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Characterizations by Automorphism Groups of Some Rank 3 Buildings – IV: Hyperbolic p-adic Moufang Buildings of Rank 3

H. VAN MALDEGHEM* and K. VAN STEEN
Department of Pure Mathematics and Computer Algebra, University of Ghent, Galglaan 2, 9000 Gent, Belgium; e-mail: hvm@cage.rug.ac.be and kvs@cage.rug.ac.be

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Abstract. In this paper, we introduce the p-adic Moufang condition for hyperbolic buildings of rank 3. It is the most obvious and simplest generalization of the p-adic Moufang condition for affine buildings, introduced in Part III of this sequence of papers. We show that p is very restricted, which confirms (but does not prove) the conjecture that no p-adic analogue is possible for the construction of Moufang (hyperbolic) buildings by Ronan and Tits.

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1. Introduction

In the previous parts of this paper, we have characterized several classes of rank 3 affine Bruhat–Tits buildings (the 'classical' examples of such buildings) by assumptions on the automorphism group. One of the conditions was the p-adic Moufang condition, which is in a way complementary to the usual Moufang condition in that it is satisfied by affine buildings over local fields with a characteristic different from its residue field, whereas the usual Moufang condition implies that the local field has the same characteristic as its residue field (see, e.g., Van Maldeghem and Van Steen [8]).

An open problem in the theory of affine buildings is whether the p-adic buildings can be constructed in a similar way as the Moufang affine buildings in Ronan and Tits [2]. Such a construction would probably give rise to 'p-adic' hyperbolic (and other types of) buildings. In the present paper, we have tried to give the simplest generalization of the notion of p-adic Moufang to compact hyperbolic rank 3 buildings (i.e., the rank 2 residues are not trees). We show that in this case p is very restricted, which makes it very unlikely that such hyperbolic buildings exist. This, in turn, makes it very unlikely that a p-adic analogue to the construction of Ronan and Tits [2] exists. At the same time, some geometric insight is gained in

* The first author is a Research Director of the Fund for Scientific Research – Flanders (Belgium).

the structure of hyperbolic buildings of rank 3, which is the second motivation for this paper.

We will view the apartments of a hyperbolic building of rank 3 as tessellations of the hyperbolic plane into congruent triangles (the latter are the chambers). The notion of a p-adic Moufang building in the affine case requires the notion of parallel walls, hence we will have to define this in the hyperbolic case, too. Also, we will view the hyperbolic plane itself as the points inside the unit circle C in a(n oriented) Euclidean real plane $\mathbb{E}$. The lines are the parts of the lines through the center of C inside C, and also the parts of the circles of $\mathbb{E}$ inside C meeting C perpendicular. This way, it will make sense to talk about clockwise progressing around a point (and it should be clear what is meant by it; once an orientation is chosen, everything below will be independent of it).

Moreover, we will say that a compact hyperbolic building of rank 3 is of type Δ_{i-j-k} if the residues of the vertices of different type are, respectively, generalized i-gons, j-gons and k-gons.

An interesting feature that comes along with our result is that apparently the primes 2 and 3, but also the hyperbolic buildings of type Δ_{3-4-6} play a special role.

2. Definitions and Statement of the Main Result

2.1. PARALLELISM IN HYPERBOLIC APARTMENTS

We denote by $St(P)$, for P a point in an apartment Σ, the set of all walls of Σ through P. Also, $cl(P, Q)$, for two points P and Q lying on the same wall N, is the intersection of all half apartments containing both P and Q. It is obviously a subset of N (the 'interval $[P, Q]$').

DEFINITION 1. Two walls M and N in an apartment Σ of a (compact) hyperbolic rank 3 building are *separating walls* if they intersect as hyperbolic lines in a hyperbolic plane.

DEFINITION 2. Suppose M and N are separating walls in some apartment Σ of a compact hyperbolic rank 3 building Δ. Then the wall M' is called *N-parallel to M* ($M'//_N M$) if and only if the following conditions are satisfied:

(i) M' separates N.

(ii) Let $\{N = h_0, \ldots, h_{r-1}\}$ be the set of walls in Σ that are incident with the vertex P determined by M and N. Let $\{N = h'_0, \ldots, h'_{s-1}\}$ be the set of walls in Σ, all incident with the vertex P' determined by M' and N. The labelling is done in such a way that if one progresses clockwise around P (respectively around P') starting from N, h_i (h'_j) is met just before h_{i+1} (h'_{j+1}) for all i, $0 \leqslant i \leqslant r-2$ (for all j, $0 \leqslant j \leqslant s-2$). If $M = h_i$ for some i, $1 \leqslant i \leqslant r-1$, then $M' = h'_j$ of $\{N = h'_0, \ldots, h'_{s-1}\}$ with $i/r = j/s$.

DEFINITION 3. Suppose M and N are separating walls of some apartment Σ of a compact hyperbolic rank 3 building Δ. Then two vertices P and Q on N that are incident (in the hyperbolic plane) with walls which are N-parallel to M are called (N, M)-*successive* if and only if none of the vertices on $N \setminus \{P, Q\}$ that are in $cl(P, Q)$ contains a wall N-parallel to M.

In view of Definition 3, we introduce a new distance $\delta'_{N,M}(P, Q)$ between two vertices P and Q on N such that P and Q are contained in walls N-parallel to some wall M, with M separating N. The value of $1 + \delta'_{N,M}(P, Q)$ is defined as the number of vertices on N in $cl(P, Q)$ that contain a wall N-parallel to M.

2.2. THE p-ADIC MOUFANG CONDITION

Let us denote by $\partial\alpha$ the boundary of the root α in some apartment Σ of Δ. We assume that roots (or, equivalently, half apartments) are closed, i.e., $\partial\alpha \subseteq \alpha$. Also, we denote by $-\alpha$ the root in Σ *opposite* α, i.e., $-\alpha = (\Sigma \setminus \alpha) \cup \partial\alpha$.

DEFINITION 4. A compact hyperbolic rank 3 building Δ satisfies the *p-adic Moufang condition* (*with respect to a given apartment* Σ), p a prime, if for every two separating walls M and N in Σ and for every root α in Σ with $\partial\alpha = M$ a group $pU(\alpha, N)$ exists such that the following conditions are satisfied (where P_0 is the intersection point of N and M).

(HPM1) Every element of $pU(\alpha, N)$ fixes every chamber having a panel in $\alpha \setminus \partial\alpha$.

(HPM2) $pU(\alpha, N)$ acts transitively on the chambers of $St(\pi) \setminus \{c\}$, with π a panel in $\partial\alpha$ and c the chamber of α in $St(\pi)$ (where $St(\pi)$ is the set of chambers in Δ containing π).

(HPM3) Suppose π is a panel in $\partial\alpha$ and suppose that c' is the chamber in $St(\pi) \cap \Sigma$ not in α. If $g \in \mathrm{Stab}_{pU(\alpha,N)}(c')$, then g fixes the wall M' of $\Sigma \setminus \alpha$ that is N-parallel to $\partial\alpha$ and for which P_0 and the vertex determined by M' and N are (N, M)-successive.

(HPM4) Suppose π is a panel in $\partial\alpha$ and suppose that c' is the chamber of $St(\pi) \cap \Sigma$ not in α. Then $(pU(\alpha, N))^p = \mathrm{Stab}_{pU(\alpha,N)}(c')$ and $\mathrm{Stab}_{pU(\alpha,N)}(c') = pU(\beta, N)$, where $\beta \supset \alpha$ is the root of Σ such that $\partial\beta$ is N-parallel to $\partial\alpha$, and such that the vertex Q determined by $\partial\beta$ and N, and the vertex P determined by $\partial\alpha$ and N are (N, M)-successive.

(HPM5) If α and β are roots in Σ with $\partial\alpha$ and $\partial\beta$ separating walls, both separating N, with $\partial\alpha \cap \partial\beta \cap N$ nonempty, and with $N \subset (\alpha \cap \beta) \cup ((-\alpha) \cap (-\beta))$, then $[pU(\alpha, N), pU(\beta, N)] \leqslant pU(\,]\alpha, \beta[\,, N)$, where $pU(]\alpha, \beta[, N)$ denotes the group generated by all groups $pU(\gamma, N)$ with γ satisfying $\alpha \cap \beta \subset \gamma$ and $(-\alpha) \cap (-\beta) \subset (-\gamma)$ and with $\gamma \notin \{\alpha, \beta\}$. If such a root γ does not exist, then $pU(]\alpha, \beta[, N)$ is, by definition, trivial.

The set of roots γ satisfying $\alpha \cap \beta \subset \gamma$ and $(-\alpha) \cap (-\beta) \subset (-\gamma)$ and $\gamma \notin \{\alpha, \beta\}$, as in (HPM5) above is sometimes itself denoted by $]\alpha, \beta[$. The condition $N \subset$

$(\alpha \cap \beta) \cup ((-\alpha) \cap (-\beta))$ in (HPM5) implies that for every $\gamma \in]\alpha, \beta[$, the wall $\partial\gamma$ separates N.

Our main result reads as follows:

THEOREM IV. *In characteristic neither* 2 *nor* 3, *p-adic Moufang hyperbolic rank* 3 *buildings of type different from* Δ_{3-4-6} *do not exist.*

Moreover, we will show that the above definition can be applied to affine buildings and that it precisely gives the definition of p-adic Moufang affine building of rank 3.

3. Proof of Theorem IV

LEMMA 5. *If* Δ *is a thick compact hyperbolic rank* 3 *building that satisfies the* p*-adic Moufang condition with respect to some apartment* Σ*, then for every vertex* P *in* Σ*, the residue* $\mathrm{Res}(P)$ *in* Δ *is a Moufang generalized polygon.*

Proof. For every root α in Σ such that $\partial\alpha$ contains P, and for any wall N in Σ that separates $\partial\alpha$ and is incident with P, the group $pU(\alpha, N)$ clearly induces in $\mathrm{Res}(P)$ a root group. We conclude that $\mathrm{Res}(P)$ satisfies the Moufang condition. □

Hence, by a result of Weiss [12] (see also Tits [5, 6]), we must only deal with hyperbolic buildings of type Δ_{i-j-k} with $i, j, k \in \{2, 3, 4, 6, 8\}$.

LEMMA 6. *If* Δ *is a* p*-adic Moufang building of type* Δ_{i-j-k}*, then* $8 \notin \{i, j, k\}$.

Proof. Since Moufang octagons only exist in characteristic 2 (see Tits [7]), it is clear that $p = 2$. But by *loc. cit.*, every Moufang octagon has root collineations of order 4, hence Condition (HPM4) can never be satisfied. □

A crucial tool is the following well-known and elementary result in group theory.

LEMMA 7. *Suppose* g *and* h *are arbitrary elements of some group* Υ*. If* $[[g, h], h] = [[g, h], g] = 1$*, then* $[g^r, h^s] = [g, h]^{rs}$.

Proof. See Gorenstein [1] (Lemma 22(i)). □

LEMMA 8. *Suppose* Δ *is a thick compact hyperbolic rank* 3 *building satisfying the* p*-adic Moufang condition with respect to some apartment* Σ *in* Δ*. Let* α *and* β *be different roots in* Σ *such that* $]\alpha, \beta[= \{\gamma\}$*, for some root* γ *in* Σ*, then* $[g^{p^i}, h^{p^j}] = [g, h]^{p^{i+j}}$*, for all* $g \in pU(\alpha, N)$*, for all* $h \in pU(\beta, N)$*, and for all natural numbers* i, j.

Proof. Since Δ satisfies (HPM5), we have the relations

$$[pU(\alpha, N), pU(\beta, N)] \leqslant pU(\gamma, N),$$

$$[pU(\gamma, N), pU(\alpha, N)] = 1 \quad \text{and} \quad [pU(\gamma, N), pU(\beta, N)] = 1.$$

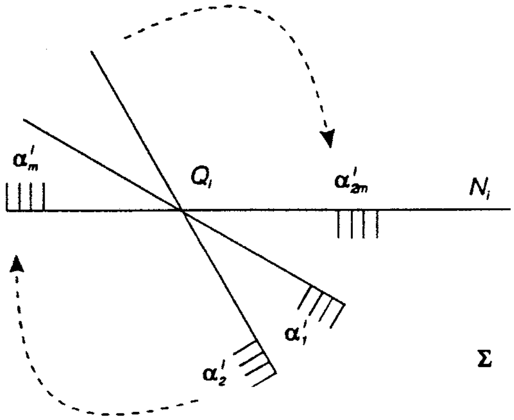

Figure 1.

Hence $[[g, h], g] = 1$ and $[[g, h], h] = 1$. The result now follows from Lemma 7.□

For the remaining proofs of this section, we assume that Σ is a given apartment of a rank 3 compact hyperbolic building Δ, and that Δ satisfies the p-adic Moufang condition with respect to Σ. We further denote by N_1 and N_2 two separating walls in Σ. The intersection point of N_1 with N_2 is denoted by P.

The *label* of a point Q_i on N_i ($i \in \{1, 2\}$) is m ($\in \mathbb{N}$) if $\text{Res}(Q_i)$ in Δ is a generalized m-gon. In the pictures, we sometimes (also) denote the points by their labels.

Suppose Q_i is a point of N_i ($i \in \{1, 2\}$) and suppose that $\text{Res}(Q_i)$ is a generalized m-gon for a specific value of $m \geqslant 2$. Then we let $\alpha_1^i, \ldots, \alpha_{2m}^i$ be roots in Σ (in a natural clockwise cyclic order) the boundary of every one of them containing Q_i, as in Figure 1.

Suppose $\text{Res}(P)$ is a generalized m-gon for some m in $\{2, 3, 4, 6\}$. Then we denote by $U_1, \ldots, U_{2m}$ the root groups (in a natural cyclic order) in the apartment of $\text{Res}(P)$ determined by Σ. Note that the walls corresponding to these root groups can be identified with the walls of Δ in Σ through P. Hence, we can choose indices in such a way that U_j corresponds with ${\alpha'}_j^i$, $1 \leqslant j \leqslant 2m$. We then note that U_j is induced by $pU(\alpha_j^i, N_i)$, and hence independent of the (suitable) wall N_i through Q_i ($i = 1, 2$).

THEOREM 9. *No p-adic Moufang rank* 3 *building of type* Δ_{3-3-4} *exists for* $p \neq 2$.

Proof. The wall N_1 contains vertices with label 3 and 4. Let P and Q be two vertices on N_1 labelled 4 and at minimum distance. Let h' and g' be arbitrary elements of $pU(\alpha_1, N_1)$ respectively $pU(\alpha_3, N_1)$ inducing nontrivial elements in, respectively, U_1 and U_3 as in Figure 2, where the shaded areas of U_1 and U_3 mark the respective roots α_1 and α_3. Then by Condition (HPM4), an element

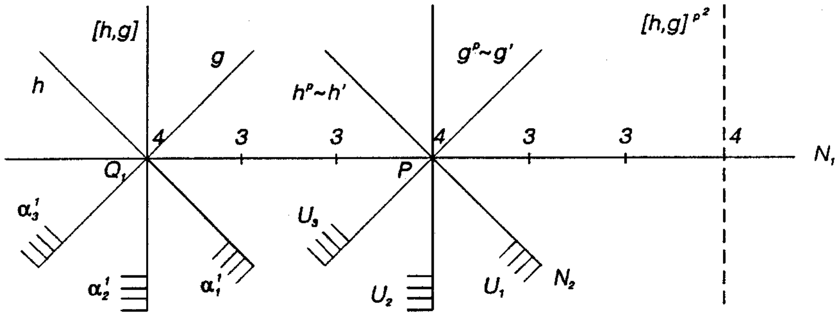

Figure 2.

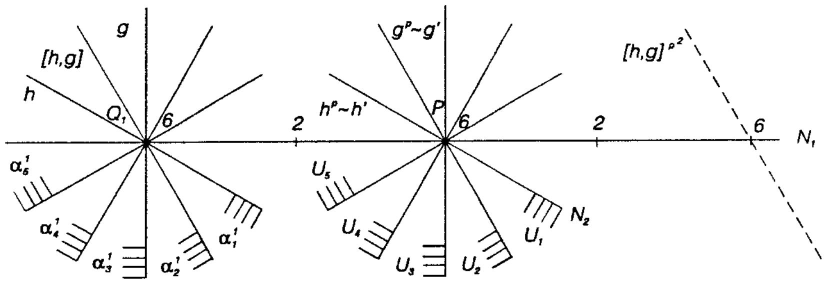

Figure 3.

$h \in pU(\alpha_1^1, N_1)$ and an element $g \in pU(\alpha_3^1, N_1)$ exist such that $h' = h^p$ and $g' = g^p$. By Lemma 8, we have $[h^p, g^p] = [h, g]^{p^2}$, hence $[h', g'] = 1$ is restricted to $\mathrm{Res}(P)$, because $[h, g]^{p^2}$ fixes the 'second' wall (drawn in dashes in Figure 2) N_1-parallel to $\partial\alpha_2^1$. By the arbitrary choice of h' and g', we conclude that $[U_1, U_3] = 1$. Similarly, replacing N_1 by N_2, one shows $[U_2, U_4] = 1$. By Tits [4], $p = 2$. □

In a similar way we prove the following theorem:

THEOREM 10. *No p-adic Moufang rank* 3 *building of type* Δ_{3-4-4} *or* Δ_{4-4-4} *exists for* $p \neq 2$.

This takes care of all cases with no residue isomorphic to a generalized hexagon.

THEOREM 11. *No p-adic Moufang rank* 3 *building of type* Δ_{2-4-6} *exists for* $p \neq 3$.

Proof. We use similar notation as in the previous proof, except that we now assume that the wall N_1 contains vertices labelled 2 and 6, and Q_1 and P are vertices of N_1 with label 6 and at minimum distance. Note that there is a unique vertex labelled 2 in $[P, Q_1]$, and through that vertex there is a unique wall distinct from N_1, and that wall is N_1-parallel to $\partial\alpha_3^1$. Let h' and g' be arbitrary elements of, respectively, $pU(\alpha_1, N_1)$ and $pU(\alpha_3, N_1)$ (corresponding with the root groups U_1 and U_3 respectively, see Figure 3).

Using Condition (HPM4), an element $h \in pU(\alpha_1^1, N_1)$ and an element $g \in pU(\alpha_3^1, N_1)$ exist such that $h' = h^p$ and $g' = g^{p^2}$ (the square of p appears here because there is an N_1-parallel wall between $\partial\alpha_3^1$ and $\partial\alpha_3$ – the latter through P corresponding with U_3 – namely one through the point labelled 2). By Lemma 8 we obtain $[h^p, g^{p^2}] = [h, g]^{p^3}$, and so, restricted to $\mathrm{Res}(P)$, we have $[h', g'] = 1$. This implies $[U_1, U_3] = 1$.

A similar argument using the wall N_2 (see Figure 3), which contains vertices labelled 6 and 4, implies $[U_2, U_4] = 1$. This implies that $p = 3$, see Tits [3]. □

In a similar way we prove the following theorem:

THEOREM 12. *No p-adic Moufang rank 3 building of type Δ_{a-6-6} ($a \in \{2, 3, 4, 6\}$) or Δ_{3-3-6} or Δ_{4-4-6} exists for $p \neq 3$.*

This shows Theorem IV.

4. The p-adic Moufang Condition Applied to Irreducible Affine Rank 3 Buildings

In the affine case, separating walls in an apartment Σ are walls that are nonparallel in the Euclidean plane. It is easily seen that N-parallel walls are just parallel walls, for all walls N. Now it is also easily seen that the convex closure of a half apartment α and a chamber having a panel in common with $\partial\alpha$ is a half apartment α' such that $\partial\alpha$ is parallel with $\partial\alpha'$ and both walls are at minimum distance (i.e., there is no wall parallel to $\partial\alpha$ in the closure of $\partial\alpha$ and $\partial\alpha'$).

In view of these remarks, it is now easy to see that condition (HPM3) is superfluous in the affine case, and that the notion of p-adic Moufang introduced in this paper and applied to affine rank 3 buildings is equivalent to the notion of affine p-adic Moufang buildings of rank 3 introduced by the second author in Part III [11].

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