line incident with R and S is fixed by α. Suppose $m \in {}^{n}\mathcal{L}$, is some line incident with R and S, and suppose m intersects l in some point Q of ${}^{n}H$. Using Property 3, for every point ${}^{2}\pi(V)$ of ${}^{2}H(O)$, $V \in {}^{n}\mathcal{P}$ and incident with l, ${}^{2}\pi(V) \neq {}^{2}\pi(Q)$, ${}^{1}\pi(V) = {}^{1}\pi(Q)$, a collineation β in ${}^{n}\Psi$ exists, fixing l and R, and mapping V to Q.
So $\beta^{-1}\alpha\beta$ is a collineation in ${}^{n}\Psi$ fixing all points that are $(n-1)$-near l, with $(\beta^{-1}\alpha\beta)^{\star_{n-1}}$ an elation with axis ${}^{n-1}\pi(l)$, and such that $(\beta^{-1}\alpha\beta)^{\star_{h-1}} = 1$. Since $(\alpha)^{\star_h} \neq 1$, one has $(\beta^{-1}\alpha\beta)^{\star_h} \neq 1$. Moreover, both ${}^{h}\pi(R)$ and ${}^{h}\pi(S)$ are incident with $\beta^{-1}\alpha\beta({}^{h}\pi(m))$, because β stabilizes the sets of points incident with m and $\beta(m)$, respectively, and S belongs to both m and $\beta(m)$ (since $\beta(R) = R$ and ${}^{1}\pi(R) = {}^{1}\pi(S)$).

There are only $q-1$ possible images for ${}^{h}\pi(R)$ incident with ${}^{h}\pi(m)$ by collineations of the form $\beta^{-1}\alpha\beta$, $(\beta^{-1}\alpha\beta)^{\star_{h-1}} = 1$, $(\beta^{-1}\alpha\beta)^{\star_h} \neq 1$. But $|\{{}^{2}\pi(V) \,|\, V \in {}^{n}\mathcal{P}, V\ {}^{n}I\, l,\ {}^{1}\pi(V) = {}^{1}\pi(Q)\}| = q$. Hence, we may assume that some points V' and V'' of ${}^{n}H$ exist with $V'\ {}^{n}I\ l\ {}^{n}I\ V''$, ${}^{2}\pi(V') \neq {}^{2}\pi(V'')$, ${}^{1}\pi(V') = {}^{1}\pi(Q) = {}^{1}\pi(V'')$, some collineation β' in ${}^{n}\Psi$ fixing l and R and mapping

V' to Q, and some collineation β'' in ${}^{n}\Psi$ fixing l and R such that $\beta''(V'') = Q$, such that $(\beta''^{-1}\alpha\beta'')({}^{h}\pi(R)) = (\beta'^{-1}\alpha\beta')({}^{h}\pi(R))$. Since $(\beta''^{-1}\alpha\beta'')^{\star_h}$ and $(\beta'^{-1}\alpha\beta')^{\star_h}$ are both elations in ${}^{h}\Psi$ with axis ${}^{h}\pi(l)$, and using Theorem 20 ($h \leq n-1$), $((\beta''^{-1}\alpha\beta'')^{-1}(\beta'^{-1}\alpha\beta'))^{\star_h}$ is again an elation in ${}^{h}\Psi$ with axis ${}^{h}\pi(l)$. Additionally, ${}^{h}\pi(R)$ is fixed for $(\beta''^{-1}\alpha\beta'')^{-1}(\beta'^{-1}\alpha\beta')$. Hence $((\beta''^{-1}\alpha\beta'')^{-1}(\beta'^{-1}\alpha\beta'))^{\star_h} = 1$.

Can we tell more about $\delta = (\beta''^{-1}\alpha\beta'')^{-1}(\beta'^{-1}\alpha\beta')$? From the previous paragraphs, it is already clear that δ fixes every point of ${}^{n}H$ that is $(n-1)$-near l, and that $(\delta)^{\star_h} = 1$. Since $(\beta'^{-1}\alpha\beta')^{\star_{n-1}}$ and $(\beta''^{-1}\alpha\beta'')^{\star_{n-1}}$ are both elations with axis ${}^{n-1}\pi(l)$, and since by Theorem 20 the set of elations with axis ${}^{n-1}\pi(l)$ forms a group, $(\delta)^{\star_{n-1}}$ is an elation with axis ${}^{n-1}\pi(l)$.

Suppose $\delta = 1$. Then δ also fixes the line w' of ${}^{n}H$ determined by R and V', and consequently $\beta''^{-1}\alpha\beta''(w') = w'$. Hence $\beta''^{-1}\alpha\beta''$ fixes two 1-neighbouring lines of ${}^{n}H$ that are incident with R: w' and the line w'' of ${}^{n}H$ defined by R and V''. Only the points of ${}^{1}H({}^{n-1}\pi(R))$ in ${}^{n}\mathcal{P}(O)$ are incident with both w' and w''. Thus $\beta''^{-1}\alpha\beta''({}^{n-1}\pi(R)) = {}^{n-1}\pi(R)$. However, $\beta''^{-1}\alpha\beta''$ is a collineation in ${}^{n}\Psi$ for which $(\beta''^{-1}\alpha\beta'')^{\star_{n-1}}$ is an elation with axis ${}^{n-1}\pi(l)$ such that $(\beta''^{-1}\alpha\beta'')^{\star_h} \neq 1$. Since $h \leq n-1$, a contradiction arises. We conclude that $\delta \neq 1$. Using the transitivity of ${}^{n}\Psi$ on the triangles of ${}^{n}H$, we can obtain a non-trivial collineation in ${}^{n}\Psi$ fixing all points $(n-1)$-near l, with $()^{\star_{n-1}}$-projection an elation with axis ${}^{n-1}\pi(l)$ and center ${}^{n-1}\pi(P)$, and with $()^{\star_h}$-projection trivial. Hence, by induction, a non-trivial ${}^{1}h$-collineation can be constructed. $\square$

Lemma 26 *The kernel of the $()^{\star_{n-1}}$-projection is not trivial. In fact, there exists a non-trivial element in ker$(()^{\star_{n-1}})$ fixing all points of ${}^{n}H$ that $(n-1)$-neighbour some point.*

Proof. Suppose ${}^{n-1}\pi(l)$, $l \in {}^{n}\mathcal{L}$, is some line of ${}^{n-1}H(O)$ and ${}^{n-1}\pi(P)$, $P \in {}^{n}\mathcal{P}$, some point of ${}^{n-1}H(O)$ incident with ${}^{n-1}\pi(l)$.

Using Lemma 24, at least one quasi-elation δ in ${}^{n}\Psi$ exists with $()^{\star_1}$-projection not trivial, such that $(\delta)^{\star_{n-1}}$ has some center ${}^{n-1}\pi(Q)\ {}^{n-1}I\ {}^{n-1}\pi(l)$, $Q \in {}^{n}\mathcal{P}$, ${}^{1}\pi(Q) \neq {}^{1}\pi(P)$, and some axis ${}^{n-1}\pi(u)\ {}^{n-1}I\ {}^{n-1}\pi(Q)$, $u \in {}^{n}\mathcal{L}$, ${}^{1}\pi(u) \neq {}^{1}\pi(l)$. Let m be one of the fixed lines in ${}^{n}\mathcal{L}$ for δ not neighbouring u (note that m exists since every quasi-elation has a quasi-axis). Notice that by Lemma 7 of Part I [11], m is $(n-1)$-near Q. Let R be some point of ${}^{n}H$ that is fixed by δ and for which ${}^{1}\pi(R) \neq {}^{1}\pi(Q)$. We can assume that $m\,{}^{n}I\,Q\,{}^{n}I\,u\,{}^{n}I\,R$.

Let us denote by Υ the group generated by all collineations δ' in ${}^{n}\Psi$ having the following properties:

(i) δ' fixes the points in ${}^{n-1}\mathcal{P}(O)$ that are incident with ${}^{n-1}\pi(u)$;

(ii) δ' fixes the lines in ${}^{n-1}\mathcal{L}(O)$ that are incident with ${}^{n-1}\pi(Q)$;

(iii) δ' fixes every point of ${}^{n}H$ that $(n-1)$-neighbours Q;

(iv) δ' fixes every line of ${}^{n}\!H$ that $(n-1)$-neighbours u;

(v) $\delta'(R) = R$.

Note that $\delta \in \Upsilon$.

Next we claim that $ker(()^{\star_1}{}_{\Upsilon}) \neq 1$. Property 3 allows collineations $\gamma_{Q'}$ in ${}^{n}\Psi$ fixing R, some point T of ${}^{n}\!H$ incident with m, ${}^{1}\pi(T) \neq {}^{1}\pi(Q)$, that map Q to any point Q' of ${}^{n}\!H$ incident with u, ${}^{n-1}\pi(Q) = {}^{n-1}\pi(Q')$. There are q possible choices for Q' incident with u, ${}^{n-1}\pi(Q) = {}^{n-1}\pi(Q')$, giving rise to q two by two different collineations $\gamma_{Q'}^{-1}\delta\gamma_{Q'}$. Indeed, suppose that Q'' and Q''' are different points of ${}^{n}\!H$ satisfying $Q''\ {}^{n}\!I\ u\ {}^{n}\!I\ Q'''$, ${}^{n-1}\pi(Q) = {}^{n-1}\pi(Q'') = {}^{n-1}\pi(Q''')$, such that $\gamma_{Q''}^{-1}\delta\gamma_{Q''} = \gamma_{Q'''}^{-1}\delta\gamma_{Q'''}$. Then two $(n-1)$-neighbouring fixed lines for $\gamma_{Q''}^{-1}\delta\gamma_{Q''}$ exist, namely $\gamma_{Q''}^{-1}(m)$ and $\gamma_{Q'''}^{-1}(m)$. Since $Q'' \neq Q'''$, some point ${}^{1}\pi(U) \in {}^{1}\mathcal{P}(O)$, $U \in {}^{n}\mathcal{P}(O)$, ${}^{1}\pi(U)\ {}^{1}\!\!\not{I}\ {}^{1}\pi(u)$, exists that is fixed by $\gamma_{Q''}^{-1}\delta\gamma_{Q''}$. This contradicts $(\gamma_{Q''}^{-1}\delta\gamma_{Q''})^{\star_1} \neq 1$. All collineations $\gamma_{Q'}^{-1}\delta\gamma_{Q'}$, $Q'\ {}^{n}\!I\ u$, ${}^{n-1}\pi(Q) = {}^{n-1}\pi(Q')$, have a non-trivial $()^{\star_1}$-projection, and are again elements of Υ. Since 1 belongs to any group, this implies that

$$|\Upsilon| > q.$$

On the other hand

$$|{}^{1}\Upsilon| \leq q.$$

Since

$$\frac{|\Upsilon|}{|ker(()^{\star_1}{}_{\Upsilon})|} = |{}^{1}\Upsilon|,$$

the claim follows.

Consequently, the existence of some non-trivial collineation $\beta \in \Upsilon$ with $(\beta)^{\star_1} = 1$ is guaranteed. We distinguish two cases.

Case 1: $(\beta)^{\star_{n-1}} = 1$.
Then since $\beta \neq 1$, the kernel of the $()^{\star_{n-1}}$-projection is not trivial.

Case 2: $(\beta)^{\star_{n-1}} \neq 1$.
Then $(\beta)^{\star_{n-1}}$ is a ${}^{k}h$-collineation (not ${}^{k-1}h$-collineation) in ${}^{n-1}\Psi$ for some k, $1 \leq k \leq n-2$, since $(\beta)^{\star_{n-1}}$ is an elation in ${}^{n-1}\Psi$ with axis ${}^{n-1}\pi(u)$ and center ${}^{n-1}\pi(Q)$, and since $(\beta)^{\star_1} = 1$ and $(\beta)^{\star_{n-1}} \neq 1$. Using Lemma 14 of Part I [11], all points of ${}^{n-1}\!H(O)$ that are k-near ${}^{n-1}\pi(u)$ are fixed by $(\beta)^{\star_{n-1}}$.

Since all elations of the Moufang projective Hjelmslev plane ${}^{n-1}\!H$ are in ${}^{n-1}\Psi$, we can consider a collineation α in ${}^{n}\Psi$ such that $(\alpha)^{\star_{n-1}}$ is an elation in ${}^{n-1}\Psi$ with center ${}^{n-1}\pi(P)$ and axis ${}^{n-1}\pi(l)$, and $(\alpha)^{\star_k} = 1$, $(\alpha)^{\star_{k+1}} \neq 1$.

Which properties does the collineation $[\alpha, \beta]$ have? It is clear that $([\alpha, \beta])^{\star_{n-1}}$ is an elation with axis ${}^{n-1}\pi(l)$ and center ${}^{n-1}\pi(Q)$. Moreover, $\beta(R) = R$ because $\beta \in \Upsilon$ (and see condition (v) above), and α maps R to some point S of ${}^{n}\!H$ (so ${}^{n-1}\!\pi(R)$ is mapped to ${}^{n-1}\!\pi(S)$) with ${}^{k}\!\pi(S) = {}^{k}\!\pi(R)$. Hence ${}^{n-1}\!\pi(S)$ is k-near ${}^{n-1}\!\pi(u)$ and is therefore fixed by β^{-1}. This implies that $[\alpha,\beta]({}^{n-1}\!\pi(R)) = {}^{n-1}\!\pi(R)$. Since $([\alpha,\beta])^{\star_{n-1}}$ is an elation with axis ${}^{n-1}\!\pi(l)$ and since ${}^{1}\!\pi(R) \not{\!{}^{1}\!I}\; {}^{1}\!\pi(l)$, we conclude $([\alpha,\beta])^{\star_{n-1}} = 1$.

Let us look at the image of R under $[\alpha,\beta]$. Applying that $(\alpha)^{\star_{n-1}}$ is an elation with axis ${}^{n-1}\!\pi(l)$ and center ${}^{n-1}\!\pi(P)$, $(\alpha)^{\star_k} = 1$ and $(\alpha)^{\star_{k+1}} \neq 1$, $S = \alpha(R)$ satisfies ${}^{k+1}\!\pi(S) \neq {}^{k+1}\!\pi(R)$, ${}^{k}\!\pi(S) = {}^{k}\!\pi(R)$. Suppose S is fixed by β. Then, using $\beta(R) = R$, any line w of ${}^{n}\!H$ that is incident with R and S is mapped by β to a line $\beta(w)$ of ${}^{n}\!H$ that is also incident with both R and S. Since ${}^{k+1}\!\pi(R) \neq {}^{k+1}\!\pi(S)$, ${}^{k}\!\pi(R) = {}^{k}\!\pi(S)$, w and $\beta(w)$ are $(n-k)$-neighbouring lines. Hence ${}^{n-k}\!\pi(w)$ is fixed by β. Since $(\beta)^{\star_{n-1}}$ is an elation in ${}^{n-1}\Psi$ with axis ${}^{n-1}\!\pi(u)$, since $n-k \leq n-1$ and using that w is not near Q, $(\beta)^{\star_{n-k}} = 1$, a contradiction. Therefore $[\alpha,\beta](R) \neq R$. In other words $[\alpha,\beta] \neq 1$.

Hence also in this case, we conclude that the kernel of the $()^{\star_{n-1}}$-projection is not trivial. $\square$

Lemma 27 *Suppose U is some point of ${}^{n}\!H$. Then $ker(()^{\star_{n-1}})$ induces all translations in ${}^{1}\!H({}^{n-1}\!\pi(U))$.*

Proof. By Lemma 26, there exists a non-trivial element in $ker(()^{\star_{n-1}})$, say δ, fixing all points of ${}^{n}\!H$ that $(n-1)$-neighbour some point Q of ${}^{n}\!H$.

Consider an arbitrary point $T \in {}^{n}\mathcal{P}$, T not neighbouring Q. Then we claim that δ cannot induce a non-trivial homology in ${}^{1}\!H({}^{n-1}\!\pi(T))$. Indeed, suppose δ induces a non-trivial homology in ${}^{1}\!H({}^{n-1}\!\pi(T))$. Then some point T' of ${}^{n}\!H$, ${}^{n-1}\!\pi(T) = {}^{n-1}\!\pi(T')$ exists such that $\delta(T') = T'$. Hence the line u determined by T' and any point $Q' \in {}^{n}\mathcal{P}$ that $(n-1)$-neighbours Q is fixed by δ. Since $(\delta)^{\star_{n-1}} = 1$, and since for all points ${}^{1}\!\pi(V)$ of ${}^{1}\!H$, $V \in {}^{n}\mathcal{P}$, ${}^{1}\!\pi(V)\ {}^{1}\!I\ {}^{1}\!\pi(u)$, ${}^{n-1}\!\pi(u) \cap {}^{n-2}\!H({}^{1}\!\pi(V))$ corresponds with lines at infinity of ${}^{1}\!H({}^{n-1}\!\pi(u))$, δ induces a collineation in ${}^{1}\!H({}^{n-1}\!\pi(u))$ with center at infinity. So δ induces in ${}^{1}\!H({}^{n-1}\!\pi(u))$ a collineation with an affine center and at the same time a center at infinity. Hence $\delta_{{}^{1}\!H({}^{n-1}\pi(u))} = 1$. As a consequence, δ induces an elation in ${}^{1}\!H({}^{n-1}\!\pi(T))$ with axis at infinity. However, additionally $\delta(T') = T'$. Hence $\delta_{{}^{1}\!H({}^{n-1}\pi(T))} = 1$.

Since $\delta \neq 1$, there consequently exists some point $U \in {}^{n}\mathcal{P}$ such that δ induces a non-trivial elation in ${}^{1}\!H({}^{n-1}\!\pi(U))$. Using Property 3, every point at infinity occurs as a center of some non-trivial translation of ${}^{1}\!H({}^{n-1}\!\pi(U))$. So $ker(()^{\star_{n-1}})$ induces at least $(q+1)(p^h-1)+1$ translations in ${}^{1}\!H({}^{n-1}\!\pi(U))$, with p^h the number of translations induced in ${}^{1}\!H({}^{n-1}\!\pi(U))$ for some fixed center at infinity. Applying $p^h > 1$ and Theorem 15, it follows that $p^h = q$. $\square$

Lemma 28 *A subgroup Υ of ${}^{n}\Psi$ exists every element of which fixes all lines of ${}^{n}\!H$ that $(n-1)$-neighbour some line $l \in {}^{n}\mathcal{L}$, all points of ${}^{n-1}\!H$ that are $(n-2)$-near ${}^{n-1}\!\pi(l)$, some line ${}^{n-1}\!\pi(m)$ of ${}^{n-1}\!H$ $(m \in {}^{n}\mathcal{L})$ that is incident with ${}^{n-1}\!\pi(P)$, ${}^{1}\!\pi(m) \neq {}^{1}\!\pi(l)$, $P\ {}^{n}\!I\ l$, such that Υ acts transitively on the points of ${}^{1}\!H({}^{n-1}\!\pi(P))$ in ${}^{n}\mathcal{P}$ that are incident with l.*

Proof. Let Σ be an apartment of Δ containing l, P, ${}^{n-1}\pi(m)$ and O. By v we denote the unique vertex in Σ at distance n from ${}^{1}\pi(l)$, corresponding with a line of ${}^{n}H({}^{1}\pi(l))$, and such that ${}^{1}\pi(P) \in cl(v, {}^{1}\pi(l))$. Then ${}^{n-1}\pi(m)$ is the vertex in Σ at distance $n-1$ from O and adjacent to both v and ${}^{n-1}\pi(v)$, where ${}^{n-1}\pi(v)$ is the unique vertex in $cl(v, {}^{1}\pi(l))$ at distance $n-1$ from ${}^{1}\pi(l)$.

Let us denote the unique vertex in $cl(l, {}^{n-1}\pi(P))$, corresponding with a point of ${}^{n-1}H({}^{1}\pi(l))$ as U. Then clearly $\alpha(U) = U$, for every potential element of Υ (if Υ exists). From this consideration, it is clear that we are done, whenever we can prove the existence of a subgroup of ${}^{n}\Psi({}^{1}\pi(l))$, consisting of collineations having a trivial action in ${}^{n-1}H({}^{1}\pi(l))$, that additionally acts transitively on the lines of ${}^{n}H({}^{1}\pi(l))$ that are incident with some chosen point X of ${}^{n}H({}^{1}\pi(l))$, and $(n-1)$-neighbour (with respect to the base-vertex ${}^{1}\pi(l)$) some chosen line of ${}^{n}H({}^{1}\pi(l))$, with X the point of ${}^{n}H({}^{1}\pi(l))$ corresponding with a vertex of Σ which has as canonical image in ${}^{1}H({}^{1}\pi(l))$ the point corresponding with the vertex O.

Shifting the problem to ${}^{n}\Psi(O)$, we need to prove the existence of a subgroup of ${}^{n}\Psi(O)$, consisting of collineations having a trivial $()^{\star_{n-1}}$-projection, acting transitively on the lines of ${}^{n}H(O)$ that are incident with some prechosen point of ${}^{n}H(O)$, and that $(n-1)$-neighbour some prechosen line of ${}^{n}H(O)$.

Dually, it suffices to prove the existence of a subgroup of $ker(()^{\star_{n-1}})$ inducing in ${}^{1}H({}^{n-1}\pi(R))$, R some point in ${}^{n}\mathcal{P}(O)$, a group of translations acting transitively on the points of ${}^{n}H(O)$ that $(n-1)$-neighbour R and that are incident with some chosen line $r \in {}^{n}\mathcal{L}(O)$, ${}^{n-1}\pi(R)$ ${}^{n-1}\mathrm{I}$ ${}^{n-1}\pi(r)$.

The existence of such a subgroup is guaranteed by Lemma 27. $\square$

Lemma 29 *At least one non-trivial ${}^{1}h$-collineation exists in ${}^{n}\Psi$.*

Proof. Using Lemma 24, at least one quasi-elation α in ${}^{n}\Psi$ exists with $()^{\star_1}$-projection not trivial. Suppose the induced elation $(\alpha)^{\star_{n-1}}$ in ${}^{n-1}H(O)$ has some axis ${}^{n-1}\pi(l)$, $l \in {}^{n}\mathcal{L}$, and some center ${}^{n-1}\pi(P)$, $P \,{}^{n}\mathrm{I}\, l$. Let $m \,{}^{n}\mathrm{I}\, P$ be one of the fixed lines for α in ${}^{n}\mathcal{L}$ not neighbouring l (m exists since α has a quasi-center).

Let Υ refer to the subgroup of ${}^{n}\Psi$ generated by all collineations β in ${}^{n}\Psi$ such that the fixed points of ${}^{n}H(O)$ for α that are $(n-1)$-near l are also fixed points for β, such that the lines of ${}^{n}H$ that $(n-1)$-neighbour l are fixed by β, and such that $(\beta)^{\star_{n-1}}$ is an elation with axis ${}^{n-1}\pi(l)$ and center ${}^{n-1}\pi(P)$. Note that $\alpha \in \Upsilon$.

By Lemma 28, a collineation γ in ${}^{n}\Psi$ exists, fixing all lines in ${}^{n}\mathcal{L}$ that $(n-1)$-neighbour l, fixing ${}^{n-1}\pi(m)$ and all points in ${}^{n-1}\mathcal{P}$ that are $(n-2)$-near ${}^{n-1}\pi(l)$, mapping P to some arbitrary point P' of ${}^{n}H$ different from P, $P' \,{}^{n}\mathrm{I}\, l$, ${}^{n-1}\pi(P') = {}^{n-1}\pi(P)$. It is clear that $[\alpha, \gamma] \in \Upsilon$. Indeed, clearly $([\alpha, \gamma])^{\star_{n-1}}$ is an elation with axis ${}^{n-1}\pi(l)$ and center ${}^{n-1}\pi(P)$, and γ stabilizes the set of all points $(n-1)$-near l fixed by α since α fixes all lines $(n-1)$-neighbouring l by Lemma 11 of Part I [11].

Suppose $[\alpha, \gamma] = 1$. Then $[\alpha, \gamma](\gamma^{-1}(m)) = \gamma^{-1}(m)$. Hence α fixes both m and $\gamma^{-1}(m)$. Since $m \cap \gamma^{-1}(m) \cap l = \emptyset$ and since m and $\gamma^{-1}(m)$ are $(n-1)$-neighbouring lines of ${}^{n}\!H$, some point ${}^{1}\pi(R)$ not incident with ${}^{1}\pi(l)$, $R \in {}^{n}\mathcal{P}$, exists such that $\alpha({}^{1}\pi(R)) = {}^{1}\pi(R)$. Since $(\alpha)^{\star_1}$ is a non-trivial elation with axis ${}^{1}\pi(l)$, a contradiction arises. Hence $[\alpha, \gamma] \neq 1$.

Since both α and γ induce in ${}^{1}\!H({}^{n-1}\pi(T))$, for all points T of ${}^{n}\!H$ incident with l, an elation with the same center at infinity (Lemma 11 of Part I [11] and Lemma 28), and as a consequence of Theorem 4.14 in Hughes & Piper [5], $[\alpha, \gamma]$ fixes every point of ${}^{n}\!H$ that $(n-1)$-neighbours l. We conclude that the non-trivial collineation $\delta = [\alpha, \gamma]$ fixes all points of ${}^{n}\!H$ that are $(n-1)$-near l, that $(\delta)^{\star_1} = 1$, and that $(\delta)^{\star_{n-1}}$ is an elation with axis ${}^{n-1}\pi(l)$ and center ${}^{n-1}\pi(P)$.

Applying Lemma 25 to δ, a non-trivial ${}^{1}h$-collineation can be constructed. $\square$

By Proposition 2, we now have that ${}^{n}\!H$ is a Moufang Hjelmslev plane of level n, and that all elations belong to ${}^{n}\Psi$. As in Theorem 19, we conclude that ${}^{n}\!H$ is desarguesian. This completes the proof of Theorem I.

5 Proof of the Main Result

By Theorem I, all projective Hjelmslev planes ${}^{i}\!H(O)$, $i \geq 1$, are desarguesian. The assertion follows from Theorem 12 of Van Maldeghem [10] and Section 14 of Tits [7].

Alternatively, we can argue as follows. Suppose l^∞ is some line of Δ^∞ and let P^∞ and Q^∞ be two different points of Δ^∞ not incident with l^∞. Then a vertex O in Δ exists such that for every $k \geq 1$, P^∞ and Q^∞ (represented as rays starting in O) determine non-neighbouring points of ${}^{k}\!H(O)$, which are not near the line of ${}^{k}\!H(O)$ determined by l^∞ (represented as a ray starting in O). Since ${}^{k}\!H(O)$ is a Moufang projective Hjelmslev plane (Theorem I) for which the 'base-vertex' O was chosen arbitrarily in Δ, it follows that an elation acting on Δ^∞ exists with axis l^∞, mapping P^∞ to Q^∞, that is the inverse limit of elations acting on projective Hjelmslev planes with base-vertex O. Hence Δ^∞ satisfies the Moufang condition. By Van Maldeghem [8], Δ^∞ is desarguesian. $\square$

References

[1] Baker C., Lane N. D. and Lorimer J. W., The Artin-Zorn theorem for finite punctually cohesive projective Klingenberg Planes, *Ars Combin.* **29** (1990), 143 – 149.

[2] Baker C., Lane N. D. and Lorimer J. W., Local alternative rings and finite alternative right chain rings, *C. R. Math. Rep. Acad. Sci. Canada* **12** (1990).

[3] Baker C., Lane N. D. and Lorimer J. W., An affine characterization of Moufang projective Klingenberg planes, *Res. in Math.* **17** (1990).

[4] Bruhat F. and Tits J., Groupes réductifs sur un corps local. I. Données radicielles valuées, *Inst. Hautes Études Sci. Publ. Math.* **41** (1972), 5 – 252.

[5] Hughes D. R. and Piper F. C., *Projective planes*, Springer, Berlin-New York (1973).

[6] Huppert B., *Endliche Gruppen I*, Springer-Verlag Berlin Heidelberg New York 1967.

[7] Tits J., Immeubles de type affine, **in** *Buildings and the Geometry of Diagrams*, *Springer Lecture Notes in Mathematics* **1181** (L. Rosati, ed.), Springer, Berlin (1986), 159 – 190.

[8] Van Maldeghem H., On locally finite alternative division rings with valuation, *J. Geometry* **30** (1987), 42 – 48.

[9] Van Maldeghem H., Valuations on PTR's induced by triangle buildings, *Geom. Dedicata* **26** (1988), 29 – 84.

[10] Van Maldeghem H., Automorphisms of non-classical triangle buildings, *Tijdsch. Belg. Wisk. Gen.* **42** (1990), 201 – 237.

[11] Van Maldeghem H. and Van Steen K., Characterizations by automorphisms of some rank 3 buildings, I. Some properties of half strongly-transitive triangle buildings, *preprint*.

[12] Van Steen K., *Non-spherical Tits-buildings of rank* 3, Ph.D.-thesis, University of Ghent, 1996.

Address of the authors:
Hendrik Van Maldeghem and Kristel Van Steen,
Department of Pure Mathematics and Computer Algebra
University of Ghent,
Galglaan 2,
9000 Gent
BELGIUM
email: hvm@@cage.rug.ac.be and kvs@@cage.rug.ac.be