Lax Embeddings of Polar Spaces in Finite Projective Spaces

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Abstract

A polar space $\mathcal{S}$ with point set P is laxly embedded in the projective space $\mathrm{PG}(d,q), d \geq 2$, if the following conditions are satisfied : (i) P is a point set of $\mathrm{PG}(d,q)$ which generates $\mathrm{PG}(d,q)$, and (ii) each line L of $\mathcal{S}$ is a subset of a line L' of $\mathrm{PG}(d,q)$, and distinct lines L_1, L_2 of $\mathcal{S}$ define distinct lines L'_1, L'_2 of $\mathrm{PG}(d,q)$. In this paper we determine all polar spaces of rank at least three which are laxly embedded in $\mathrm{PG}(d,q)$, where $d \geq 4$ if $\mathcal{S}$ is isomorphic to the polar space $W(2m+1,s), m \geq 2$ and s odd, arising from a symplectic polarity in $\mathrm{PG}(2m+1,s)$, and where $d \geq 3$ in all other cases. Laxly embedded generalized quadrangles were considered in a foregoing paper.

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1 Introduction

The geometry of points and lines of a non-singular quadric of projective Witt-index at least one in $\mathrm{PG}(n,q)$ is a polar space denoted by $Q^+(n,q)$ if n is odd and the quadric is hyperbolic, by $Q^-(n,q)$ if n is odd and the quadric is elliptic, and by $Q(n,q)$ if n is even; the quadric will be denoted by Q^+, Q^- and Q, respectively. The geometry of all points of $\mathrm{PG}(2m+1,q)$, $m \geq 1$, together with all totally isotropic lines of a symplectic polarity in $\mathrm{PG}(2m+1,q)$ is a polar space denoted by $W(2m+1,q)$. Finally, the geometry of points and lines of a non-singular hermitian variety H of projective Witt-index at least one in $\mathrm{PG}(n,q^2)$ is a polar space $H(n,q^2)$. For q even, the polar space $W(2m+1,q)$ is isomorphic to the polar space $Q(2m+2,q)$.

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A polar space $\mathcal{S}$ isomorphic to one of $Q^+(n,q)$, $Q^-(n,q)$, $Q(n,q)$, $W(2m+1,q)$, $H(n,q^2)$ is called a *finite classical polar space.* Any non-degenerate finite polar space (non-degenerate means that no point of the polar space is collinear with all points of the polar space) of projective Witt-index at least two, is classical; for more details see Buekenhout and Cameron [1995]. The following table gives the *ranks* (the rank is one more than the projective Witt-index) of these classical polar spaces.

$\mathcal{S}$ is isomorphic to	rank
$Q^+(2m+1,q)$	$m+1$
$Q^-(2m+1,q)$	m
$Q(2m,q)$	m
$H(2m+1,q^2)$	$m+1$
$H(2m,q^2)$	m
$W(2m+1,q)$	$m+1.$

If $\mathcal{S}$ is isomorphic to either one of $Q^+(n,q)$, $Q^-(n,q)$, $Q(n,q)$, $H(n,q^2)$, or to $W(n,q)$ with q odd, then n is called the *universal embedding dimension* of $\mathcal{S}$; if $\mathcal{S}$ is isomorphic to $W(n,q)$ with q even, then $n+1$ is the *universal embedding dimension* of $\mathcal{S}$. The universal embedding dimension of $\mathcal{S}$ will be denoted by $\mathrm{ued}(\mathcal{S})$.

Any polar space considered in this paper is assumed to be non-degenerate, unless explicitly mentioned otherwise.

In this paper we will identify a line L of a polar space $\mathcal{S}$ with the set of points incident with L. This way, we view the lines of $\mathcal{S}$ as subsets of the point set P of $\mathcal{S}$. This will be especially convenient for the purpose of this paper. Likewise, we view the lines of any projective space as subsets of the point set.

Definition 1 A polar space $\mathcal{S}$ with point set P is *laxly embedded* in the projective space $\mathrm{PG}(d,q)$, $d \geq 2$, if the following conditions are satisfied:

(*i*) P is a point set of $\mathrm{PG}(d,q)$ which generates $\mathrm{PG}(d,q)$;

(*ii*) each line L of $\mathcal{S}$ is a subset of a line L' of $\mathrm{PG}(d,q)$, and distinct lines L_1, L_2 of $\mathcal{S}$ define distinct lines L'_1, L'_2 of $\mathrm{PG}(d,q)$.

A lax embedding is called *full* if in (*ii*) above $L = L'$. Note that the description of the finite classical polar spaces above yields full embeddings of these. We call these full embeddings *natural embeddings* of the finite classical polar spaces (and consequently $W(2n+1,q) \cong (2n+2,q)$, q even, has two non-isomorphic natural embeddings).

Definition 2 A lax embedding of a polar space $\mathcal{S}$ is called *weak* if the set of points of $\mathcal{S}$ collinear in $\mathcal{S}$ with any given point is contained in a hyperplane of $\mathrm{PG}(d, q)$.

In [1981a] Lefèvre-Percsy defined "weakly embedded polar space" and in [1996] Thas and Van Maldeghem defined "sub-weakly embedded polar space". Further, in [1996], Thas and Van Maldeghem proved that these two notions are equivalent. In Section 2 we will show that a weakly embedded polar space in the sense of Definition 2 is the same as a weakly embedded (or equivalently sub-weakly embedded) polar space in the sense of Thas and Van Maldeghem [1996]

All full embeddings of finite polar spaces were classified by Buekenhout and Lefèvre [1974], [1976] and Lefèvre-Percsy [1977]. Only the natural embeddings of the classical finite polar spaces turn up. All weak embeddings in $\mathrm{PG}(3, q)$ of finite thick generalized quadrangles (a generalized quadrangle is a polar space of rank 2) are classified by Lefèvre-Percsy [1981a] and in $\mathrm{PG}(d, q)$, with $d > 3$, by Thas and Van Maldeghem [19**]; all weak embeddings in $\mathrm{PG}(d, q)$ of polar spaces of rank at least 3 were determined by Thas and Van Maldeghem [1996]. Every weak embedding in $\mathrm{PG}(d, q)$ of a polar space either turns out to be full in a subspace $\mathrm{PG}(d, q')$ over a subfield $\mathrm{GF}(q')$ of $\mathrm{GF}(q)$, or is the universal weak embedding of $W(3, 2)$ in a projective 4-space over an odd characteristic finite field.

In [19**] Thas and Van Maldeghem consider lax embeddings of generalized quadrangles in finite projective spaces. In particular they classify all finite thick Moufang generalized quadrangles, that is, all finite thick classical and dual classical generalized quadrangles, laxly embedded in $\mathrm{PG}(d, q)$, $d > 2$, with the exception of the lax embeddings of $W(3, s)$, s odd, in $\mathrm{PG}(3, q)$. We just mention here the part of their Main Theorem that will be used in the present paper.

If the generalized quadrangle $\mathcal{S}$ with parameters s and t (that is, each line contains $s + 1$ points and each point is on $t + 1$ lines), $s \neq 1$, is laxly embedded in $\mathrm{PG}(d, q)$, then $d \leq 5$. Also,

(*i*) If $d = 5$, then $\mathcal{S} \cong Q^-(5, s)$. Also, if $s \neq 2$ for q odd, then $\mathcal{S}$ is fully and naturally embedded in some subspace $\mathrm{PG}(5, s)$ of $\mathrm{PG}(5, q)$.

(*ii*) If $d = 4$, then $s \leq t$.

  (a) If $s = t$, then $\mathcal{S} \cong Q(4, s)$. Also, if $s \neq 2$ for q odd, and $s \neq 3$ for $q \equiv 1 \pmod 3$, then $\mathcal{S}$ is fully and naturally embedded in some subspace $\mathrm{PG}(4, s)$ of $\mathrm{PG}(4, q)$.

  (b) If $t^2 = s^3$, then $\mathcal{S} \cong H(4, s)$ and it is fully and naturally embedded in some subspace $\mathrm{PG}(4, s)$ of $\mathrm{PG}(4, q)$.

(c) If $\mathcal{S} \cong Q^-(5,s)$ and $s \neq 2$ for q odd, then there exists a PG$(5,q)$ containing PG$(4,q)$ and a point $x \in$PG$(5,q)\backslash$PG$(4,q)$ such that $\mathcal{S}$ is the projection from x onto PG$(4,q)$ of a generalized quadrangle $\widetilde{\mathcal{S}} \cong Q(5,s)$ which is fully and naturally embedded in some subspace PG$(5,s)$ of PG$(5,q)$.

(iii) $d = 3$.

(a) If $t = 1$, then $\mathcal{S}$ is a subgrid of order $(s,1)$ of some $Q^+(3,q)$ (trivial case).

(b) If $s = t^2$, then $\mathcal{S} \cong H(3,s)$ and $\mathcal{S}$ is fully and naturally embedded in some subspace PG$(3,s)$ of PG$(3,q)$.

(c) If $\mathcal{S} \cong H(4,s)$, then there exists a PG$(4,q)$ containing PG$(3,q)$ and a point $x \in$PG$(4,q)\backslash$PG$(3,q)$ such that $\mathcal{S}$ is the projection from x onto PG$(3,q)$ of a generalized quadrangle $\widetilde{\mathcal{S}} \cong H(4,s)$ which is fully and naturally embedded in a subspace PG$(4,s)$ of PG$(4,q)$.

(d) If $\mathcal{S} \cong Q(4,s)$, with $s \neq 2$ for q odd and $s \neq 3$ for $q \equiv 1 \pmod 3$, then there exists a PG$(4,q)$ containing PG$(3,q)$ and a point $x \in$PG$(4,q)\backslash$PG$(3,q)$ such that $\mathcal{S}$ is the projection from x onto PG$(3,q)$ of a generalized quadrangle $\widetilde{\mathcal{S}} \cong Q(4,s)$ which is fully and naturally embedded in a subspace PG$(4,s)$ of PG$(4,q)$.

(e) If $\mathcal{S} \cong Q^-(5,s)$, with $s \neq 2$ for q odd, then there exists a PG$(5,q)$ containing PG$(3,q)$ and a line L of PG$(5,q)$ skew to PG$(3,q)$ such that $\mathcal{S}$ is the projection from L onto PG$(3,q)$ of a generalized quadrangle $\widetilde{\mathcal{S}} \cong Q^-(5,s)$ which is fully and naturally embedded in a subspace PG$(5,s)$ of PG$(5,q)$.

Remark 1 The lax embeddings of the generalized quadrangles $\mathcal{S} \cong W(s)$, s odd, in PG$(3,q)$ are not yet classified.

Definition 3 A *full polar subspace* $\mathcal{S}'$ of a polar space $\mathcal{S}$ is a polar subspace of $\mathcal{S}$ whose lines are full lines of $\mathcal{S}$ and any two points of which are collinear in $\mathcal{S}'$ if and only if they are collinear in $\mathcal{S}$.

Main Result. *Assume that $\mathcal{S}$ is a polar space of rank at least three which is laxly embedded in* PG(d,q).

(i) If $d \geq 3$ and if $\mathcal{S}$ is isomorphic either to one of $Q^+(n,s)$, $Q^-(n,s)$, $Q(n,s)$, $H(n,s)$, or to $W(n,s)$ with s even, then there exists a PG(n,q) *containing* PG(d,q) *and a* PG$(n-d-1,q)$ *of* PG(n,q) *skew to* PG(d,q) *such that $\mathcal{S}$ is the projection from* PG$(n-d-1,q)$ *onto* PG(d,q) *of a polar space $\widetilde{\mathcal{S}} \cong \mathcal{S}$ which is fully and naturally embedded in a subspace* PG(n,s) *of* PG(n,q).

(*ii*) *If* $d \geq 4$ *and if* $\mathcal{S}$ *is isomorphic to* $W(2m+1,s)$, $m \geq 2$ *and* s *odd, then there exists a* $\mathrm{PG}(2m+1,q)$ *containing* $\mathrm{PG}(d,q)$ *and a* $\mathrm{PG}(2m-d,q)$ *of* $\mathrm{PG}(2m+1,q)$ *skew to* $\mathrm{PG}(d,q)$ *such that* $\mathcal{S}$ *is the projection from* $\mathrm{PG}(2m-d,q)$ *onto* $\mathrm{PG}(d,q)$ *of a polar space* $\widetilde{\mathcal{S}} \cong W(2m+1,s)$ *which is fully and naturally embedded in a subspace* $\mathrm{PG}(2m+1,s)$ *of* $\mathrm{PG}(2m+1,q)$.

We prove the Main Result in a sequence of theorems and lemmas.

2 Weak is weak

If $\mathcal{S}$ is a laxly embedded polar space in $\mathrm{PG}(d,q)$, then for each line L of $\mathcal{S}$, we denote by L' the (set of points on the) corresponding line of $\mathrm{PG}(d,q)$. In particular, we have $L \subseteq L'$.

Lemma 1 *If the polar space* $\mathcal{S}$ *is laxly embedded in* $\mathrm{PG}(d,q)$, $d \geq 2$, *and if* L *is any line of* $\mathcal{S}$, *then the points of* L *are the only points of* $\mathcal{S}$ *on the corresponding line* L' *of* $\mathrm{PG}(d,q)$.

Proof. Assume, by way of contradiction, that x is a point of $\mathcal{S}$ on $L' \backslash L$. If M is a line of $\mathcal{S}$ through x and concurrent with L, then also $M \subseteq L'$, contradicting (*ii*) in the definition of lax embedding. □

In Thas and Van Maldeghem [1996] a polar space $\mathcal{S}$ is said to be weakly embedded in $\mathrm{PG}(d,q)$, $d \geq 2$, if the following three conditions are satisfied:

(WE1) $\mathcal{S}$ is laxly embedded in $\mathrm{PG}(d,q)$,

(WE2) for any point x of $\mathcal{S}$, the subspace generated by $x^{\perp} = \{$ all points of $\mathcal{S}$ which are collinear in $\mathcal{S}$ with $x\}$ meets $\mathcal{S}$ precisely in $x^{\perp}$,

(WE3) if for two lines L_1 and L_2 of $\mathcal{S}$ the corresponding lines L_1' and L_2' of $\mathrm{PG}(d,q)$ meet in some point x, then x belongs to $\mathcal{S}$.

This is nothing else than the original definition of Lefèvre-Percsy [1981a], [1981b]. Thas and Van Maldeghem [1996] say that the polar space $\mathcal{S}$ is sub-weakly embedded in $\mathrm{PG}(d,q)$ if (WE1) and (WE2) are satisfied; in the same paper they prove that $\mathcal{S}$ is weakly embedded in $\mathrm{PG}(d,q)$ if and only if it is sub-weakly embedded in $\mathrm{PG}(d,q)$.

In the Introduction the polar space $\mathcal{S}$ was defined to be weakly embedded in $\mathrm{PG}(d,q)$, $d \geq 2$, if it is laxly embedded in $\mathrm{PG}(d,q)$ and for any point x of $\mathcal{S}$ the set $x^{\perp}$ is contained in a hyperplane of $\mathrm{PG}(d,q)$. In the next theorem we will prove that the two definitions of weak are equivalent.

Theorem 1 *Let $\mathcal{S}$ be a polar space which is laxly embedded in* $\mathrm{PG}(d, q)$*, $d \geq 2$. Then for any point x of $\mathcal{S}$ the subspace generated by $x^\perp$ meets $\mathcal{S}$ precisely in $x^\perp$ if and only if for any point x of $\mathcal{S}$ the set $x^\perp$ is contained in a hyperplane of* $\mathrm{PG}(d, q)$.

Proof. First, assume that for any point x of $\mathcal{S}$ the subspace $\langle x^\perp \rangle$ generated by x meets $\mathcal{S}$ precisely in $x^\perp$. If $x^\perp$ is not contained in a hyperplane of $\mathrm{PG}(d, q)$, then $\langle x^\perp \rangle$ is not contained in a hyperplane, so $\langle x^\perp \rangle = \mathrm{PG}(d, q)$, hence $\langle x^\perp \rangle$ contains $\mathcal{S}$, a contradiction.

Next, suppose that for any point x of $\mathcal{S}$ the set $x^\perp$ is contained in a hyperplane of $\mathrm{PG}(d, q)$. Assume, by way of contradiction, that the subspace $\langle x^\perp \rangle$ generated by x contains a point y of $\mathcal{S}$ not in $x^\perp$. As every line of $\mathcal{S}$ through y contains a point of $x^\perp$, we have that $y^\perp \subseteq \langle x^\perp \rangle$. Let z be a point of $\mathcal{S}$ not in $x^\perp \cup y^\perp$ and assume that there is a line of $\mathcal{S}$ through z which does not contain a point of $x^\perp \cap y^\perp$. As L contains exactly one point of $x^\perp$ and exactly one point of $y^\perp$, it follows that L belongs to $\langle x^\perp \rangle$, so $z \in \langle x^\perp \rangle$. Next, assume that z is a point of $\mathcal{S}$ not in $x^\perp \cup y^\perp$ and that every line of $\mathcal{S}$ through z contains a point of $x^\perp \cap y^\perp$. Let M be a line of $\mathcal{S}$ containing z and let m be the unique point of $x^\perp \cap y^\perp$ on M. If $\mathcal{S}$ has rank 2, then for any point $u \in (x^\perp \cap y^\perp) \setminus \{m\}$ we have $u \notin m^\perp$; if $\mathcal{S}$ has rank at least 3, then $x^\perp \cap y^\perp$ together with the lines of $\mathcal{S}$ in $x^\perp \cap y^\perp$ form a classical polar space, and so $(x^\perp \cap y^\perp) \setminus \{m\}$ contains a point u not belonging to $m^\perp$. Assume, by way of contradiction, that u and z are not collinear in $\mathcal{S}$. The line uy of $\mathcal{S}$ contains exactly one point u_1 collinear with z. The line $u_1 z$ of $\mathcal{S}$ contains exactly one point u_2 of $x^\perp \cap y^\perp$. As $u_1 \neq u_2$ the point z belongs to $y^\perp$ a contradiction. Hence u and z are collinear in $\mathcal{S}$. It follows that any point $r \neq z$ of $\mathcal{S}$ on the line mz, is not collinear with u. By a preceding case we now have $r \in \langle x^\perp \rangle$ (the line of $\mathcal{S}$ through r and containing a point of mx does not contain a point of $x^\perp \cap y^\perp$). As also $m \in \langle x^\perp \rangle$, the line mr of $\mathcal{S}$ belongs to $\langle x^\perp \rangle$, and so $z \in \langle x^\perp \rangle$. Hence every point of $\mathcal{S}$ belongs to $\langle x^\perp \rangle$, a contradiction as $\langle x^\perp \rangle \neq \mathrm{PG}(d, q)$. We conclude that $\langle x^\perp \rangle$ does not contain a point y of $\mathcal{S}$ not in $x^\perp$. □

Lemma 2 *If the polar space $\mathcal{S}$ is weakly embedded in* $\mathrm{PG}(d, q)$*, $d \geq 2$, then for any point x of $\mathcal{S}$ the subspace $\langle x^\perp \rangle$ is a hyperplane of* $\mathrm{PG}(d, q)$.

Proof. Assume that the polar space $\mathcal{S}$ is weakly embedded in $\mathrm{PG}(d, q)$, $d \geq 2$, and that x is a point of $\mathcal{S}$ for which $\langle x^\perp \rangle$ is not a hyperplane of $\mathrm{PG}(d, q)$. Let y be a point of $\mathcal{S}$ not in $x^\perp$, and let $\mathrm{PG}(d-1, q)$ be a hyperplane of $\mathrm{PG}(d, q)$ containing $\langle x^\perp \rangle$ and y. As in the second part of the proof of Theorem 1 it now follows that $\mathcal{S}$ is contained in $\mathrm{PG}(d-1, q)$, clearly a contradiction. We conclude that for any point x of $\mathcal{S}$ the subspace $\langle x^\perp \rangle$ is a hyperplane. □

3 The classical polar space is not isomorphic to $W(n, s)$, with s odd

In this section we assume that the finite classical polar space $\mathcal{S}$ has rank at least three, is laxly embedded in $\mathrm{PG}(d, q)$, with $d \geq 3$, and is not isomorphic to $W(n, s)$, with s odd. Further, let $\mathrm{ued}(\mathcal{S}) = n$. Here no full polar subspace of $\mathcal{S}$ is of type $W(l, s)$, with s odd.

Lemma 3 *Any line L of $\mathcal{S}$ is a subline over $\mathrm{GF}(s)$ of the corresponding line L' of $\mathrm{PG}(d, q)$. In particular, $\mathrm{GF}(s)$ is a subfield of $\mathrm{GF}(q)$.*

Proof. Let L be a line of $\mathcal{S}$. Now consider a plane π of $\mathcal{S}$ containing L. Then π is a subplane of a plane $\langle \pi \rangle = \pi'$ of $\mathrm{PG}(d, q)$, and so π is a plane over $\mathrm{GF}(s)$ of π'. It follows that L is a subline over $\mathrm{GF}(s)$ of L'. In particular $\mathrm{GF}(s)$ is a subfield of $\mathrm{GF}(q)$. □

Theorem 2 *We have $d \leq n$.*

Proof. Let $\mathcal{S}'$ be a full polar subspace of $\mathcal{S}$ with $\mathrm{ued}(\mathcal{S}') = n - 1$.
If $\mathcal{S}'$ is a (classical) generalized quadrangle, then by Section 1 we have that $\mathcal{S}'$ generates a $\mathrm{PG}(d', q)$ with $d' \leq n - 1$; if $\mathcal{S}'$ has rank at least three, then we asssume by induction on n that $\mathcal{S}'$ generates a $\mathrm{PG}(d', q)$ with $d' \leq n - 1$. Let x be a point of $\mathcal{S}$ not in $\mathcal{S}'$. If z is a point of $\mathcal{S}$ with $z \in x^\perp \backslash \{x\}$, then the line zx of $\mathcal{S}$ contains a point u of $\mathcal{S}'$, and so $z \in \langle \mathrm{PG}(d', q), x \rangle$. Now assume that y is a point of $\mathcal{S}$ not in $\mathcal{S}'$, with $y \notin x^\perp$. First, suppose that $x^\perp \cap y^\perp$ does not belong to $\mathcal{S}'$. Let v be a point of $x^\perp \cap y^\perp$ which is not contained in $\mathcal{S}'$. Then the line vy of $\mathcal{S}$ contains a point w of $\mathcal{S}'$ and so vy belongs to $\langle \mathrm{PG}(d', q), x \rangle$. Hence y is a point of $\langle \mathrm{PG}(d', q), x \rangle$. Next, suppose that $x^\perp \cap y^\perp$ is contained in $\mathcal{S}'$. Let L be any line of $\mathcal{S}$ through y and let u be the common point of $\mathcal{S}'$ and L. Then $u \in x^\perp$. Now let $t \in L \backslash \{y, u\}$. Assume, by way of contradiction, that $t^\perp \cap x^\perp$ belongs to $\mathcal{S}'$. Then any point of $\mathcal{S}'$ in $x^\perp$, is also in $t^\perp$ and in $y^\perp$, so belongs to $u^\perp$. So the polar space formed by the points of $x^\perp$ in $\mathcal{S}'$, together with the lines of $\mathcal{S}'$ in $x^\perp$, is degenerate, a contradiction. Hence $t^\perp \cap x^\perp$ is not contained in $\mathcal{S}'$. By a preceding case $t \in \langle \mathrm{PG}(d', q), x \rangle$. Now it is clear that also $y \in \langle \mathrm{PG}(d', q), x \rangle$. It follows that $\mathcal{S}$ is contained in $\langle \mathrm{PG}(d', q), x \rangle$, and consequently $d' \geq d - 1$, that is, $d \leq n$. □

Theorem 3 *If $d = n$, then $\mathcal{S}$ is fully and naturally embedded in some subspace $\mathrm{PG}(d, s)$ of $\mathrm{PG}(d, q)$.*

Proof. Let y be any point of $\mathcal{S}$. Then on each line of $\mathcal{S}$ through y we can choose a point z, such that the set P of these points z together with the lines of $\mathcal{S}$ in P form a full polar subspace $\mathcal{S}'$ of $\mathcal{S}$ (of the same type as $\mathcal{S}$), with $\mathrm{ued}(\mathcal{S}') = n - 2$. By Theorem

2 the point set P generates a subspace $\mathrm{PG}(d', q)$ of $\mathrm{PG}(d, q)$, with $d' \leq n-2$. Hence the projective space $\langle y^\perp \rangle$ is at most $(n-1)$-dimensional. It follows that $\mathcal{S}$ is weakly embedded in $\mathrm{PG}(d, q)$, and the result follows. □

Lemma 4 *If $d < n$, then $\mathcal{S}$ contains a full polar subspace $\mathcal{S}'$ with* $\mathrm{ued}(\mathcal{S}') = n-1$ *and which generates* $\mathrm{PG}(d, q)$*.*

Proof. Let $\mathcal{S}'$ be a full polar subspace of $\mathcal{S}$ with $\mathrm{ued}(\mathcal{S}') = n-1$. If x is a point of $\mathcal{S}$ not in $\mathcal{S}'$, then analogously as in the proof of Theorem 2 we show that $\mathcal{S}$ is contained in $\langle \mathrm{PG}(d', q), x\rangle$, with $\mathrm{PG}(d', q)$ the projective space generated by $\mathcal{S}'$. Hence $d' \geq d-1$. If $d' = d$ we are done, so assume that $d' = d-1$. By induction on d we may assume that $\mathcal{S}'$ contains a full polar subspace $\mathcal{S}''$ with $\mathrm{ued}(\mathcal{S}'') = n-2$ and which generates $\mathrm{PG}(d-1, q)$. Let y be a point of $\mathcal{S}''$.

First, suppose that all lines of $\mathcal{S}$ containing y are in $\mathrm{PG}(d-1, q)$. Let z be a point of $\mathcal{S}$ collinear with y, but not contained in $\mathcal{S}'$. Similarly as in the proof of Theorem 2 we then obtain that $\mathcal{S}$ is contained in $\langle \mathrm{PG}(d-1, q), z\rangle$ $=\mathrm{PG}(d-1, q)$, a contradiction. So there is a line L of $\mathcal{S}$ through y not in $\mathrm{PG}(d-1, q)$. Let π be a plane of $\mathcal{S}$ containing L which intersects $\mathcal{S}'$ in a line M which does not belong to $\mathcal{S}''$. Then in the plane π there is a line $N \neq M$ of $\mathcal{S}$ through y, such that $\mathcal{S}''$ and N define a non-degenerate full polar subspace $\mathcal{S}'_1$ of $\mathcal{S}$, with $\mathrm{ued}(\mathcal{S}'_1) = n-1$. Clearly $\mathcal{S}'_1$ generates $\mathrm{PG}(d, q)$.

As first step in the induction let us consider a polar space $\mathcal{S}'$ which is laxly embedded in $\mathrm{PG}(3, q)$. Let $\mathcal{S}''$ be a full polar subspace of $\mathcal{S}'$, with $\mathrm{ued}(\mathcal{S}'') = \mathrm{ued}(\mathcal{S}') - 1$. If $\mathcal{S}''$ generates $\mathrm{PG}(3, q)$ we are done. So assume that $\mathcal{S}''$ generates a plane $\mathrm{PG}(2, q)$. If u is a point of $\mathcal{S}''$, then there is a line L of $\mathcal{S}'$ through u not contained in $\mathrm{PG}(2, q)$. Now we choose a line M of $\mathcal{S}''$ in π such that no plane of $\mathcal{S}'$ containing L intersects M. Then any full polar subspace $\mathcal{S}''_1$ of $\mathcal{S}'$, with $\mathrm{ued}(\mathcal{S}''_1) = \mathrm{ued}(\mathcal{S}') - 1$, generates $\mathrm{PG}(3, q)$ (by the choice of L and M such a polar subspace $\mathcal{S}''_1$ exists).

We conclude that $\mathcal{S}$ always contains a full polar subspace $\mathcal{S}'$, with $\mathrm{ued}(\mathcal{S}') = n-1$ and which generates $\mathrm{PG}(d, q)$. □

Theorem 4 *If $d < n$, then there exists a* $\mathrm{PG}(n, q)$ *containing* $\mathrm{PG}(d, q)$ *and a* $\mathrm{PG}(n-d-1, q)$ *of* $\mathrm{PG}(n, q)$ *skew to* $\mathrm{PG}(d, q)$ *such that $\mathcal{S}$ is the projection from* $\mathrm{PG}(n-d-1, q)$ *onto* $\mathrm{PG}(d, q)$ *of a polar space $\widetilde{\mathcal{S}} \cong \mathcal{S}$ which is fully and naturally embedded in a subspace* $\mathrm{PG}(n, s)$ *of* $\mathrm{PG}(n, q)$*.*

Proof. By Lemma 4 the polar space $\mathcal{S}$ contains a full polar subspace $\mathcal{S}'$ with $\mathrm{ued}(\mathcal{S}') = n-1$ and where $\mathcal{S}'$ generates $\mathrm{PG}(d, q)$. Proceeding by induction on n and also relying on Theorem 3, we may assume that there is a $\mathrm{PG}(n-1, q)$ containing $\mathrm{PG}(d, q)$ and a

PG$(n-d-2,q)$ of PG$(n-1,q)$ skew to PG(d,q) such that $\mathcal{S}'$ is the projection from PG$(n-d-2,q)$ onto PG(d,q) of a polar space $\widetilde{\mathcal{S}}' \cong \mathcal{S}'$ which is fully and naturally embedded in a subspace PG$(n-1,s)$ of PG$(n-1,q)$.

Let L be a line of $\mathcal{S}$ not in $\mathcal{S}'$, and let y be the common point of L and $\mathcal{S}'$. Suppose that the point $\widetilde{y}$ of $\widetilde{\mathcal{S}}'$ is projected from PG$(n-d-2,q)$ onto y. Now we embed PG$(n-1,q)$ in a PG(n,q). Further, let PG$(n-d-1,q)$ be a projective subspace of PG(n,q) containing PG$(n-d-2,q)$, with PG$(n-d-1,q) \not\subset$PG$(n-1,q)$. In PG$(n-d+1,q) = \langle$PG$(n-d-1,q), L\rangle$ we now consider a line $\widetilde{L}$ over GF(s) containing $\widetilde{y}$ and not contained in PG$(n-1,q)$, whose projection from PG$(n-d-1,q)$ onto PG(d,q) coincides with L. Then PG$(n-1,s)$ and $\widetilde{L}$ are contained in a unique subspace PG(n,s) of PG(n,q).

Let u be a point of $\mathcal{S}$ not in $\mathcal{S}'$, with $u \notin L$ and $u \notin y^{\perp}$. Further, let v be the unique point of L which is collinear with u, and let u' be the unique common point of $\mathcal{S}'$ and the line uv of $\mathcal{S}$. The point of $\widetilde{L}$ which is projected from PG$(n-d-1,q)$ onto v is denoted by $\widetilde{v}$, and the point of $\widetilde{\mathcal{S}}'$ which is projected from PG$(n-d-1,q)$ (or PG$(n-d-2,q)$) onto u' is denoted by $\widetilde{u}'$. Now let us define $\widetilde{u}$ as the intersection of $\langle \widetilde{u}', \widetilde{v}\rangle$ and $\langle$PG$(n-d-1,q), u\rangle$.

Next, let u be a point of $\mathcal{S}$ not in $\mathcal{S}'$, with $u \notin L$ and where u and L are in a common plane π of $\mathcal{S}$. If M is the common line of $\mathcal{S}'$ and π, then let $\widetilde{M}$ be the line of $\widetilde{\mathcal{S}}'$ which is projected onto M. Then, by definition, $\{\widetilde{u}\} = \langle \widetilde{L}, \widetilde{M}\rangle \cap \langle$PG$(n-d-1,q), u\rangle$. It is clear that $\widetilde{u}$ belongs to the plane over GF(s) containing $\widetilde{L}$ and $\widetilde{M}$. Hence $\widetilde{u}$ is a point of PG(n,s).

Let u be again a point of $\mathcal{S}$ not in $\mathcal{S}'$, with $u \notin L$ and $u \notin y^{\perp}$. As before, let v be the unique point of L which is collinear with u, and let u' be the unique point of $\mathcal{S}'$ and the line uv of $\mathcal{S}$. Now choose planes π, π' of $\mathcal{S}$ through v, where π contains L, $\pi \cap \pi'$ is a line U of $\mathcal{S}$, and π' contains u. Then $u' \in \pi'$. Let $\widetilde{U}$ be the line of PG(n,s) consisting of all points $\widetilde{w}$, with $\widetilde{w}$ corresponding to $w \in U$, and let $\widetilde{T}$ be the line of $\widetilde{\mathcal{S}}'$ which is projected onto the common line T of π' and $\mathcal{S}'$. The plane $\widetilde{\pi}'$ over GF(s) defined by $\widetilde{T}$ and $\widetilde{U}$, is projected onto the plane π'; clearly $\widetilde{\pi}'$ is a plane of PG(n,s). As $u', v, u \in \pi'$ and $\widetilde{u}', \widetilde{v} \in \widetilde{\pi}'$, it is now easily follows that $\widetilde{u} \in \widetilde{\pi}'$. It follows that $\widetilde{u} \in$PG(n,s). Let π_1 be a plane of $\mathcal{S}$ containing L, let U be the common line of $\mathcal{S}'$ and π_1,and let L_1 be a line of $\mathcal{S}$ in π_1 distinct from L and U. Further, let y_1 be the point of $\mathcal{S}'$ on L_1. We will show that if r is any point of $\mathcal{S}$ not in $\mathcal{S}'$, and not in $L \cup L_1$, for which r and L are in a common plane of $\mathcal{S}$ if $r \in y^{\perp}$, and for which r and L_1 are in a common plane of $\mathcal{S}$ if $r \in y_1^{\perp}$, then interchanging roles of L and L_1 the point $\widetilde{r}$ is the same element of PG(n,s) (this claim is clear if $r \in L \cup L_1$). First, assume that r and L , and also r and L_1, are in a common plane of $\mathcal{S}$. If $r \in \pi_1$, then it is clear that $\widetilde{r}$ remains the same if the roles of L and L_1 are interchanged. So we may assume that $r \notin \pi_1$. Let γ be the plane of $\mathcal{S}$ containing r and L, and let γ_1 be the plane of $\mathcal{S}$ containing r and L_1. Further, let N be the line of $\mathcal{S}'$ in γ, and let N_1 be the line of $\mathcal{S}'$ in γ_1. If $\{t\} = N \cap N_1$ and $\{w\} = L \cap L_1$, then $r \in \langle t, w\rangle$ and $\widetilde{r}$ is the unique common point of $\langle \widetilde{t}, \widetilde{w}\rangle$ and $\langle$PG$(n-d-1,q), r\rangle$. So $\widetilde{r}$ remains the same by interchanging L and L_1. Next, assume that r and L are in a common plane γ of $\mathcal{S}$, and that $r \notin y_1^\perp$. If $\{w\} = L \cap L_1$ and t is the common point of $\mathcal{S}'$ and the line wr of $\mathcal{S}$, then $\widetilde{r}$ is the unique common point of $\langle \widetilde{t}, \widetilde{w} \rangle$ and $\langle \mathrm{PG}(n-d-1,q), r \rangle$. If we interchange roles of L and L_1, then $\widetilde{r}$ is also obtained by intersecting $\langle \widetilde{t}, \widetilde{w} \rangle$ and $\langle \mathrm{PG}(n-d-1,q), r \rangle$. Finally, assume that $r \notin y^\perp$ and $r \notin y_1^\perp$. Let $L \cap L_1 = \{w\}$. If $r \in w^\perp$, then it is clear that $\widetilde{r}$ remains the same if we interchange the roles of L and L_1. So assume that $r^\perp \cap L = \{z\}$ and $r^\perp \cap L_1 = \{z_1\}$, with $z \neq z_1$. Then $\widetilde{r}$ is the same if it is constructed using either L or the line zz_1 of $\mathcal{S}$, and also it is the same if it is constructed using either L_2 or the line zz_1 of $\mathcal{S}$. Hence again we may interchange L and L_1.

Next, let L_1 be a line of $\mathcal{S}$ which has a point w in common with L, with w not in $\mathcal{S}'$, and assume that L and L_1 are not in a common plane of $\mathcal{S}$. Let y_1 be the point of L_1 in $\mathcal{S}'$. Again we will show that if r is any point of $\mathcal{S}$ not in $\mathcal{S}'$, and not in $L \cup L_1$, for which r and L are in a common plane of $\mathcal{S}$ if $r \in y^\perp$, and for which r and L_1 are in a common plane of $\mathcal{S}$ if $r \in y_1^\perp$, then interchanging roles of L and L_1 the point $\widetilde{r}$ is the same element of $\mathrm{PG}(n,s)$ (this claim is clear if $r \in L \cup L_1$). Choose a plane π of $\mathcal{S}$ containing L and a plane π_1 of $\mathcal{S}$ containing L_1, in such a way that $\pi \cap \pi_1$ is a line L_2 of $\mathcal{S}$. Then $w \in L_2$. The common point of L_2 and $\mathcal{S}'$ is denoted by y_2. Assume first that if $r \in y_2^\perp$, then also $r \in w^\perp$. Applying two times the previous section, we then see that $\widetilde{r}$ is the same if we interchange the roles of L and L_1. Now assume that $r \in y_2^\perp$ and $r \notin w^\perp$. Then choose a plane $\pi^* \neq \pi$ of $\mathcal{S}$ containing L and a plane $\pi_1^* \neq \pi_1$ of $\mathcal{S}$ containing L_1 , in such a way that $\pi^* \cap \pi_1^*$ is a line L_2^* of $\mathcal{S}$. The common point of L_2^* and $\mathcal{S}'$ is denoted by y_2^*. The planes π_1^* and π^* can be chosen in such a way that $y_2^* \notin y_2^\perp$. If $r \notin y_2^{*\perp}$, then interchanging L_2 and L_2^* we are done. So let $r \in y_2^{*\perp}$. Choose a point $z \neq y_1, y_2$ on the line y_1y_2 of $\mathcal{S}$, and let z^* be the common point of $z^\perp$ and the line yy_2^* of $\mathcal{S}$. Let M be the line zw of $\mathcal{S}$, and let M^* be the line z^*w of $\mathcal{S}$. As $r \notin y_1^\perp$ we have $r \notin z^\perp$, and as $v \notin y^\perp$ we have $r \notin z^{*\perp}$. Then $\widetilde{r}$ is the same if it is constructed using either L or M^*, using either M^* or M, and using either M or L_1. So again $\widetilde{r}$ is the same if we interchange the roles of L and L_1.

Now let u be a point of $\mathcal{S}$ not in $\mathcal{S}'$, with $u \notin L$, u and L not in a common plane of $\mathcal{S}$, and $u \in y^\perp$. Now we choose a line L_1 of $\mathcal{S}$ not in $\mathcal{S}'$ in such a way that L and L_1 intersect, and $u \notin y_1^\perp$ with y_1 the unique common point of L_1 and $\mathcal{S}'$ (it is easy to show that such a line L_1 of $\mathcal{S}$ always exists). Now we construct $\widetilde{u}$ with respect to the line L_1 and we show that $\widetilde{u}$ is independent of the choice of L_1. So we choose a second line L_2 of $\mathcal{S}$ not in $\mathcal{S}'$, in such a way that L and L_2 intersect, and $u \notin y_2^\perp$ with y_2 the unique common point of L_2 and $\mathcal{S}'$. If $L \cap L_1 = L \cap L_2$, then from the foregoing two sections it is clear that L_1 and L_2 define the same point $\widetilde{u}$. Next, let $L \cap L_1 \neq L \cap L_2$, but assume that $L_1 \cap L_2 \neq \emptyset$. If $L_1 \cap L_2 \neq \{y_1\}$, then again L_1 and L_2 define the same point $\widetilde{u}$. So let $L_1 \cap L_2 = \{y_1\}$. Choose a line L_3 of $\mathcal{S}$ not in $\mathcal{S}'$, intersecting L_1 and L_2, and intersecting yy_1 in a point distinct from y, y_1. Then L_1 and L_3 define the same point $\widetilde{u}$, and also L_2 and L_3 define the same point $\widetilde{u}$. Hence L_1 and L_2 define the same point $\widetilde{u}$. Now, we suppose that $L \cap L_1 \neq L \cap L_2$ and $L_1 \cap L_2 = \emptyset$. Let π be a plane of $\mathcal{S}$ containing L, let $L_3 \neq L$ be a line of $\mathcal{S}$ in π for which $L \cap L_1 = L \cap L_3$, let $L_4 \neq L$ be a line of $\mathcal{S}$ in π for which $L \cap L_2 = L \cap L_4$, and choose L_3 and L_4 in such a way that the common point of these lines does not belong to $u^\perp$. By foregoing cases the lines L_1 and L_3 define the same point $\widetilde{u}$, the lines L_2 and L_4 define the same point $\widetilde{u}$, and also the lines L_3 and L_4 define the same point $\widetilde{u}$. It follows that L_1 and L_2 define the same point $\widetilde{u}$. We conclude that $\widetilde{u}$ is independent of the choice of the line $\widetilde{u}$. Also, $\widetilde{u}$ is a point of $\mathrm{PG}(n, s)$.

So for any point u of $\mathcal{S}$ the point $\widetilde{u}$ is defined, and $\widetilde{u}$ always belongs to $\mathrm{PG}(n, s)$.

Clearly $\theta : u \mapsto \widetilde{u}$ is an injection of the point set of $\mathcal{S}$ into $\mathrm{PG}(n, s)$. Now let u_1, u_2, u_3 be distinct collinear points of $\mathcal{S}$.

If at least two of these points belong to $\mathcal{S}'$, then clearly $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3$ are also collinear in $\mathrm{PG}(n, s)$.

Next, assume that u_1 is in $\mathcal{S}'$, and that u_2, u_3 are not in $\mathcal{S}'$. If the line M of $\mathcal{S}$ containing u_1, u_2, u_3 also contains a point of $L \backslash \{y\}$, then it is clear that $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3$ are collinear in $\mathrm{PG}(n, s)$. So we may assume that the line M does not contain a point of $L \backslash \{y\}$. First, suppose that $u_1 \notin y^\perp$. Then we may assume that also $u_2 \notin y^\perp$. Let L_1 be the line of $\mathcal{S}$ which contains u_2 and a point of L. By the foregoing we may construct $\widetilde{u}_2, \widetilde{u}_3$ with respect to L_1, and then it is clear that $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3$ are collinear in $\mathrm{PG}(n, s)$. Next, suppose that $u_1 \in y^\perp \backslash \{y\}$. If $u_2 \notin y^\perp$, then let L_1 be again the line of $\mathcal{S}$ which contains u_2 and a point of L. Again we may construct $\widetilde{u}_2, \widetilde{u}_3$ with respect to L_1, and so it is clear that $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3$ are collinear in $\mathrm{PG}(n, s)$. If $u_2 \in y^\perp$, then let N be a line of $\mathcal{S}$ through u_1 in the plane π of $\mathcal{S}$ containing y, u_1, u_2, u_3, but distinct from the lines yu_1 and u_1u_2 of $\mathcal{S}$. Now let $\pi_1 \neq \pi$ be a plane of $\mathcal{S}$ through N, but not in a 3-dimensional projective subspace of $\mathcal{S}$ containing N. Let N_1 be the common line of π_1 and $\mathcal{S}'$. Then neither N_1, y, u_1, nor N_1, u_1, u_2, are in a common plane of $\mathcal{S}$. Further, let $y_1 \in N_1 \backslash \{u_1\}$ and let L_1 be the line of $\mathcal{S}$ which contains y_1 and a point of L. Now we may construct $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3$ with respect to L_1, and by a foregoing case we conclude that $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3$ are collinear in $\mathrm{PG}(n, s)$. Finally, we assume that $u_1 = y$. If L, u_2, u_3 are in a common plane of $\mathcal{S}$, then it is clear that $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3$ are collinear in $\mathrm{PG}(n, s)$. So assume that $u_2^\perp$ does not contain L. Let π be a plane of $\mathcal{S}$ containing y, u_2, u_3, and let D be the common line of π and $\mathcal{S}'$. Further, let $y_2, y_3 \in D \backslash \{y\}$, $y_2 \neq y_3$, and let $y_2u_2 \cap y_3u_3 = \{u\}$. Then $\widetilde{y}, \widetilde{y}_2, \widetilde{y}_3$, respectively $\widetilde{y}_2, \widetilde{u}_2, \widetilde{u}$, respectively $\widetilde{y}_3, \widetilde{u}_3, \widetilde{u}$, are collinear in $\mathrm{PG}(n, s)$. So $\widetilde{y}, \widetilde{y}_2, \widetilde{y}_3, \widetilde{u}_2, \widetilde{u}_3, \widetilde{u}$ are in a common plane $\widetilde{\pi}$ of $\mathrm{PG}(n, s)$. Now we consider a second plane π_1 of $\mathcal{S}$ containing y, u_2, u_3, and let D_1 be the common line of π_1 and $\mathcal{S}'$. As the line $\widetilde{D}_1$ of $\mathrm{PG}(n, s)$ has no point in common with the line $\widetilde{y}_2\widetilde{u}_2$ of $\mathrm{PG}(n, s)$, the plane $\widetilde{\pi}_1$ of $\mathrm{PG}(n, s)$ which corresponds with the plane π_1 is distinct from the plane $\widetilde{\pi}$. As $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3$ belong to $\widetilde{\pi} \cap \widetilde{\pi}_1$, the points $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3$ are collinear in $\mathrm{PG}(n, s)$.

Finally, assume that u_1, u_2, u_3 are not in $\mathcal{S}'$. Let u be the common point of $\mathcal{S}'$ and the line u_2u_3 of $\mathcal{S}$. By the foregoing cases the points $\widetilde{u}, \widetilde{u}_1, \widetilde{u}_2$ are collinear in $\mathrm{PG}(n, s)$, and also the points $\widetilde{u}, \widetilde{u}_1, \widetilde{u}_3$ are collinear in $\mathrm{PG}(n,s)$. It follows that $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3$ are collinear in $\mathrm{PG}(n,s)$.

If $\widetilde{\mathcal{S}}$ is the incidence structure having as points all points $\widetilde{u}$, with u any point of $\mathcal{S}$ and having as lines all lines $\widetilde{N}$, with N any line of $\mathcal{S}$ (incidence is the natural one), then it is now clear that $\widetilde{\mathcal{S}} \cong \mathcal{S}$ is a polar space which is fully (and hence naturally) embedded in $\mathrm{PG}(n,s)$, whose projection from $\mathrm{PG}(n-d-1,q)$ onto $\mathrm{PG}(d,q)$ is the given polar space $\mathcal{S}$.

Now by induction, by Theorem 3, and by the fact that Theorems 3 and 4 are valid for rank two, the theorem is completely proved. □

4 The polar space is isomorphic to $W(2m+1,s)$, $m \geq 2$ and s odd

In this section we assume that the finite classical polar space $\mathcal{S}$ is laxly embedded in $\mathrm{PG}(d,q)$, with $d \geq 4$, and is isomorphic to $W(2m+1,s)$, with $m \geq 2$ and s odd.

Lemma 5 *Any line L of $\mathcal{S}$ is a subline over $\mathrm{GF}(s)$ of the corresponding line L' of $\mathrm{PG}(d,q)$. In particular, $\mathrm{GF}(s)$ is a subfield of $\mathrm{GF}(q)$.*

Proof. See proof of Lemma 3. □

Theorem 5 *We have $d \leq 2m+1$.*

Proof. Let $\mathcal{S}'$ be a full polar subspace of $\mathcal{S}$, with $\mathcal{S}' \cong W(2m-1,s)$. If $m=2$, then by Section 1 we have that $\mathcal{S}'$ generates a $\mathrm{PG}(d',q)$ with $d' \leq 3$; if $\mathcal{S}'$ has rank at least three, then we assume by induction on m that $\mathcal{S}'$ generates a $\mathrm{PG}(d',q)$ with $d' \leq 2m-1$. Let x and y be distinct points of $\mathcal{S}$, where $x^\perp$ and $y^\perp$ contain all points of $\mathcal{S}'$. Let z be a point of $\mathcal{S}$ not in $\mathcal{S}'$, with $z^\perp$ not containing all points of $\mathcal{S}'$. Then there is a line L of $\mathcal{S}$ containing z, but containing no point of $\mathcal{S}'$. This line L intersects $x^\perp$ and $y^\perp$ in distinct points, and so $z \in \langle \mathrm{PG}(d',q), x, y \rangle$. Let V be the set of all points v of $\mathcal{S}$, for which $v^\perp$ contains all points of $\mathcal{S}'$. Then $|V| = s+1$, and no two points of V are collinear in $\mathcal{S}$. Let $w \in V \backslash \{x,y\}$ and let M be a line of $\mathcal{S}$ through w. Then at least two points of M are not in V, so M belongs to $\langle \mathrm{PG}(d',q), x, y \rangle$, and hence w is a point of $\langle \mathrm{PG}(d',q), x, y \rangle$. It follows that $\mathcal{S}$ is contained in $\langle \mathrm{PG}(d',q), x, y \rangle$, and so $d' \geq d-2$, that is, $d \leq 2m+1$. □

Theorem 6 *If $d = 2m+1$, then $\mathcal{S}$ is fully und naturally embedded in some subspace $\mathrm{PG}(d,s)$ of $\mathrm{PG}(d,q)$.*

Proof. Let y be any point of $\mathcal{S}$. Then on each line of $\mathcal{S}$ through y we can choose a point z, such that the set of these points z together with the lines of $\mathcal{S}$ in P form a full polar subspace $\mathcal{S}' \cong W(2m-1, s)$ of $\mathcal{S}$. By Theorem 5 the point set P generates a subspace $\mathrm{PG}(d', q)$ of $\mathrm{PG}(d, q)$, with $d' \leq 2m-1$. Hence the projective space $\langle y^\perp \rangle$ is at most $2m$-dimensional. It follows that $\mathcal{S}$ is weakly embedded in $\mathrm{PG}(d, q)$ and the result follows. □

Theorem 7 *If $(d, m) = (4, 2)$, then there exists a $\mathrm{PG}(5, q)$ containing $\mathrm{PG}(4, q)$ and a point p of $\mathrm{PG}(5, q)$ not in $\mathrm{PG}(4, q)$ such that $\mathcal{S}$ is the projection from p onto $\mathrm{PG}(4, q)$ of a polar space $\widetilde{\mathcal{S}} \cong \mathcal{S}$ which is fully and naturally embedded in a subspace $\mathrm{PG}(5, s)$ of $\mathrm{PG}(5, q)$.*

Proof. Assume that $\mathcal{S}'' \cong W(3, s)$ is a full polar subspace of $\mathcal{S}$ which contains given intersecting lines L, M of $\mathcal{S}$, with L and M not belonging to a common plane of $\mathcal{S}$. Then $\mathcal{S}''$ either generates a plane or a 3-dimensional space in $\mathrm{PG}(4, q)$. Assume that $\mathcal{S}''$ generates a plane π of $\mathrm{PG}(4, q)$. Choose a point x of $\mathcal{S}''$ on $L \backslash M$. By the second part of the proof of Theorem 1, the set $x^\perp$ is not contained in π. Let L_1 be a line of $\mathcal{S}$ through x, but not contained in the plane π. Assume that every plane of $\mathcal{S}$ containing L_1, also contains a line of $\mathcal{S}''$ through x. Then let M_1 be a line of $\mathcal{S}''$ through x, distinct from L, and let L_2 be a line of $\mathcal{S}$ through x in the plane of $\mathcal{S}$ defined by L_1 and M_1, where $M_1 \neq L_2 \neq L_1$. Then L and L_2 are not in a common plane of $\mathcal{S}$. So without loss of generality we may assume that L and L_1 do not belong to a common plane of $\mathcal{S}$. Then $M^\perp \cap L_1 = \emptyset$, with $M^\perp = \cap_{z \in M} z^\perp$. From the theory of classical polar spaces it now follows that L_1 and M are lines of a full polar subspace $\mathcal{S}' \cong W(3, s)$ of $\mathcal{S}$. Clearly L is a line of $\mathcal{S}'$.

From the foregoing section it follows that $\mathcal{S}$ always contains a full polar subspace $\mathcal{S}' \cong W(3, s)$ which contains the given lines L, M of $\mathcal{S}$ and generates a subspace $\mathrm{PG}(3, q)$ of $\mathrm{PG}(4, q)$.

So let $\mathcal{S}' \cong W(3, s)$ be a full polar subspace of $\mathcal{S}$ which generates a 3-dimensional subspace $\mathrm{PG}(3, q)$ of $\mathrm{PG}(4, q)$. Let x be a point of $\mathcal{S}'$ and let L, M, N be three distinct lines of $\mathcal{S}'$ containing x. Assume, by way of contradiction, that L, M, N are not coplanar in $\mathrm{PG}(3, q)$. Consider now a full polar subspace $\mathcal{S}'' \cong W(3, s)$ of $\mathcal{S}$, with $\mathcal{S}'' \neq \mathcal{S}'$, containing L, M, N as lines. Then $\mathcal{S}'$ and $\mathcal{S}''$ are contained in a common $\mathrm{PG}(3, q)$. It follows that the points and lines of $\mathcal{S}$ in $\mathrm{PG}(3, q)$ form a degenerate full polar subspace of $\mathcal{S}$. The point set of this degenerate polar space is necessarily a set of type $y^\perp$, with y some point of $\mathcal{S}$. Let $\overline{\mathcal{S}}$ be a naturally embedded polar space isomorphic to $W(5, s)$ and let γ be an isomorphism of $\mathcal{S}$ onto $\overline{\mathcal{S}}$. Then $(\mathcal{S}')^\gamma$ and $(\mathcal{S}'')^\gamma$ belong to the $\mathrm{PG}(4, s)$ with point set $(y^\perp)^\gamma$. Hence every full polar subspace of $\overline{\mathcal{S}}$ isomorphic to $W(3, s)$ and containing the coplanar lines $L^\gamma, M^\gamma, N^\gamma$ belongs to the hyperplane $\mathrm{PG}(4, s)$, clearly a contradiction. It follows that

L, M, N are coplanar in PG$(3, q)$. Consequently, $\mathcal{S}'$ is weakly embedded in PG$(3, q)$. From Section 1 we now conclude that the point set of $\mathcal{S}'$ is a subspace PG$(3, s)$ of PG$(3, q)$.

Let V be the set of the $s + 1$ points z of $\mathcal{S}$, for which $z^\perp$ contains each point of $\mathcal{S}'$. As $\mathcal{S}$ is not contained in PG$(3, q)$, the set V contains at most one point of PG$(3, q)$. Let y be a point of V which is not contained in PG$(3, q)$. Further, let L be a line of $\mathcal{S}$ containing y and let PG$(4, s)$ be the subspace of PG$(4, q)$ defined by PG$(3, s)$ and L. Consider a line M of $\mathcal{S}$ through y , with $M \neq L$. If L, M belong to a common plane π of $\mathcal{S}$, then π contains a line of $\mathcal{S}'$, so π is a plane of PG$(4, s)$, hence M is a line of PG$(4, s)$. Now suppose that L, M do not belong to a common plane of $\mathcal{S}$. Let u be the common point of L and $\mathcal{S}'$, let v be the common point of M and $\mathcal{S}'$, and let w be a point of $\mathcal{S}'$ in $u^\perp \cap v^\perp$. The line of $\mathcal{S}$ containing w and y will be denoted by N. By the foregoing N is a line of PG$(4, s)$, and then, by interchanging roles of L and N, also M is a line of PG$(4, s)$. It follows that $y^\perp$ is a point set of PG$(4, s)$. As $|y^\perp| = s^4 + s^3 + s^2 + s + 1$, the set $y^\perp$ coincides with the point set of PG$(4, s)$.

Let $y_1 \in V$, with $y_1 \neq y$ and y_1 not in PG$(3, q)$, and let L_1 be the line of $\mathcal{S}$ through u and y_1. Now we consider a projective space PG$(5, q)$ containing PG$(4, q)$ and a line $\widetilde{L}_1$ over GF(s) in PG$(5, q)$ which passes through u but is not contained in PG$(4, q)$, such that L_1 is the projection of $\widetilde{L}_1$ from some point $p \in$PG$(5, q)\backslash$PG$(4, q)$ onto PG$(4, q)$. Then there is exactly one subspace PG$(5, s)$ of PG$(5, q)$ which contains $\widetilde{L}_1$ and PG$(4, s)$. As $y_1 \notin$ PG$(4, s)$ it is clear that $p \notin$ PG$(5, s)$. The subspace PG$^{(1)}(4, s)$ defined by $\widetilde{L}_1$ and PG$(3, s)$ is projected from p onto the subspace with point set $y_1^\perp$ of PG$(4, q)$. Let r_1, r_2 be points of $\mathcal{S}'$, with $r_2 \notin r_1^\perp$, and such that $\langle r_1, r_2, y, y_1\rangle$ is 3-dimensional. Then $\mathcal{S}$ contains a full polar subspace $\mathcal{S}^* \cong W(3, s)$ which contains r_1, r_2, y, y_1. From the foregoing it follows that $\mathcal{S}^*$ is naturally embedded in some 3-dimensional projective space over GF(s). The set $V = \{y, y_1\}^{\perp\perp} = \cap_{x\in y^\perp\cap y_1^\perp} x^\perp$ of $\mathcal{S}$ coincides with the set $\{y, y_1\}^{\perp^*\perp^*}$ of $\mathcal{S}^*$ (here "$\perp^*$" means perpendicularity in $\mathcal{S}^*$). Hence V is a subline over GF(s). Let $z \in V\backslash\{y, y_1\}$ and consider the line Z of $\mathcal{S}$ passing through z and r, with r any point of $\mathcal{S}'$. Putting $r = r_1$ and considering again $\mathcal{S}^*$, it follows that the lines ry, ry_1, rz of $\mathcal{S}$ belong to a common plane π_1 over GF(s). Let $\widetilde{y}_1$ be the point of $\widetilde{L}_1$ which corresponds to y_1 and let $\widetilde{\pi}_1$ be the subplane of PG$(5, s)$ containing $r, y, \widetilde{y}_1$. As $\widetilde{\pi}_1$ is projected from p onto π_1, the plane $\widetilde{\pi}_1$ contains a point $\widetilde{z}$ which is projected from p onto z; the line $r\widetilde{z}$ of $\widetilde{\pi}_1$ is projected from p onto the line rz of $\mathcal{S}$. The point set of the polar space $\mathcal{S}$ is the union of the lines rz of $\mathcal{S}$, with r in $\mathcal{S}'$ and z in V. Hence each point of $\mathcal{S}$ is the projection from p onto PG$(4, q)$ of a point of PG$(5, s)$. As the number of points of PG$(5, s)$ is also the number of points of $\mathcal{S}$, it is clear that for each point w of $\mathcal{S}$ there is exactly one point $\widetilde{w}$ of PG$(5, s)$ which is projected from p onto w.

Let N be any line of $\mathcal{S}$ and let $\widetilde{N}$ be the corresponding point set of PG$(5, s)$. If $\langle N, p\rangle\cap$ PG$(5, s)$ is not a line over GF(s), then it is a subplane over GF(s). So if $\langle N, p\rangle\cap$ PG$(5, s)$ is not a line over GF(s), then on the line N' of PG$(4, q)$ containing N there are s^2 points of $\mathcal{S}$ not on N, contradicting Lemma 1. So $\langle N, p\rangle \cap$ PG$(5, s)$ is a line over GF(s), that is, $\widetilde{N}$ is a line over GF(s). It follows that $\mathcal{S}$ is the projection from p onto PG$(4, q)$ of a $\widetilde{S} \cong W(5, s)$ which is fully (and hence naturally) embedded in PG$(5, s)$. $\square$

Theorem 8 *If $d = 2m$, with $m > 2$, then there exists a* PG$(2m+1, q)$ *containing* PG$(2m, q)$ *and a point p of* PG$(2m+1, q)$ *not in* PG$(2m, q)$ *such that $\mathcal{S}$ is the projection from p onto* PG$(2m, q)$ *of a polar space $\widetilde{\mathcal{S}} \cong \mathcal{S}$ which is fully and naturally embedded in a subspace* PG$(2m+1, s)$ *of* PG$(2m+1, q)$.

Proof. Let L and M be distinct intersecting lines of $\mathcal{S}$, where L and M do not lie in a common plane of $\mathcal{S}$. Further, let $\mathcal{S}''$ be a full polar subspace of $\mathcal{S}$ which contains L and M, and which is isomorphic to $W(2m-1, s)$. Assume that $\mathcal{S}''$ generates the projective subspace PG(l, q) of PG$(2m, q)$. By Theorem 5 we have $l \leq 2m-1$. Let x and y be distinct points of $\mathcal{S}$ for which $x^{\perp}$ and $y^{\perp}$ contain all points of $\mathcal{S}''$. Then by the second part of the proof of Theorem 1 $\mathcal{S}$ is contained in $\langle x^{\perp}, y^{\perp}\rangle = \langle \text{PG}(l, q), x, y\rangle$. Hence $l \geq 2m-2$. It follows that $l \in \{2m-1, 2m-2\}$. Suppose that $l = 2m-2$. Proceeding by induction on m, we assume that $\mathcal{S}''$ contains a full polar subspace $\mathcal{S}''' \cong W(2m-3, s)$ which contains L and M, and generates a PG$(2m-3, s)$. Let x be a point of $\mathcal{S}''$ for which $x^{\perp}$ contains each point of $\mathcal{S}'''$. If any such point x would be a point of PG$(2m-3, q)$, then by the second part of the proof of Theorem 1 the polar space $\mathcal{S}''$ would be contained in PG$(2m-3, q)$, clearly a contradiction. Hence we may assume that $x \notin$ PG$(2m-3, q)$. Now consider a full polar subspace $\mathcal{S}''_1 \cong W(2m-1, s)$ of $\mathcal{S}$, $\mathcal{S}''_1 \neq \mathcal{S}''$, which contains x and $\mathcal{S}'''$. Assume that $\mathcal{S}''_1$ does not generate a $(2m-1)$-dimensional space. So it necessarily generates PG$(2m-2, q)$. As $\mathcal{S}''$ and $\mathcal{S}''_1$ belong to PG$(2m-2, q)$, it follows that all points and lines of $\mathcal{S}$ in PG$(2m-2, q)$ form a necessarily degenerate full polar subspace $\mathcal{S}^*$ of $\mathcal{S}$; the point set of $\mathcal{S}^*$ is a set $z^{\perp}$, with z some point of $\mathcal{S}$. Let $\overline{\mathcal{S}}$ be a naturally embedded polar space isomorphic to $W(2m+1, s)$ and let γ be an isomorphism of $\mathcal{S}$ onto $\overline{\mathcal{S}}$. Then $(\mathcal{S}'')^{\gamma}$ and $(\mathcal{S}''_1)^{\gamma}$ belong to the PG$(2m, s)$ with point set $(z^{\perp})^{\gamma}$. Now we choose a full polar subspace $\mathcal{S}''_2 \cong W(2m-1, s)$ of $\mathcal{S}$ which contains x and $\mathcal{S}'''$, such that $(\mathcal{S}''_2)^{\gamma}$ does not belong to PG$(2m, s)$. Then $\mathcal{S}''_2$ generates a $(2m-1)$-dimensional space. Also, $\mathcal{S}''_2$ contains the lines L and M. Relying on the first part of the proof of Theorem 7, we now conclude that any two intersecting lines L and M of $\mathcal{S}$, with L and M not lying in a common plane of $\mathcal{S}$, are contained in a full polar subspace $\mathcal{S}' \cong W(2m-1, s)$ of $\mathcal{S}$ which generates a PG$(2m-1, q)$. By Theorem 6 $\mathcal{S}'$ is fully and hence naturally embedded in some subspace PG$(2m-1, s)$ of PG$(2m-1, q)$, and so the point set of $\mathcal{S}'$ is the point set of PG$(2m-1, s)$.

Let V be the set of the $s+1$ points z of $\mathcal{S}$, for which $z^{\perp}$ contains each point of $\mathcal{S}'$. As $\mathcal{S}$ is not contained in PG$(2m-1, q)$, the set V contains at most one point of PG$(2m-1, q)$. Let y be a point of V which is not contained in PG$(2m-1, q)$. Further, let L be a line of

$\mathcal{S}$ containing y and let $\mathrm{PG}(2m,s)$ be the subspace of $\mathrm{PG}(2m,q)$ defined by $\mathrm{PG}(2m-1,s)$ and L. Exactly as in the proof of Theorem 7 we now have that $y^\perp$ coincides with the point set of $\mathrm{PG}(2m,s)$.

Let $y_1 \in V$, with $y_1 \neq y$ and y_1 not in $\mathrm{PG}(2m-1,q)$, and let L_1 be the line of $\mathcal{S}$ through u and y_1, where u is the common point of L and $\mathcal{S}'$. Now we consider a projective space $\mathrm{PG}(2m+1,q)$ containing $\mathrm{PG}(2m,q)$ and a line $\widetilde{L}_1$ over $\mathrm{GF}(s)$ in $\mathrm{PG}(2m+1,q)$ which passes through u but is not contained in $\mathrm{PG}(2m,q)$, such that L_1 is the projection of $\widetilde{L}_1$ from some point $p \in \mathrm{PG}(2m+1,q) \backslash \mathrm{PG}(2m,q)$ onto $\mathrm{PG}(2m,q)$. Then there is exactly one subspace $\mathrm{PG}(2m+1,s)$ of $\mathrm{PG}(2m+1,q)$ which contains $\widetilde{L}_1$ and $\mathrm{PG}(2m,s)$. As $y_1 \notin \mathrm{PG}(2m,s)$, it is clear that $p \notin \mathrm{PG}(2m+1,s)$. The subspace $\mathrm{PG}^{(1)}(2m,s)$ defined by $\widetilde{L}_1$ and $\mathrm{PG}(2m-1,s)$ is projected from p onto the subspace with point set $y_1^\perp$ of $\mathrm{PG}(2m,q)$. Let r be any point of $\mathcal{S}'$. By the first part of the proof the lines ry and ry_1 of $\mathcal{S}$ are contained in a full polar subspace $\mathcal{S}^* \cong W(2m-1,s)$ of $\mathcal{S}$ which is naturally embedded in some subspace $\mathrm{PG}^*(2m-1,s)$ of $\mathrm{PG}(2m,q)$. The set $V = \{y,y_1\}^{\perp\perp} = \bigcap_{x \in y^\perp \cap y_1^\perp} x^\perp$ of $\mathcal{S}$ coincides with the set $\{y,y_1\}^{\perp^*\perp^*}$ of $\mathcal{S}^*$ (here "$\perp^*$" means perpendicularity in $\mathcal{S}^*$). Hence V is a subline over $\mathrm{GF}(s)$. Exactly as in the proof of Theorem 7 we now show that for each point w of $\mathcal{S}$ there is exactly one point $\widetilde{w}$ of $\mathrm{PG}(2m+1,s)$ which is projected from p onto w.

Finally, let N be any line of $\mathcal{S}$ and let $\widetilde{N}$ be the corresponding point set of $\mathrm{PG}(2m+1,s)$. As in the proof of Theorem 7 we prove that $\widetilde{N}$ is a line of $\mathrm{PG}(2m+1,s)$. It follows that $\mathcal{S}$ is the projection from p onto $\mathrm{PG}(2m,q)$ of a $\widetilde{\mathcal{S}} \cong W(2m+1,s)$ which is fully (and hence naturally) embedded in $\mathrm{PG}(2m+1,s)$. $\square$

Lemma 6 *If $d \leq 2m-1$, with $m \geq 3$, then $\mathcal{S}$ contains a full polar subspace $\mathcal{S}' \cong W(2m-1,s)$ which generates $\mathrm{PG}(d,q)$ and which contains two given intersecting lines of $\mathcal{S}$, not contained in a common plane of $\mathcal{S}$.*

Proof. Let L and M be two given intersecting lines of $\mathcal{S}$ which are not contained in a common plane of $\mathcal{S}$. Now choose a full polar subspace $\mathcal{S}' \cong W(2m-1,s)$ of $\mathcal{S}$ for which L and M are lines. Assume that $\mathcal{S}'$ generates a projective subspace $\mathrm{PG}(l,q)$ of $\mathrm{PG}(d,q)$. Let x and y be distinct points of $\mathcal{S}$ for which $x^\perp$ and $y^\perp$ contain all points of $\mathcal{S}'$. Then by the reasoning in the second part of the proof of Theorem 1 $\mathcal{S}$ is contained in $\langle x^\perp, y^\perp \rangle = \langle \mathrm{PG}(l,q), x, y \rangle$. Hence $l \geq d-2$.

If $l = d$, then $\mathcal{S}'$ generates $\mathrm{PG}(d,q)$ and we are finished.

Next, assume that $l = d-1 = 2m-2$. Then by the first part of proofs of Theorems 7 and 8 $\mathcal{S}'$ contains a full polar subspace $\mathcal{S}'' \cong W(2m-3,s)$ which contains L and M and which generates a subspace $\mathrm{PG}(2m-3,q)$ of $\mathrm{PG}(d-1,q) = \mathrm{PG}(2m-2,q)$. If V is the set of all points x in $\mathcal{S}$ such that $x^\perp$ contains all points of $\mathcal{S}''$, then V together with all lines of $\mathcal{S}$ in V is a full polar subspace $\mathcal{S}''' \cong W(3,s)$ of $\mathcal{S}$. Also, as every point of $\mathcal{S}$ belongs to a line of $\mathcal{S}$ intersecting $\mathcal{S}''$ and $\mathcal{S}'''$, we have that the union of the point sets of $\mathcal{S}''$ and $\mathcal{S}'''$ generates $\mathrm{PG}(d,q)$. Consider a point p of $\mathcal{S}'''$ which does not belong to $\mathrm{PG}(2m-3,q) = \mathrm{PG}(d-2,q)$; then p and $\mathcal{S}''$ generate a $\mathrm{PG}(d-1,q)$. Assume, by way of contradiction, that every point p' of $\mathcal{S}'''$ which does not belong to $\mathrm{PG}(d-1,q)$ is in $p^\perp$. Choose hyperbolic lines R and R' in $\mathcal{S}'''$ with $R' \subset R^\perp$ (in $\mathcal{S}$), such that $p \notin R \cup R'$. Then p is on a unique line N of $\mathcal{S}'''$ intersecting R and R'. As each point of $\mathcal{S}'''$ is on a line of $\mathcal{S}'''$ intersecting R and R', it follows easily that each point of $\mathcal{S}'''$ not on N is contained in $\mathrm{PG}(d-1,q)$, clearly a contradiction. Hence $\mathcal{S}'''$ contains a point p' not in $\mathrm{PG}(d-1,q)$ with $p' \notin p^\perp$. Now $\mathcal{S}$ has a unique full polar subspace $\mathcal{S}'_1 \cong W(2m-1,s)$ containing $\mathcal{S}''$ and $\{p,p'\}$; clearly $\mathcal{S}'_1$ generates $\mathrm{PG}(d,q)$.

Next, assume that $l = d-1 \leq 2m-3$, with $d \geq 5$. Now we proceed by induction on d. So we may assume that $\mathcal{S}'$ contains a full polar subspace $\mathcal{S}'' \cong W(2m-3,s)$ which generates $\mathrm{PG}(d-1,q)$ and for which L and M are lines. If V is the set of all points x in $\mathcal{S}$ such that $x^\perp$ contains all points of $\mathcal{S}''$, then V together with all lines of $\mathcal{S}$ in V is a full polar subspace $\mathcal{S}''' \cong W(3,s)$ of $\mathcal{S}$. Also, the union of the point sets of $\mathcal{S}''$ and of $\mathcal{S}'''$ generates $\mathrm{PG}(d,q)$. Consider a point of $\mathcal{S}'''$ which does not belong to $\mathrm{PG}(d-1,q)$; then p and $\mathcal{S}''$ generate $\mathrm{PG}(d,q)$. Now let $\mathcal{S}'_1$ be any full polar subspace of $\mathcal{S}$ isomorphic to $W(2m-1,s)$ which contains $\mathcal{S}''$ and p. Clearly any such $\mathcal{S}'_1$ generates $\mathrm{PG}(d,q)$.

Now, assume that $l = d-2$. Then $l \leq 2m-3$. Suppose that $d \geq 6$ and proceed by induction on d. So we may assume that $\mathcal{S}'$ contains a full polar subspace $\mathcal{S}'' \cong W(2m-3,s)$ which generates $\mathrm{PG}(d-2,q)$ and for which L and M are lines. Then by a previous case (the case $l = d-1 = 2m-2$) $\mathcal{S}$ has a full polar subspace $\mathcal{S}'_1 \cong W(2m-1,s)$ containing $\mathcal{S}''$ and generating $\mathrm{PG}(d,q)$.

Finally, we consider the "small" cases $d = 4,5$. First, let $d = 4$. We have to handle the cases $l = 3$ and $l = 2$. We must prove that $\mathcal{S}$ contains a full polar subspace $\mathcal{S}'_1 \cong W(2m-1,s)$ which generates $\mathrm{PG}(4,q)$ and contains the lines L and M. First, let $l = 3$. As in the section on the case $l = d-1 \leq 2m-3$, with $d \geq 5$, it is sufficient to prove that $\mathcal{S}'$ contains a full polar subspace $\mathcal{S}'' \cong W(2m-3,s)$ which generates $\mathrm{PG}(3,q)$ and for which L and M are lines. Let x be the common point of L and M, and assume, by way of contradiction, that each point of $\mathcal{S}'$ not in $x^\perp$ belongs to the plane $\langle L, M\rangle$. Let z be a point of $\mathcal{S}'$ not in $x^\perp$ and let N be a line of $\mathcal{S}'$ through x but not in $\langle L, M\rangle$ (N exists as $\mathcal{S}'$ generates $\mathrm{PG}(3,q)$). Let U be the unique line of $\mathcal{S}'$ containing z and a point of N. Then on U there are s points not in $\langle L, M\rangle$ or $x^\perp$, a contradiction. So $\mathcal{S}'$ has a point y not in $\langle L, M\rangle$ or $x^\perp$. If $\{u\} = L \cap y^\perp$ and $\{v\} = M \cap y^\perp$, then $\{u,v\}^{\perp\perp} \cup \{x,y\}^{\perp\perp}$ belongs to a full polar subspace $\mathcal{S}'' \cong W(2m-3,s)$ of $\mathcal{S}'$ which generates $\mathrm{PG}(3,q)$ and for which L and M are lines. Next, let $l = 2$. Suppose that $\mathcal{S}'' \cong W(2m-3,s)$ is a full polar subspace of $\mathcal{S}'$ for which L and M are lines. Now we proceed as in the case $l = d-2$, with $l \leq 2m-3$ and $d \geq 6$. Finally, the case $d = 5$ with $l = 3$ is completely similar to the case $d = 4$ with $l = 3$ at the beginning of this section.

Now the lemma is completely proved. $\square$

Theorem 9 *If $d \leq 2m - 1$, with $d \geq 4$, then there exists a* $\mathrm{PG}(2m+1, q)$ *containing* $\mathrm{PG}(d, q)$ *and a* $\mathrm{PG}(2m - d, q)$ *of* $\mathrm{PG}(2m+1, q)$ *skew to* $\mathrm{PG}(d, q)$ *such that $\mathcal{S}$ is the projection from* $\mathrm{PG}(2m - d, q)$ *onto* $\mathrm{PG}(d, q)$ *of a polar space $\widetilde{\mathcal{S}} \cong \mathcal{S}$ which is fully and naturally embedded in a subspace* $\mathrm{PG}(2m+1, s)$ *of* $\mathrm{PG}(2m+1, q)$.

Proof. By Lemma 6, $\mathcal{S}$ contains a full polar subspace $\mathcal{S}' \cong W(2m-1, s)$ which generates $\mathrm{PG}(d, q)$. Proceeding by induction on m and taking account of Theorems 6, 7 and 8, we may assume that there exists a $\mathrm{PG}(2m-1, q)$ containing $\mathrm{PG}(d, q)$ and a subspace $\mathrm{PG}(2m - d - 2, q)$ of $\mathrm{PG}(2m-1, q)$ skew to $\mathrm{PG}(d, q)$ such that $\mathcal{S}'$ is the projection from $\mathrm{PG}(2m - d - 2, q)$ onto $\mathrm{PG}(d, q)$ of a polar space $\widetilde{\mathcal{S}'} \cong \mathcal{S}'$ which is fully and naturally embedded in a subspace $\mathrm{PG}(2m-1, s)$ of $\mathrm{PG}(2m-1, q)$ (if $d = 2m-1$, then we put $\mathcal{S}' = \widetilde{\mathcal{S}'}$). Clearly the point sets of $\widetilde{\mathcal{S}'}$ and $\mathrm{PG}(2m-1, s)$ coincide.

Let V be the set of all points u of $\mathcal{S}$ such that $u^\perp$ contains the point set of $\mathcal{S}'$; then $|V| = s + 1$. Let $x \in V$, let $z \in x^\perp$ with z a point of $\mathcal{S}'$, and let $\widetilde{z}$ be the point of $\widetilde{\mathcal{S}'}$ which is projected from $\mathrm{PG}(2m - d - 2, q)$ onto z. The line of $\mathcal{S}$ containing z and x will be denoted by L. Embed $\mathrm{PG}(2m-1, q)$ into a projective space $\mathrm{PG}(2m, q)$, and let $\mathrm{PG}(2m - d - 1, q)$ be a subspace of $\mathrm{PG}(2m, q)$ which contains $\mathrm{PG}(2m - d - 2, q)$ but does not belong to $\mathrm{PG}(2m-1, q)$. In $\mathrm{PG}(2m - d + 1, q) = \langle \mathrm{PG}(2m - d - 1, q), L \rangle$ we now consider a line $\widetilde{L}$ over $\mathrm{GF}(s)$ containing $\widetilde{z}$ whose projection from $\mathrm{PG}(2m - d - 1, q)$ onto $\mathrm{PG}(d, q)$ is L, and which is not in $\mathrm{PG}(2m-1, q)$. Then there is a unique subspace $\mathrm{PG}(2m, s)$ of $\mathrm{PG}(2m, q)$ which contains $\mathrm{PG}(2m-1, s)$ and $\widetilde{L}$.

Let u be a point of $\mathcal{S}'$, with $u \in z^\perp \backslash \{z\}$. With u there corresponds the point $\widetilde{u}$ of $\widetilde{\mathcal{S}'}$. If π is the plane of $\mathcal{S}$ containing the points u, z, x, and if $\widetilde{\pi}$ is the plane of $\mathrm{PG}(2m, s)$ containing the points $\widetilde{u}, \widetilde{z}, \widetilde{x}$, then π is the projection of $\widetilde{\pi}$ from $\mathrm{PG}(2m - d - 1, q)$ onto $\mathrm{PG}(d, q)$. So the line $\widetilde{u}\widetilde{x}$ of $\mathrm{PG}(2m, s)$ is projected from $\mathrm{PG}(2m - d - 1, q)$ onto the line of $\mathcal{S}$ containing the points u and x. Next, let u_1 be a point of $\mathcal{S}$ which does not belong to $z^\perp$. Further, let u_2 be a point of $\mathcal{S}$ with $u_2 \in u_1^\perp \cap z^\perp$, and let $\widetilde{u}_2$ be the corresponding point of $\widetilde{\mathcal{S}'}$. Interchanging roles of z and u_2 we then see that the line of $\mathcal{S}$ through u_1 and x is the projection of the line $\widetilde{u}_1\widetilde{x}$ of $\mathrm{PG}(2m, s)$ from $\mathrm{PG}(2m - d - 1, q)$ onto $\mathrm{PG}(d, q)$. It follows that the point set of $\mathrm{PG}(2m, s)$ is projected from $\mathrm{PG}(2m - d - 1, q)$ onto $x^\perp$.

Now let $y \in V \backslash \{x\}$; the line of $\mathcal{S}$ through y and z will be denoted by M. Embed $\mathrm{PG}(2m, q)$ in a projective space $\mathrm{PG}(2m+1, q)$ and let $\mathrm{PG}(2m - d, q)$ be a subspace of $\mathrm{PG}(2m+1, q)$ containing $\mathrm{PG}(2m - d - 1, q)$, with $\mathrm{PG}(2m - d, q) \not\subset \mathrm{PG}(2m, q)$. In $\mathrm{PG}(2m - d + 2, q) = \langle \mathrm{PG}(2m - d, q), M \rangle$ we now consider a line $\widetilde{M}$ over $\mathrm{GF}(s)$ containing $\widetilde{z}$ whose projection from $\mathrm{PG}(2m - d, q)$ onto $\mathrm{PG}(d, q)$ is M, and which is not in $\mathrm{PG}(2m, q)$. Then there is a unique subspace $\mathrm{PG}(2m+1,s)$ of $\mathrm{PG}(2m+1,q)$ containing $\mathrm{PG}(2m,s)$ and $\widetilde{M}$. As in the previous section we show that the point set of the projective subspace $\mathrm{PG}^{(1)}(2m,s)$ of $\mathrm{PG}(2m+1,q)$ containing $\mathrm{PG}(2m-1,s)$ and $\widetilde{M}$, is projected from $\mathrm{PG}^{(1)}(2m-d-1,q)$ onto $y^{\perp}$, where $\mathrm{PG}^{(1)}(2m-d-1,q) = \mathrm{PG}^{(1)}(2m,q) \cap \mathrm{PG}(2m-d,q)$ with $\mathrm{PG}^{(1)}(2m,q) = \langle \mathrm{PG}(2m-1,q), \widetilde{M} \rangle$; hence the point set of $\mathrm{PG}^{(1)}(2m,s)$ is also projected from $\mathrm{PG}(2m-d,q)$ onto $y^{\perp}$.

By Lemma 6 $\mathcal{S}$ contains a full polar subspace $\mathcal{S}^* \cong W(2m-1,s)$ which generates $\mathrm{PG}(d,q)$ and contains the lines L and M and $\mathcal{S}$. Again by induction and the Theorems 6, 7 and 8, $\mathcal{S}^*$ can be obtained as projection of some naturally embedded $\widetilde{\mathcal{S}}^*$. So the set $\{x,y\}^{\perp^*\perp^*}$ in $\mathcal{S}^*$ is the projection of some line over $\mathrm{GF}(s)$, hence also $\{x,y\}^{\perp^*\perp^*}$ is a line over $\mathrm{GF}(s)$; also, $L, M, \{x,y\}^{\perp^*\perp^*}$ and all lines of $\mathcal{S}^*$ containing z and a point of $\{x,y\}^{\perp^*\perp^*}$, are lines of a common plane over $\mathrm{GF}(s)$. As V coincides with $\{x,y\}^{\perp^*\perp^*}$, we see that V is a line over $\mathrm{GF}(s)$ and that L, M, V and all lines of $\mathcal{S}$ containing z and a point of V are lines of a common plane on $\mathrm{GF}(s)$.

Let w be any point of $\mathcal{S}$ not in $\mathcal{S}'$. Then w is on a line tv of $\mathcal{S}$, with $t \in V$ and v in $\mathcal{S}'$. Interchanging roles of z and v, we see that the lines vx, vy, vt of $\mathcal{S}$ belong to a common plane π over $\mathrm{GF}(s)$. Let $\widetilde{v}$ be the point of $\widetilde{\mathcal{S}}'$ which corresponds to v. Then the line vx, respectively vy, of $\mathcal{S}$ is the projection from $\mathrm{PG}(2m-d,q)$ onto $\mathrm{PG}(d,q)$ of the line $\widetilde{v}\widetilde{x}$, respectively $\widetilde{v}\widetilde{y}$, of $\mathrm{PG}(2m+1,s)$. So π is the projection from $\mathrm{PG}(2m-d,q)$ onto $\mathrm{PG}(d,q)$ of the plane $\widetilde{\pi}$ of $\mathrm{PG}(2m+1,s)$ containing $\widetilde{v},\widetilde{x},\widetilde{y}$. As w is a point of π, it is the projection of a point of $\widetilde{\pi}$, so the projection of a point of $\mathrm{PG}(2m+1,s)$. As the number of points of $\mathrm{PG}(2m+1,s)$ is also the number of points of $\mathcal{S}$, it is clear that for each point w of $\mathcal{S}$ there is exactly one point $\widetilde{w}$ of $\mathrm{PG}(2m+1,s)$ which is projected from $\mathrm{PG}(2m-d,q)$ onto w.

Let N be any line of $\mathcal{S}$ and let w_1 and w_2 be distinct points of N. Then the line $\widetilde{w_1}\widetilde{w_2}$ of $\mathrm{PG}(2m+1,s)$ is projected from $\mathrm{PG}(2m-d,q)$ onto a set R of size $s+1$ of $\mathcal{S}$. As the line $N' = \langle N \rangle$ of $\mathrm{PG}(d,q)$ intersects the point set of $\mathcal{S}$ exactly in N, we necessarily have $N = R$. It follows that $\mathcal{S}$ is the projection from $\mathrm{PG}(2m-d,q)$ onto $\mathrm{PG}(d,q)$ of a $\widetilde{\mathcal{S}} \cong W(2m+1,s)$ which is fully (and hence naturally) embedded in $\mathrm{PG}(2m+1,s)$.

Finally, the induction argument is complete by Theorems 6, 7 and 8 (the smallest values of m for given d). $\square$

5 The open cases

For laxly embedded polar spaces of rank $r \geq 3$, the following cases are still open :

(a) determine all polar spaces $\mathcal{S}$ of rank $r \geq 3$, with $\mathcal{S} \ncong W(n,s)$ for s odd, which are laxly embedded in PG$(2, q)$;

(b) determine all polar spaces $\mathcal{S} \cong W(n, s)$, with $n \geq 5$ and s odd, which are laxly embedded in PG$(d, q), d \in \{2, 3\}$.

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