Generalized quadrangles weakly embedded of degree > 2 in projective space

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Abstract

In this paper, we classify all generalized quadrangles weakly embedded in projective space of degree > 2. More exactly, given a (possibly infinite) generalized quadrangle $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ and a map π from $\mathcal{P}$ (respectively $\mathcal{L}$) to the set of points (respectively lines) of a projective space $\mathbf{PG}(d, \mathbb{K})$, $\mathbb{K}$ some skewfield, $d \geq 2$ (but not necessarily finite), such that

(i) π is injective on points,

(ii) if $x \in \mathcal{P}$ and $L \in \mathcal{L}$ with $x \, \mathbf{I} \, L$, then x^π is incident with L^π in $\mathbf{PG}(d, \mathbb{K})$,

(iii) the set of points $\{x^\pi \mid x \in \mathcal{P}\}$ generates $\mathbf{PG}(d, \mathbb{K})$,

(iv) if $x, y \in \mathcal{P}$ such that y^π is contained in the subspace of $\mathbf{PG}(d, \mathbb{K})$ generated by the set $\{z^\pi \mid z$ is collinear with x in $\Gamma\}$, then y is collinear with x in Γ,

(v) there exists a line of $\mathbf{PG}(d, \mathbb{K})$ not in the image of π and which meets Γ in at least 3 points,

then we show that Γ is a Moufang quadrangle and we can explicitly describe the weak embedding of Γ in $\mathbf{PG}(d, \mathbb{K})$.

1 Introduction

Weakly embedded polar spaces were introduced by Lefèvre-Percsy, see e.g. [4] (although she had a stronger notion of weak embedding, but it was proved to be equivalent with the present one by Thas & Van Maldeghem [11]). In the same paper, she proves that the number of points of a weakly embedded polar space Γ on a secant line (i.e., a

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line of the projective space not belonging to the polar space and meeting Γ in at least two points) is a constant (and hence does not depend on that line). Following THAS & VAN MALDEGHEM [10], we call this constant the *degree* of the weak embedding. In [3], LEFÈVRE-PERCSY classifies the finite weakly embedded generalized quadrangles (which are the non-degenerate polar spaces of rank 2) in $\mathbf{PG}(3,q)$. All those weak embeddings have automatically degree > 2. In THAS & VAN MALDEGHEM [11], all weakly embedded quadrangles in finite projective space are classified. In the present paper, we extend the results of these papers to the infinite case, on the condition that the weak embedding has degree > 2. Notice that, when the weak embedding is a full embedding, i.e., every point in $\mathbf{PG}(d,\mathbb{K})$ of every line of the quadrangle is also a point of the quadrangle, then the embedding is one of the known ones by DIENST [2] (the result is that only the classical Moufang quadrangles turn up with their natural embedding in a (possibly degenerate) polarity, see TITS [15]). Hence, our Main Result is also a partial generalization of Dienst's result. There is yet no hope of further generalization to degree 2 by the methods we use in this paper.

Note that results of STEINBACH [7] and THAS & VAN MALDEGHEM [10] treat the same kind of question for polar spaces with some additional conditions. In all cases, the assumptions imply that the polar space is classical. In the present paper, we hypothesize an arbitrary quadrangle and prove that it must belong to the class of so-called Moufang quadrangles. Then we have to treat several classes (amongst them the classical cases). An alternative approach not using the classification of Moufang quadrangles is developed in STEINBACH [8], though only a partial answer is given there.

So the eventual determination of all weakly embedded quadrangles of degree > 2 requires some knowledge about the classification of Moufang quadrangles. We will introduce notation and repeat some known results in the next section.

2 Definitions and Notation

A *generalized quadrangle* $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a point-line incidence geometry (where $\mathcal{P}$ is the set of points and $\mathcal{L}$ the set of lines) satisfying the following two axioms:

(i) Each point is incident with $t+1$ lines; each line is incident with $s+1$ points; two distinct points are never incident with two distinct lines (here $s, t \geq 1$, possibly infinite).

(ii) If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{L}$ for which $x \,\mathbf{I}\, M \,\mathbf{I}\, y \,\mathbf{I}\, L$.

The pair (s,t) is usually called the *order* of Γ. If $s, t > 1$, then the quadrangle is said to be *thick*. Furthermore, we use standard terminology such as *collinear points, concurrent* *lines,* etc. Also, there is a *duality* for generalized quadrangles: every statement has a dual, i.e., if one interchanges the names point and line (and the numbers s and t), then a (usually new) statement is obtained. The dual of Γ is denoted by Γ^D. Further, the line M (respectively the point y) of (ii) is called the *projection* of L onto x (respectively of x onto L).

Generalized quadrangles were introduced by TITS in [12]. For more information, we refer to the monograph of PAYNE & THAS [6], to THAS [9], or VAN MALDEGHEM [19] (in the latter also the infinite case is covered).

There is no hope of classifying all generalized quadrangles (the situation is more or less the same as for projective planes), as there are (many variations of) free constructions of such geometries, see e.g. TITS [14]. Nevertheless, if one imposes some extra conditions, then classification is possible. Two such conditions are related to our Main Result, namely, the Moufang condition, and the condition of being weakly embedded in a projective space.

2.1 Moufang quadrangles

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a thick generalized quadrangle. We denote by $\Gamma(a)$ the set of elements of Γ incident with the element a (point or line). A *point-elation* is an automorphism of Γ fixing the set $\Gamma(x) \cup \Gamma(y) \cup \Gamma(L)$ elementwise, where x, y, $x \neq y$, are two collinear points incident with, say, the line L. Such a collineation is also called an (x, L, y)-elation. If for some line $M \mathbf{I} x$, $M \neq L$, the group of all (x, L, y)-elations acts transitively on $\Gamma(M) \setminus \{x\}$, then we say that (x, L, y) is a *Moufang path.* Dually, one defines *line-elations* and *Moufang paths* (L, x, M). Let $x, y \in \mathcal{P}$, $L, M \in \mathcal{L}$. If the paths (x, L, y), for all choices of $x \mathbf{I} L \mathbf{I} y$, $x \neq y$ (respectively the paths (L, x, M) for all choices of $L \mathbf{I} x \mathbf{I} M$, $L \neq M$) are Moufang paths, then we say that Γ is a *half-Moufang quadrangle* and that all point-elation groups (respectively line-elation groups) *act transitively.* If all paths (x, L, y) and all paths (L, x, M) are Moufang paths, then we say that Γ is a *Moufang quadrangle.*

For fixed x, y, L as above, the group of all (x, L, y)-elations of a Moufang quadrangle is also called a *root group.* Let $x \mathbf{I} L \mathbf{I} y \mathbf{I} M \mathbf{I} z \mathbf{I} N$, with $x \neq y \neq z$, $L \neq M \neq N$. Let U_1 respectively U_2, U_3, U_4 be the group of all (x, L, y)-elations, respectively (L, y, M)-elations, (y, M, z)-elations, (M, z, N)-elations in a Moufang quadrangle Γ. By TITS [17], the following situations can occur, up to duality.

- (i) $[U_1, U_3] = [U_2, U_4] = \{1\}$. Then we call the corresponding quadrangle a *mixed quadrangle.*
- (ii) $[U_1, U_3] = \{1\}$ and $[U_2, U_4] = U_3$. Then we say that the quadrangle is *strictly of type* C_2.

(*iii*) $[U_2, U_4] = V_3 \subset U_3$, with $\{1\} \neq V_3 \neq U_3$. These quadrangles are called *Moufang quadrangles of type BC_2.*

It also follows from Tits [17] that the Moufang quadrangles Γ of type BC_2 have subquadrangles strictly of type C_2 such that the groups U_2 respectively U_4 in these quadrangles coincide with each other (with the above notation), and such that the group U_3 of the subquadrangle is equal to the group V_3 of Γ. We say that Γ extends that subquadrangle Γ'. Note that in that case Γ' is an *ideal subquadrangle* of Γ, i.e., all lines of Γ incident with a point of Γ' belong to Γ' as well. Dually, one defines a *full subquadrangle.*

The standard examples of Moufang quadrangles are the *classical quadrangles*, i.e. generalized quadrangles corresponding with (σ, ϵ)-hermitian or pseudo-quadratic forms (both called σ-quadratic forms in Tits [17]), see Tits [15, § 8]. When $\sigma \neq 1$, then we will call such a quadrangle a *hermitian quadrangle*; when $\sigma = 1$, then we have an *orthogonal quadrangle.* The duality class is fixed by requiring that the points of the quadrangle correspond with the 1-dimensional singular subspaces of the corresponding form.

When Γ is an orthogonal quadrangle or a hermitian quadrangle, we may assume that Γ is associated to a (left) vector space W over some skewfield $\mathbb{L}$ and to one of the following forms:
(a) a pseudo-quadratic form q on W,
(b) a (σ, ϵ)-hermitian form f on W with $\Lambda_{min} := \{c - \epsilon c^\sigma \mid c \in \mathbb{L}\} = \{c \in \mathbb{L} \mid \epsilon c^\sigma = -c\} =: \Lambda_{max}$.

The assumption in (b) on $\mathbb{L}, \sigma, \epsilon$ is harmless; in the case where it is not satisfied (which may only happen when $\mathbb{L}$ is a non-perfect field of characteristic 2 or a non-commutative skewfield of characteristic 2) we pass to an isomorphic quadrangle associated to a pseudo-quadratic form, see Cohen [1, (3.23), (3.27)]. For example, from the symplectic quadrangle in characteristic 2 ($\dim W = 4$, f an alternating form) we pass to an isomorphic quadrangle associated to an ordinary quadratic form on a vector space of dimension $4 + \dim_{\mathbb{L}^2}\mathbb{L}$.

The mixed quadrangles are certain subquadrangles of orthogonal quadrangles defined over a (non-perfect) field of characteristic 2, see (6.1.1). In fact, orthogonal quadrangles are either strictly of type C_2, or isomorphic to mixed quadrangles. Hermitian quadrangles in vector spaces of dimension 4 are strictly of type C_2, the other hermitian quadrangles are Moufang quadrangles of type BC_2 extending hermitian quadrangles in vector spaces of dimension 4. Finally, *exceptional quadrangles* are Moufang quadrangles (not related to σ-quadratic forms) of type BC_2 extending orthogonal quadrangles which are not mixed ones. For all these properties, we refer to Tits [17].

An orthogonal quadrangle in a vector space of dimension $2n$ will be called a *D_n-quadrangle.* Over the quadratic closure of the base field, it is part of a building of type D_n. Note that by Tits [17], the exceptional Moufang quadrangles of type E_6, E_7, E_8 contain ideal D_n-quadrangles (and that fixes their duality class for the rest of the paper) for $n = 5, 6, 8$, respectively of type E_6, E_7, E_8. Finally, the exceptional Moufang quadrangles of type F_4 contain as full and as ideal subquadrangles orthogonal quadrangles with the property that the “anisotropic part” of the (σ, ϵ)-hermitian form f is degenerate, see for instance Van Maldeghem [19](5.5.5), or Tits & Weiss [18].

Let Γ be a generalized quadrangle and p a point in Γ. If a collineation fixes every point collinear with p, then we call that collineation a *central collineation* or a *central elation.* Dually, one defines an *axial elation* or *axial collineation.* Every Moufang quadrangle contains, up to duality, non-trivial central elations. This can easily be deduced from the main result of Tits [16].

2.2 Weak embedding of quadrangles

Let $\mathbf{PG}(d, \mathbb{K})$ be some d-dimensional projective space $d \geq 2$ (but not necessarily finite), $\mathbb{K}$ any skewfield. Let Γ be a generalized quadrangle with point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathbf{I}$. Then we say that Γ *is weakly embedded in* $\mathbf{PG}(d, \mathbb{K})$ if there exists a map π from $\mathcal{P}$ (respectively $\mathcal{L}$) to the set of points (respectively lines) of $\mathbf{PG}(d, \mathbb{K})$, such that the following conditions are satisfied:

(*i*) π is injective on points,

(*ii*) if $x \in \mathcal{P}$ and $L \in \mathcal{L}$ with $x \,\mathbf{I}\, L$, then x^π is incident with L^π in $\mathbf{PG}(d, \mathbb{K})$,

(*iii*) the set of points $\{x^\pi \mid x \in \mathcal{P}\}$ generates $\mathbf{PG}(d, \mathbb{K})$,

(*iv*) if $x, y \in \mathcal{P}$ such that y^π is contained in the subspace of $\mathbf{PG}(d, \mathbb{K})$ generated by the set $\{z^\pi \mid z \text{ is collinear with } x \text{ in } \Gamma\}$, then y is collinear with x in Γ.

The map π is called the *weak embedding.* It will sometimes be convenient to see a weak embedding as an injective morphism from the point-line geometry Γ to the geometry of 1- and 2-dimensional subspaces of a vector space (and to write $\pi(x)$ instead of x^π for a point x). Also, for a given weak embedding π, we will denote by Γ^π the quadrangle whose points and lines are the images under π of the points and lines of Γ. The quadrangle Γ^π is a subgeometry of $\mathbf{PG}(d, \mathbb{K})$.

Let π be a weak embedding of Γ. A line of $\mathbf{PG}(d, \mathbb{K})$ which intersects the set of points of Γ^π in at least two elements, and which is not a line of Γ^π, is called a *secant line.* It has been shown by Lefèvre-Percsy [4] that the number of points of Γ^π on a secant line is a constant, and we call that constant the *degree.* In this paper, we will mainly be concerned with weakly embedded quadrangles of degree > 2.

A *full* embedding π of a generalized quadrangle Γ in $\mathbf{PG}(d,\mathbb{K})$ is a weak embedding such that all points of $\mathbf{PG}(d,\mathbb{K})$ on a line of Γ^π are also points of Γ^π.

3 Main Result

In order to state our Main Result, we need a couple of definitions.

Let $\mathbb{L}'$ be a quaternion skewfield, and let $\mathbb{L}$ be its center. Let σ be the standard (anti-)-involution in $\mathbb{L}'$. The set of points of $\mathbf{PG}(3,\mathbb{L}')$ whose coordinates (X_0, X_1, X_2, X_3) satisfy the relation

$$X_0^\sigma X_1 - X_1^\sigma X_0 + X_2^\sigma X_3 - X_3^\sigma X_2 = 0,$$

together with all lines of $\mathbf{PG}(3,\mathbb{L}')$ contained in that set, constitutes a hermitian quadrangle Γ, which we call the *quaternion quadrangle* over $\mathbb{L}$. The points of a line are parametrized by the skewfield $\mathbb{L}' \cup \{\infty\}$. We may write $\mathbb{L}' = \mathbb{L} + \mathbb{L}x + \mathbb{L}y + \mathbb{L}xy$ in a standard way. Then there is an ideal subquadrangle for which the points on some line are parametrized by $\mathbb{L}+\mathbb{L}x+\mathbb{L}y\cup\{\infty\}$. We call such a subquadrangle a *special subquadrangle* of Γ and the corresponding weak embedding a *standard weak embedding.*

We now state our Main Result.

Main Result. *Let Γ be a thick generalized quadrangle weakly embedded of degree > 2 in the projective space $\mathbf{PG}(d,\mathbb{K})$, $d \geq 2$ (but not necessarily finite), $\mathbb{K}$ some skewfield. Then Γ is a Moufang quadrangle. Up to isomorphism, Γ is one of the following:*

(1) Γ *is an orthogonal quadrangle or a hermitian quadrangle as in* Subsection 2.1 *and the weak embedding is induced by a semi-linear mapping (see below).*

(2) Γ *is a quaternion quadrangle and the composition of some automorphism of* Γ *and the weak embedding is induced by a semi-linear mapping (see* (5.4.2)*).*

(3) Γ *is a mixed quadrangle and an explicit description of the weak embedding can be given (see* Section 6*).*

(4) Γ *is a special subquadrangle of some quaternion quadrangle and the weak embedding is a standard weak embedding of* Γ *in a subspace* $\mathbf{PG}(3,\mathbb{D})$*, where* $\mathbb{D}$ *is a quaternion subskewfield inside* $\mathbb{K}$.

In particular this means that Γ can never be the dual of a hermitian quadrangle, nor can Γ be isomorphic or dual to an exceptional quadrangle.

In the case where Γ is an orthogonal quadrangle or a hermitian quadrangle weakly embedded (of degree > 2) in $\mathbf{PG}(d,\mathbb{K})$, let Γ be associated to a (left) vector space W over some skewfield $\mathbb{L}$ and to a pseudo-quadratic form q on W or to a (σ, ϵ)-hermitian form f on W such that $\Lambda_{min} = \Lambda_{max}$. Further, let V be a (left) vector space over K such that $\mathbf{PG}(d, \mathbb{K}) \simeq \mathbf{PG}(V)$. Apart from the exceptions mentioned in the case (2) of the Main Result, there exists an embedding $\alpha : \mathbb{L} \to \mathbb{K}$ and a (with respect to α) semi-linear mapping $\varphi : W \to V$ such that $\pi(\mathbb{L}w) = \mathbb{K}\varphi(w)$ for all points $\mathbb{L}w$ of Γ (which means that the weak embedding $\pi : \Gamma \to \mathbf{PG}(V)$ is induced by a semi-linear mapping). In particular, Γ is fully embedded in the projective space $\mathbf{PG}(\varphi(W))$, where $\varphi(W)$ is a vector space over the subskewfield $\mathbb{L}^\alpha$ of $\mathbb{K}$.

In the case where Γ is a dual orthogonal quadrangle (but not a mixed quadrangle) weakly embedded of degree > 2 in $\mathbf{PG}(d, \mathbb{K})$, let Γ^D have a standard embedding in a d'-dimensional projective space. Then $d = 3$ and we have the following four possibilities: $d' = 4$ and Γ is a symplectic quadrangle, $d' = 5$ and Γ is a hermitian quadrangle, $d' = 7$ and Γ is a quaternion quadrangle and, finally, $d' = 6$ and Γ is a special subquadrangle of some quaternion quadrangle.

We emphasize the fact that the Main Result is a gluing together of several independent results which are usually stronger than stated above; for instance once we reduced the general case to the Moufang case, we must treat classical quadrangles, but at the same time, we handle classical polar spaces and degree 2.

The paper is organized as follows. In the next section we reduce the problem to Moufang quadrangles. In Section 5 we classify the weakly embedded orthogonal and hermitian polar spaces. We remark that also the degree 2 weak embeddings are included here, and that we consider the more general case of polar spaces, since this does not make the proof more difficult or longer. In fact, it suffices to generalize the results of STEINBACH [7] and that is also the way we will state the result. The general idea is to prove here that the weak embedding is full over some subfield, and indeed this is always true except if the polar space is the unique generalized quadrangle of order $(2, 2)$. For this exceptional weak embedding, we refer to THAS & VAN MALDEGHEM [11]. Section 6 deals with the mixed quadrangles. Again the more general case of degree ≥ 2 is treated. Next, in Section 7, we classify all weak embeddings of degree > 2 of dual hermitian and dual orthogonal Moufang quadrangles, where the special subquadrangle of a quaternion quadrangle turns up. Finally, Section 8 takes care of the exceptional Moufang quadrangles.

We remark that the dual orthogonal and exceptional case could also be treated for degree 2, since every weakly embedded quadrangle of degree 2 has *regular* lines (a line L is called regular if for every line M not meeting L, the two lines L and M are contained in a full subquadrangle with two lines per point). Hence dual orthogonal weakly embedded quadrangles of degree 2 are mixed quadrangles. Also, the exceptional Moufang quadrangles do not have regular lines nor regular points (as one can deduce from the commutation relations of these quadrangles). But we consider these non-existing theorems as minor remarks (since yet, we cannot reduce the classification of weakly embedded quadrangles of degree 2 to the Moufang ones), and hence we do not insist on these results.

4 Reduction to Moufang quadrangles

The next lemma is for $d = 3$ contained in Van Maldeghem [19].

4.0.1 Lemma. *Let Γ be a generalized quadrangle weakly embedded of degree $\delta > 2$ in $\mathbf{PG}(d, \mathbb{K})$, for some skewfield $\mathbb{K}$. Then Γ is a half-Moufang quadrangle. More precisely, all point-elation groups act transitively.*

PROOF. To simplify notation, we identify Γ with Γ^π in this proof.

Let p be any point of Γ and let x be any point of Γ opposite p. Then it follows directly from Lefèvre-Percsy [4] that the group of central collineations of Γ with center p acts transitively on the set of points of Γ on the line px, except for p. Moreover, every such central collineation is induced by a perspectivity of $\mathbf{PG}(d, \mathbb{K})$. Now let L_1 and L_2 be two distinct lines of Γ incident with p and pick a point y of Γ on L_1, $y \neq p$, and pick two points z, z' of Γ on L_2, $z \neq p \neq z'$. We establish a (p, L_1, y)-elation mapping z to z'. Let L_3 be any line of Γ incident with y, $L_3 \neq L_1$. Let a be any point of Γ on the secant yz, $y \neq a \neq z$. Also, let z^* be the projection in Γ of z onto L_3. Since y, z belong to the hyperplane spanned by the points of Γ collinear in Γ with z^*, we have $a \perp z^*$. Let z^{**} be the projection of z' onto az^*, and let y' be the projection of z^{**} onto L_1. Since a, z' and y' all lie in the plane spanned by p, y, z, and since p is not collinear in Γ with z^{**}, we must have that a, y', z' are collinear in $\mathbf{PG}(d, \mathbb{K})$. Now we consider the central collineation θ_y with center y and mapping z to a. Also, we have a central collineation $\theta_{y'}$ with center y' and mapping a to z'. The collineation $\theta' = \theta_y \theta_{y'}$ fixes all points of L_1 and it also fixes the line pz. Moreover, it maps z to z'. Now we consider the action of θ' on the elements of $\mathbf{PG}(d, \mathbb{K})$, and we still denote that extension by θ'. If we look at the restriction of θ' to the projective plane pyz, then we see that it is the composition of two elations with axis py; hence that restriction is an elation itself, clearly with center p since pz is fixed. Hence all lines of Γ through p inside the plane pyz are fixed. Now suppose some line L of Γ through p is not fixed by θ'. Then we consider the 4-dimensional subspace U of $\mathbf{PG}(d, \mathbb{K})$ generated by L, L_1, L_2, L_3. We look at the restriction $\theta'|U$ of θ' to U. Notice that $\Gamma' = \Gamma \cap U$ is a generalized quadrangle weakly embedded of degree δ in U. Now we claim that we can choose a different point a_1 on the secant yz, i.e., a_1 is a point of Γ' on the line yz and $z \neq a_1 \neq y$ and $a \neq a_1$. Indeed, otherwise $\delta = 3$ and hence, there is a unique line L' of Γ in the plane LL_1. Clearly $L^{\theta_y} = L^{\theta_{y'}} = L'$ and so L is fixed under θ'. So we may assume that a_1 exists. We now replace a in the previous reasoning by a_1 and obtain (with "corresponding" notation) a collineation $\theta'_1 = (\theta_y)_1 (\theta_{y''})_1$, where y'' is the intersection of L_1 and $a_1 z'$. Now we consider $\theta'' = {\theta'}^{-1} \theta'_1 = \theta_{y'}^{-1} [(\theta_y)^{-1} (\theta_y)_1] (\theta_{y''})_1$. Since both θ' and θ'_1 induce translations with center p on the projective line pz mapping z to z', it is clear that θ'' induces the identity on the plane pyz. We now show that θ'' does not fix all points of Γ in U collinear with p. Let, for any point x of Γ, η_x denote the tangent hyperplane in x. Suppose that $\eta_y \cap \eta_p \cap U = \eta_{y'} \cap \eta_p \cap U$. Then $\theta'|(U \cap \eta_p)$ would be an elation with axis $\eta_y \cap \eta_p \cap U$ and center p (since pz is fixed), hence also L would be fixed, a contradiction. Denote $\eta_y \cap \eta_p \cap U$ by π; denote $\eta_{y'} \cap \eta_p \cap U$ by π'. We know $\pi \neq \pi'$, but $\pi \cap \pi' = L_1$. We look at the action of θ'' on the plane π. Since both θ_y and $(\theta_y)_1$ induce the identity in π, this action is given by $\theta_{y'}^{-1}(\theta_{y''})_1$. Since $\pi \neq \pi'$, $\theta_{y'}$ induces a non-trivial elation in π with axis L_1. On the other hand, $(\theta_{y''})_1$ induces a (not necessarily non-trivial) elation in π with axis L_1 and center $y'' \neq y'$. Hence θ'' induces in π a non-trivial elation with axis L_1. So not all points in U collinear with p in Γ can be fixed by θ''. Now we use a central elation with center p to map $L_3^{\theta''}$ to L_3, and, composing with θ'', we obtain a collineation θ^* that clearly fixes the 3-space $L_1L_2L_3$ pointwise, but does not fix all points in U collinear with p in Γ. So θ^* is a non-trivial elation in U with axis $L_1L_2L_3$ and some center c. Clearly θ^* maps L onto a line in the plane cL (note that c is not on L otherwise L is preserved by θ^* and hence also every point on L, hence θ^* is trivial, a contradiction), and so $c \in \eta_p$. Similarly, $c \in \eta_y$ and $c \in \eta_{y'}$. Hence $c \in \pi \cap \pi' = L_1$. But similarly also $c \in \eta_z$ and this implies that $c = p$, contradicting an earlier remark that c does not lie on L.

Hence we have shown that θ' fixes all lines through p. Now suppose that $\theta_{y'}$ maps L_3 to L'_3. If we denote by z'' the projection of z onto L'_3, then similarly as above, one shows that p, z'' and z^* are collinear (in $\mathbf{PG}(d, \mathbb{K})$). Hence there exists a central collineation θ_p with center p mapping z'' to z^*. The collineation $\theta = \theta'\theta_p$ fixes all lines through p, all points on L_1 and it maps z to z'. Similarly as above, one shows that it also fixes all lines through y.

This shows the result. □

4.0.2 Lemma. *Let Γ be a generalized quadrangle weakly embedded of degree $\delta > 2$ in $\mathbf{PG}(d, \mathbb{K})$, for some skewfield $\mathbb{K}$. Then Γ is a Moufang quadrangle and the little projective group of Γ is induced by $\mathbf{PSL}(d, \mathbb{K})$.*

PROOF. Let $L_1 \,\mathbf{I}\, p \,\mathbf{I}\, L_2$, with L_1, L_2 lines of Γ and p a point of Γ. Let a and b be two points on L_1 and L_2 respectively with $a \neq p \neq b$. Let q, q', $q \neq p \neq q'$, be two points of Γ collinear with both a and b. We show that there is an (L_1, p, L_2)-elation mapping q to q'. Together with the preceding lemma, this will imply the result. For this, let z be any point on L_2, $p \neq z \neq b$ (and z exists by the thickness of Γ). Let x be the projection of z onto aq', and let x' be the projection of x onto bq. Further, let x'' be the projection of x' onto L_1. Also, let L be the projection of xx' onto p and let y be the intersection of L and xx'.

Now let θ_1 be the (b, L_2, p)-elation mapping q to x'. This exists by the preceding lemma, and by the proof of the preceding lemma we have that θ_1 induces on the projective line L_1 an elation mapping a to x''. Furthermore, θ_1 fixes L_2 pointwise.

Now let θ_2 be the (p, L, y)-elation mapping x' to x. This again exists by the preceding lemma, and by the proof of the preceding lemma we have that θ_2 induces on the projective line L_1 an elation mapping x'' to a. Hence $\theta_1\theta_2$ fixes L_1 pointwise. Furthermore, θ_2 induces on the projective line L_2 an elation mapping b to z.

Finally let θ_3 be the (p, L_1, a)-elation mapping x to q'. By the proof of the preceding lemma we again have that θ_3 induces on the projective line L_2 an elation mapping z to b. Hence $\theta_1\theta_2\theta_3$ fixes L_2 pointwise. Furthermore, θ_3 fixes L_1 pointwise, hence $\theta_1\theta_2\theta_3$ fixes L_1 pointwise.

Clearly $\theta_1\theta_2\theta_3$ fixes all lines through p, and it maps q to q'. So we obtain an (L_1, p, L_2)-elation mapping q to q'. The lemma is proved. $\square$

So we have shown that, in order to prove the Main Result, we have to classify the weak embeddings of degree > 2 of Moufang quadrangles. We will consider the different classes of Moufang quadrangles separately.

5 Orthogonal and hermitian quadrangles

In this section, we are concerned with polar spaces associated to a (σ, ϵ)-hermitian form or a pseudo-quadratic form. The main purpose of Theorem (5.1.1) is about generalized quadrangles, but the generalization to polar spaces does not make the proof more difficult or longer. We generalize the result of STEINBACH [7] to the case that $\mathrm{Rad}(W, f) \neq 0$, also including the case where $\dim W/\mathrm{Rad}(W, f) = 4$.

5.1 Introduction and statement of the theorem

Before we can state Theorem (5.1.1) we need some preparations. Let $\mathbb{L}$ be a skewfield and W be a (left) vector space over $\mathbb{L}$ endowed with a (σ, ϵ)-hermitian form or a pseudo-quadratic form q (with associated (σ, ϵ)-hermitian form f) in the sense of TITS [15, §8]. We may assume that $\epsilon = \pm 1$ and $\sigma^2 = 1$. We let

$$\begin{aligned}
\mathrm{Rad}(W, f) &= \{w \in W \mid f(w, x) = 0 \text{ for all } x \in W\}, \\
x^{\perp} &= \{w \in W \mid f(w, x) = 0\} \text{ for } x \in W, \\
\Lambda := \Lambda_{min} &= \{c - \epsilon c^{\sigma} \mid c \in \mathbb{L}\}, \\
\Lambda_{max} &= \{c \in \mathbb{L} \mid \epsilon c^{\sigma} = -c\}.
\end{aligned}$$

A subspace U of W is called *singular*, if $f(u, u') = 0$ resp. $q(u) = 0$ for all $u, u' \in U$. The 1-, 2- and 3-dimensional subspaces of W are called *points, lines, planes* respectively. The geometry $\mathcal{S}$ of singular subspaces of W is usually called a *classical polar space* (that includes the ordinary non-degenerate and/or non-singular polar spaces). We say that $\mathcal{S}$ is the *polar space associated with W and f resp. q*, see COHEN [1, Section 3] for example.

Let S be the set of singular points of W. For each subspace U of W, we denote by $U \cap S$ the set of singular points in U. The subspace of a vector space which is spanned by a subset M is denoted by $\langle M \rangle$. If $x, y \in V$ are singular with $f(x, y) = 1$ then we call (x, y) a *hyperbolic pair* and $\langle x, y \rangle$ a *hyperbolic line.* By a 4^+-*space* we mean the orthogonal sum of two hyperbolic lines.

For skewfields $\mathbb{L}$ and $\mathbb{K}$, a mapping $\alpha : \mathbb{L} \to \mathbb{K}$ is called an *embedding* (resp. an *anti-embedding*) if α is injective, α respects addition and $(cd)^\alpha = c^\alpha d^\alpha$ (resp. $(cd)^\alpha = d^\alpha c^\alpha$) for $c, d \in \mathbb{L}$.

Weak embeddings of classical polar spaces in projective space are defined as for generalized quadrangles, see Subsection 2.2. In particular, there is an injective mapping π from S into the set of points of V. We set $\pi(U \cap S) := \{\pi(u) \mid u \in U \cap S\}$ for each subspace U of W. If N is a singular line of W, then $\langle \pi(N \cap S) \rangle$ is a line in V. If x, y are singular points of W with $\pi(y) \subseteq \langle \pi(x^\perp \cap S) \rangle$, then $y \subseteq x^\perp$.

We prove the following result:

5.1.1 Theorem. *Let $\mathbb{L}$ and $\mathbb{K}$ be skewfields and let W be a vector space over $\mathbb{L}$. We assume that there is either a (σ, ϵ)-hermitian form f on W such that $\Lambda_{min} = \Lambda_{max}$ or a pseudo-quadratic form q on W with corresponding (σ, ϵ)-hermitian form f. We suppose $W = U \perp \mathrm{Rad}(W, f)$ with U containing singular lines. Further, let V be a vector space over $\mathbb{K}$.*

We exclude the following special cases: (1) $\dim W = 4$ *and q is an (ordinary) quadratic form (this case corresponds to non-thick quadrangles) or* (2) $\dim W = 4$ *and $\mathbb{L}$ is a quaternion skewfield or* (3) *the quadrangle is isomorphic to the symplectic quadrangle over* $\mathbf{GF}(2)$.

If π is a weak embedding of the associated polar space $\mathcal{S}$ in $\mathbf{PG}(V)$, then there exists an embedding $\alpha : \mathbb{L} \to \mathbb{K}$ and a semi-linear (with respect to α) mapping $\varphi : W \to V$ such that $\pi(\mathbb{L}x) = \mathbb{K}\varphi(x)$ for all $0 \neq x \in W$, x singular (i. e. π is induced by a semi-linear mapping).

The condition on U above just means that $\mathcal{S}$ always contains generalized quadrangles as subgeometries. Theorem (5.1.1) shows that $\mathcal{S}$ is fully embedded in the projective space $\mathbf{PG}(\varphi(W))$, where $\varphi(W)$ is a vector space over the subskewfield $\mathbb{L}^\alpha$ of $\mathbb{K}$. Different as in STEINBACH [7], here the semi-linear mapping $\varphi : W \to V$ is not necessarily injective.

Theorem (5.1.1) does not require finite dimension or rank, commutative fields or non-degeneracy of forms.

The idea of the proof is to apply the result in STEINBACH [7] to the mapping π restricted to $U \cap S$ in the case that $\dim U \geq 5$. Hence there exists an embedding $\alpha : \mathbb{L} \to \mathbb{K}$ and an injective semi-linear mapping $\varphi : U \to V$ with $\pi(\mathbb{L}w) = \mathbb{K}\varphi(w)$ for all $0 \neq w \in U$, w singular. We extend φ to W as follows: Let (x_1, y_1) be a hyperbolic pair in U. If $0 \neq r \in \mathrm{Rad}(W, f)$ and $q_r \in \mathbb{L}$ with $\epsilon {q_r}^\sigma = -q_r$ resp. $q_r + \Lambda = q(r)$ (depending on whether there is a (σ, ϵ)-hermitian form or a pseudo-quadratic form on W) then there exists a unique $r' \in \langle \pi(\langle x_1, y_1\rangle^\perp \cap S)\rangle$ such that $\pi(q_r x_1 - y_1 + r) = \langle {q_r}^\alpha \varphi(x_1) - \varphi(y_1) + r'\rangle$. We extend φ to W by $\varphi(u + r) := \varphi(u) + r'$, if $u \in U$, $0 \neq r \in \mathrm{Rad}(W, f)$ and r' as above. Then φ is semi-linear with respect to α and satisfies $\pi(\mathbb{L}w) = \mathbb{K}\varphi(w)$ for $0 \neq w \in W$, w singular.

If $\dim U = 4$, then we first have to construct a semi-linear mapping $\varphi : U \to V$ which induces π. Here the cases excluded in Theorem (5.1.1) play a special role, see the introduction to Subsection 5.4.

5.2 General lemmas

5.2.1 Lemma. *Let a, b, c be singular points, such that $\langle a, b\rangle$ is a singular line with $\langle a, b\rangle \cap \mathrm{Rad}(W, f) = 0$ and $c \not\subseteq b^\perp$, and set $E := \langle a, b, c\rangle$. Then $\langle \pi(E \cap S)\rangle = \langle \pi(a), \pi(b), \pi(c)\rangle$.*

PROOF. Let $E' := \langle \pi(E \cap S)\rangle$. We may write $E = \langle b, c\rangle \perp d$ for some singular point $d \subseteq \langle a, b\rangle$. Let e be a singular point such that $Q := \langle b, c\rangle \perp \langle d, e\rangle$ is a 4^+-space. Then E' is properly contained in $\langle \pi(Q \cap S)\rangle$, since otherwise $\pi(e) \subseteq E' \subseteq \langle \pi(d^\perp \cap S)\rangle$ and $e \subseteq d^\perp$, a contradiction. Hence E' has dimension at most 3 by STEINBACH [7, (2.4)]. Further, $\pi(a)$ is not contained in $\langle \pi(b), \pi(c)\rangle$, since otherwise $\pi(a) \subseteq \langle \pi(e^\perp \cap S)\rangle$ and $d \subseteq \langle a, b\rangle \subseteq e^\perp$, a contradiction. Thus $E' = \langle \pi(a), \pi(b), \pi(c)\rangle$. □

5.2.2 Lemma. *If a, b are singular points in W with $H := \langle a, b\rangle$ a hyperbolic line, then $\langle \pi(H \cap S)\rangle = \langle \pi(a), \pi(b)\rangle$.*

PROOF. Since the line $\langle \pi(a), \pi(b)\rangle$ is contained in $\langle \pi(H \cap S)\rangle$, we have to show that $\langle \pi(H \cap S)\rangle$ is a line. Let $H = \langle x_1, y_1\rangle \subseteq \langle x_1, y_1\rangle \perp \langle x_2, y_2\rangle =: Q$ with (x_i, y_i) a hyperbolic pair $(i = 1, 2)$. With $E := \langle x_1, y_1, x_2\rangle$ and $E_1 := \langle x_1, y_1, y_2\rangle$ we obtain that $\langle \pi(H \cap S)\rangle \subseteq \langle \pi(E \cap S)\rangle \cap \langle \pi(E_1 \cap S)\rangle$. By (5.2.1) $\langle \pi(E \cap S)\rangle$ and $\langle \pi(E_1 \cap S)\rangle$ are different planes of V, hence the claim holds. □

5.2.3 Lemma. *Let (x, y) be a hyperbolic pair in W and $H = \langle x, y\rangle$. Then $\langle \pi(x), \pi(y)\rangle \cap \langle \pi(H^\perp \cap S)\rangle = 0$.*

PROOF. Let $\pi(x) = \langle x' \rangle, \pi(y) = \langle y' \rangle$ and $c, d \in \mathbb{K}$ with $cx' + dy' \in \langle \pi(H^\perp \cap S) \rangle$. If $c \neq 0$, then $x' \in \langle \pi(H^\perp \cap S) \rangle + \pi(y) \subseteq \langle \pi(y^\perp \cap S) \rangle$. This yields $x \in y^\perp$, a contradiction. Hence $c = 0$ and similarly $d = 0$. $\square$

5.3 The extension of the semi-linear mapping

Because of $W = U \perp \mathrm{Rad}(W, f)$ we have $\mathrm{Rad}(U, f) = 0$. If $\dim U \geq 5$, then by STEINBACH [7] there exists a semi-linear mapping $\varphi : U \to V$ (with respect to the embedding $\alpha : \mathbb{L} \to \mathbb{K}$) such that $\pi(\mathbb{L}w) = \mathbb{K}\varphi(w)$ for $0 \neq w \in U$, w singular. We suppose this to be true also in the case $\dim U = 4$. This assumption will be justified in Subsection 5.4.

5.3.1 Lemma. *Let $0 \neq r \in \mathrm{Rad}(W, f)$ and $q_r \in \mathbb{L}$ with $\epsilon q_r{}^\sigma = -q_r$ resp. $q_r + \Lambda = q(r)$. If (x, y) is a hyperbolic pair in U and $H := \langle x, y \rangle$, then there exists a unique $r' \in \langle \pi(H^\perp \cap S) \rangle$ such that the following holds:*

(a) *We have $\pi(q_r x - y + r) = \langle q_r{}^\alpha \varphi(x) - \varphi(y) + r' \rangle$.*

(b) *We have $\pi(q_r \widetilde{x} - \widetilde{y} + r) = \langle q_r{}^\alpha \varphi(\widetilde{x}) - \varphi(\widetilde{y}) + r' \rangle$ for each hyperbolic pair $(\widetilde{x}, \widetilde{y})$ in $U \cap H^\perp$.*

(c) *We have $\pi(cx - y + r) = \langle c^\alpha \varphi(x) - \varphi(y) + r' \rangle$ for each $c \in \mathbb{L}$ with $\epsilon c^\sigma = -c$ resp. $c + \Lambda = q(r)$. In particular, r' is independent of the choice of q_r.*

(d) *If r is singular, then $\pi(r) = \langle r' \rangle$.*

(e) *We have $\pi(q_r x - \widetilde{y} + r) = \langle q_r{}^\alpha \varphi(x) - \varphi(\widetilde{y}) + r' \rangle$ for each hyperbolic pair $(x, \widetilde{y})$ in U.*

PROOF. (a), (b) For $a := q_r x - y + r$, $\widetilde{a} := q_r \widetilde{x} - \widetilde{y} + r$, we see that a is contained in the singular line $\langle a - \widetilde{a}, \widetilde{a} \rangle$ with $a - \widetilde{a} = q_r x - y - q_r \widetilde{x} + \widetilde{y} \in U$. Since π is injective on singular points, there exists $v \in V$ such that $\pi(\widetilde{a}) = \langle v \rangle$, $\pi(a) = \langle q_r{}^\alpha \varphi(x) - \varphi(y) - q_r{}^\alpha \varphi(\widetilde{x}) + \varphi(\widetilde{y}) + v \rangle$. With $r' := -q_r{}^\alpha \varphi(\widetilde{x}) + \varphi(\widetilde{y}) + v$ the existence of r' is clear. The uniqueness of r' follows with (5.2.3).

(c) Let $\widetilde{r} \in \langle \pi(H^\perp \cap S) \rangle$ such that $\pi(cx - y + r) = \langle c^\alpha \varphi(x) - \varphi(y) + \widetilde{r} \rangle$. Since $cx - y + r \in \langle q_r x - y + r, x \rangle$, there exist $\lambda, \mu \in \mathbb{K}$ with $c^\alpha \varphi(x) - \varphi(y) + \widetilde{r} = \lambda(q_r{}^\alpha \varphi(x) - \varphi(y) + r') + \mu \varphi(x)$ by (5.2.2). Now (5.2.3) yields $\widetilde{r} = r'$.

(d) If r is singular, we may choose $q_r := 0$. Since r is contained in the singular line $\langle -y + r, y \rangle$, there exist $v \in V$, $A \in \mathbb{K}$ such that $\pi(r) = \langle v \rangle$, $v = -\varphi(y) + r' + A\varphi(y)$. Because of $v \in \langle \pi(H^\perp \cap S) \rangle$, (5.2.3) yields $v = r'$.

(e) We first handle the case $y \neq \widetilde{y} \in y^\perp$. Let $\widetilde{H} = \langle x, \widetilde{y} \rangle$, $a := q_r x - y + r$ and $\widetilde{a} := q_r x - \widetilde{y} + r$. By (a) there exists $\widetilde{r} \in \langle \pi(\widetilde{H}^\perp \cap S) \rangle$ with $\pi(\widetilde{a}) = \langle q_r{}^\alpha \varphi(x) - \varphi(\widetilde{y}) + \widetilde{r} \rangle$. Since $\widetilde{a}$ is contained in the singular line $\langle a, y-\widetilde{y}\rangle$, there exist $\lambda, \mu \in \mathbb{K}$ such that $q_r{}^\alpha\varphi(x) - \varphi(\widetilde{y}) + \widetilde{r} = \lambda(q_r{}^\alpha\varphi(x) - \varphi(y) + r') + \mu\varphi(y-\widetilde{y})$. This yields $\lambda\varphi(y) - \varphi(\widetilde{y}) \in \langle\pi(x^\perp \cap S)\rangle$ and $\lambda = 1$, since $y - \widetilde{y} \in x^\perp$. Hence $\widetilde{r} - r' = (\mu-1)\varphi(y-\widetilde{y})$.

Let $x^* \in U$ be singular such that $(x^*, y-\widetilde{y})$ is a hyperbolic pair in $\widetilde{H}^\perp$ and set $H_0 := \langle x^*, y-\widetilde{y}\rangle$. Then $\widetilde{r} \in \langle\pi(H_0{}^\perp \cap S)\rangle$. If $q_r = 0$, then (d) yields $r' \in \pi(r) \subseteq \langle\pi(x^{*\perp} \cap S)\rangle$, hence $(\mu-1)\varphi(y-\widetilde{y}) = \widetilde{r} - r' \in \langle\pi(x^{*\perp}\cap S)\rangle$ and $\mu = 1$. So we may assume $q_r \neq 0$. Since $q_r x \in \langle q_r x^* - (y-\widetilde{y}) + r, q_r(x^*-x) - (y-\widetilde{y}) + r\rangle$, (b) yields that there exist $s, t \in \mathbb{K}$ with

$$q_r{}^\alpha\varphi(x) = s(q_r{}^\alpha\varphi(x^*) - \varphi(y-\widetilde{y}) + \widetilde{r}) + t(q_r{}^\alpha\varphi(x^*-x) - \varphi(y-\widetilde{y}) + r').$$

Now (5.2.3) for H_0 yields $\mu = 1$ and $\widetilde{r} = r'$.

If $\widetilde{y} \notin y^\perp$, then there exists $y^* \in U \cap y^\perp \cap \widetilde{y}^\perp$ such that (x, y^*) is a hyperbolic pair. We may apply the first part of the proof twice and the result follows. □

We extend the mapping φ to $W = U \perp \mathrm{Rad}(W,f)$ as follows. Let (x_1, y_1) be a hyperbolic pair in U and $H_1 = \langle x_1, y_1\rangle$. For $0 \neq r \in \mathrm{Rad}(W,f)$, we set $\varphi(r) := r'$ with r' of (5.3.1). Further, let $\varphi(u+r) = \varphi(u) + \varphi(r)$ for $u \in U, r \in \mathrm{Rad}(W,f)$.

5.3.2 Lemma. *The mapping $\varphi : W \to V$ defined above is semi-linear (with respect to α).*

PROOF. First, we show that $\varphi : \mathrm{Rad}(W,f) \to V$ respects scalars. Let $0 \neq c \in \mathbb{L}$, $0 \neq r \in \mathrm{Rad}(W,f)$ and $q_r \in \mathbb{L}$ with $\epsilon q_r{}^\sigma = -q_r$ respectively $q_r + \Lambda = q(r)$. Let (x_2, y_2) be a hyperbolic pair in $U \cap H_1{}^\perp$. For $a := q_r x_2 - y_2 + r$ and $z := q_r c^\sigma x_1 - c^{-1}y_1 + r$, we see that z is contained in the singular line $\langle a, z-a\rangle$. Hence by (5.3.1)(b), there exists $\lambda \in \mathbb{K}$ such that $\pi(z) = \langle\varphi(a) + \lambda\varphi(z-a)\rangle$. Applying (5.2.3) for $\langle x_2, y_2\rangle$ yields $\lambda = 1$. Further, $\pi(z) = \pi(cq_r c^\sigma x_1 - y_1 + cr) = \langle(cq_r c^\sigma)^\alpha\varphi(x_1) - \varphi(y_1) + \varphi(cr)\rangle$; hence $\varphi(cr) = c^\alpha\varphi(r)$.

Next, we show that $\varphi : \mathrm{Rad}(W,f) \to V$ respects addition. Let $r_1, r_2 \in \mathrm{Rad}(W,f)$. We may assume $r_1, r_2, r_1 + r_2 \neq 0$. Let $q_{r_i} \in \mathbb{L}$ with $\epsilon q_{r_i}{}^\sigma = -q_{r_i}$ respectively $q_{r_i} + \Lambda = q(r_i)$ $(i = 1,2)$. We set $a_1 := q_{r_1}x_1 - y_1 + r_1$, $a_2 := q_{r_2}x_2 - y_2 + r_2$. Then $(q_{r_1} - q_{r_2})x_1 - y_1 + r_1 + r_2 \in \langle x_1 + x_2, y_2 + a_1, a_2\rangle$. We apply (5.2.1) and (5.3.1)(b). Since $\varphi(r_i) \in \langle\pi(H_1{}^\perp \cap S)\rangle \cap \langle\pi(H_2{}^\perp \cap S)\rangle$, we may compare coefficients by (5.2.3) and we obtain $\varphi(r_1 + r_2) = \varphi(r_1) + \lambda\varphi(r_2)$ for some $\lambda \in \mathbb{K}$ with $\lambda = 1$ if $q_{r_2} \neq 0$. Similarly, $(q_{r_2} - q_{r_1})x_2 - y_2 + r_1 + r_2 \in \langle x_1 + x_2, y_1 + a_2, a_1\rangle$ and $\varphi(r_1 + r_2) = \mu\varphi(r_1) + \varphi(r_2)$ for some $\mu \in \mathbb{K}$ with $\mu = 1$ if $q_{r_1} \neq 0$. Hence we are left with the case $q_{r_1} = q_{r_2} = 0$. Since we may assume $\langle r_1\rangle \neq \langle r_2\rangle$, the vectors $\varphi(r_1)$ and $\varphi(r_2)$ are linearly independent and $\lambda = \mu = 1$.

This yields the lemma. □

5.3.3 Lemma. *We have $\pi(\mathbb{L}w) = \mathbb{K}\varphi(w)$ for all $0 \neq w \in W$, w singular.*

PROOF. Let $0 \neq w \in W$ be singular, $w = u + r$ with $u \in U$, $r \in \mathrm{Rad}(W, f)$. We may assume $r \neq 0$. Let $q_r \in \mathbb{L}$ with $\epsilon {q_r}^\sigma = -q_r$ respectively $q_r + \Lambda = q(r)$.

First, we assume $u \notin H_1 = \langle x_1, y_1 \rangle$. Let $\langle x \rangle$ be a singular point in $U \cap {H_1}^\perp$ with $f(x, u) = -1$. For $y := q_r x - u$, we have $w = q_r x - y + r$ with (x, y) a hyperbolic pair. We choose $\widetilde{y} \in U \cap {H_1}^\perp$ such that $(x, \widetilde{y})$ is a hyperbolic pair. Then the definition of $\varphi(r)$ and (5.3.1)(b) for $(x, \widetilde{y})$, (5.3.1)(e) for (x, y) yields $\pi(w) = \langle \varphi(w) \rangle$.

So we are left with the case $u \in H_1$. If $u = 0$, then $w = r$ and $\pi(r) = \langle \varphi(r) \rangle$ by (5.3.1)(d). If $u = dx_1$ with $0 \neq d \in \mathbb{L}$, then we may choose $q_r = 0$. Since U contains singular lines, we obtain $\pi(q_r(-(\epsilon d^\sigma)^{-1})y_1 + dx_1 + r) = \langle \varphi(dx_1) + \varphi(r) \rangle$ applying (5.3.1)(b) twice.

If finally $cu = dx_1 - y_1$ with $c, d \in \mathbb{L}$, then there exists $\lambda \in \mathbb{L}$ such that $d = cq_r c^\sigma + \lambda =: q_{cr}$ with $\epsilon {q_{cr}}^\sigma = -q_{cr}$ respectively $q(cr) = q_{cr} + \Lambda$. This yields $cw = q_{cr} x_1 - y_1 + cr$ and $\pi(cw) = \langle {q_{cr}}^\alpha \varphi(x_1) - \varphi(y_1) + \varphi(cr) \rangle = \langle \varphi(cw) \rangle$. $\square$

5.4 The construction of a semi-linear mapping on a 4^+-space

In this subsection, we assume that $W = U \perp \mathrm{Rad}(W, f)$ with U a 4^+-space. Our aim is to show that the weak embedding π restricted to $U \cap S$ is induced by a semi-linear mapping. If $\mathbb{L}$ is a quaternion skewfield, we possibly have to apply an automorphism of the quadrangle first, see (5.4.2). The case where q is not an (ordinary) quadratic form, may be handled as in Tits [15, (8.19.7)], using translations of a projective line. For quadratic forms, a 4^+-space is just a grid, which does not supply enough structure. In this case, we assume that $\mathrm{Rad}(W, f) \neq 0$. By methods inspired by the first case, we construct a semi-linear mapping $\varphi : U \to V$ which induces π (except for the case that the quadrangle is isomorphic to the symplectic quadrangle over $\mathbf{GF}(2)$, see the example in Thas & Van Maldeghem [11]).

5.4.1 Remark. Let $\mathbb{L}$ be a quaternion skewfield with σ its standard (anti-)involution. Then the center of $\mathbb{L}$ is $Z(\mathbb{L}) = \{c + c^\sigma \mid c \in \mathbb{L}\}$. Let $U := \{(x_1, x_2, x_3, x_4) \mid x_i \in \mathbb{L}\}$ and $q : U \to \mathbb{L}/\Lambda$ be the pseudo-quadratic form (with associated $(\sigma, -1)$-hermitian form f) defined by $q(x_1, x_2, x_3, x_4) = x_1 {x_3}^\sigma + x_2 {x_4}^\sigma + \Lambda$. The mapping δ with

$$\begin{array}{lcl}
\langle (0,0,1,0) \rangle \delta & = & \langle (0,0,1,0) \rangle, \\
\langle (0,0,a,1) \rangle \delta & = & \langle (0,0,a^\sigma,1) \rangle, \\
\langle (0,1,b,m) \rangle \delta & = & \langle (0,1,b^\sigma,m) \rangle, \\
\langle (1,a,l+aa'^\sigma,a') \rangle \delta & = & \langle (1,a^\sigma,l+a^\sigma a',a'^\sigma) \rangle
\end{array}$$

for $a, b, a' \in \mathbb{L}$, $l, m \in Z(\mathbb{L}) = \Lambda$ yields an automorphism of the generalized quadrangle associated to U and q. This automorphism is the one constructed in Tits [15, (8.15)] (for right vector spaces).

5.4.2 Lemma. *We exclude the case that q is an (ordinary) quadratic form. Let $\langle x_1, x_2\rangle$ be a singular line in U and let $x_1{}', x_2{}' \in V$ such that $\pi(x_1) = \langle x_1{}'\rangle$, $\pi(x_2) = \langle x_2{}'\rangle$, $\pi(x_1 + x_2) = \langle x_1{}' + x_2{}'\rangle$, We define $\alpha : \mathbb{L} \to \mathbb{K}$ by $\pi(cx_1 + x_2) = \langle \alpha(c)x_1{}' + x_2{}'\rangle$ for $c \in \mathbb{L}$. Then one of the following holds:*

(a) *The mapping α is an embedding and there exists a semi-linear (with respect to α) mapping $\varphi : U \to V$ such that $\pi(\mathbb{L}w) = \mathbb{K}\varphi(w)$ for $0 \neq w \in U$, w singular.*

(b) *The mapping α is an anti-embedding, $\mathbb{L}$ is a quaternion skewfield, σ is its standard (anti-)involution and there exists a semi-linear (with respect to $\alpha\sigma$) mapping $\varphi : U \to V$ such that $\pi\delta(Lw) = \mathbb{K}\varphi(w)$ for $0 \neq w \in U$, w singular, where δ is as in* (5.4.1).

PROOF. The proof is similar to Tits [15, (8.19.7)]. □

In Lemma (5.4.9) below, we show that (5.4.2)(b) does not occur when $\mathrm{Rad}(W, f) \neq 0$. In the following, we handle the case that q is a quadratic form and $\mathrm{Rad}(W, f) \neq 0$.

5.4.3 Notation. Let q be a quadratic form. Let $U = \langle u_1, v_1\rangle \perp \langle u_2, v_2\rangle$ with (u_i, v_i) a hyperbolic pair ($i = 1, 2$) and set $H_i := \langle u_i, v_i\rangle$ ($i = 1, 2$). For $0 \neq r \in \mathrm{Rad}(W, f)$ with $q(r) \neq 0$ and

$$a_1 := -q(r)u_1 + v_1 - r, \qquad a_2 := -u_2 - q(r)v_2 + r,$$

$\langle a_1, a_2\rangle$ is a singular line. We choose $u_1{}', v_2{}', u_2{}', v_1{}' \in V$ such that

$$\begin{aligned} \pi(u_1) &= \langle u_1{}'\rangle, & & \\ \pi(v_2) &= \langle v_2{}'\rangle, & \pi(u_1 + v_2) &= \langle u_1{}' + v_2{}'\rangle, \\ \pi(u_2) &= \langle u_2{}'\rangle, & \pi(u_1 + u_2) &= \langle u_1{}' + u_2{}'\rangle, \\ \pi(v_1) &= \langle v_1{}'\rangle, & \pi(u_2 - v_1) &= \langle u_2{}' - v_1{}'\rangle. \end{aligned}$$

Then $u_1{}', u_2{}', v_1{}', v_2{}'$ are linearly independent by (5.2.3). For $c \in \mathbb{L}$, there exists a unique $c' \in \mathbb{K}$ with $\pi(cu_1 + u_2) = \langle c'u_1{}' + u_2{}'\rangle$. We set $q' := q(r)'$.

5.4.4 Lemma. *We use the notation of* (5.4.3). *Then the following holds:*

(a) *We have $\pi(v_1 - cv_2) = \langle v_1{}' - c'v_2{}'\rangle$ for $c \in \mathbb{L}$.*

(b) *We have $\pi(cu_1 + u_2 - (v_1 - cv_2)) = \langle c'u_1{}' + u_2{}' - (v_1{}' - c'v_2{}')\rangle$ for $c \in \mathbb{L}$.*

(c) *There exists $r' \in V$ such that*

$$\pi(a_1) = \langle -q'u_1{}' + v_1{}' - r'\rangle, \qquad \pi(a_2) = \langle -u_2{}' - q'v_2{}' + r'\rangle.$$

In particular, $r' \in \langle \pi(H_1{}^\perp \cap S)\rangle \cap \langle \pi(H_2{}^\perp \cap S)\rangle$.

(d) *If* $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{K}$ *with* $\alpha u_1{}' + \beta u_2{}' + \gamma v_1{}' + \delta v_2{}' + \epsilon r' = 0$*, then* $\alpha = \beta = \gamma = \delta = 0$.

We set $a_1{}' := -q' u_1{}' + v_1{}' - r'$*,* $a_2{}' := -u_2{}' - q' v_2{}' + r'$.

PROOF. (a) For $c \in \mathbb{L}$, $z := v_1 - cv_2$ is contained in $\langle cu_1 + u_2, cu_1 + u_2 - (v_1 - cv_2)\rangle$ with $cu_1 + u_2 - (v_1 - cv_2) \in \langle u_1 + v_2, u_2 - v_1\rangle$. Hence $\pi(z)$ is contained in $\langle c' u_1{}' + u_2{}', u_1{}' + v_2{}', u_2{}' - v_1{}'\rangle$ and in $\langle v_1{}', v_2{}'\rangle$. Comparing coefficients yields (a).

(b) For $c \in \mathbb{L}$, $cu_1 + u_2 - (v_1 - cv_2)$ is contained in the two singular lines $\langle cu_1 + u_2, v_1 - cv_2\rangle$ and $\langle u_1 + v_2, u_2 - v_1\rangle$. We apply π and obtain (b).

(c) For $a := -(a_1 + a_2)$, we have $a_2 \in \langle a, a_1\rangle$ and hence $\pi(a_2) \subseteq \langle \pi(a), \pi(a_1)\rangle$. Since π is injective on singular points, there exists $v \in V$ such that $\pi(a_1) = \langle v\rangle$ and $\pi(a_2) = \langle -(q' u_1{}' + u_2{}' - v_1{}' + q' v_2{}') - v\rangle$. The claim follows with $r' := -q' u_1{}' + v_1{}' - v$.

(d) This follows from (5.2.3) and (c). $\square$

5.4.5 Lemma. *We use the notation of* (5.4.3) *and* (5.4.4)*. For* $0 \neq c \in \mathbb{L}$*, we have:*

$$\begin{array}{lcl}
\pi(cu_1 - q(r)c^{-1}v_1 + r) & = & \langle c' u_1{}' - q' {c'}^{-1} v_1{}' + r'\rangle, \\
\pi(cu_1 + a_2) & = & \langle c' u_1 + a_2{}'\rangle, \\
\pi(u_2 - q(r)c^{-1}v_1) & = & \langle u_2{}' - q' {c'}^{-1} v_1{}'\rangle, \\
\pi(u_1 + q(r)c^{-1}v_2) & = & \langle u_1{}' + q' {c'}^{-1} v_2{}'\rangle, \\
\pi((c - q(r))u_1 + a_2) & = & \langle (c' - q')u_1{}' + a_2{}'\rangle.
\end{array}$$

PROOF. Because of $z := cu_1 - q(r)c^{-1}v_1 + r \in \langle a_2, (cu_1 + u_2) - q(r)c^{-1}(v_1 - cv_2)\rangle$ and $z \in H_2{}^{\perp}$, (5.2.3) yields the first claim. Since $cu_1 + a_2$ is contained in the two singular lines $\langle -u_2 + z, v_1 - cv_2\rangle$ and $\langle u_1, a_2\rangle$, (5.4.4)(d) yields the second one. Similarly, $u_2 - q(r)c^{-1}v_1 \in \langle -q(r)v_2 + z, cu_1 + a_2\rangle$ and $u_1 + q(r)c^{-1}v_2 \in \langle u_1 + u_2 - q(r)c^{-1}(v_1 - v_2), u_2 - q(r)c^{-1}v_1\rangle$, so we may calculate the image points under π.

Since $\mathbb{L}$ is commutative, $w := q(r)u_1 + u_2 - q(r)c^{-1}(v_1 - q(r)v_2)$ is contained in $\langle q(r)u_1 + u_2, v_1 - q(r)v_2\rangle$ and in $\langle u_2 - q(r)c^{-1}v_1, u_1 + q(r)c^{-1}v_2\rangle$. We obtain $\pi(w) = \langle q' u_1{}' + u_2{}' - q' {c'}^{-1} v_1{}' + q' q' {c'}^{-1} v_2{}'\rangle$, since $\mathbb{K}$ is not necessarily commutative, Finally, $(c - q(r))u_1 + a_2 \in \langle z, v_2, w\rangle$ and we may use (5.2.1). $\square$

5.4.6 Lemma. *If* $\mathbb{L} \neq \mathbf{GF}(2)$*, then* $\operatorname{char} \mathbb{K} = 2$ *and* $\pi((c + q(r))u_1 + a_2) = \langle (c' + q')u_1{}' + a_2{}'\rangle$.

PROOF. By (5.4.5) we have $(c - q(r))' = c' - q'$ for $0 \neq c \in \mathbb{L}$. Because of $q(r) \neq 0$, we have $\operatorname{char} \mathbb{L} = 2$. If $\mathbb{L} \neq \mathbf{GF}(2)$, then there exists $0, q \neq c \in \mathbb{L}$ and we obtain $c' = ((c - q(r)) - q(r))' = c' - q' - q'$. This shows $\operatorname{char} \mathbb{K} = 2$. The second claim follows from (5.4.5). $\square$

5.4.7 Lemma. *We assume that q is a quadratic form and $\mathbb{L} \neq \mathbf{GF}(2)$. If there exists $r \in \mathrm{Rad}(W, f)$ with $q(r) \neq 0$, then there exists an embedding $\alpha : \mathbb{L} \to \mathbb{K}$ and a semi-linear (with respect to α) mapping $\varphi : U \to V$ such that $\pi(\mathbb{L}w) = \mathbb{K}\varphi(w)$ for all $0 \neq w \in U$, w singular.*

PROOF. Let $L_1 := \langle x_1, x_2\rangle$ be a singular line in U and choose $x_1', x_2' \in V$ such that $\pi(x_1) = \langle x_1'\rangle$, $\pi(x_2) = \langle x_2'\rangle$, $\pi(x_1+x_2) = \langle x_1'+x_2'\rangle$. By $P(L_1)$ we denote the set of points of L_1, and similarly for other lines. We define $\tau : \mathbb{L} \cup \{\infty\} \to P(L_1)$ by $\mu \mapsto \langle \mu x_1 + x_2\rangle$ for $\mu \in \mathbb{L}$, $\infty \mapsto \langle x_1\rangle$ and similarly $\tau' : \mathbb{K} \cup \{\infty\} \to P(L_1')$, where $L_1' = \langle x_1', x_2'\rangle$. For $\alpha := \tau'^{-1}\pi\tau : \mathbb{L} \to \mathbb{K}$, we obtain $\alpha(0) = 0$, $\alpha(1) = 1$ and $\alpha(\infty) = \infty$.

We denote by $\mathbf{PGL}_2(\mathbb{L})$ the set of all invertible mappings $\gamma : \mathbb{L}\cup\{\infty\} \to \mathbb{L}\cup\{\infty\}$, where γ is of the form $\gamma : x \mapsto (xc+d)^{-1}(xa+b)$, $x \in \mathbb{L}\cup\{\infty\}$ with $a, b, c, d \in \mathbb{L}$. The elements of $T := \{\tau(\gamma\beta\gamma^{-1})\tau^{-1} \mid \gamma \in \mathbf{PGL}_2(\mathbb{L})\}$, where $\beta : x \mapsto x + 1$, are called translations of $P(L_1)$. Similarly, we define T' for $P(L_1')$.

Let $t \in T$ and let $\beta_0, \gamma \in \mathbf{PGL}_2(\mathbb{L})$ with $\beta_0 : x \mapsto x+q(r)$, $\gamma : x \mapsto (xc+d)^{-1}(xa+b)$, such that $t = \tau\gamma\beta_0\gamma^{-1}\tau^{-1}$. We set $u_1 := ax_1 + cx_2$, $u_2 := bx_1 + dx_2$. Then $\{u_1, u_2\}$ is a basis of L_1 and we have $\tau\gamma : \mathbb{L} \cup \{\infty\} \to P(L_1)$, $\mu \mapsto \langle \mu u_1 + u_2\rangle$ $(\mu \in \mathbb{L})$, $\infty \mapsto \langle u_1\rangle$. For u_1', u_2' as in (5.4.3), there are $a', b', c', d' \in \mathbb{K}$ such that $u_1' = a'x_1'+c'x_2'$, $u_2' = b'x_1'+d'x_2'$. We set $\gamma' : x \mapsto (xc' + d')^{-1}(xa' + b')$ and $\beta_0' : x \mapsto x + q'$ for $x \in \mathbb{K} \cup \{\infty\}$.

We use the notation of (5.4.3), (5.4.4). Let $L_2 := \langle u_1, a_2\rangle$, $L_2' := \langle u_1', a_2'\rangle$. We define $\rho_1 : P(L_1) \to P(L_2)$ by $\rho_1 : \langle cu_1 + u_2\rangle \mapsto \langle (c + q(r))u_1 + a_2\rangle$ $(c \in \mathbb{L})$, $\langle u_1\rangle \mapsto \langle u_1\rangle$ and $\rho_2 : P(L_2) \to P(L_1)$ by $\rho_2 : \langle cu_1 + a_2\rangle \mapsto \langle cu_1 + u_2\rangle$ $(c \in \mathbb{L})$, $\langle u_1\rangle \mapsto \langle u_1\rangle$. Similarly, we define $\rho_1' : P(L_1') \to P(L_2')$ and $\rho_2' : P(L_2') \to P(L_1')$. Then we have $\pi\rho_2 = \rho_2'\pi$ on $P(L_2)$ by (5.4.5) and $\pi\rho_1 = \rho_1'\pi$ on $P(L_1)$ by (5.4.6). Hence $\pi\rho_2\rho_1 = \rho_2'\rho_1'\pi$ on $P(L_1)$. For $t' := (\tau'\gamma')\beta_0'\gamma'^{-1}\tau'^{-1} \in T'$, we have $t = \rho_2\rho_1$, $t' = \rho_2'\rho_1'$. Hence $\pi t = t'\pi$ on $P(L_1)$. Since $\mathbb{L}$ is commutative, we obtain that α is an embedding as in Tits [15, (8.12.3)].

In (5.4.3) we use the hyperbolic pairs (x_i, y_i) $(i = 1, 2)$, where $U = \langle x_1, y_1\rangle \perp \langle x_2, y_2\rangle$. Then $\pi(cx_1 + x_2) = \langle \alpha(c)x_1' + x_2'\rangle$, $\pi(y_1 - cy_2) = \langle y_1' - \alpha(c)y_2'\rangle$ by the definition of α and (5.4.4)(a). Since $q(r) \neq 0$ and α is an embedding, (5.4.5) yields $\pi(y_1 + cx_2) = \langle y_1' + \alpha(c)x_2'\rangle$ and $\pi(y_2 - cx_1) = \langle y_2' - \alpha(c)x_1'\rangle$ for $c \in L$. If z is a singular point in U, which is not contained in $\langle x_1, x_2\rangle$, $\langle x_1, y_2\rangle$, then there exist $c, d \in \mathbb{L}$ such that $z = \langle cdx_1 + y_1 + cx_2 - dy_2\rangle$. Hence $z \subseteq \langle dx_1 + x_2, y_1 - dy_2\rangle \cap \langle y_1 + cx_2, y_2 - cx_1\rangle$. We apply π and obtain $\pi(z) = \langle \alpha(c)\alpha(d)x_1' + \alpha(c)x_2' + y_1' - \alpha(d)y_2'\rangle$. The claim follows with $\varphi : U \to V$ defined by $\varphi(c_1x_1+c_2x_2+d_1y_1+d_2y_2) = \alpha(c_1)x_1'+\alpha(c_2)x_2'+\alpha(d_1)y_1'+\alpha(d_2)y_2'$. □

5.4.8 Lemma. *We assume that q is a quadratic form. If there exists $0 \neq r \in \mathrm{Rad}(W, f)$ with $q(r) = 0$, then there exists an embedding $\alpha : \mathbb{L} \to \mathbb{K}$ and a semi-linear (with respect to α) mapping $\varphi : U \to V$ such that $\pi(\mathbb{L}w) = \mathbb{K}\varphi(w)$ for all $0 \neq w \in U$, w singular.*

PROOF. : Let $r' \in V$ such that $\pi(r) = \langle r' \rangle$. Let $L_1 := \langle x_1, x_2 \rangle$ be a singular line in U and choose ${x_1}', {x_2}' \in V$ such that

$$\begin{aligned} \pi(x_2) &= \langle {x_2}' \rangle, & \pi(x_2 + r) &= \langle {x_2}' + r' \rangle, \\ \pi(x_1) &= \langle {x_1}' \rangle, & \pi(x_1 + x_2) &= \langle {x_1}' + {x_2}' \rangle. \end{aligned}$$

We define $\tau, \tau', \alpha, T, T'$ as in the proof of (5.4.7). Let $t \in T$ and let $\beta, \gamma \in \mathbf{PGL}_2(\mathbb{L})$ with $\beta : x \mapsto x+1$, $\gamma : x \mapsto (xc+d)^{-1}(xa+b)$, such that $t = (\tau\gamma)\beta\gamma^{-1}\tau^{-1}$. For $u_1 := ax_1 + cx_2$, $u_2 := bx_1 + dx_2$, $\tau\gamma$ is as in the proof of (5.4.7). We let $L_2 := \langle u_1, u_2 + r \rangle$, ${L_2}' = \langle \pi(L_2 \cap S) \rangle$. For $\rho_1 : P(L_1) \to P(L_2)$ defined by $z \mapsto \langle z, u_1 + r \rangle \cap L_2$, we have

$$\rho_1 : \langle \mu u_1 + u_2 \rangle \mapsto \langle (\mu + 1)u_1 + u_2 + r \rangle \ \ (\mu \in \mathbb{L}), \quad \langle u_1 \rangle \mapsto \langle u_1 \rangle.$$

Similarly, for $\rho_2 : P(L_2) \to P(L_1)$ defined by $z \mapsto \langle z, r \rangle \cap L_1$, we have

$$\rho_2 : \langle \mu u_1 + u_2 + r \rangle \mapsto \langle \mu u_1 + u_2 \rangle \ \ (\mu \in \mathbb{L}), \quad \langle u_1 \rangle \mapsto \langle u_1 \rangle.$$

This yields $t = \rho_2\rho_1$. We choose ${u_1}', {u_2}' \in V$ such that

$$\begin{aligned} \pi(u_2) &= \langle {u_2}' \rangle, & \pi(u_2 + r) &= \langle {u_2}' + r' \rangle, \\ \pi(u_1) &= \langle {u_1}' \rangle, & \pi(u_1 + r) &= \langle {u_1}' + r' \rangle. \end{aligned}$$

We define γ' as in the proof of (5.4.7) and ${\rho_1}'$, ${\rho_2}'$ similarly as ρ_1, ρ_2. With $\beta' : x \mapsto x + 1$ and $t' := (\tau'\gamma')\beta'{\gamma'}^{-1}{\tau'}^{-1} \in T'$ we obtain $t' = {\rho_2}'{\rho_1}'$.

For $a \in P(L_2)$, we have $\pi\rho_2(a) = \pi(\langle a, r \rangle \cap L_1) = \langle \pi(a), r' \rangle \cap {L_1}' = {\rho_2}'\pi(a)$ and similarly, $\pi\rho_1(z) = {\rho_1}'\pi(z)$ for $z \in P(L_1)$. This shows $\pi t = t'\pi$ on $P(L_1)$. Now α is an embedding as in the proof of (5.4.7).

Let $U = \langle x_1, y_1 \rangle \perp \langle x_2, y_2 \rangle$ with hyperbolic pairs (x_i, y_i) $(i = 1, 2)$. We choose ${y_1}', {y_2}' \in V$ such that

$$\begin{aligned} \pi(y_2) &= \langle {y_2}' \rangle, & \pi(y_2 - r) &= \langle {y_2}' - r' \rangle, \\ \pi(y_1) &= \langle {y_1}' \rangle, & \pi(y_1 - y_2) &= \langle {y_1}' - {y_2}' \rangle. \end{aligned}$$

Since $r' \neq 0$, ${x_1}', {x_2}', {y_1}', {y_2}', r'$ are linearly independent as in (5.4.4)(d). Let $0 \neq c \in \mathbb{L}$. Because of $y_2 - cx_1 \in \langle cx_1 + x_2, x_2 + r, y_2 - r \rangle$ and (5.2.1), we obtain $\pi(y_2 - cx_1) = \langle {y_2}' - \alpha(c){x_1}' \rangle$. Similarly, $y_1 + cx_2 \in \langle x_1 + x_2, y_1 - y_2, y_2 - cx_1 \rangle$ and $\pi(y_1 + cx_2) = \langle {y_1}' + \alpha(c){x_2}' \rangle$. Further, $y_1 - cy_2 \in \langle y_1 + cx_2, y_2 - cx_1, cx_1 + x_2 \rangle$ and $\pi(y_1 - cy_2) = \langle {y_1}' - \alpha(c){y_2}' \rangle$. We now finish the proof as in (5.4.7). $\square$

5.4.9 Lemma. *If* $\dim U = 4$ *and* $\mathrm{Rad}(W, f) \neq 0$*, then one of the following holds:*

(a) *There exists an embedding* $\alpha : \mathbb{L} \to \mathbb{K}$ *and a semi-linear (with respect to* α*) mapping* $\varphi : U \to V$ *such that* $\pi(\mathbb{L}w) = \mathbb{K}\varphi(w)$ *for all* $0 \neq w \in U$*,* w *singular.*

(b) *We have* $\mathbb{L} = \mathbf{GF}(2)$, $\dim W = 5$ *and* q *is a quadratic form. The weak embedding is the so-called universal weak embedding of the symplectic quadrangle over* $\mathbf{GF}(2)$ *described in* Thas & Van Maldeghem [11].

PROOF. Let $U = \langle x_1, y_1\rangle \perp \langle x_2, y_2\rangle$ with hyperbolic pairs (x_i, y_i) $(i = 1, 2)$ and let $0 \neq r \in \mathrm{Rad}(W, f)$. We lead the assumption that the mapping $\alpha : \mathbb{L} \to \mathbb{K}$ defined in (5.4.2) is an anti-embedding to a contradiction. For $x \in \{x_1, y_1, x_2, y_2\}$ let $x' := \varphi(x)$ with φ of (5.4.2)(b). Let $q_r \in \mathbb{L}$ with $q_r + \Lambda = q(r)$. We first handle the case $q_r \neq 0$. For $a_1 := q_r x_1 - y_1 + r$ and $a_2 := {q_r}^{\sigma} x_2 - y_2 + r$, there exists a unique $r' \in \langle \pi(\langle x_1, y_1\rangle^{\perp} \cap S)\rangle$ with $\pi(a_1) = \langle \alpha(q_r){x_1}' - {y_1}' + r'\rangle$, $\pi(a_2) = \langle \alpha({q_r}^{\sigma}){x_2}' - {y_2}' + r'\rangle$ as in (5.3.1). Let $0 \neq c \in \mathbb{L}$ and $z := q_r c^{\sigma} x_1 - c^{-1} y_1 + r$. Because of $z \in \langle a_2, z - a_2\rangle$, there exist $z' \in V$, $A, B \in \mathbb{K}$ such that $\pi(z) = \langle z'\rangle$ with

$$z' = A(\alpha({q_r}^{\sigma}){x_2}' - {y_2}' + r') + B(\alpha(c^{\sigma})\alpha(q_r){x_1}' - \alpha(c)^{-1}{y_1}' - \alpha(c^{\sigma})\alpha({q_r}^{\sigma})\alpha(c^{-\sigma}){x_2}' + {y_2}').$$

Since $z \in \langle x_2, y_2\rangle^{\perp}$, (5.2.3) yields $\alpha({q_r}^{\sigma}) = \alpha(c^{\sigma})\alpha({q_r}^{\sigma})\alpha(c^{-\sigma})$. Hence $q_r = c q_r c^{-1}$ for $c \in \mathbb{L}$, i. e. $q_r \in Z(\mathbb{L}) = \Lambda$ and r is singular.

We now handle the case that r is singular. Let $\pi(r) = \langle r'\rangle$ with $\pi(x_1 - r) = \langle {x_1}' - r'\rangle$. For $0 \neq c, d \in \mathbb{L}$, we have $cx_2 + dy_1 \in \langle cx_2 + r, dy_1 - r\rangle$ with $cx_2 + r \in \langle x_1 + cx_2, x_1 - r\rangle$, $dy_1 - r \in \langle dy_1 - y_2, y_2 - r\rangle$, $y_2 - r \in \langle x_1 - r, y_2 - x_1\rangle$. This yields $\pi(cx_2 + dy_1) = \langle \alpha(c){x_2}' + \alpha(d){y_1}'\rangle$. Further, $\pi(x_2 + c^{-1} d y_1) = \langle {x_2}' + \alpha(c^{-1} d){y_1}'\rangle$ by (5.4.2)(b). Hence $\alpha(c)^{-1}\alpha(d) = \alpha(c^{-1} d) = \alpha(d)\alpha(c)^{-1}$ for $0 \neq c, d \in \mathbb{L}$ and α is an embedding, a contradiction.

By (5.4.2), (5.4.7), (5.4.8) we are left with the case where q is a quadratic form, $\mathbb{L} = \mathbf{GF}(2)$ and $\dim W = 5$ (recall that U is a 4^+-space). Hence the polar space associated to W and q is isomorphic to the symplectic quadrangle over $\mathbf{GF}(2)$. If char $\mathbb{K} = 2$, then (a) holds as in the proof of (5.4.7). It is possible that char $\mathbb{K} \neq 2$. In this case the weak embedding π is as described in Thas & Van Maldeghem [11]. The proof of this can be taken over without notable change from *loc. cit.* □

6 Mixed quadrangles

6.1 Introduction and statement of the Theorem

In this section, we show that every weak embedding of any mixed quadrangle $Q(\mathbb{L}', \mathbb{L}^2; \Lambda', \Lambda^2)$ in a projective space is induced by an embedding $\alpha : \Lambda' \to \mathbb{K}$ and a so-called semi-linear mapping $\varphi : \Lambda \to V$; for definitions see (6.1.3).

6.1.1 Definition of mixed quadrangles. Let $\mathbb{L}$ be a (commutative) field of characteristic 2 and let

$$\mathbb{L}^2 \subseteq \Lambda' \subseteq \mathbb{L}' \subseteq \Lambda \subseteq \mathbb{L},$$

where $\mathbb{L}'$ is a subfield of $\mathbb{L}$, Λ is a subspace of $\mathbb{L}$ considered as vector space over $\mathbb{L}'$ and Λ' is a subspace of $\mathbb{L}'$ considered as vector space over $\mathbb{L}^2$. We suppose that $\mathbb{L}$ respectively $\mathbb{L}'$ are generated as rings by Λ respectively Λ'.

Let $W(\mathbb{L}')$ be the symplectic quadrangle associated to the vector space $M := \mathbb{L}' \times \mathbb{L}' \times \mathbb{L}' \times \mathbb{L}'$ and the symplectic form $b : M \times M \to \mathbb{L}'$ defined by $b(x, y) = x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3$ for $x = (x_1, x_2, x_3, x_4), y = (y_1, y_2, y_3, y_4) \in M$.

Let $Q(E, f)$ be the orthogonal quadrangle associated to the vector space $E := \mathbb{L}' \times M$ with scalar multiplication $c(x_0; x_1, x_2, x_3, x_4) := (c^2x_0; cx_1, cx_2, cx_3, cx_4)$ for $c, x_i \in \mathbb{L}'$ ($i = 0, \dots, 4$) and the quadratic form $f : E \to \mathbb{L}'$ defined by $f((x_0; x_1, x_2, x_3, x_4)) = x_0 + x_1x_2 + x_3x_4$. The dimension of the subspace $\mathbb{L}'$ in the first factor is the dimension of the field extension $\mathbb{L}' : \mathbb{L}'^2$. Then

$$Q(E, f) \simeq W(\mathbb{L}'), \tag{1}$$

and the isomorphism is induced by the projection of E on the second factor, see Cohen [1, (3.27)].

Let $Q(E_0, f)$ be the subquadrangle $Q(E, f)$ belonging to the subspace $E_0 := \mathbb{L}^2 \times M$ of E. Then

$$Q(E_0, f) \simeq Q(\mathbb{L} \times M, q), \tag{2}$$

where the latter is the orthogonal quadrangle associated to the vector space $\mathbb{L} \times M$ over $\mathbb{L}'$ with usual scalar multiplication and the (non-degenerate) quadratic form $q : \mathbb{L} \times M \to \mathbb{L}'$ defined by $q((x_0; x_1, x_2, x_3, x_4)) = {x_0}^2 + x_1x_2 + x_3x_4$ for $x_0 \in \mathbb{L}$, $x_i \in \mathbb{L}'$ ($i = 1, \dots, 4$). The isomorphism is induced by the bijective linear mapping $t : E_0 \to \mathbb{L} \times M$ with $t(({x_0}^2; x_1, x_2, x_3, x_4)) = (x_0; x_1, x_2, x_3, x_4)$.

Every point in the symplectic quadrangle $W(\mathbb{L}')$ is spanned by a vector of the following form

$$(1, 0, 0, 0), \;\; (a, 0, 1, 0), \;\; (b, 0, k, 1), \;\; (l + aa', 1, a', a),$$

where $a, b, a', k, l \in \mathbb{L}'$.

Restricting to $a, b, a' \in \Lambda'$, $k, l \in \Lambda^2$ yields a generalized quadrangle, the so-called *mixed quadrangle* $Q(\mathbb{L}', \mathbb{L}^2; \Lambda', \Lambda^2)$ first defined in Tits [13], see Van Maldeghem [19, (3.4.2)]. (We may assume that $\mathbb{L}$ is not perfect, since otherwise $\mathbb{L}^2 = \Lambda' = \mathbb{L}' = \Lambda = \mathbb{L}$.)

The image of $Q(\mathbb{L}', \mathbb{L}^2; \Lambda', \Lambda^2)$ in $Q(E, f)$ under the isomorphism in (1) is contained in $Q(E_0, f)$. The image in $Q(\mathbb{L} \times M, q)$ under the isomorphism in (2) yields the points spanned by vectors of the form

$$(0; 1, 0, 0, 0), \;\; (0; a, 0, 1, 0), \;\; (k; b, 0, k^2, 1), \;\; (l; l^2 + aa', 1, a', a),$$

where $a, b, a' \in \Lambda'$, $k, l \in \Lambda$.

6.1.2 Notation. We always regard the mixed quadrangle $\mathcal{Q} := Q(\mathbb{L}', \mathbb{L}^2; \Lambda', \Lambda^2)$ as subquadrangle of $Q(\mathbb{L} \times M, q)$ as in (6.1.1). Let S be the set of points of $\mathcal{Q}$ and $W := \mathbb{L} \times M$. For each subspace U of W, we denote by $U \cap S$ the set of points of $\mathcal{Q}$, which are contained in U.

The 1-, 2- and 3-dimensional subspaces of W are called *points, lines, planes* respectively. The subspace of a vector space which is spanned by a subset X is denoted by $\langle X \rangle$. For each subspace U of W, we let $U^\perp = \{w \in W \mid (w, u) = 0 \text{ for } u \in U\}$, where $(\, , \,)$ is the bilinear form associated to q. A subspace U of W is called *singular*, if $q(u) = 0$ for $u \in U$. A *hyperbolic line* of W is a line $\langle x, y \rangle$, where x, y are singular points and $y \not\subseteq x^\perp$.

The lines of $\mathcal{Q}$ are the singular lines $\langle a, b \rangle$ of W, where a and b are points of $\mathcal{Q}$.

6.1.3 Definition. Let $\mathbb{L}, \mathbb{L}', \Lambda, \Lambda'$ be as in (6.1.1) and let $\mathbb{K}$ be a skewfield and V be a vector space over $\mathbb{K}$. A mapping $\alpha : \Lambda' \to \mathbb{K}$ is called an *embedding*, if α has the following properties:

(a) α is injective, α respects addition,

(b) $\alpha(l^2 c) = \alpha(l^2)\alpha(c)$ for $l \in \mathbb{L}$, $c \in \Lambda'$,

(c) $\alpha(c^2) = \alpha(c)^2$ for $c \in \Lambda'$,

(d) $\alpha(c)\alpha(d) = \alpha(d)\alpha(c)$ for $c, d \in \Lambda'$.

Let $\alpha : \Lambda' \to \mathbb{K}$ be an embedding. A mapping $\varphi : \Lambda \to V$ is called a *semi-linear mapping* (with respect to α), if $\varphi(l + k) = \varphi(l) + \varphi(k)$ and $\varphi(cl) = \alpha(c)\varphi(l)$ for $l, k \in \Lambda$, $c \in \Lambda'$.

6.1.4 Remark. If $\alpha : \Lambda' \to \mathbb{K}$ is an embedding, then we have $\alpha(c^2 d) = \alpha(c)^2 \alpha(d)$ and $\alpha(c^{-1}) = \alpha(c)^{-1}$ for $c, d \in \Lambda'$, $c \neq 0$. When we regard Λ' as vector space over $\mathbb{L}^2$, then $\alpha : \Lambda' \to \mathbb{K}$ is a semi-linear mapping (with respect to the embedding $\alpha \mid_{\mathbb{L}^2} : \mathbb{L}^2 \to \mathbb{K}$).

Let $\varphi : \Lambda \to V$ be a semi-linear mapping (with respect to α). If there exists some $l_0 \in \Lambda$ with $\varphi(l_0) \neq 0$, then it is possible to extend $\alpha : \Lambda' \to \mathbb{K}$ to $\mathbb{L}' = \langle \Lambda' \rangle$ because of the equation $\varphi(cl) = \alpha(c)\varphi(l)$ for $c \in \Lambda'$, $l \in \Lambda$. Then $\varphi : \Lambda \to V$ is a semi-linear mapping with respect to the embedding $\alpha : \mathbb{L}' \to \mathbb{K}$.

We prove the following result:

6.1.5 Theorem. *We use the notation of* (6.1.2) *with* $\mathbb{L} \neq \mathbf{GF}(2)$. *Let* $\mathbb{K}$ *be a skewfield and let* V *be a vector space over* $\mathbb{K}$. *We assume that* π *is a weak embedding of the mixed quadrangle* $\mathcal{Q} := Q(\mathbb{L}', \mathbb{L}^2; \Lambda', \Lambda^2)$ *into the projective space* $\mathbf{PG}(V)$. *Then there exists an* *embedding* $\alpha : \Lambda' \to \mathbb{K}$*, a decomposition* $V = V_1 \times \mathbb{K}^4$*, where* $\mathbb{K}^4 = \mathbb{K} \times \mathbb{K} \times \mathbb{K} \times \mathbb{K}$*, and a semi-linear mapping* $\varphi : \Lambda \to V_1$ *(in the sense of* (6.1.3)*) such that*

$$\begin{aligned}
\pi(\mathbb{L}'(0;1,0,0,0)) &= \mathbb{K}(1,0,0,0),\\
\pi(\mathbb{L}'(0;a,0,1,0)) &= \mathbb{K}(\alpha(a),0,1,0),\\
\pi(\mathbb{L}'(k;b,0,k^2,1)) &= \mathbb{K}(\varphi(k) + (\alpha(b),0,\alpha(k^2),1)),\\
\pi(\mathbb{L}'(l;l^2+aa',1,a',a)) &= \mathbb{K}(\varphi(l) + (\alpha(l^2)+\alpha(a')\alpha(a),1,\alpha(a'),\alpha(a))),
\end{aligned}$$

for $a,b,a' \in \Lambda'$*,* $k,l \in \Lambda$*. Further, the subspace* $\mathbb{K}^4$ *of* V *is unique and the basis of* $\mathbb{K}^4$ *used in the above description,* α *and* φ *are unique up to scalar.*

6.1.6 Remark. Note that, if the dimension of V is equal to 4, and if $\Lambda' = \mathbb{L}'$ (and hence Λ' is a field), then the weak embedding of $\mathcal{Q}$ into $\mathbf{PG}(V)$ is full over the subfield $\alpha(\mathbb{L}')$ of $\mathbb{K}$. We will use that result later on in the proof of Theorem 7.2.2.

6.1.7 Main idea of the proof. The idea of the proof of Theorem 6.1.5 is as follows: If $\mathbb{L}'w$ is a point of $\mathcal{Q}$, then we often write $\pi(w)$ instead of $\pi(\mathbb{L}'w)$. We set $\pi(U \cap S) := \{\pi(u) \mid u \in U \cap S\}$ for each subspace U of W. Let

$$x_1 = (0;1,0,0,0),\ \ y_1 = (0;0,1,0,0),\ \ x_2 = (0;0,0,1,0),\ \ y_2 = (0;0,0,0,1).$$

We choose $x_1{}', x_2{}' \in V$ such that

$$\pi(x_1) = \langle x_1{}' \rangle, \quad \pi(x_2) = \langle x_2{}' \rangle, \quad \pi(x_1+x_2) = \langle x_1{}' + x_2{}' \rangle.$$

For each $c \in \Lambda'$ there exists a unique scalar $\alpha(c) \in \mathbb{K}$ such that $\pi(cx_1+x_2) = \langle \alpha(c)x_1{}'+x_2{}' \rangle$. This mapping $\alpha : \Lambda' \to \mathbb{K}$ yields the desired embedding. The semi-linear mapping $\varphi : \Lambda \to V$ is defined by $\varphi(l) \in \langle \pi(\langle x_1, y_1 \rangle^\perp \cap S) \rangle$ and $\pi((l;l^2,1,0,0)) = \langle \varphi(l) + \alpha(l^2)x_1{}' + y_1{}' \rangle$. The proofs use some ideas of the description of orthogonal quadrangles weakly embedded in a projective space in Section 5.

6.2 The calculation of some image points

In this subsection, we deduce some first properties of the mapping $\alpha : \Lambda' \to \mathbb{K}$ defined in (6.1.7). The first two of the following lemmas and the properties of the weak embedding π (see Subsection 2.2) are used throughout Section 6 for the calculation of images under π of points of $\mathcal{Q}$. We must take care of the fact that at the beginning we do not know whether $\mathbb{K}$ is commutative or char $\mathbb{K} = 2$.

6.2.1 Lemma. *If* a*,* b *are points of* $\mathcal{Q}$ *with* $N = \langle a, b \rangle$ *a line of* $\mathcal{Q}$*, then* $\langle \pi(N \cap S) \rangle = \langle \pi(a), \pi(b) \rangle$*.*

PROOF. Since π is injective on singular points, we see that $\langle \pi(a), \pi(b) \rangle$ is a line which is contained in the line $\langle \pi(N \cap S) \rangle$. $\square$

6.2.2 Lemma. *The following holds:*

(a) *If x, y are points of $\mathcal{Q}$ such that $H = \langle x, y \rangle$ is a hyperbolic line of W, then $\langle \pi(x), \pi(y) \rangle \cap \langle \pi(H^\perp \cap S) \rangle = 0$.*

(b) *Let $H_1 = \langle x_1, y_1 \rangle$, $H_2 = \langle x_2, y_2 \rangle$ be hyperbolic lines in W with $H_2 \subseteq {H_1}^\perp$. For $z \in \{x_1, y_1, x_2, y_2\}$, we choose $z' \in V$ such that $\pi(z) = \langle z' \rangle$. If $v \in \langle \pi({H_1}^\perp \cap S) \rangle \cap \langle \pi({H_2}^\perp \cap S) \rangle$ and $a, b, c, d \in \mathbb{K}$ with $a{x_1}' + b{y_1}' + c{x_2}' + d{y_2}' = 0$, then $a = b = c = d = 0$.*

PROOF. The proof of (a) is the same as in (5.2.3) and (a) implies (b). $\square$

6.2.3 Notation. Let the vectors $x_1, y_1, x_2, y_2, {x_1}', {x_2}'$ and the mapping $\alpha : \Lambda' \to \mathbb{K}$ be as introduced in (6.1.7) and set $H_1 = \langle x_1, y_1 \rangle$, $H_2 = \langle x_2, y_2 \rangle$. Let $0 \neq \lambda \in \Lambda'$ be fixed throughout Subsection 6.2. We choose ${y_1}', {y_2}' \in V$ such that

$$\begin{aligned} \pi(y_1) &= \langle {y_1}' \rangle, & \pi(y_1 - \lambda x_2) &= \langle {y_1}' - \alpha(\lambda){x_2}' \rangle, \\ \pi(y_2) &= \langle {y_2}' \rangle, & \pi(\lambda x_1 + y_2) &= \langle \alpha(\lambda){x_1}' + {y_2}' \rangle. \end{aligned}$$

Then ${x_1}', {y_1}', {x_2}', {y_2}'$ are linearly independent by (6.2.2). For $c_1, c_2, c_3, c_4 \in \mathbb{K}$, we write $(c_1, c_2, c_3, c_4) := c_1{x_1}' + c_2{y_1}' + c_3{x_2}' + c_4{y_2}'$.

6.2.4 Lemma. *For $c \in \Lambda'$, we have*

(a) $\pi((0; 0, 1, 0, -c)) = \langle (0, 1, 0, -\alpha(\lambda)\alpha(c)\alpha(\lambda)^{-1}) \rangle$,

(b) $\pi((0; -\lambda c, 1, -\lambda, -c)) = \langle (-\alpha(\lambda)\alpha(c), 1, -\alpha(\lambda), -\alpha(\lambda)\alpha(c)\alpha(\lambda)^{-1}) \rangle$.

PROOF. We have $y_1 - cy_2 \in \langle cx_1 + x_2, y_1 - \lambda x_2 - c(y_2 + \lambda x_1) \rangle$. Hence there exists $A \in \mathbb{K}$ such that $\pi(y_1 - cy_2)$ is contained in $\langle {y_1}', {y_2}' \rangle$ and in $\langle \alpha(c){x_1}' + {x_2}', {y_1}' - \alpha(\lambda){x_2}' - A({y_2}' + \alpha(\lambda){x_1}') \rangle$. Comparing coefficients yields (a). For (b) we use that $-\lambda cx_1 + y_1 - \lambda x_2 - cy_2$ is contained in $\langle cx_1 + x_2, y_1 - cy_2 \rangle$ and in $\langle y_1 - \lambda x_2, y_2 + \lambda x_1 \rangle$. $\square$

6.2.5 Lemma. *There exists $r' \in \langle \pi({H_1}^\perp \cap S) \rangle \cap \langle \pi({H_2}^\perp \cap S) \rangle$ such that*

$$\begin{aligned} \pi(a_1) &= \langle -\alpha(\lambda)r' + (-\alpha(\lambda)^2, 1, 0, 0) \rangle, \quad \textit{for } a_1 := (-\lambda; -\lambda^2, 1, 0, 0), \\ \pi(a_2) &= \langle -r' + (0, 0, 1, 1) \rangle, \quad \textit{for } a_2 := (-1; 0, 0, 1, 1). \end{aligned}$$

PROOF. For $a := (0; -\lambda^2, 1, -\lambda, -\lambda)$, we have $a_1 = a + \lambda a_2$. Hence by (6.2.4)(b) there exists $v \in V$ such that $\pi(a_2) = \langle v \rangle$, $\pi(a_1) = \langle (-\alpha(\lambda)^2, 1, -\alpha(\lambda), -\alpha(\lambda)) + \alpha(\lambda)v \rangle$. The claim follows with $r' := (0, 0, 1, 1) - v$. □

6.2.6 Lemma. *Let a_1, a_2, r' be as in* (6.2.5). *For $c \in \Lambda'$, we have*

(a) $\pi((-c; -c^2, 1, 0, 0)) = \langle -\alpha(\lambda)\alpha(c)\alpha(\lambda)^{-1} r' + (-\alpha(\lambda)\alpha(c)\alpha(\lambda)^{-1}\alpha(c), 1, 0, 0) \rangle$,

(b) $\pi((1; c, 0, -1, -1)) = \langle r' + (\alpha(c), 0, -1, -1) \rangle$,

(c) $\pi((0; 0, 1, -c, 0)) = \langle (0, 1, -\alpha(\lambda)\alpha(c)\alpha(\lambda)^{-1}, 0) \rangle$,

(d) $\pi((0; c, 0, 0, 1)) = \langle (\alpha(\lambda)\alpha(c)\alpha(\lambda)^{-1}, 0, 0, 1) \rangle$.

PROOF. We may assume $c \neq 0$. Since $z := (-c; -c^2, 1, 0, 0)$ is contained in the line $\langle a_2, -c(cx_1 + x_2) + (y_1 - cy_2) \rangle$ of $\mathcal{Q}$, there exist by (6.2.4), (6.2.5) $z' \in V$, $A, B \in \mathbb{K}$ such that $\pi(z) = \langle z' \rangle$, $z' = A\alpha(c)(x_2{}' + y_2{}' - r') - B(\alpha(c)x_1{}' + x_2{}') + (y_1{}' - \alpha(\lambda)\alpha(c)\alpha(\lambda)^{-1} y_2{}')$. We use (6.2.2)(a) for H_2 and (a) follows.

For (b), we use that $cx_1 - a_2$ is contained in the line $\langle z + cx_2, y_1 - cy_2 \rangle$ of $\mathcal{Q}$; for (c) that $y_1 - cx_2 \in \langle cx_1 - a_2, z + cy_2 \rangle$; for (d) that $cx_1 + y_2 \in \langle c(x_1 + x_2) - (y_1 - y_2), y_1 - cx_2 \rangle$. □

6.2.7 Lemma. *We have $\alpha(c)\alpha(\lambda) = \alpha(\lambda)\alpha(c)$ for $c \in \Lambda'$.*

PROOF. Since $y_1 - cy_2$ is contained in the line $\langle cx_1 + x_2, -c(x_1 + y_2) + (y_1 - x_2) \rangle$ of $\mathcal{Q}$, there exist $A, B, C \in \mathbb{K}$ such that

$$(0, 1, 0, -\alpha(\lambda)\alpha(c)\alpha(\lambda)^{-1}) = A(\alpha(c), 0, 1, 0) + B(C, 1, -1, C)$$

by (6.2.4)(a) and (6.2.6)(c), (d) for scalar 1. Comparing coefficients yields the claim. □

6.2.8 Lemma. *Let a_1, a_2, r' be as in* (6.2.5). *For $0 \neq c \in \Lambda'$, we have*

(a) $\pi((0; -c\lambda, 1, -c, -\lambda)) = \langle (-\alpha(c)\alpha(\lambda), 1, -\alpha(c), -\alpha(\lambda)) \rangle$,

(b) $\pi((1; c - \lambda, 0, -1, -1)) = \langle r' + (\alpha(c) - \alpha(\lambda), 0, -1, -1) \rangle$.

PROOF. We apply (6.2.7), (6.2.6) and (6.2.4)(a). For (a), we use that $z := -c\lambda x_1 + y_1 - cx_2 - \lambda y_2$ is contained in the two lines $\langle \lambda x_1 + x_2, y_1 - \lambda y_2 \rangle$ and $\langle cx_1 + y_2, y_1 - cx_2 \rangle$ of $\mathcal{Q}$. For $z_0 := (-c; -c^2, 1, 0, 0)$, $(c - \lambda)x_1 - a_2$ is contained in the line $\langle z, z_0 + (c - \lambda)y_2 \rangle$ of $\mathcal{Q}$. □

6.3 The embedding $\alpha : \Lambda' \to \mathbb{K}$

We use the notation of (6.2.3) with scalar $\lambda = 1$.

6.3.1 Lemma. *The mapping* $\alpha : \Lambda' \to \mathbb{K}$ *introduced in* (6.1.7) *respects addition. In particular* char $\mathbb{K} = 2$. *Further,* $\alpha(c)\alpha(\lambda) = \alpha(\lambda)\alpha(c)$ *for* $c, \lambda \in \Lambda'$.

PROOF. By (6.2.7), (6.2.6)(c), (d) we have $\pi(cx_1 + y_2) = \langle \alpha(c){x_1}' + {y_2}' \rangle$, $\pi(y_1 - cx_2) = \langle {y_1}' - \alpha(c){x_2}' \rangle$ for $c \in \Lambda'$. This shows that ${y_1}', {y_2}'$ are such that we may apply the results of Subsection 6.2 for arbitrary $0 \neq \lambda \in \Lambda'$. The last statement hence is (6.2.7). By (6.2.6)(b), (6.2.8)(b) we may calculate $\pi((1; c - \lambda, 0, -1, -1))$ in two ways, hence $\alpha(c - \lambda) = \alpha(c) - \alpha(\lambda)$ for $0 \neq c, \lambda \in \Lambda'$. Let $0, 1 \neq c \in \Lambda'$ (c exists, since $\mathbb{L} \neq \mathbf{GF}(2)$). Then $\alpha(c) = \alpha((c-1)-1) = \alpha(c-1) - 1 = \alpha(c) - 1 - 1$. Hence char $\mathbb{K} = 2$ and α respects addition. $\square$

6.3.2 Lemma. *We have* $\alpha(\lambda^2) = \alpha(\lambda)^2$ *for* $\lambda \in \Lambda'$.

PROOF. For $0 \neq \lambda \in \Lambda'$, we have $z := (\lambda; 0, 0, \lambda^2, 1) \in \langle (\lambda; \lambda^2, 1, 0, 0), (0; \lambda^2, 1, \lambda^2, 1) \rangle$. Hence there exists $z' \in V$, $A \in \mathbb{K}$, such that $\pi(z) = \langle z' \rangle$,

$$z' = \alpha(\lambda)r' + (\alpha(\lambda)^2, 1, 0, 0) + A(\alpha(\lambda^2), 1, \alpha(\lambda^2), 1)$$

by (6.2.6)(a), (6.2.8). We apply (6.2.2)(a) for H_1 and obtain the claim. $\square$

6.4 The semi-linear mapping $\varphi : \Lambda \to V$

We use the notation of (6.2.3) with scalar $\lambda = 1$.

6.4.1 Lemma. *For* $l \in \Lambda$, *there exists a unique* $\varphi(l) \in \langle \pi({H_1}^{\perp} \cap S) \rangle$ *with* $\pi((l; l^2, 1, 0, 0)) = \langle \varphi(l) + (\alpha(l^2), 1, 0, 0) \rangle$. *Further,* $\pi((l; 0, 0, l^2, 1)) = \langle \varphi(l) + (0, 0, \alpha(l^2), 1) \rangle$.

PROOF. For $z_1 := (l; l^2, 1, 0, 0)$, $z_2 := (l; 0, 0, l^2, 1)$, we see that z_1 is contained in the line $\langle z_1 - z_2, z_2 \rangle$ of $\mathcal{Q}$. Hence by (6.2.8)(a) there exists $v \in V$ such that $\pi(z_2) = \langle v \rangle$, $\pi(z_1) = \langle (\alpha(l^2), 1, \alpha(l^2), 1) + v \rangle$. With $\varphi(l) := (0, 0, \alpha(l^2), 1) + v$ the existence of $\varphi(l)$ is clear. The uniqueness follows with (6.2.2), thus the claim. $\square$

6.4.2 Lemma. *For* $l \in \Lambda$, $c, d \in \Lambda'$, *we have* $\alpha(l^2 d) = \alpha(l^2)\alpha(d)$ *and*

(a) $\pi((l; c, 0, l^2, 1)) = \langle \varphi(l) + (\alpha(c), 0, \alpha(l^2), 1) \rangle$,

(b) $\pi((l;l^2,1,c,0)) = \langle\varphi(l) + (\alpha(l^2),1,\alpha(c),0)\rangle$,

(c) $\pi((l;l^2+cd,1,c,d)) = \langle\varphi(l) + (\alpha(l^2)+\alpha(c)\alpha(d),1,\alpha(c),\alpha(d))\rangle$.

PROOF. We may assume $l \neq 0$. Since $(l;c,0,l^2,1)$ is contained in the line

$$\langle(l^{-1};l^{-2},1,0,0) + l^{-2}c(0;0,0,0,1),(0;l^{-2}c+l^{-2},1,1,l^{-2}c+l^{-2})\rangle$$

of $\mathcal{Q}$, we obtain that $\pi(l;c,0,l^2,1)) = \langle\varphi(l)+(\alpha(l^2)\alpha(l^{-2}c),0,\alpha(l^2),1)\rangle$. Further, $(l;l^2,1,c,0)$ is contained in the line $\langle l;c,0,l^2,1),(0;c+l^2,1,c+l^2,1)\rangle$ of $\mathcal{Q}$. This yields $\alpha(l^2)\alpha(l^{-2}c) = \alpha(c)$ and (a), (b). For (c), we use that $(l;l^2+cd,1,c,d)$ is contained in the lines

$$\langle(l;l^2,1,c,0),(0;c,0,0,1)\rangle \text{ and } \langle(l;l^2d+c,0,l^2,1)+(0;(l^2+c)(d+1),1,l^2+c,d+1)\rangle$$

of $\mathcal{Q}$. This yields $\alpha(l^2d) = \alpha(l^2)\alpha(d)$ and (c). $\square$

6.4.3 Remark. Since $\mathbb{L} = \langle\Lambda\rangle$, we may extend the result of (6.4.2) to $\alpha(l^2c) = \alpha(l^2)\alpha(c)$ for $l \in \mathbb{L}$, $c \in \Lambda'$.

6.4.4 Lemma. *For $l,k \in \Lambda$, $c \in \Lambda'$, we have $\varphi(l+k) = \varphi(l)+\varphi(k)$ and $\varphi(cl) = \alpha(c)\varphi(l)$.*

PROOF. The first claim follows from $(l+k;l^2+k^2,1,0,0) \in \langle(k;0,0,k^2,1),(l;l^2+k^2,1,k^2,1)\rangle$ and the second one from $(cl;c^2l^2,1,0,0) \in \langle(l;0,0,l^2,1),(0;c^2l^2,1,cl^2,c)\rangle$, using (6.4.2). $\square$

6.4.5 Proof of Theorem 6.1.5

Let $\alpha : \Lambda' \to \mathbb{K}$, $\varphi : \Lambda \to V$ be the mappings introduced in (6.1.7). By (6.3.1), (6.3.2), (6.4.3) α is an embedding. Let V_1 be a complement of $\langle x_1', y_1', x_2', y_2'\rangle$ in V which contains $\pi(\langle x_1,y_1\rangle^\perp \cap S) \cap \pi(\langle x_2,y_2\rangle^\perp \cap S)$. By (6.4.4) $\varphi : \Lambda \to V_1$ is a semi-linear mapping in the sense of (6.1.3). The image points under π are as stated in Theorem (6.1.5), see (6.4.2)(a), (c).

For the uniqueness of α and φ, we first observe that the subspace $\mathbb{K}^4$ in Theorem (6.1.5) is $\mathbb{K}^4 = \langle\pi(x_1),\pi(y_1),\pi(x_2),\pi(y_2)\rangle$. Let $\mathcal{B} = \{\widetilde{x_1},\widetilde{y_1},\widetilde{x_2},\widetilde{y_2}\}$ be a second basis of $\mathbb{K}^4$ and $\beta : \Lambda' \to \mathbb{K}$ be an embedding, $\psi : \lambda \to V$ be a semi-linear mapping such that the conclusion of Theorem (6.1.5) holds for $\mathcal{B},\beta,\psi$. Since $\langle x_1'\rangle = \langle\widetilde{x_1}\rangle, \langle x_2'\rangle = \langle\widetilde{x_2}\rangle, \langle x_1'+x_2'\rangle = \langle\widetilde{x_1}+\widetilde{x_2}\rangle$, there exists $0 \neq c \in \mathbb{K}$ such that $\widetilde{x_1} = cx_1'$, $\widetilde{x_2} = cx_2'$. Similarly, $\widetilde{y_1} = cy_1'$, $\widetilde{y_2} = cy_2'$. Since $\langle\alpha(a)x_1'+x_2'\rangle = \langle\beta(a)\widetilde{x_1}+\widetilde{x_2}\rangle$, this implies that $\beta(a) = c\alpha(a)c^{-1}$ for $a \in \Lambda'$. Since $\langle\varphi(l)+\alpha(l^2)x_1'+y_1'\rangle = \langle\psi(l)+\beta(l^2)\widetilde{x_1}+\widetilde{y_1}\rangle$ and $\varphi(l),\psi(l) \in \langle\pi(H_1{}^\perp \cap S)\rangle$, we obtain $\psi(l) = c\varphi(l)$ for $l \in \Lambda$ by (6.2.2)(a). This proves the Theorem 6.1.5.

7 Dual hermitian and dual orthogonal quadrangles

7.1 Dual hermitian quadrangles

From now on, it will be convenient to identify a weakly embedded quadrangle Γ with its image Γ^π in $\mathbf{PG}(d, \mathbb{K})$. By abuse of language, we will say that Γ is weakly embedded in $\mathbf{PG}(d, \mathbb{K})$.

7.1.1 Lemma. *Let Γ be any generalized quadrangle weakly embedded of degree > 2 in $\mathbf{PG}(d, \mathbb{K})$, for some skewfield $\mathbb{K}$. Then Γ admits non-trivial central line-elations.*

PROOF. Let p, q, q' be three mutually opposite points of Γ which are collinear in $\mathbf{PG}(d, \mathbb{K})$. According to Lefèvre-Percsy [4], there exists a central collineation with center p mapping q to q'. $\square$

Lemma 7.1.1 implies that, if Γ is a dual hermitian quadrangle weakly embedded of degree > 2 in $\mathbf{PG}(d, \mathbb{K})$, then the hermitian quadrangle Γ^D admits non-trivial axial point-elations. Now consider the following description of the hermitian quadrangles, see Tits [17].

Let V be a right vector space over some skewfield $\mathbb{K}$, let $g : V \times V \to \mathbb{K}$ be a $(\sigma, 1)$-linear form for some anti-automorphism σ of $\mathbb{K}$ whose square is the identity. Put

$$\begin{cases} \mathbb{K}_\sigma &= \{t^\sigma - t : t \in \mathbb{K}\}, \\ q &: V \to \mathbb{K}/\mathbb{K}_\sigma : x \mapsto g(x,x) + \mathbb{K}_\sigma, \\ f &: V \times V \to \mathbb{K} : (x,y) \mapsto g(x,y) + g(y,x)^\sigma \end{cases}$$

and suppose that q is non-degenerate and has Witt index 2, cp. Tits [15], Section 8. We know that we can write V as

$$V = e_{-2}\mathbb{K} \bigoplus e_{-1}\mathbb{K} \bigoplus V_0 \bigoplus e_1\mathbb{K} \bigoplus e_2\mathbb{K},$$

such that

$$q(x_{-2}, x_{-1}, x_0, x_1, x_2) = x_{-2}^\sigma x_2 + x_{-1}^\sigma x_1 + q_0(x_0),$$

with $x_i \in e_i\mathbb{K}$, $i = -2, -1, 1, 2$ and $x_0 \in V_0$, and where q_0 is a non-degenerate anisotropic σ-quadratic form (so $q_0^{-1}(0) = 0 \in V_0$).

Let $R_1 = \mathbb{K}$ and put $R_2 = \{(k_0, k_1) \in V_0 \times \mathbb{K} : k_1 \in -q_0(k_0)\}$. We define an addition in R_2, and a scalar multiplication as follows. For $(k_0, k_1), (l_0, l_1) \in R_2$ and $a \in \mathbb{K}$, we put:

$$\begin{aligned} (k_0, k_1) \oplus (l_0, l_1) &= (k_0 + l_0, k_1 + l_1 - f(k_0, l_0)), \\ a \otimes (k_0, k_1) &= (k_0 a, a^\sigma k_1 a). \end{aligned}$$

It is straightforward to check that all these operations are well defined. Also, $R_2, \oplus$ is a group, not necessarily commutative. Now we can introduce intrinsic coordinates for the dual of the corresponding hermitian quadrangle Γ (see VAN MALDEGHEM [19]). The points of Γ^D are the elements (∞), (a), (k,b), and (a,l,a'), where $a,b,a' \in R_1$, $k,l \in R_2$ (with $k=(k_0,k_1)$ and $l=(l_0,l_1)$; we will assume this for every element of R_2 from now on) and ∞ is a new symbol; the lines of Γ^D are the elements $[\infty]$, $[k]$, $[a,l]$ and $[k,b,k']$ with $a,l \in R_1$ and $k,l,k' \in R_2$. Incidence is given by

$$[k,b,k']\,\mathbf{I}\,(k,b)\,\mathbf{I}\,(k)\,\mathbf{I}\,(\infty)\,\mathbf{I}\,[\infty]\,\mathbf{I}\,(a)\,\mathbf{I}\,[a,l]\,\mathbf{I}\,(a,l,a'),$$

with obvious notation, and by $(a,l,a')\,\mathbf{I}\,[k,b,k']$ if and only if

$$\left\{\begin{array}{rcl} (k'_0,k'_1) & = & (l_0,l_1)\oplus(a^\sigma\otimes(k_0,k_1))\oplus(0,aa'^\sigma - a'a^\sigma), \\ b & = & a' - ak_1 + f(l_0,k_0). \end{array}\right.$$

One can easily check that the $((0),[\infty],(\infty))$-elation which maps $(0,0)$ to $(0,B)$ has the following action on the lines concurrent with $[\infty]$:

$$[a,(l_0,l_1)] \mapsto [a,(l_0,l_1 - aB^\sigma + Ba^\sigma)].$$

For an axial elation, we must have $-aB^\sigma + Ba^\sigma = 0$, for all $a \in R_1$ and this implies readily that σ is the identity, a contradiction. Hence we have shown:

7.1.2 Lemma. *No dual hermitian quadrangle is weakly embedded of degree > 2 in projective space.*

Note that the previous lemma is certainly false for degree 2 as there are orthogonal quadrangles which are the dual of certain hermitian quadrangles (and every orthogonal quadrangle has a standard embedding of degree 2 in some projective space).

7.2 Dual orthogonal quadrangles

7.2.1 Lemma. *A generalized quadrangle Γ which is weakly embedded in $\mathbf{PG}(d,\mathbb{K})$ and for which there exists a line L (of Γ) such that all points of L in $\mathbf{PG}(d,\mathbb{K})$ are also points of Γ is fully embedded. In other words: if a weakly embedded quadrangle contains at least one full line, then all lines are full and the quadrangle is fully embedded.*

PROOF. Suppose first that the degree of the weak embedding is equal to 2. Let L be a fully embedded line. Since Γ is supposed to be thick, we may assume that there is some line M of Γ opposite L which is not full. The space generated by L and M meets Γ in a weak quadrangle (a grid), and so L and M belong to a regulus $\mathcal{R}$. Every line in the opposite regulus belongs to Γ because every point in $\mathbf{PG}(d,\mathbb{K})$ on L belongs to Γ. Hence also every point in $\mathbf{PG}(d,\mathbb{K})$ on M belongs to Γ.

Now suppose that the degree δ satisfies $\delta > 2$. We may assume that there is a full line L and a non-full line M of Γ meeting in a point p of Γ. Let N be a third line of Γ in the plane $\alpha := LM$ of $\mathbf{PG}(d,\mathbb{K})$ (then N necessarily contains p). Consider the group U of root elations fixing every point on N and fixing every line through p and q, where q is any point of Γ on N different from p. By Lemma 4.0.2, the group U is a subgroup of $\mathbf{PSL}_{d+1}(\mathbb{K})$, and it acts simply-transitively on the points of L distinct from p. Hence U can be seen as the group of translations in π with axis N and center p. Hence it acts transitively on the points of M distinct from p. Hence all these points belong to Γ, since at least one of them does. $\square$

By Section 6, we may assume that the dual orthogonal quadrangle is not a mixed quadrangle, i.e., the corresponding bilinear form f_0 (see Tits [17]; in fact, in the description of the hermitian quadrangles in Subsection 7.1 above, we put $\sigma = 1$ and then f_0 is the restriction of f to $V_0 \times V_0$) is not identical zero. Hence it follows that f_0 is surjective, so, with the usual notation, $[U_1, U_3] = U_2$, where U_2 is a root group of (central) line-elations. Now let p be the center of the line-elations belonging to U_2; let U_1 be the set of all (q_1, L_1, p)-elations, and let U_3 be the set of all (p, L_3, q_3)-elations. Let M be any line of Γ through q_1. Let x be the projection of q_3 onto M. We remark that both U_1 and U_3 preserve the 3-dimensional space W generated by M, L_1, L_3. Hence also U_2 preserves W. Hence $\{q_1, q_3\}^\perp$ must be contained in W, which is clearly only possible when $W = \mathbf{PG}(d,\mathbb{K})$, hence $d = 3$.

Definition. Let us call an orthogonal quadrangle *of dimension* d' if it has a standard embedding in d'-dimensional projective space, and if it is not a mixed quadrangle.

7.2.2 Theorem. *If Γ^D is a d'-dimensional orthogonal quadrangle, with $d' = 4, 5, 7$, and Γ is weakly embedded of degree > 2 in $\mathbf{PG}(d,\mathbb{K})$, then $d = 3$ and Γ is a symplectic, hermitian or quaternion quadrangle (and hence fully embedded over some sub(skew)field of $\mathbb{K}$) by Section 5).*

PROOF. If $d' = 4$, then Γ is a symplectic quadrangle and the result follows.

Now let $d' = 5$. Let $q_0(x_1, x_2) = Ax_1^2 + Bx_1x_2 + Cx_2^2$ be the associated quadratic form. Then we have as corresponding bilinear form $f_0((x_1, x_2), (y_1, y_2)) = 2Ax_1y_1 + B(x_1y_2 + x_2y_1) + 2Cx_2y_2$. Since Γ is not a mixed quadrangle, f_0 is not identical 0, which means that $B \neq 0$ in characteristic 2. This implies that $q_0(x, 1)$ defines always a quadratic Galois extension $\mathbb{L}$ of the ground field $\mathbb{F}$ over which Γ is defined. It is now easy to see that the 3-dimensional hermitian quadrangle over $\mathbb{L}$ with as corresponding involutory automorphism the unique non-trivial element of the Galois group $\mathrm{Gal}(\mathbb{L}/\mathbb{F})$ is dual to Γ.

Note that, if Γ is mixed in the previous paragraph, then $q_0(x,1)$ defines an inseparable quadratic field extension $\mathbb{L}$ of $\mathbb{F}$ and we have $\mathbb{L}^2 \subseteq \mathbb{F} \subseteq \mathbb{L}$. Hence the weak embedding of Γ ($= \mathcal{Q}(\mathbb{L},\mathbb{F};\mathbb{L},\mathbb{F}) \simeq \mathcal{Q}(\mathbb{L}^2,\mathbb{F}^2;\mathbb{L}^2,\mathbb{F}^2)$ in the notation of (6.1.1)) in $\mathbf{PG}(3,\mathbb{K})$ is by Remark (6.1.6) full over a subfield $\mathbb{L}'$ of $\mathbb{K}$ isomorphic to $\mathbb{L}$.

Finally let $d' = 7$. Since Γ is not mixed, Γ^D contains a 5-dimensional subquadrangle $(\Gamma^*)^D$ which is not a mixed quadrangle. Indeed, Γ^D being mixed is equivalent with all points of Γ^D being regular. So we may assume that, by transitivity, there is no regular point. This implies that there are points x_1, x_2, x_3, y_1, y_2, y_3 with $x_i \perp y_j$ if and only if $(i,j) \neq (3,3)$. The 5-dimensional space (in the standard embedding of Γ^D) generated by these six points intersects Γ^D in a 5-dimensional subquadrangle which is not a mixed quadrangle. Every 4-dimensional subquadrangle of $(\Gamma^*)^D$ is the dual of a symplectic one. We fix such a 4-dimensional subquadrangle Γ_0^D. Then Γ_0 is a symplectic quadrangle which is fully embedded in some subspace $\mathbf{PG}(3,\mathbb{F})$ of $\mathbf{PG}(3,\mathbb{K})$ for some subfield $\mathbb{F}$ of $\mathbb{K}$ (see Section 5). Also, Γ^* is fully embedded in some subspace $\mathbf{PG}(3,\mathbb{L}^*)$ for some subfield $\mathbb{L}^*$ of $\mathbb{K}$ and $\mathbb{L}^*$ is a quadratic Galois extension of $\mathbb{F}$. Let $\{1,x\}$ be a basis of $\mathbb{L}^*$ over $\mathbb{F}$. Let $(\Gamma^{**})^D$ be a second subquadrangle of Γ^D containing Γ_0^D and such that $(\Gamma^{**})^D$ is the intersection of Γ^D with a 5-dimensional projective subspace in its standard 7-dimensional orthogonal embedding. Then Γ^{**} is fully embedded in some subspace $\mathbf{PG}(3,\mathbb{L}^{**})$ of $\mathbf{PG}(3,\mathbb{K})$ over some subfield $\mathbb{L}^{**}$, and $\mathbb{L}^{**}$ is a quadratic (not necessarily Galois) extension of $\mathbb{F}$. Let $\{1,y\}$ be a basis of $\mathbb{L}^{**}$ over $\mathbb{F}$.

We now show that $\mathbb{F}, x, y$ generate a non-commutative subskewfield $\mathbb{D}$ of $\mathbb{K}$ which is 4-dimensional over $\mathbb{F}$. Hence $\mathbb{D}$ is a standard quaternion division algebra over $\mathbb{F}$.

We fix a line L of $\mathbf{PG}(3,\mathbb{K})$ which is also a line of Γ_0. We can coordinatize L with $\mathbb{K} \cup \{\infty\}$ in such a way that the points of Γ_0, respectively Γ^*, Γ^{**}, on L are coordinatized with $\mathbb{F} \cup \{\infty\}$, respectively $\mathbb{L}^* \cup \{\infty\}$, $\mathbb{L}^{**} \cup \{\infty\}$. Now note that, if $a, b \in \mathbb{K}$ are the coordinates on L of points of Γ, then also $a+b$ is the coordinate of a point on L of Γ (using the Moufang condition). Also, by considering suitable 5-dimensional subquadrangles of Γ^D containing Γ_0^D, one sees that a^{-1} and also every element of $\mathbb{F}a + \mathbb{F}b$ corresponds with a point of Γ, and that every element of $\mathbb{F}$ commutes with a (and hence $\mathbb{F}$ is in the center of $\mathbb{D}$). Moreover, $\mathbb{F} + \mathbb{F}a$ is a subfield of $\mathbb{K}$ and hence every element a of $\mathbb{K}$ which corresponds with a coordinate of a point of L belonging to Γ is quadratic over $\mathbb{F}$, i.e., a satisfies a quadratic equation with coefficients in $\mathbb{F}$. Now note that the formula

$$(a^{-1} + (b^{-1} - a)^{-1})^{-1} = a - aba$$

of Mendelsohn [5] is also true in the non-commutative case. Applied to $a = x$ and $b = y$, this shows that $xyx \in \mathbb{F} + \mathbb{F}x + \mathbb{F}y$. It is easily seen that the coefficient of y is not equal to zero (otherwise $y^{-1} \in \mathbb{F} + \mathbb{F}x$), hence we can write $xyx \in \mathbb{F} + \mathbb{F}x + \mathbb{F}^\times y$, where $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$. We now show that $\mathbb{D}$ is equal to $\mathbb{F} + \mathbb{F}x + \mathbb{F}y + \mathbb{F}xy$. Suppose that x satisfies $x^2 - Ax - B = 0$, with $A \in \mathbb{F}$ and $B \in \mathbb{F}^\times$. Then

$$yx = (y(xx))x^{-1} = Ay + Byx^{-1} \in \mathbb{F}y + \mathbb{F}^\times yx^{-1}. \tag{3}$$

Also, we have

$$xy = (xyx)x^{-1} \in \mathbb{F}x^{-1} + \mathbb{F} + \mathbb{F}yx^{-1} = \mathbb{F} + \mathbb{F}x + \mathbb{F}^{\times}yx^{-1}. \tag{4}$$

Combining Equations (3) and (4), we see that $yx \in \mathbb{F} + \mathbb{F}x + \mathbb{F}y + \mathbb{F}xy$. It now follows easily that $\mathbb{D} = \mathbb{F} + \mathbb{F}x + \mathbb{F}y + \mathbb{F}xy$. Now we show that $xy \notin \mathbb{F} + \mathbb{F}x + \mathbb{F}y$. Suppose by way of contradiction that $xy = C + Dx + Ey$, with $C, D, E \in \mathbb{F}$. Multiplying this equation at the left with x, and substituting $Ax + B$ for x^2, we obtain

$$Axy + By = (C + DA)x + DB + Exy,$$

hence $(A - E)xy = DB + (C + DA)x - By$. If $A = E$, then since $1, x, y$ are linearly independent over $\mathbb{F}$ (otherwise $\Gamma^* = \Gamma^{**}$), $B = 0$, a contradiction to $x \notin \mathbb{F}$. Hence $A - E \neq 0$, and we have

$$E = \frac{-B}{A - E},$$

which implies $E^2 = AE + B$. Hence the quadratic equation $u^2 - Au - B = 0$ in the unknown u over the field $\mathbb{L}^*$ has two solutions in $\mathbb{F}$ and consequently $x \in \mathbb{F}$, a contradiction. Hence $\mathbb{D}$ is 4-dimensional over $\mathbb{F}$. It remains to show that $\mathbb{D}$ is non-commutative. Suppose on the contrary that $\mathbb{D}$ is commutative. Then $xy = yx$. Multiplying both sides at the left with x, we see that $x^2y = Axy + By = xyx \in \mathbb{F} + \mathbb{F}x + \mathbb{F}y$, hence $A = 0$. So $x^2 \in \mathbb{F}$. Similarly, $y^2 \in \mathbb{F}$. By interchanging the roles of y and $x + y$, we also have $(x + y)^2 \in \mathbb{F}$. This implies $2xy \in \mathbb{F}$, hence the characteristic of F is equal to 2. This means that $\mathbb{L}^*$ is a non-Galois extension of $\mathbb{F}$, a contradiction.

It is clear that the points of L with coordinates in $\mathbb{F} + \mathbb{F}x + \mathbb{F}y \cup \{\infty\}$ are precisely the points on L of a subquadrangle Γ' of Γ with Γ'^D a 6-dimensional subquadrangle of Γ^D (this follows immediately from consideration of the subquadrangle generated by $(\Gamma^*)^D$ and the line M of Γ^D corresponding to the point with coordinate y, noting that every line of that subquadrangle which is incident with the point of Γ^D corresponding with the line L of Γ, can be obtained from the line corresponding with the coordinate 0 by applying an elation fixing the line corresponding with the coordinate ∞ and generated by the elations in $(\Gamma^*)^D$ and $(\Gamma^{**})^D$). Hence there exists a coordinate z corresponding with a point of Γ on L with $z \notin \mathbb{F} + \mathbb{F}x + \mathbb{F}y$. If $z \in \mathbb{D}$, then it follows easily that the set of coordinates of points on L belonging to Γ is precisely $\mathbb{D} \cup \{\infty\}$. Suppose now $z \notin \mathbb{D}$. We seek a contradiction. Note that $\mathbb{F}, x, z$ generate a skewfield which is 4-dimensional over its center $\mathbb{F}$, and similarly for $\mathbb{F}, y, z$. Consider the subskewfield $\mathbb{O}$ of $\mathbb{K}$ generated by $\mathbb{D}$ and z. We claim that $\mathbb{O}$ is equal to the subspace $\mathbb{S}$ over $\mathbb{F}$ with

$$\mathbb{S} = \mathbb{F} + \mathbb{F}x + \mathbb{F}y + \mathbb{F}xy + \mathbb{F}z + \mathbb{F}xz + \mathbb{F}yz + \mathbb{F}xyz.$$

For this, we only have to show that $x\mathbb{S} = y\mathbb{S} = z\mathbb{S} = \mathbb{S}x = \mathbb{S}y = \mathbb{S}z$. Since $x^2yz = (ax + B)yz$, we immediately have $x\mathbb{S} = \mathbb{S}$. Similarly for $\mathbb{S}z = \mathbb{S}$. For $y\mathbb{S}$, we note that

$y(xz) = (yx)z \in \mathbb{D}z \subseteq \mathbb{S}$ and also $y(xyz) = (yxy)z \in \mathbb{D}z$, hence $y\mathbb{S} = \mathbb{S}$. Similarly $\mathbb{S}y = \mathbb{S}$. For $\mathbb{S}x$, we have to show that $yzx \in \mathbb{S}$ and $xyzx \in \mathbb{S}$. But $yzx = y(zx) \in y\mathbb{S} = \mathbb{S}$ and $xyzx = x(yzx) \in x\mathbb{S} = \mathbb{S}$. Similarly for $z\mathbb{S} = \mathbb{S}$. Hence we have shown that $\mathbb{O} = \mathbb{S}$.

Similarly as above, one shows that $xyz \notin \mathbb{F} + \mathbb{F}x + \mathbb{F}y + \mathbb{F}xy + \mathbb{F}z + \mathbb{F}xz + \mathbb{F}yz$. Hence the dimension of $\mathbb{O}$ over $\mathbb{F}$ is equal to 8. Since $\mathbb{O}$ is a skewfield, and $\mathbb{F}$ is easily seen to be the center of $\mathbb{O}$, this is a contradiction (the dimension should be a perfect square).

Hence we have proved that $z \in \mathbb{D}$ and so the set of all coordinates of points of L in Γ is equal to $\mathbb{D} \cup \{\infty\}$.

Now let L' be a line of Γ_0 opposite L, and let M, M' be the lines of Γ_0 concurrent with L' and meeting L in the points with respective coordinates 0 and ∞. Then L, M, L', M' are the sides of an apartment of Γ_0. We can take as points of a reference system the intersections $L \cap M = e_1$, $M \cap L' = e_2$, $L' \cap M' = e_3$ and $M' \cap L = e_4$. We choose the unit point e in the space $\mathbf{PG}(3, \mathbb{F})$ in which Γ_0 is fully embedded. Since the dual of every 5-dimensional subquadrangle of Γ^D containing Γ_0 is fully embedded in some $\mathbf{PG}(3, \mathbb{L})$ over some subfield $\mathbb{L}$ containing $\mathbb{F}$ and such that $\mathbf{PG}(s, \mathbb{F})$ is contained in $\mathbf{PG}(3, \mathbb{L})$, we see that the points of Γ on L together with e_2, e_3 and e generate a subspace $\mathbf{PG}(3, \mathbb{D})$ which contains all points of Γ on the lines L, L', M, M'. Let p be an arbitrary point of Γ not collinear with e_i, $i = 1, 2, 3, 4$. Let N be the line of $\mathbf{PG}(3, \mathbb{K})$ meeting both L and L' and incident with p. Put $L \cap N = \{q\}$ and $L' \cap N = \{q'\}$. Let r respectively r' be the point on L respectively L' collinear in Γ with p. Let q_0 respectively q_0' be the point of Γ on L respectively L' collinear in Γ with r' respectively r. Then clearly p must lie in the planes q_0, q_0', r and q_0, q_0', r'. Hence q_0, q_0' and p are collinear in $\mathbf{PG}(3, \mathbb{K})$ and so $q_0 = q$ and $q_0' = q'$. Similarly, p lies on a line which meets both M and M' in points of $\mathbf{PG}(3, \mathbb{D})$. Hence p lies in $\mathbf{PG}(3, \mathbb{D})$. It is now easily seen that all points of Γ lie in $\mathbf{PG}(3, \mathbb{D})$ (by varying the points with coordinates 0 and ∞ on L) and hence Γ is weakly embedded in $\mathbf{PG}(3, \mathbb{D})$. But it has at least one full line, namely, L. Hence it is fully embedded in $\mathbf{PG}(3, \mathbb{D})$. Clearly, Γ is a quaternion quadrangle. $\square$

7.2.3 Theorem. *If Γ^D is a d'-dimensional orthogonal quadrangle, with $d' \geq 4$, and Γ is weakly embedded of degree > 2 in $\mathbf{PG}(d, \mathbb{K})$, then $d = 3$ and $d' \leq 7$.*

PROOF. We have already shown that $d = 3$. Suppose now $d' > 7$. By taking a suitable subquadrangle, we may assume that $d' = 8$. Let L be as in the proof of Theorem 7.2.2, and also choose x and y similarly. Since $d' > 7$, we can now find z not belonging to $\mathbb{F} + \mathbb{F}x + \mathbb{F}y + \mathbb{F}xy$, where $\mathbb{F}$ is also defined similarly as in the previous proof. But, as in that proof, this leads to a contradiction (a skewfield of dimension 8 over its center). $\square$

The last case that remains is the case $d' = 6$. We use the notation of Section 3.

7.2.4 Theorem. *If Γ^D is a 6-dimensional orthogonal quadrangle over some field $\mathbb{F}$ and Γ is weakly embedded of degree > 2 in $\mathbf{PG}(d, \mathbb{K})$, then $d = 3$ and Γ is a standard embedding* *in subspace* $\mathbf{PG}(3,\mathbb{L})$ *of a special subquadrangle of some quaternion quadrangle over* $\mathbb{F}$, $\mathbb{F}$ *a subfield of* $\mathbb{K}$, $\mathbb{L}$ *a quaternion skewfield over* $\mathbb{F}$ *inside* $\mathbb{K}$*, i.e., there exists a* 7*-dimensional orthogonal (dual quaternion) quadrangle* Γ_* *over* $\mathbb{F}$ *with corresponding quaternion skewfield* $\mathbb{L}$ *such that the points and lines of* Γ *are points and lines of a full embedding of* Γ_*^D *over a subskewfield* $\mathbb{D}$ *of* $\mathbb{K}$ *isomorphic to* $\mathbb{L}$.

PROOF. We can copy the proof of Theorem 7.2.2, case $d' = 7$, up to the points where we obtain a 6-dimensional subquadrangle Γ'^D, which coincides now with Γ. Also the last part of that proof can be copied: Γ lies in $\mathbf{PG}(3,\mathbb{D})$, which is a subspace of $\mathbf{PG}(d,\mathbb{K})$ (implying $d = 3$) over the subskewfield $\mathbb{D}$, which is a quaternion skewfield over $\mathbb{F}$. We take the same notation as in that last paragraph. If we have two collinear points r and r' of Γ with r on L and r' on L', then a little calculation inside the dual of the orthogonal or mixed quadrangle defined by the 4-dimensional orthogonal quadrangle Γ_0 and the line of Γ^D corresponding with r shows that, if $(1,0,0,1)$ and $(0,1,1,0)$ are two collinear points in Γ (and we can always choose the coordinates as such), the coordinates $(x_1,0,0,x_4)$ and $(0,x_2,x_3,0)$ of r and r' are related by (if $x_1 \neq 0 \neq x_4$) $x_2 = x_4^{-\sigma}$ and $x_3 = x_1^{-\sigma}$, where σ is the identity in $\mathbb{F}$, and also in every field $\mathbb{F}(t)$ if $\mathbb{F}(t)$ is a non-Galois extension of $\mathbb{F}$; and where σ is the unique non-trivial element of the Galois group of $\mathbb{F}(t)$ if the latter is a Galois extension of $\mathbb{F}$. For this calculation, see also Dienst [2], or Van Maldeghem [19]. But then one sees that σ is the restriction of the standard involution in $\mathbb{D}$. Also, under the same assumptions, the coordinates $(x_1,x_2,0,0)$, $x_1 \neq 0 \neq x_2$, and $(0,0,x_3,x_4)$ of collinear points on M respectively M' satisfy $x_3 = x_1^{-\sigma}$ and $x_4 = -x_2^{-\sigma}$. So if $p = (x_1,x_2,x_3,x_4)$ is a point in $\mathbf{PG}(3,\mathbb{D})$ of Γ, then, by the argument of the last paragraph of the proof of Theorem 7.2.2, and since p lies on the lines determined by $(x_1,0,0,x_4)$, $(0,x_2,x_3,0)$ respectively $(x_1,x_2,0,0)$, $(0,0,x_3,x_4)$, p is in Γ collinear with the points $(0,x_4^{-\sigma},x_1^{-\sigma},0)$, $(x_3^{-\sigma},0,0,x_2^{-\sigma})$, $(0,0,x_1^{-\sigma},-x_2^{-\sigma})$ and with $(x_3^{-\sigma},-x_3^{-\sigma},0,0)$. All points collinear with p must lie in a plane. If we use (x_1,x_2,x_3,x_4), $(0,x_4^{-\sigma},x_1^{-\sigma},0)$, $(0,0,x_1^{-\sigma},-x_2^{-\sigma})$ and $(x_3^{-\sigma},-x_4^{-\sigma},0,0)$ to express this, then we obtain after a short calculation

$$x_1^\sigma x_3 - x_3^\sigma x_1 + x_2^\sigma x_4 - x_4^\sigma x_2 = 0, \qquad (*)$$

for all x_1,x_2,x_3,x_4 all different from 0. But this relation is easily extended to the other cases (if only one coordinate is zero, e.g., $x_3 = 0$, then the above calculation still holds noting that $(0,0,x_3,x_4) = (0,0,0,x_4)$ is collinear in Γ with $(1,0,0,0)$; if at least two coordinates are zero, then either p lies on $L \cup L' \cup M \cup M'$ and the result follows, or p has some coordinates $(x_1,0,x_3,0)$ or $(0,x_2,0,x_4)$. Assume for instance $p = (x_1,0,x_3,0)$. Then p is collinear with $(0,1,0,0)$ in Γ, and hence there is some point $p' = (x_1,x_2,x_3,0)$ of Γ with $x_2 \neq 0$. The assertion now follows by applying the previous results to p').

So we have shown that Γ is a subquadrangle of the hermitian quadrangle defined by the equation $(*)$ above, which is clearly the quaternion quadrangle over $\mathbb{D}$. The points on the line L are parametrized by $\mathbb{F} + \mathbb{F}x + \mathbb{F}y \cup \{\infty\}$ (with the notation of the previous proof). This completes the proof. $\square$

8 Exceptional Moufang quadrangles

8.1 Exceptional quadrangles of type E_i, $i = 6, 7, 8$

The exceptional quadrangles will be treated as a class of Moufang quadrangles of type BC_2 extending the orthogonal quadrangles which are not mixed quadrangles. As such, our approach is independent of the classification of exceptional Moufang quadrangles, except that we will assume that there is no exceptional quadrangle extending an orthogonal quadrangle of dimension $d \leq 7$. Indeed, the classification implies that only orthogonal quadrangles of dimension $d \geq 8$ can be extended, see Tits & Weiss [18]); for the types E_i, $u = 6, 7, 8$, this is obvious, for type F_4, this follows from the observation that the ideal full subquadrangles are on a C_4-building, see Van Maldeghem [19], Appendix C.

Let Γ be an exceptional Moufang quadrangle. Then Γ contains an ideal orthogonal subquadrangle Γ' of dimension $d > 7$. Since ${\Gamma'}^D$ is not weakly embedded of degree > 2 in some projective space by the previous section, and since, if ${\Gamma'}^D$ is weakly embedded of degree 2 in some projective space, then obviously, all the lines of ${\Gamma'}^D$ are regular and thus Γ' is a mixed quadrangle (a contradiction), Γ^D cannot be weakly embedded of degree > 2 in some projective space. If Γ is of type F_4, the dual argument implies that neither Γ can be weakly embedded of degree > 2 in some projective space. Also, since the point-elation groups of the exceptional quadrangles of type E_i, $i = 6, 7, 8$, are non-commutative, Γ cannot be weakly embedded of degree > 2 in projective space (because the point-elation groups can be seen as groups of elations in projective planes with a fixed axis and fixed center, see above).

So we have shown:

8.1.1 Theorem. *If Γ is an exceptional Moufang quadrangle, then neither Γ nor Γ^D admits a weak embedding of degree > 2 in some projective space.*

References

[1] A. M. Cohen, *Point–line spaces related to buildings.* In: F. Buekenhout, editor, *Handbook of Incidence Geometry, Buildings and Foundations*, Chapter 12, North-Holland, Amsterdam (1995), 647 – 737.

[2] K. J. Dienst, Verallgemeinerte Vierecke in projektiven Räumen, *Arch. Math.* (Basel) **35** (1980), 177 – 186.

[3] C. Lefèvre-Percsy, Quadrilatères généralisés faiblement plongés dans $\mathbf{PG}(3, q)$, *European J. Combin.* **2** (1981), 249 – 255.

[4] C. Lefèvre-Percsy, Projectivités conservant un espace polaire faiblement plongé, *Acad. Roy. Belg. Bull. Cl. Sci.* **67** (1981), 45 – 50.

[5] N. S. Mendelsohn, (problem in problem section) *Amer. Math. Monthly* **51** (1944), 171.

[6] S. E. Payne & J. A. Thas, *Finite Generalized Quadrangles*, Pitman, Boston London Melbourne, 1984.

[7] A. Steinbach, Classical polar spaces (sub-)weakly embedded in projective spaces, *Bull. Belg. Math. Soc. Simon Stevin* **3** (1996), 477 – 490.

[8] A. Steinbach, Generalized quadrangles arising from groups generated by abstract transvection groups, submitted.

[9] J. A. Thas, *Generalized Polygons.* In: F. Buekenhout, editor, *Handbook of Incidence Geometry, Buildings and Foundations*, Chapter 9, North-Holland, Amsterdam (1995), 383 – 431.

[10] J. Thas & H. Van Maldeghem, Orthogonal, symplectic and unitary polar spaces sub-weakly embedded in projective space, *Compositio Math.* **103** (1996), 75 – 93.

[11] J. Thas & H. Van Maldeghem, Generalized quadrangles weakly embedded in finite projective space, to appear in *J. Statist. Plan. Inference.*

[12] J. Tits, Sur la trialité et certains groupes qui s'en déduisent, *Inst. Hautes études Sci. Publ. Math.* **2** (1959), 13 – 60.

[13] J. Tits, Quadrangles de Moufang, I. Preprint, 1976.

[14] J. Tits, Endliche Spiegelungsgruppen, die als Weylgruppen auftreten, *Invent. Math.* **43** (1977), 283 – 295.

[15] J. Tits, *Buildings of spherical type and finite BN-pairs*, *Lecture Notes in Math.* **386**. Springer–Verlag, Berlin Heidelberg New York, 1974.

[16] J. Tits, Moufang polygons, I. Root data, *Bull. Belg. Math. Soc. Simon Stevin* **1** (1994), 455 – 468.

[17] J. Tits, Résumé de cours, *Annuaire du Collège de France*, 95e année, 1994-1995, 79 – 95.

[18] J. Tits & R. Weiss, *Classification of Moufang Polygons*, book in preparation.

[19] H. Van Maldeghem, *Generalized Polygons, A Geometric Approach*, Birkhäuser, to appear.

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