Two remarks on generalized hexagons

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Abstract

We characterize the point-distance-2-regular hexagons as the only hexagons for which the intersection sets have size one, and containing on ovoidal subspace all the points of which are 3-regular. We also give a characterization of the finite split Cayley hexagon of even order.

1 Introduction

A *weak generalized n-gon* Γ is a point-line incidence geometry whose incidence graph has girth $2n$ and diameter n, for some natural number n, $n \geq 2$. A weak generalized n-gon is called a *generalized n-gon* if it is thick (i.e. if every vertex in the incidence graph has valency > 2). Generalized polygons were introduced by Tits [4]. For an extensive survey including most proofs, we refer the reader to [5].
We say that Γ has *order* (s, t) if every line contains $s + 1$ points, and every point is incident with exactly $t+1$ lines. Distances are measured in the incidence graph, the distance function is denoted by δ. Elements at maximal distance are called *opposite*. For any element x, we denote by $\Gamma_{[i]}(x)$ the set of elements at distance i from x, and by $x^{\perp\!\!\perp}$ the set of elements not opposite x. In this paper, we only deal with hexagons ($n = 6$). We denote by $H(q)$ the split Cayley hexagon over the field GF(q) (for a description, see [5]).

Let Γ be a generalized hexagon. A subhexagon Γ' of Γ is a subgeometry which is itself a (weak) generalized hexagon. A subhexagon Γ' is called *ideal* if every line pencil of Γ' coincides with the corresponding line pencil of Γ. A subhexagon Γ' is called *full* if every point row of Γ' coincides with the corresponding point row of Γ. If two elements u, v of Γ are at distance 4, then the unique element of $\Gamma_2(u) \cap \Gamma_2(v)$ will be denoted by $u \bowtie v$. If two elements u and v are not opposite, then there is a unique element incident with u and nearest to v; we denote this element by $\mathrm{proj}_u v$ and call it the *projection of v onto u*. For two opposite elements x and y of Γ, we denote by x^y the set $\Gamma_2(x) \cap \Gamma_4(y)$. Also, we write $\langle x, y\rangle = \Gamma_3(x) \cap \Gamma_3(y)$ (and such a set is called a *regulus*). For two opposite points x and y, we say that the pair (x, y) is 3-regular if the set $\langle x, y\rangle$ is determined by two of its lines. A point x is 3-regular if the pair (x, z) is 3-regular for all points z opposite x. If all points of the generalized hexagon are 3-regular, then Γ is said to be 3-*regular*. If for two arbitrary opposite points x and y, the set x^y is determined by any two of its points, then Γ is said to be *point-regular*.

2 Preparation to the theorems

An *ovoidal subspace* $\mathcal{O}$ in a generalized hexagon Γ is a proper subset of the point set of Γ such that each point of Γ not in $\mathcal{O}$ is collinear with a unique point of $\mathcal{O}$. An ovoidal subspace in which all the points are opposite each other is called an *ovoid*. By [2], an ovoidal subspace is either an ovoid, a full subhexagon, or the set of points at distance 1 or 3 from a given line M.

Let x be a point of a generalized hexagon Γ, and z, z' two points opposite x such that $\delta(z, z') = 4$ and $\delta(x, z \bowtie z') = 4$. Then the set $x^z \cap x^{z'}$ is called an *intersection set* if $x^z \neq x^{z'}$.

The following result is well-known :

Theorem (**Ronan** [3]). *Let Γ be a 3-regular generalized hexagon. If all intersection sets have size 1, then Γ is point-regular, and hence a Moufang hexagon.*

In the previous theorem, one can weaken the condition on the intersection sets (which is done in [2]) or the condition of 3-regularity, which is the aim of Theorem 3.1.

Let x and y be two opposite points in a generalized hexagon Γ. Then the set $I(x, y) = \{x, y\}^{\perp\!\!\perp\perp\!\!\perp}$ is called an *imaginary line*. In the finite case, a *long imaginary line* is an imaginary line which coincides with every regulus containing two of its elements. Then we have the following characterization (which also exists in the infinite case):

Theorem (**van Bon, Cuypers & Van Maldeghem** [1]). *If in a finite generalized hexagon Γ, all imaginary lines are long, then $\Gamma \cong H(q)$, q even.*

Theorem 3.2 weakens the conditions of the previous theorem.

3 Theorems

Theorem 3.1 *Let Γ be a generalized hexagon in which all intersection sets have size 1. If Γ contains an ovoidal subspace $\mathcal{O}$ all the points of which are 3-regular, then Γ is 3-regular and hence a Moufang hexagon.*

Proof. The condition about the intersection sets is equivalent with the fact that every two opposite points x and y are contained in a (unique) thin ideal subhexagon, which we denote by $\mathcal{D}(x,y)$. Let x and y be two opposite points : we prove that the pair (x,y) is 3-regular. It is sufficient to find a point z at distance 3 from two lines of $\langle x,y\rangle$ such that one of the pairs (x,z) or (y,z) is 3-regular. We may assume that neither x nor y belongs to $\mathcal{O}$.

($*$) Suppose there is a point $a \in \mathcal{D}(x,y)$ opposite x and at distance 4 from y such that the pair (a,x) is 3-regular. Let L_a be the line through $a \bowtie y$ at distance 3 from x and let z be an arbitrary point in $\langle L_a, M\rangle$, $L_a \neq M \in \langle x,y\rangle$, $z \neq x$. We show that $x^y = x^z$. Put $u = \mathrm{proj}_{L_a} z$, $u' = \mathrm{proj}_M a$, $X = \mathrm{proj}_{u'} a$ and z' the point of $\langle L_a, X\rangle$ collinear with u. Then $x^y = x^a = x^{z'} = x^z$.

($**$) Suppose in addition to ($*$) there is a point $b \in \mathcal{D}(x,y)$ opposite y and at distance 4 from x such that the pair (b,y) is 3-regular. Then the pair (x,y) is 3-regular. Indeed, let L_b be the line through $b \bowtie x$ at distance 3 from y. If $L_a \neq L_b$, put $M = L_a$ and $N = L_b$. If $L_a = L_b$, put $M = L_a$ and $N \in \langle x,y\rangle$, $N \neq L_a$. Applying ($*$), we see that for an arbitrary point $z \in \langle M,N\rangle$, $x \neq z \neq y$, $x^y = x^z$ and $y^x = y^z$, so z lies at distance 3 from every line of $\langle x,y\rangle$, and the pair (x,y) is 3-regular.

Let first $\mathcal{O}$ be an ovoid not containing x or y. Then $\mathcal{D}(x,y)$ contains 0, 1 or 2 points of $\mathcal{O}$. Suppose first $\mathcal{D}(x,y)$ contains two points a and b of $\mathcal{O}$. Up to interchanging x and y, one of the following situations occurs :

- $\delta(a,y) = 4 = \delta(b,x)$ and $\delta(a,x) = 6 = \delta(b,y)$.
  It immediately follows from ($**$) that the pair (x,y) is 3-regular.

- $\delta(a,x) = 2$, $\delta(b,x) = 4$ and $\delta(b,y) = 6$.
  Note that b lies at distance 4 from the point $a \bowtie y$. Let L_b be the line of $\langle x,y\rangle$ at distance 3 from b and L_a the line of $\langle x,y\rangle$ through a. Then ($*$) shows that

  (1) $y^x = y^{z'}$, for all points $z' \in \langle L_a, L_b\rangle$, $z' \neq y$.

  Consider the point v of y^x on L_b. Suppose first that the unique point o of $\mathcal{O}$ collinear with v does not lie on vy. Put $u = \mathrm{proj}_{L_a} o$. Note that $u \neq a$ since $o \notin \mathcal{D}(x,y)$. Let finally $z = u \bowtie (\mathrm{proj}_{L_b} u)$. Then applying ($*$), we obtain

  (2) $z^y = z^x = z^w$, for all $w \in \langle L_a, L_b\rangle$, $w \neq z$.

  Combining (1) and (2) as in ($**$), we see that the pair (x,y) is 3-regular. So we may now assume that o lies on vy. Consider an arbitrary point p of $\mathcal{D}(x,y)$ collinear with v, different from y or $v \bowtie x$. Since the line vp does not contain a point of $\mathcal{O}$, we can apply the previous argument to obtain that the pair (x,p) is 3-regular. But now again applying ($**$) shows that also the pair (x,y) is 3-regular.

- $\delta(a,x) = 2 = \delta(b,y)$.
  Let p be a point of $\mathcal{D}(x,y)$ collinear with $a \bowtie y$, different from a and y, and p' a point

of $\mathcal{D}(x,y)$ collinear with $b \bowtie x$, different from b and x. Then the previous paragraph shows that both (x,p) and (y,p') are 3-regular pairs, so applying $(**)$ gives that also the pair (x,y) is 3-regular.

Suppose now $\mathcal{D}(x,y)$ contains exactly 1 point a of $\mathcal{O}$. Then we have the following cases to consider :

- $\delta(a,x) = 2$.
  Let v be a point of x^y different from a and put $w = v \bowtie y$, $w' = a \bowtie y$. Denote by o the unique point of $\mathcal{O}$ collinear with v. If o lies on vw, then put $z = o \bowtie (\mathrm{proj}_{aw'} o)$. If o does not lie on vw, then put $u = \mathrm{proj}_{aw'} o$ and $z = u \bowtie (\mathrm{proj}_{vw} u)$. Now $\mathcal{D}(x,z)$ contains two points of $\mathcal{O}$, so the pair (x,z) (and hence the pair (x,y)) is 3-regular.

- $\delta(a,x) = 4$ and $\delta(a,y) = 6$.
  Let again L_a be the line of $\langle x,y \rangle$ at distance 3 from a. We already know that

  (3) $y^x = y^z$, for all points z, $y \neq z \in \langle L_a, M \rangle$, $M \in \langle x,y \rangle$, $M \neq L_a$.

  Choose a point $v \in y^x$, v not on L_a such that the unique point $o \in \mathcal{O}$ collinear with v does not lie on the line vy. Let L_v be the line of $\langle x,y \rangle$ through v. If o lies on L_v, then put $z = o \bowtie (\mathrm{proj}_{L_a} o)$. From the previous case, it is then clear that (x,z) is a 3-regular pair. If o does not lie on L_v, put $u = \mathrm{proj}_{L_a} o$ and $z = u \bowtie \mathrm{proj}_{L_v} u$. Then

  (4) $z^y = z^x = z^w$, for all points w, $z \neq w \in \langle L_v, M \rangle$, $M \in \langle x,y \rangle$, $M \neq L_v$.

  Combining (3) and (4), we again see that (x,y) is a 3-regular pair.

Suppose finally $\mathcal{D}(x,y)$ does not contain any point of $\mathcal{O}$. Similarly as before, we can find a point z at distance 3 from two lines of $\langle x,y \rangle$ for which the hexagon $\mathcal{D}(x,z)$ contains a point of $\mathcal{O}$, from which the result.

Suppose now $\mathcal{O} = \Gamma_1(M) \cup \Gamma_3(M)$, M a line of Γ not incident with x or y. Then one easily shows that either $\mathcal{D}(x,y)$ contains the line M, or it intersects $\mathcal{O}$ in either 0 points or 2 collinear points. As before, the case that $\mathcal{D}(x,y)$ contains no point of $\mathcal{O}$ can be reduced to one of the other cases. If $\mathcal{D}(x,y)$ contains M, then the pair (x,y) is 3-regular because of $(**)$. So we only have to consider the case that $\mathcal{D}(x,y)$ contains exactly two collinear points a and b of $\mathcal{O}$. We consider the following situations :

- $\delta(a,x) = 2 = \delta(b,y)$.
  Since M is concurrent with the line ab, we can find a point $z \in \mathcal{O}$ at distance 3 from ab and another line of $\langle x,y \rangle$, hence the result.

- $\delta(a,y) = 6 = \delta(b,x)$ (hence $\delta(a,x) = 4 = \delta(b,y)$).
  Clear because of $(**)$.

- $\delta(a,x) = 2$, $\delta(b,x) = 4$ and $\delta(b,y) = 6$.
  Note that the line ab is concurrent with M, and that a and b lie at distance 3 from M. Let L_a be the line of $\langle x,y \rangle$ through a and L' an arbitrary line of $\langle x,y \rangle$ different from

L_a. Let u be the projection onto L' of the intersection of M and ab. Let finally z be the unique point of $\langle L_a, L' \rangle$ collinear with u. Clearly, $\mathcal{D}(x,z)$ contains M, hence the pair (x,z) is 3-regular, and so is (x,y).

Suppose finally that $\mathcal{O}$ is a full subhexagon. If $\mathcal{D}(x,y)$ contains 1 point of $\mathcal{O}$, then it has at least an ordinary hexagon in common with $\mathcal{O}$, and (x,y) is 3-regular because of $(**)$. Again, the case that $\mathcal{D}(x,y)$ contains no point of $\mathcal{O}$ can be reduced to the previous one. $\square$

Consider the following property in a finite generalized hexagon Γ :

(I) Let L and M be two arbitrary opposite lines, x, y different points of $\langle L, M \rangle$ and $x' = \mathrm{proj}_M x$, $y' = \mathrm{proj}_L y$. Let N be an arbitrary line concurrent with xx', not through x or x'. Then $\mathrm{proj}_N y = \mathrm{proj}_N z$, for all $z \in \langle L, M \rangle \setminus \{x\}$.

Theorem 3.2 *A finite generalized hexagon Γ satisfies condition (I) if and only if Γ is isomorphic to $H(q)$, q even.*

Proof. If $\Gamma \cong H(q)$, q even, then condition (I) follows from the fact that all imaginary lines are long. Suppose now Γ is a finite generalized hexagon in which (I) holds. It is enough to proof that Γ is 3-regular. Indeed, combining the 3-regularity and (I), we obtain that an imaginary line coincides with a regulus containing two of its points. Since Γ is finite, this implies that all imaginary lines are long, hence the result.

So let L and M be two opposite lines, x, y, z different points of $\langle L, M \rangle$ and $N \in \langle x, y \rangle$. We have to proof that $\delta(z, N) = 3$. Put $p = \mathrm{proj}_L y$, $p' = \mathrm{proj}_M y$, $x' = \mathrm{proj}_N x$, $y' = \mathrm{proj}_N y$ and $z' = \mathrm{proj}_{xx'} z$. We first show that $z' = x'$. Suppose by way of contradiction that $z' \neq x'$ and put $z'' = \mathrm{proj}_{py} z'$. Suppose first $\mathrm{proj}_{z'} z \neq \mathrm{proj}_{z'} z''$. But this contradicts (I) since the projections of x and z onto the line through z'' and $z' \bowtie z''$ do not coincide, so $\mathrm{proj}_{z'} z = \mathrm{proj}_{z'} z''$. Note that $z' \bowtie z \neq z' \bowtie z''$ since otherwise, there would be an ordinary pentagon through the points z'', p, $\mathrm{proj}_L z$, z and $z \bowtie z'$. Let u be the projection of z' onto yp'. But now, noting that $\mathrm{proj}_{z'} u \neq \mathrm{proj}_{z'} z$, the projections of x and z onto the line through u and $u \bowtie z'$ do not coincide, the final contradiction, so $x' = z'$. Interchanging the roles of x and y, we see that $y' = \mathrm{proj}_{yy'} z$. But this creates an ordinary pentagon containing x', y' and z, unless $\delta(z, N) = 3$. $\square$

Consider the following weaker version of condition (I) :

(I') Let L and M be two arbitrary opposite lines, x, y different points of $\langle L, M \rangle$ and $x' = \mathrm{proj}_M x$, $y' = \mathrm{proj}_L y$. Let N be an arbitrary line concurrent with xx', not through x or x' and at distance 4 from yy'. Then $\mathrm{proj}_N y = \mathrm{proj}_N z$, for all $z \in \langle L, M \rangle \setminus \{x\}$.

Corollary 1 *A finite generalized hexagon Γ of order (s,t), $t \leq s$, satisfies condition (I') if and only if Γ is isomorphic to $H(q)$, q even.*

Proof. Suppose Γ satisfies (I'), and let L, M, x, x', y, y' be as in (I'). Let z be a point of $\langle L, M \rangle$, $x \neq z \neq y$, and put $z' = \mathrm{proj}_L z$. Let v be an arbitrary point on xx', $x \neq v \neq x'$.

Because of condition (I'), $\mathrm{proj}_v yy' \neq \mathrm{proj}_v zz'$. This shows that $t \geq s$, so $t = s$ and (I) holds. $\square$

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