On Embeddings of the Flag Geometries of Projective Planes in Finite Projective Spaces

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Abstract

The flag geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ of a finite projective plane Π of order s is the generalized hexagon of order $(s, 1)$ obtained from Π by putting $\mathcal{P}$ equal to the set of all flags of Π, by putting $\mathcal{L}$ equal to the set of all points and lines of Π and where $\mathbf{I}$ is the natural incidence relation (inverse containment), i.e., Γ is the dual of the double of Π in the sense of Van Maldeghem [7]. Then we say that Γ is fully and weakly embedded in the finite projective space $\mathbf{PG}(d, q)$ if Γ is a subgeometry of the natural point-line geometry associated with $\mathbf{PG}(d, q)$, if $s = q$, if the set of points of Γ generates $\mathbf{PG}(d, q)$, and if the set of points of Γ not opposite any given point of Γ does not generate $\mathbf{PG}(d, q)$. Preparing the classification of all such embeddings, we construct in this paper the classical examples, prove some generalities and show that the dimension d of the projective space belongs to $\{6, 7, 8\}$.

Mathematics Subject Classification 1991: 51E15, 51A45, 51E12.
Key words and phrases: generalized hexagons, projective planes, projective embeddings.

1 Introduction

The problem that we consider in this paper stems from an attempt to characterize the "natural" embeddings of all finite Moufang classical generalized hexagons (generalized hexagons were introduced by Tits [6]; for more details on the "natural" embeddings, see Section 3 below). In fact, it is well-known that a finite Moufang hexagon of order (s, t) contains a subhexagon of order either $(1, t)$ or $(s, 1)$ (or both). In order to distinguish between these two (non-disjoint) cases, one sometimes calls a finite Moufang hexagon

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with a subhexagon of order $(1,t)$ *classical*, and one with a subhexagon of order $(s,1)$ *dual classical*. The natural embeddings in $\mathbf{PG}(d,q)$ of all classical finite hexagons of order (q,t) have been characterized in several ways by Thas & Van Maldeghem [4, 5]. The main tool in these cases is the fact that all lines of the generalized hexagon Γ through a point of Γ belong to a plane of $\mathbf{PG}(d,q)$. The "natural" embeddings of the dual classical hexagons (see also Section 3) in general no longer have that property. Hence one needs new techniques to handle these embeddings. In the present paper, we introduce such a technique, namely, we look first at the embeddings of the hexagons of order $(q,1)$ in $\mathbf{PG}(d,q)$. Our Main Conjecture is that the embeddings of such geometries Γ of order $(q,1)$ arising from the "natural" embeddings of the dual classical hexagons of order (q,q) and (q,q^3) are characterized by one simple property: it must be a *weak embedding*, i.e., the points of Γ not opposite a given point of Γ do not generate the ambient projective space $\mathbf{PG}(d,q)$ (for precise definitions, see below). We will show here that an embedded generalized hexagon Γ of order $(q,1)$ in $\mathbf{PG}(d,q)$ satisfying this property must arise from a Desarguesian projective plane as described in the abstract (see also Section 2 below), and that $d \in \{6,7,8\}$. The distinct cases $d = 6,7,8$ will be treated in detail elsewhere, since they are quite involved. At present, our Main Conjecture is proved in the cases $d = 6,7$.

2 Preliminaries

Let Π be a (finite) projective plane of order s. We define the *flag geometry* Γ of Π as follows. The points of Γ are the flags of Π (i.e., the incident point-line pairs of Π); the lines of Γ are the points and lines of Π. Incidence between points and lines of Γ is reverse containment. It follows that Γ is a (finite) generalized hexagon of order $(s,1)$ (see (1.6) of Van Maldeghem [7]). The advantage of viewing Γ rather as a generalized hexagon than as the flag geometry of a projective plane is that one can apply results from the general theory of generalized hexagons. We will call Γ a *thin hexagon* (since there are only 2 lines through every point of Γ).

Throughout, we assume that Γ is a thin hexagon of order $(s,1)$ with corresponding projective plane Π. We introduce some further notation. For a point x of Γ, we denote by $x^\perp$ the set of points of Γ collinear with x (two points are *collinear* if they are incident with a common line); we denote by $x^{\perp\!\!\!\perp}$ the set of points of Γ not opposite x (i.e., not at distance 6 from x in the incidence graph of Γ). For a line L of Γ, we write $L^{\perp\!\!\!\perp}$ for the intersection of all sets $p^{\perp\!\!\!\perp}$ with p a point incident with L (in this notation, we view L as the set of points incident with it). For an element x of Γ (point or line), we denote by $\Gamma_i(x)$ the set of all elements of Γ at distance i from x in the incidence graph of Γ. In this notation, we have $p^\perp = \Gamma_0(p) \cup \Gamma_2(p)$, $p^{\perp\!\!\!\perp} = \Gamma_0(p) \cup \Gamma_2(p) \cup \Gamma_4(p)$ and $L^{\perp\!\!\!\perp} = \Gamma_1(L) \cup \Gamma_3(L)$.

Let $\mathbf{PG}(d,q)$ be the d-dimensional projective space over the Galois field $\mathbf{GF}(q)$. We say that Γ is *weakly embedded in* $\mathbf{PG}(d,q)$ if the point set of Γ is a subset of the point set of $\mathbf{PG}(d,q)$ which generates $\mathbf{PG}(d,q)$, if the line set of Γ is a subset of the line set of $\mathbf{PG}(d,q)$, if the incidence relation in Γ is the restriction of the incidence relation in $\mathbf{PG}(d,q)$, and if for every point x of Γ, the set $x^{\perp\!\!\perp}$ does not generate $\mathbf{PG}(d,q)$. If moreover $s=q$, then we say that the weak embedding is also *full*.

The only examples of weak full embeddings of finite thin hexagons in $\mathbf{PG}(d,q)$ known to us arise from full embeddings of the dual classical generalized hexagons of order (q,q) and (q,q^3) (the fully embedded dual classical generalized hexagon of order (q,q) considered here is a subhexagon of the fully embedded dual classical generalized hexagon of order (q,q^3)). In the next paragraph we will briefly describe how this is done, and we will give a detailed description of the embeddings. In Section 4 we will show our Main Result:

Main Result. *If* Γ *is a thin generalized hexagon weakly and fully embedded in some projective space* $\mathbf{PG}(d,q)$*, and if* Γ *is the flag geometry of the projective plane* Π*, then* Π *is Desarguesian and* $d\in\{6,7,8\}$*.*

In the next section it will be shown that the cases $d=6,7$ actually occur.

3 The examples

3.1 Some background

The background of the construction concerns the dual classical hexagons. For definitions and properties used in this subsection, we refer to Thas [3] or Van Maldeghem [7] (since this is not essential for the sequel).

Let $\mathsf{H}(q)$ denote the classical hexagon of order q (the *split Cayley hexagon* in the terminology of Van Maldeghem [7]). It has a natural standard embedding in $\mathbf{PG}(6,q)$ with the following properties. On top of being a weak and full embedding, for any point x of $\mathsf{H}(q)$, the set of points $x^{\perp}$ is the point set of a projective subplane $\mathbf{PG}(2,q)$ of $\mathbf{PG}(6,q)$. In other words, the set of lines of $\mathsf{H}(q)$ through x is a plane line pencil in $\mathbf{PG}(6,q)$. Hence, on the Grassmann variety of the lines of $\mathbf{PG}(6,q)$, the lines of $\mathsf{H}(q)$ become points, and the points of $\mathsf{H}(q)$ — which can be identified with the corresponding plane line pencil — become lines. Hence the dual of $\mathsf{H}(q)$ is embedded in some $\mathbf{PG}(d,q)$; this embedding will be called *natural*. Moreover, one can check that this embedding is weak. Since we will not use that fact, we wil not prove it here. Hence also the thin subhexagons of order $(q,1)$ of the dual of $\mathsf{H}(q)$ are weakly and fully embedded in some $\mathbf{PG}(d',q)$. The example we describe in the next section is the result of a computation along the lines we just explained. But we will present an independent description.

Note that if q is a power of 3, then $\mathsf{H}(q)$ is self dual, and hence it has a natural embedding in $\mathbf{PG}(6,q)$, but also a "natural" embedding in some $\mathbf{PG}(d,q)$. These two embeddings are non-isomorphic.

3.2 Coordinates

Let V be a 3-dimensional vector space over $\mathbf{GF}(q)$, and let V^* be the dual space. We choose dual bases. Then the vector lines of the tensor product $V \otimes V^*$ can be seen as the point-line pairs of the projective plane $\mathbf{PG}(2,q)$. Indeed, it is easily calculated that the pair $\{(x_0,x_1,x_2),[a_0,a_1,a_2]\}$ (we use parentheses for the coordinates of points and brackets for those of lines) corresponds to the vector line generated by the vector $(a_0x_0,a_0x_1,a_0x_2,a_1x_0,a_1x_1,a_1x_2,a_2x_0,a_2x_1,a_2x_2)$. In fact, the point-line pairs of $\mathbf{PG}(2,q)$ are bijectively mapped (and we denote this bijection by θ) onto the Segre variety $\mathcal{S}_{2;2}$ in $\mathbf{PG}(8,q)$; see Hirschfeld & Thas [1], §25.5. We denote coordinates in $\mathbf{PG}(8,q)$ by $X_{00},X_{01},X_{02},X_{10},\ldots,X_{22}$. It then is easily seen that the incident point-line pairs of $\mathbf{PG}(2,q)$ are mapped into the hyperplane $\mathbf{PG}(7,q)$ of $\mathbf{PG}(8,q)$ with equation $X_{00}+X_{11}+X_{22}=0$, and that the image under θ of the set of flags of $\mathbf{PG}(2,q)$ is a set of points which generates $\mathbf{PG}(7,q)$ (this follows from the fact that $\mathcal{S}_{2;2}$ generates $\mathbf{PG}(8,q)$). Now consider the flag $F=\{(x_0,x_1,x_2),[a_0,a_1,a_2]\}$ of $\mathbf{PG}(2,q)$. Any flag of $\mathbf{PG}(2,q)$ not opposite F (viewed as a point of the thin hexagon Γ corresponding with $\mathbf{PG}(2,q)$) has the form $\{(y_0,y_1,y_2),[b_0,b_1,b_2]\}$ with $b_0y_0+b_1y_1+b_2y_2=0$ and either

$$b_0x_0+b_1x_1+b_2x_2 \quad = \quad 0, \tag{1}$$

or

$$a_0y_0+a_1y_1+a_2y_2=0. \tag{2}$$

Hence the corresponding point $p=(b_iy_j)_{i,j=0,1,2}$ of $\mathbf{PG}(8,q)$ satisfies the equation $X_{00}+X_{11}+X_{22}=0$ and (by multiplying Equations (1) and (2) with y_0,y_1,y_2 and b_0,b_1,b_2 respectively) either also $x_0X_{0j}+x_1X_{1j}+x_2X_{2j}=0$, $j=0,1,2$, or $a_0X_{i0}+a_1X_{i1}+a_2X_{i2}=0$, $i=0,1,2$. Making the appropriate linear combinations (multiplying with a_j and x_i, $i,j=0,1,2$), we see that the coordinates of p satisfy the equations $X_{00}+X_{11}+X_{22}=0$ and

$$\sum_{i,j=0}^{2} a_jx_iX_{ij}=0. \tag{3}$$

Now we note that the hyperplane with equation (3) is always distinct from $\mathbf{PG}(7,q)$. Indeed, the conditions $a_jx_i=0$, $i,j=0,1,2$, $i\neq j$, readily imply that, without loss of generality, we may assume $a_0=x_0=1$ and $a_1=a_2=x_1=x_2=0$, contradicting the fact that we have a flag. Also, we remark that the set of flags containing one fixed point (respectively line) of $\mathbf{PG}(2,q)$ is mapped under θ onto the set of points of a line of $\mathbf{PG}(7,q)$; this is immediately checked with an elementary calculation. Hence identifying every flag of $\mathbf{PG}(2,q)$ with its image under θ, we obtain a weak and full embedding of Γ in $\mathbf{PG}(7,q)$. We call this embedding (and every equivalent one with respect to the linear automorphism group of $\mathbf{PG}(7,q)$) a *natural embedding of* Γ *in* $\mathbf{PG}(7,q)$.

By another elementary calculation, one easily sees that the intersection of all hyperplanes with equation (3) is the point k with coordinates $x_{ii}=1$, $x_{ij}=0$, $i,j\in\{0,1,2\}$, $j\neq i$. This point lies in $\mathbf{PG}(7,q)$ if and only if the characteristic of $\mathbf{GF}(q)$ is equal to 3. Hence, in this case, we can project the weakly embedded thin hexagon Γ from k onto some hyperplane $\mathbf{PG}(6,q)$ of $\mathbf{PG}(7,q)$ not containing k to obtain a weak and full embedding of Γ in the 6-dimensional projective space $\mathbf{PG}(6,q)$. We call this embedding also a *natural embedding of* Γ.

The exceptional behaviour over fields with characteristic 3 is in conformity with the special behaviour of classical generalized hexagons over such fields (the hexagons related to Dickson's group $G_2(q)$, $q=3^e$, are at the same time classical and dual classical, as remarked before).

Hence we see that with every Desarguesian projective plane $\Pi\cong\mathbf{PG}(2,q)$, there corresponds a weak full embedding of the corresponding thin hexagon Γ in $\mathbf{PG}(7,q)$, and if $q=3^e$, then there is an additional weak full embedding of Γ in $\mathbf{PG}(6,q)$. In the next section, we will show that any weakly fully embedded thin hexagon has as corresponding projective plane a Desarguesian one and that the dimension of the ambient projective space is either 6, 7 or 8. This is about as far as one can go in the classification of weakly fully embedded thin hexagons using only synthetic arguments.

Remark. Everything in this section can be generalized to the infinite case without notable change.

4 Proof of the Main Result

Standing hypotheses. From now on we suppose that $\Gamma=(\mathcal{P},\mathcal{L},\mathbf{I})$ is a generalized hexagon (with point set $\mathcal{P}$ and line set $\mathcal{L}$) of order $(q,1)$ weakly embedded in $\mathbf{PG}(d,q)$. We denote by Π the projective plane for which the dual of the double of it is isomorphic to Γ.

Let $x\in\mathcal{P}$. The set $x^{\perp\!\!\!\perp}$ does not generate $\mathbf{PG}(d,q)$; hence it generates some (proper) subspace of $\mathbf{PG}(d,q)$ which we will denote by ζ_x. Since there are only two lines through every point of Γ, the embedding is also flat (i.e., all lines of Γ through any fixed point x are contained in a plane of $\mathbf{PG}(d,q)$), and hence Lemma 3 of Thas & Van Maldeghem [4] holds (it does not matter that Γ is thin). This implies that ζ_x is a hyperplane which does not contain any point of Γ opposite x. Also, as a consequence we have that $\zeta_x \neq \zeta_y$ for $x \neq y$, $x, y \in \mathcal{P}$. This implies that for any line L of Γ, the set $L^{\perp\!\!\perp}$ is contained in a $(d-2)$-dimensional space, obtained by intersecting ζ_x and ζ_y for two arbitrary but distinct points on L. We denote by ξ_L the subspace of $\mathbf{PG}(d,q)$ generated by $\Gamma_3(L)$. By the foregoing, it has dimension at most $d-2$.

We summarize the results of the previous paragraph for further reference.

Lemma 1 *For every $x \in \mathcal{P}$, the space $\zeta_x = \langle x^{\perp\!\!\perp} \rangle$ is a hyperplane which does not contain any point of $\Gamma_6(x)$. In particular, $\zeta_x \neq \zeta_y$ for $x, y \in \mathcal{P}$ with $x \neq y$. Also, for every line $L \in \mathcal{L}$, the space $\xi_L = \langle L^{\perp\!\!\perp} \rangle$ is at most $(d-2)$-dimensional.*

We will actually see that the dimension of ξ_L has to be at least $d-3$ (Lemma 5 below).

An almost direct consequence of Lemma 1 is the following lemma.

Lemma 2 *Every apartment Σ of Γ generates a 5-dimensional subspace of $\mathbf{PG}(d,q)$.*

Proof. Suppose Σ generates a projective subspace of dimension < 5. Then there is a point x of Σ which is contained in the subspace generated by the other five points of Σ. Hence, if y is opposite x in Σ, it follows that $x \in \zeta_y$, contradicting Lemma 1. $\square$

Since Γ is a full weak and flat embedding, we can use Lemma 1 of Thas & Van Maldeghem [4]. Since Γ is thin, we must modify it as follows.

Lemma 3 *Let Γ be weakly embedded in $\mathbf{PG}(d,q)$. Let U be any subspace of $\mathbf{PG}(d,q)$ containing an apartment Σ of Γ. Then the points x of Γ in U for which $\Gamma_1(x) \subseteq U$ together with the lines of Γ in U form a (thin) subhexagon Γ' of Γ. Let L, M be two concurrent lines of Σ and let x, y be two points not contained in Σ but incident with respectively L and M. If U contains $\Gamma_1(x)$ and $\Gamma_1(y)$, then Γ' is of some order $(s,1)$, $1 < s \leq q$.*

Note that in general the condition on M and y in the last statement of the previous lemma can not be deleted, since Γ' could correspond in Π with a degenerate subplane.

Proposition 4 *Let Γ be weakly embedded in $\mathbf{PG}(d,q)$. Then $d \geq 6$.*

Proof. By Lemma 2, we must have $d \geq 5$. Suppose now $d = 5$. For any line L, the space ξ_L is at most 3-dimensional. Suppose it is 2-dimensional. Then two lines at distance 4 in Γ and concurrent with L meet in ξ_L, a contradiction. Hence ξ_L is 3-dimensional, for all lines L of Γ.

Now let L and M be two opposite lines of Γ. If $\xi_L \cup \xi_M$ is contained in a 4-space, then the intersection $\xi_L \cap \xi_M$ contains a plane, which meets L in some point of Γ; hence ξ_M contains a point of L, which must then be non-opposite every point of M, a contradiction. Hence $\xi_L \cup \xi_M$ generates $\mathbf{PG}(5,q)$ and hence $\xi_L \cap \xi_M$ is a line K. Note that the set of points of $\mathbf{PG}(5,q)$ on K is precisely the set $\Gamma_3(L) \cap \Gamma_3(M)$. Let N be a line of $\Gamma_4(M) \cap \Gamma_6(L)$. Then $\Gamma_3(L) \cap \Gamma_3(N)$ forms a line K' in $\mathbf{PG}(5,q)$ which meets K in a point (on $\Gamma_2(M) \cap \Gamma_2(N)$), and which meets every element of $\Gamma_2(L)$ in precisely one point. For exactly one element, this point coincides with the intersection of K and K'. Hence at least q elements of $\Gamma_2(L)$ are contained in the plane π generated by K and K', and consequently $\Gamma_2(L)$ contains a pair of concurrent lines, a contradiction.

The lemma is proved. $\square$

We have actually seen examples in the case $d = 6$, so the previous lemma is the best we can do. Before establishing an upper bound for d, we will show that Π is always Desarguesian. To that aim, we need the following lemma.

Lemma 5 *Let L be any line of Γ. Then ξ_L has dimension either $d-2$ or $d-3$, and it contains no point of $\Gamma_5(L)$. Also, there is a unique $(d-2)$-space $\widetilde{\xi_L}$ contained in all ζ_x, $x \,\mathbf{I}\, L$.*

Proof. Clearly ξ_L does not contain a point x of Γ at distance 5 from L since such a point is opposite at least one point $y \,\mathbf{I}\, L$ and this would imply $x \in \xi_L \subseteq \zeta_y$, contradicting Lemma 1.

If ξ_L is $(d-2)$-dimensional, then we put $\widetilde{\xi_L} = \xi_L$ and the result follows. So suppose that ξ_L has dimension $\ell < d-2$. Consider two distinct lines M, M' of Γ opposite L (then $M' \in \Gamma_4(M)$) and let U be the subspace of $\mathbf{PG}(d,q)$ generated by ξ_L, M, M'. Since there is a unique line L' of Γ meeting ξ_L, M and M', the space U has dimension at most $\ell+3$. By Lemma 3, U induces a subhexagon Γ' of Γ, which must be of order $(q,1)$ in view of $\xi_L \subseteq U$. Hence $\Gamma' = \Gamma$ and $U = \mathbf{PG}(d,q)$. Whence $\ell \geq d-3$, and so, by our assumption, $\ell = d-3$.

Consider any line M of Γ opposite L. Put $U_M = \langle \xi_L, M \rangle$. Then U_M must have dimension $d-1$. Indeed, otherwise M has at least one point in common with ξ_L and such a point belongs to $\Gamma_5(L)$, a contradiction. Hence U_M is a hyperplane. Clearly $U_{M'} \neq U_M$ for $M \neq M'$, M' opposite L in Γ, for otherwise Γ is induced in a hyperplane U_M. Also, $U_M \neq \zeta_x$, for all $x \,\mathbf{I}\, L$ because M contains points opposite x. Hence the set of hyperplanes

$$\{U_M \mid M \in \Gamma_6(L)\} \cup \{\zeta_x \mid x \,\mathbf{I}\, L\}$$

is the complete set of q^2+q+1 hyperplanes through ξ_L. They form the lines of the residual plane $\pi(\xi_L)$ in $\mathbf{PG}(d,q)$ of ξ_L. Let K be a line of Γ at distance 4 from L. It is clear that $\langle \xi_L, K\rangle$ is $(d-2)$-dimensional and that it is contained in every hyperplane U_M, $M \in \Gamma_2(K) \setminus \Gamma_2(L)$, and in ζ_x, for the unique $x \in \Gamma_1(L) \cap \Gamma_3(K)$. Hence each U_M contains $q+1$ points of $\pi(\xi_L)$, each of which is on a unique line of $\pi(\xi_L)$ of type ζ_x. In total, we have q^2+q such points on lines U_M of $\pi(\xi_L)$, so the only missing point can be denoted $\widetilde{\xi}_L$ and belongs to all lines of type ζ_x.

The lemma is proved. $\square$

We can now show that Π is Desarguesian.

Proposition 6 *The projective plane* Π *is isomorphic to* $\mathbf{PG}(2,q)$.

Proof. Consider two arbitrary lines M and M' of Γ which are at distance 4. Let L be some line of Γ opposite both M and M'. Next, let (x_0,x_1,x_2,x_3) be an arbitrary sequence of points on M, containing at least three different elements. Now, let y_i, $i=0,1,2,3$, be the unique point on L not opposite x_i. Then ζ_{y_i} contains x_i, but not x_j for $j \neq i$. By the previous lemma we now have the following equality of cross-ratios:

$$(x_0,x_1;x_2,x_3) = (\zeta_{y_0},\zeta_{y_1};\zeta_{y_2},\zeta_{y_3}).$$

If we denote by x_i', $i=0,1,2,3$, the point on M' not opposite y_i, then obviously $(x_0,x_1;x_2,x_3) = (\zeta_{y_0},\zeta_{y_1};\zeta_{y_2},\zeta_{y_3}) = (x_0',x_1';x_2',x_3')$. Denote by N_i respectively N_i' the unique line of Γ containing x_i respectively x_i', and not opposite L, $i=0,1,2,3$. Without loss of generality we may assume that the lines N_i, N_i' are points of the projective plane Π. Then in Π the perspectivity with center L from the line M onto the line M' maps the point N_i onto the point N_i', $i=0,1,2,3$. By the foregoing, with this perspectivity there corresponds a projectivity of the line M of $\mathbf{PG}(d,q)$ onto the line M' of $\mathbf{PG}(d,q)$ (as cross-ratios are preserved). It follows that the group of projectivities of any line M of Π acts semi-regularly on the set of all ordered triples of distinct points of Π on M. Hence this group is sharply 3-transitive on the line M of Π, and so Π is the plane $\mathbf{PG}(2,q)$ (see e.g. PICKERT [2], p. 139). $\square$

We now look for an upper bound for d.

Proposition 7 *Let* Γ *be weakly embedded in* $\mathbf{PG}(d,q)$. *Then* $d \leq 8$.

Proof. Consider an apartment Σ in Γ. We consider a coordinatization over $\mathbf{GF}(q)$ of Π with respect to the triangle T in Π corresponding to Σ (we use homogeneous coordinates). The point in Π with coordinates $(1,1,1)$ is some line E in $\mathbf{PG}(d,q)$. Now let r be some generator of the multiplicative group of $\mathbf{GF}(q)$, and let R be the line of $\mathbf{PG}(d,q)$ which corresponds to the point of Π with coordinates $(1,r,0)$. It is easily seen that the triangle T and the points $(1,1,1)$ and $(1,r,0)$ generate the whole plane Π. Hence, Γ is induced in the subspace generated by Σ, E, R. Consequently $\mathbf{PG}(d,q)$ must be generated by Σ, E, R. Since R meets Σ in some point, and since Σ generates a subspace of dimension 5, we see that $d \leq 5+2+1 = 8$. $\square$

We end with some important and useful consequences of the previous results.

Corollary 8 *Let L and M be two arbitrary opposite lines of Γ. Let $L_0, L_1, \ldots, L_k$ be $k+1$ distinct elements of $\Gamma_2(L)$, $1 \leq k \leq q$, and put $\Gamma_2(M) \cap \Gamma_2(L_i) = \{M_i\}$, $0 \leq i \leq k$. Then the dimension of the subspace U of $\mathbf{PG}(d,q)$ generated by $L_0, L_1, \ldots, L_k$ is equal to the dimension of the subspace V generated by $M_0, M_1, \ldots, M_k$. In particular, the dimension of ξ_L is independent of the line L of Γ.*

Proof. Let U have dimension ℓ. Then by Lemma 5 $\langle U, M\rangle$ has dimension $\ell+2$. But $\langle U, M\rangle = \langle V, L\rangle$. Hence, by Lemma 5 again, V has dimension ℓ. $\square$

Corollary 9 *Let L_0, L_1, L_2 be three distinct lines of Γ concurrent with some line $L \in \mathcal{L}$. Then $U := \langle L_0, L_1, L_2\rangle$ has dimension 4.*

Proof. If the dimension of U is not 4, then it must be 3. In that case, let M_0, M_1, M_2 (respectively M_0', M_1', M_2') be three lines of Γ concurrent with respectively L_0, L_1, L_2 and all belonging to $\Gamma_2(M)$ (respectively $\Gamma_2(M')$), for some line M (respectively M') opposite L of Γ. By Corollary 8, the dimension of $\langle M_0, M_1, M_2\rangle$ (respectively $\langle M_0', M_1', M_2'\rangle$) is equal to 3. Now note that in Π the mapping $M_i' \mapsto M_i$, $i = 0, 1, 2$, is the restriction to $\{M_0', M_1', M_2'\}$ of a perspectivity from M' onto M. Since the group of projectivities of any line of Π is (sharply) 3-transitive, we conclude that for any three distinct lines K_0, K_1, K_2 concurrent with some line K of Γ the space $W := \langle K_0, K_1, K_2\rangle$ has dimension 3. We can thus choose K in $\Gamma_2(M_0) \cap \Gamma_4(M) \cap \Gamma_6(L)$, $K_0 = M_0$ and K_i concurrent with L_i, $i = 1, 2$. We now have that $\langle U, M\rangle$ is 5-dimensional (by Lemma 5), hence $U \cap V$ is a line. But similarly, $U \cap W$ is a line (which meets $U \cap V$ in the unique element of $\Gamma_1(L_0) \cap \Gamma_1(M_0)$), and this implies that L_1 and L_2 are contained in the plane spanned by $U \cap V$ and $U \cap W$. This contradicts the fact that the lines L_1 and L_2 do not meet in $\mathbf{PG}(d,q)$. $\square$

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