Journal of Combinatorial Theory, Series A **90**, 241–256 (2000)
doi:10.1006/jcta.1999.3024, available online at http://www.idealibrary.com on IDEAL®

Classification of Embeddings of the Flag Geometries of Projective Planes in Finite Projective Spaces, Part 2

Joseph A. Thas and Hendrik Van Maldeghem[1]

Department of Pure Mathematics and Computer Algebra, University of Ghent, Galglaan 2, B-9000 Gent, Belgium
E-mail: {jat, hvm}@cage.rug.ac.be

Communicated by Francis Buekenhout

Received May 1, 1999

The flag geometry $\Gamma = (\mathscr{P}, \mathscr{L}, \mathbf{I})$ of a finite projective plane Π of order s is the generalized hexagon of order $(s, 1)$ obtained from Π by putting $\mathscr{P}$ equal to the set of all flags of Π, by putting $\mathscr{L}$ equal to the set of all points and lines of Π, and where $\mathbf{I}$ is the natural incidence relation (inverse containment), i.e., Γ is the dual of the double of Π in the sense of H. Van Maldeghem (1998, "Generalized Polygons," Birkhäuser Verlag, Basel). Then we say that Γ is fully and weakly embedded in the finite projective space $\mathbf{PG}(d, q)$ if Γ is a subgeometry of the natural point-line geometry associated with $\mathbf{PG}(d, q)$, if $s = q$, if the set of points of Γ generates $\mathbf{PG}(d, q)$, and if the set of points of Γ not opposite any given point of Γ does not generate $\mathbf{PG}(d, q)$. In two earlier papers we have shown that the dimension d of the projective space belongs to $\{6, 7, 8\}$, that the projective plane Π is Desarguesian, and we have classified the full and weak embeddings of Γ (Γ as above) in the case that there are two opposite lines L, M of Γ with the property that the subspace $U_{L,M}$ of $\mathbf{PG}(d, q)$ generated by all lines of Γ meeting either L or M has dimension 6 (which is automatically satisfied if $d = 6$). In the present paper, we partly handle the case $d = 7$; more precisely, we consider for $d = 7$ the case where for all pairs (L, M) of opposite lines of Γ, the subspace $U_{L,M}$ has dimension 7 and where there exist four lines concurrent with L contained in a 4-dimensional subspace of $\mathbf{PG}(7, q)$.
© 2000 Academic Press

Key Words: generalized hexagons; projective planes; projective embeddings.

1. DEFINITIONS AND STATEMENT OF THE MAIN RESULT

We continue our program of determining all full weak embeddings of generalized hexagons of order $(q, 1)$ in the projective space $\mathbf{PG}(d, q)$. Let us briefly recall that this is motivated by an attempt to characterize the

[1] The second author is a Research Director of the Fund of Scientific Research–Flanders (Belgium).

0097-3165/00 $35.00
Copyright © 2000 by Academic Press
All rights of reproduction in any form reserved.

"natural" embeddings of all finite Moufang classical hexagons. For more details, we refer to Part 1 of this paper (Thas and Van Maldeghem [5]).

The problem we consider may be stated as follows. Let Π be a (finite) projective plane of order s. We define the *flag geometry* Γ of Π as follows. The points of Γ are the flags of Π (i.e., the incident point-line pairs of Π); the lines of Γ are the points and lines of Π. Incidence between points and lines of Γ is reverse containment. It follows that Γ is a (finite) generalized hexagon of order $(s, 1)$ (see (1.6) of Van Maldeghem [8]). The advantage of viewing Γ rather as a generalized hexagon than as a flag geometry of a projective plane is that one can apply results from the general theory of generalized hexagons. We will call Γ a *thin generalized hexagon* (since there are only 2 lines through every point of Γ).

Throughout, we assume that Γ is a thin generalized hexagon of order $(s, 1)$ with corresponding projective plane $\pi(\Gamma) = \Pi$. We introduce some further notation. For a point x of Γ, we denote by $x^{\perp}$ the set of points of Γ collinear with x (two points are *collinear* if they are incident with a common line); we denote by $x^{\perp\!\!\!\perp}$ the set of points of Γ not opposite x (i.e., not at distance 6 from x in the incidence graph of Γ). For a line L of Γ, we write $L^{\perp\!\!\!\perp}$ for the intersection of all sets $p^{\perp\!\!\!\perp}$ with p a point incident with L (in this notation we view L as the set of points incident with it). For an element x of Γ (point or line), we denote by $\Gamma_i(x)$ the set of elements of Γ at distance i from x in the incidence graph of Γ. In this notation, we have $p^{\perp} = \Gamma_0(p) \cup \Gamma_2(p)$, $p^{\perp\!\!\!\perp} = \Gamma_0(p) \cup \Gamma_2(p) \cup \Gamma_4(p)$ and $L^{\perp\!\!\!\perp} = \Gamma_1(L) \cup \Gamma_3(L)$, with p any point and L any line of Γ. Furthermore, an apartment of Γ is a thin subhexagon of order $(1, 1)$. It corresponds with a triangle in $\pi(\Gamma)$.

Let $\mathbf{PG}(d, q)$ be the d-dimensional projective space over the Galois field $\mathbf{GF}(q)$. We say that Γ is *weakly embedded in* $\mathbf{PG}(d, q)$ if the point set of Γ is a subset of the point set of $\mathbf{PG}(d, q)$ which generates $\mathbf{PG}(d, q)$, if the line set of Γ is a subset of the line set of $\mathbf{PG}(d, q)$, if the incidence relation in $\mathbf{PG}(d, q)$ restricted to Γ is the incidence relation in Γ, and if for every point of Γ, the set $x^{\perp\!\!\!\perp}$ does not generate $\mathbf{PG}(d, q)$. If moreover $s = q$, then we say that the weak embedding is also *full*.

The only previously known examples of weak full embeddings of finite thin hexagons in $\mathbf{PG}(d, q)$ arise from full embeddings of the dual classical generalized hexagons of order (q, q), see Thas and Van Maldeghem [4]. Let us call these examples *classical*. In the course of our classification, we have discovered a new class of weak full embeddings of finite thin hexagons in $\mathbf{PG}(8, q)$, and we call these examples *semi-classical* (note that this disproves the conjecture that was stated by us in Thas and Van Maldeghem [4]). They will not turn up in the present paper, but we will give a description and characterization of them in the last part of this work (see Thas and Van Maldeghem [6]).

The following result is proved in [4, 5].

THEOREM. *If Γ is a thin generalized hexagon weakly and fully embedded in some projective space* $\mathbf{PG}(d, q)$, *and if Γ is the flag geometry of the projective plane $\pi(\Gamma)$, then $\pi(\Gamma)$ is Desarguesian and $d \in \{6, 7, 8\}$. If moreover $L^{\perp\!\!\!\perp}$ is contained in a* 4-*dimensional subspace of* $\mathbf{PG}(d, q)$, *for some line L of Γ, then the embedding is one of the classical examples.*

For $d=6$ the set $L^{\perp\!\!\!\perp}$, with L any line of Γ, is always contained in a 4-dimensional subspace of $\mathbf{PG}(d, q)$ and for $d=8$ the set $L^{\perp\!\!\!\perp}$ is never contained in a 4-dimensional subspace of $\mathbf{PG}(d, q)$; see [4].

It is our aim to show that for $d=7$ the condition "$L^{\perp\!\!\!\perp}$ is contained in a 4-dimensional subspace of $\mathbf{PG}(d, q)$, for some line L of Γ" can be dropped. This will be achieved here and in Part 3 of this paper. All classical examples satisfy that condition; hence we must show that a weak and full embedding of Γ in $\mathbf{PG}(7, q)$ with the property that for all lines L of Γ, the set $L^{\perp\!\!\!\perp}$ generates a subspace of dimension at least 5 (and hence exactly 5 by [4]) cannot exist. It turns out that there are two main cases to distinguish. In the present paper, we consider one of them. More exactly, we show the following theorem.

MAIN RESULT. *Let Γ be a thin generalized hexagon of order* $(q, 1)$ *weakly embedded in* $\mathbf{PG}(7, q)$. *If for some line L of Γ there exist four distinct lines $L_1, L_2, L_3, L_4 \in \Gamma_2(L)$ such that the subspace generated by L_1, L_2, L_3, L_4 is* 4-*dimensional, then $L^{\perp\!\!\!\perp}$ is contained in a* 4-*dimensional space and the embedding is one of the classical examples.*

The final case, where every 4 distinct lines L_1, L_2, L_3, $L_4 \in \Gamma_2(L)$ generate a 5-dimensional space will be treated in Part 3.

We remark that, for some proofs, we do not restrict ourselves to $d=7$. Indeed, many of our results will also hold for $d=8$. Only at the end, it will be necessary to assume $d=7$, since there, the case $d=8$ is completely different (there do arise examples for $d=8$).

2. PRELIMINARY RESULTS

2.1. *Some Known Results*

STANDING HYPOTHESES. *From now on we suppose that* $\Gamma = (\mathscr{P}, \mathscr{L}, \mathbf{I})$ *is a generalized hexagon of order* $(q, 1)$ *weakly embedded in* $\mathbf{PG}(d, q)$, *and we denote by $\pi(\Gamma)$ the projective plane for which the dual of the double is isomorphic to Γ.*

We now recall some facts and definitions from [4].

Let $x \in \mathscr{P}$. The set $x^{\perp\!\!\!\perp}$ does not generate $\mathbf{PG}(d, q)$; hence it generates some (proper) subspace of $\mathbf{PG}(d, q)$ which we will denote by ζ_x. For any line L of Γ, we denote by ξ_L the subspace of $\mathbf{PG}(d, q)$ generated by $\Gamma_3(L)$.

LEMMA 1. *For every* $x \in \mathscr{P}$, *the space* $\zeta_x = \langle x^{\perp\!\!\perp} \rangle$ *is a hyperplane which does not contain any point of* $\Gamma_6(x)$. *In particular,* $\zeta_x \neq \zeta_y$ *for* $x, y \in \mathscr{P}$ *with* $x \neq y$. *Also, there is a unique* $(d-2)$*-space* $\tilde{\xi}_L$ *contained in all* ζ_x, $L \in \mathscr{L}$ *and* $x\mathbf{I}L$.

LEMMA 2. *For every line* $L \in \mathscr{L}$, *the space* $\xi_L = \langle L^{\perp\!\!\perp} \rangle$ *has dimension either* $d-3$ *or* $d-2$, *and it contains no point of* $\Gamma_5(L)$.

LEMMA 3. *Every apartment* Σ *of* Γ *generates a 5-dimensional subspace of* $\mathbf{PG}(d, q)$.

LEMMA 4. *Let* U *be any subspace of* $\mathbf{PG}(d, q)$ *containing an apartment* Σ *of* Γ. *Then the points* x *of* Γ *in* U *for which* $\Gamma_1(x) \subseteq U$ *together with the lines of* Γ *in* U *form a (weak) subhexagon* Γ' *of* Γ. *Let* L, M *be two concurrent lines of* Σ *and let* x, y *be two points not contained in* Σ *but incident with respectively* L *and* M. *If* U *contains* $\Gamma_1(x)$ *and* $\Gamma_1(y)$, *then* Γ' *has some order* $(s, 1)$, $1 < s \leqslant q$.

LEMMA 5. *Let* Γ *be weakly and fully embedded in* $\mathbf{PG}(d, q)$. *Then* $6 \leqslant d \leqslant 8$.

LEMMA 6. *The projective plane* $\pi(\Gamma)$ *is isomorphic to* $\mathbf{PG}(2, q)$.

LEMMA 7. *Let* L *and* M *be two arbitrary opposite lines of* Γ. *Let* $L_0, L_1, ..., L_k$ *be* $k+1$ *distinct elements of* $\Gamma_2(L)$, $1 \leqslant k \leqslant q$, *and put* $\Gamma_2(M) \cap \Gamma_2(L_i) = \{M_i\}$, $0 \leqslant i \leqslant k$. *Then the dimension of the subspace* U *of* $\mathbf{PG}(d, q)$ *generated by* $L_0, L_1, ..., L_k$ *is equal to the dimension of the subspace* V *generated by* $M_0, M_1, ..., M_k$.

LEMMA 8. *Let* L_0, L_1, L_2 *be three distinct lines of* Γ *concurrent with some line* $L \in \mathscr{L}$. *Then* $U := \langle L_0, L_1, L_2 \rangle$ *has dimension* 4.

LEMMA 9. *Let* L *be any line of* Γ, *and let* x_0, x_1, x_2, x_3 *be four distinct points on* L. *Without loss of generality, we may assume that* L *corresponds with a line* L' *of* $\pi(\Gamma)$. *Let* x_i, $0 \leqslant i \leqslant 3$, *correspond in* $\pi(\Gamma)$ *with the flag* $\{x_i', L'\}$. *Let* θ *be any self-projectivity of* L' *in* $\pi(\Gamma)$, *that is,* θ *is induced by perspectivities of* $\pi(\Gamma)$, *and suppose that the point* y_i *of* Γ *corresponds with the flag* $\{x_i'^{\theta}, L'\}$ *of* $\pi(\Gamma)$. *Then the cross-ratios* $(x_0, x_1; x_2, x_3)$ *and* $(y_0, y_1; y_2, y_3)$ *(considered as cross-ratios of points in* $\mathbf{PG}(d, q)$*) are equal. Moreover, if* M *is a line of* Γ *opposite* L, *and if* $z_i\mathbf{I}M$ *is not opposite* x_i, $i = 0, 1, 2, 3$, *then the cross-ratios* $(x_0, x_1; x_2, x_3)$ *and* $(\zeta_{z_0}, \zeta_{z_1}; \zeta_{z_2}, \zeta_{z_3})$ *are equal.*

The last lemma follows directly from Lemma 5 and the proof of Proposition 6 in [4].

An immediate consequence of Lemma 7 is the following.

COROLLARY 10. *Let L and M be two arbitrary lines of Γ. Let L_0, L_1, L_2 be three distinct elements of $\Gamma_2(L)$, and let M_0, M_1, M_2 be three distinct elements of $\Gamma_2(M)$. Then the number of elements of $\Gamma_2(L)$ contained in the space $\langle L_0, L_1, L_2\rangle$ is equal to the number of elements of $\Gamma_2(M)$ contained in the space $\langle M_0, M_1, M_2\rangle$.*

2.2. *Case Distinction*

Suppose that $\langle L^{\perp\!\!\perp}\rangle$ is a subspace of dimension $\rho_L \geqslant 5$, for all lines L of Γ. By Lemma 7, ρ_L is independent of L, and we write $\rho_L = \rho$. Clearly $\rho \leqslant d-2$, hence $d=7$, 8. If $d=7$, then $\rho=5$ and we distinguish the following cases (where NE stands for non-existence):

NE(7.1) $d=7$, $\rho=5$, and for every line L of Γ, there exists a set $\{L_1, L_2, L_3, L_4\} \subseteq \Gamma_2(L)$ of cardinality 4 such that the subspace generated by L_1, L_2, L_3, L_4 has dimension 4.

NE(7.2) $d=7$, $\rho=5$, and for every line L of Γ, and every set $\{L_1, L_2, L_3, L_4\} \subseteq \Gamma_2(L)$ of cardinality 4, the subspace generated by L_1, L_2, L_3, L_4 has dimension 5.

If $d=8$, then we have $\rho=5$ or $\rho=6$. Here, we distinguish the following cases (where EX stands for existence of examples):

NE(8.1) $d=8$ and $\rho=6$.

EX(8.2) $d=8$ and $\rho=5$.

In this paper, we treat Case NE(7.1) and we prepare Case EX(8.2). Henceforth, we assume that there is a line L of Γ such that there exist four distinct lines L_1, L_2, L_3, $L_4 \in \Gamma_2(L)$ with $\langle L_1, L_2, L_3, L_4\rangle$ 4-dimensional; then, by Lemma 7, this property holds for every line L of Γ.

3. SOME GENERAL RESULTS FOR $\rho=5$

Let M be a line of Γ opposite L. It is clear that $\xi_L \neq \xi_M$. Hence the space $\eta_{L,M} = \xi_L \cap \xi_M$ has dimension either 4 or 3. Suppose that the dimension of $\eta_{L,M}$ is equal to 4. Then here is a point x of L which belongs to $\eta_{L,M}$, and hence to ξ_M. This contradicts Lemma 2.

Hence the dimension of $\eta_{L,M}$ is 3. Now let $\mathscr{A}_{L,M}$ be the set of points of Γ in $\eta_{L,M}$; then $\mathscr{A}_{L,M} = \Gamma_3(L) \cap \Gamma_3(M)$. From our assumptions, it follows that there is a plane π of $\eta_{L,M}$ containing at least four points of $\mathscr{A}_{L,M}$. By Corollary 10, we conclude that every plane π of $\eta_{L,M}$ meeting $\mathscr{A}_{L,M}$ in at least three points has at least four points with $\mathscr{A}_{L,M}$ in common.

LEMMA 11. *Let, with the notation above, π be any plane of $\eta_{L,M}$ containing at least three points of $\mathscr{A}_{L,M}$. Let P_π be the set of points of Γ incident with L and collinear with a point of Γ in π. Then P_π is a projective subline of L—where we consider L as a projective line of* $\mathbf{PG}(d, q)$ *over* $\mathbf{GF}(q)$*—over some subfield* $\mathbf{GF}(q')$ *of* $\mathbf{GF}(q)$ *(and hence isomorphic to* $\mathbf{PG}(1, q')$*). Also, q' is independent of the choice of π.*

Proof. By Corollary 10, the number $q'+1 := |P_\pi|$ is independent of L and π. Note that by assumption $q' > 2$.

Let L_0, L_1, L_2, L_3 be four (distinct) arbitrary but fixed lines of $\Gamma_2(L)$ contained in the space $U := \langle L, \pi\rangle$. Let L_i, L_j, L_k be three arbitrary but distinct lines of $\Gamma_2(L)$ in $\langle L, \pi\rangle$. Let x_ℓ be the intersection of L and L_ℓ, $\ell \in \{0, 1, 2, 3, i, j, k\}$. Since by Lemma 8 the lines L_i, L_j, L_k generate U, Lemma 7 combined with Lemma 9 implies that the line $L_m \in \Gamma_2(L)$ through the point $x_m \in \Gamma_1(L)$ determined by $(x_0, x_1; x_2, x_3) = (x_i, x_j; x_k, x_m)$ belongs to U. Now let $F = \{(x_0, x_1; x_2, x) \mid x \in P_\pi \backslash \{x_0\}\}$. Then F has q' elements and the lemma will be proved if we show that F is a subfield of $\mathbf{GF}(q)$. Let a be an arbitrary element of F, $0 \neq a \neq 1$. Then there is a point $x \in P_\pi$ such that $(x_0, x_1; x_2, x) = a$. Then the point y defined by $(x_0, x_1; x, x_2) = (x_0, x_1; x_2, y)$ belongs to P_π. So $(x_0, x_1; x_2, y) = 1/a$ and $1/a \in F$. Now let $b \in F \backslash \{0, 1\}$ be arbitrary and suppose that $z \in P_\pi$ is such that $(x_0, x_1; x_2, z) = b$. Then the point u defined by $(x_0, x_1; x_2, u) = (x_0, x_1; y, z)$ belongs to P_π. So $(x_0, x_1; x_2, u) = ab$ and $ab \in F$. Hence F is closed under multiplication. Consequently $F \backslash \{0\}$ is a cyclic multiplicative group generated by some element $r \in \mathbf{GF}(q)$. Let r^i and r^j be two arbitrary elements of $F \backslash \{0\}$. It remains to show that $r^i + r^j \in F$.

First we assume that i and j are minimal and that $i > j$. Let $v, w \in P_\pi$ be such that $(x_0, x_1; x_2, v) = r^{i-j}$ and $(x_0, x_1; x_2, w) = r^{2(i-j)}$. Let t be defined by $(x_0, x_2; v, w) = (x_0, x_1; x_2, t)$. Then $t \in P_\pi$. So $(x_0, x_1; x_2, t) = r^{i-j} + 1$ and $r^{i-j} + 1 \in F$. Multiplying with r^j gives the desired result.

Now consider $t + t$ with $t \in F$. As $t + t = 2t$ we have to show that $2 \in F$. Clearly we may assume that q is odd. Choose $b, c \in F \backslash \{0\}$ with $b \neq c$. If $a = b + c$, then $a \in F$. As $a + b \neq c$, we have $a + a = a + (b + c) = (a + b) + c \in F$. So $2a \in F$. If $a \neq 0$, then this implies that $2 \in F$ and we are done. If $a = 0$ for every choice of $b, c \in F \backslash \{0\}$, $b \neq c$, then $|F| = 3$ and hence $F = \{0, 1, -1\}$. Now $(x_0, x_1; x_2, e) = (x_0, x_2; x_1, e) = -1$, with $P_\pi = \{x_0, x_1, x_2, e\}$. So $2 = -1 \in F$. ∎

LEMMA 12. *Let $q' > 2$ be as in* Lemma 11. *Then, with the above notation, every set $x_0, x_1, \ldots, x_{q'}$ of $q'+1$ points of $\mathscr{A}_{L,M}$ contained in a plane π of* $\mathbf{PG}(d, q)$ *forms a conic $\mathscr{C}$ in a subplane π' of π isomorphic to* $\mathbf{PG}(2, q')$*. Moreover, if $s \neq x_0$ is any point neither on L nor on M, but collinear in Γ*

with x_0, *then the* (*extension of the*) *tangent line in* π' *of* $\mathscr{C}$ *at* x_0 *coincides with the intersection of* π *and* ζ_s.

Proof. We put $\Gamma_2(L)=\{L_0, L_1, ..., L_q\}$ and $\Gamma_2(L_i)\cap\Gamma_2(M)=\{M_i\}$, $0\leqslant i\leqslant q$. Also, we denote by x_i the intersection of L_i and M_i, and we put $\{y_i\}=\Gamma_2(x_i)\cap\Gamma_1(L)$, $0\leqslant i\leqslant q$. We assume that $x_0, x_1, ..., x_{q'}$ are contained in a plane π, and we may suppose that $s\in\Gamma_3(L)$ (interchanging the roles of L and M if necessary).

We first show that $x_0, x_1, ..., x_{q'}$ form a conic $\mathscr{C}$ in a subplane π' of π isomorphic to $\mathbf{PG}(2, q')$. Indeed, if $0<i<j<k<m<q'$, then an argument similar to the one in the third paragraph of Section 4 of Thas and Van Maldeghem [5] shows that the cross-ratios $(x_0x_i, x_0x_j; x_0x_k, x_0x_m)$ and $(x_{q'}x_i, x_{q'}x_j; x_{q'}x_k, x_{q'}x_m)$ are equal and belong to $\mathbf{GF}(q')$. It follows that the points $x_0, x_1, ..., x_{q'}$ are contained in a subplane π' of π over $\mathbf{GF}(q')$ (for $q'=3$ this is trivial, for $q'=4$ this follows from the fact that the cross-ratio $(x_{i_0}x_{i_1}, x_{i_0}x_{i_2}; x_{i_0}x_{i_3}, x_{i_0}x_{i_4})$ belongs to $\mathbf{GF}(4)$, for $\{i_0, i_1, i_2, i_3, i_4\}=\{0, 1, 2, 3, 4\}$). For $q'=3, 4$, it is trivial that $\mathscr{C}:=\{x_0, x_1, ..., x_{q'}\}$ is a conic in π'. For $q'>4$, this follows from the above equality of cross-ratios.

Now let s be any point of $\Gamma_2(x_0)\cap\Gamma_3(L)$. Let N be the unique line of Γ incident with s and distinct from L_0. Put $\Gamma_3(N)\cap\Gamma_2(x_i)=\{s_i\}$, $1\leqslant i\leqslant q'$. Then $s_i\in\Gamma_3(M)\cap\Gamma_4(y_i)$, $1\leqslant i\leqslant q'$, and so by Lemma 7 we see that $x_0, s_1, s_2, ..., s_{q'}$ are contained in a plane π_s of $\mathbf{PG}(d, q)$. If we denote by z_i, $i\in\{0, 1, 2, ..., q\}$, the unique point on M collinear in Γ with x_i, then Lemma 11 implies that $z_0, z_1, ..., z_{q'}$ form a projective subline of M over $\mathbf{GF}(q')$. If we denote by t_i, $i\in\{1, 2, ..., q'\}$, the unique point on N collinear in Γ with s_i, then, since ζ_{t_i} meets M in z_i, $1\leqslant i\leqslant q'$, and ζ_s meets M in z_0, the hyperplanes $\zeta_s, \zeta_{t_1}, \zeta_{t_2}, ..., \zeta_{t_{q'}}$ form a dual projective line over $\mathbf{GF}(q')$ in $\mathbf{PG}(d, q)$. Putting $T:=\pi\cap\zeta_s$, we deduce that T, x_0x_1, x_0x_2, ..., $x_0x_{q'}$ form a dual projective line over $\mathbf{GF}(q')$ in π, and hence, since $q'>2$, this must define a pencil of lines in π' (since at least q' elements of that pencil belong to π'). We now easily deduce that T is the (extension of the) tangent line of $\mathscr{C}$ in π' at x_0.

The lemma is proved. ∎

Now let $s'\neq x_1$ be a point on L_1, but not incident with L. Let s and N be as in the proof of Lemma 12. Let N' be the unique line of Γ distinct from L_1 and incident with s'. Recall that for any line X of Γ, the $(d-2)$-dimensional space $\tilde{\xi}_X$ is the intersection of all ζ_x with x incident with X. We have the following lemma.

LEMMA 13. *With the above notation, we have that* $L_N:=\tilde{\xi}_N\cap\eta_{L,M}$ *and* $L_{N'}:=\tilde{\xi}_{N'}\cap\eta_{L,M}$ *are lines of* $\mathbf{PG}(d, q)$. *Also, the map* Θ: $\zeta_v\cap\eta_{L,M}\mapsto\zeta_{v'}\cap\eta_{L,M}$, *where* v *varies over* $\Gamma_1(N)$ *and* v' *is the unique element of* $\Gamma_1(N')$

with $\Gamma_4(v') \cap \Gamma_1(M) = \Gamma_4(v) \cap \Gamma_1(M)$, is a linear projectivity in $\eta_{L,M}$ from the pencil of planes through the line L_N to the pencil of planes through the line $L_{N'}$.

Proof. Clearly L_N does not coincide with $\eta_{L,M}$ (because $\eta_{L,M}$ contains points opposite at least one point of N). Suppose that L_N is a plane. Then $H_N := \langle \eta_{L,M}, \tilde{\xi}_N \rangle$ is a $(d-1)$-dimensional subspace of $\mathbf{PG}(d,q)$. Remark that $\xi_M \subseteq H_N$; in particular M belongs to H_N. Since $\tilde{\xi}_N$ is $(d-2)$-dimensional, it has a point in common with M, a contradiction since M is opposite N. Hence L_N (and similarly also $L_{N'}$) is a line.

Now let v vary over $\Gamma_1(N)$. Put $\{w\} = \Gamma_1(M) \cap \Gamma_4(v)$ and $\{v'\} = \Gamma_1(N') \cap \Gamma_4(w)$. Using the last assertion of Lemma 9, we see that the composition

$$\zeta_v \cap \eta_{L,M} \mapsto \zeta_v \mapsto w \mapsto \zeta_{v'} \mapsto \zeta_{v'} \cap \eta_{L,M}$$

is a linear projectivity.

The lemma is proved. ▮

We keep the same notation as in the previous two lemmas and their proofs. Note that, if $v \neq s$, then the plane $\zeta_v \cap \eta_{L,M}$ meets $\mathscr{A}_{L,M}$ in exactly two points, including x_0 (and the second point is the unique point of $\mathscr{A}_{L,M}$ collinear in Γ with w). Hence $\gamma_{x_0} := \zeta_s \cap \eta_{L,M}$ is the unique plane of $\eta_{L,M}$ containing L_N and meeting $\mathscr{A}_{L,M}$ only in x_0.

Every set of three points of $\mathscr{A}_{L,M}$ is contained in a unique (plane) conic $\mathscr{C}$ defined over $\mathbf{GF}(q')$ (see Lemma 12), and if $\mathscr{C}$ contains x_0, then by the same lemma, the tangent line of $\mathscr{C}$ at x_0 is contained in ζ_s, hence in γ_{x_0}. Hence γ_{x_0} is the unique plane in $\eta_{L,M}$ containing all tangent lines at x_0 of all conics over $\mathbf{GF}(q')$ through x_0 contained in $\mathscr{A}_{L,M}$. Naturally, we call γ_{x_0} the *tangent plane of* $\mathscr{A}_{L,M}$ at x_0 and we denote the tangent plane of $\mathscr{A}_{L,M}$ at x_i by γ_{x_i}, $i = 0, 1, ..., q$. As γ_{x_0} is contained in ζ_s and ζ_{y_0}, it is also contained in ζ_{x_0}; analogously γ_{x_i} is contained in ζ_{x_i}, $i = 0, 1, ..., q$.

Finally, note that the projectivity Θ of Lemma 13 maps the plane $\langle L_N, x_i \rangle$ to the plane $\langle L_{N'}, x_i \rangle$, $i = 2, 3, ..., q$; it maps γ_{x_0} to $\langle L_{N'}, x_0 \rangle$ and $\langle L_N, x_1 \rangle$ to γ_{x_1}.

Our next aim is to prove the following lemma:

LEMMA 14. *With the above notation, the lines L_N and $L_{N'}$ do not have any point in common.*

Proof. Suppose that L_N and $L_{N'}$ have a point y in common. Then by Lemma 13, the set of lines $\{\langle L_N, x_i \rangle \cap \langle L_{N'}, x_i \rangle \mid i = 2, 3, ..., q\} \cup \{L_N, L_{N'}\}$ is the set of generators of a quadratic cone $\mathscr{Q}$ in $\eta_{L,M}$. Let x, x', x'' be three points of $\mathscr{A}_{L,M}$. Then γ_x, $\gamma_{x'}$ and $\gamma_{x''}$ clearly are tangent planes of $\mathscr{Q}$. If $\mathscr{C}$

is the unique conic over $\mathbf{GF}(q')$ on $\mathscr{A}_{L,M}$ containing x, x', x'', then the lines $\gamma_x \cap \langle x, x', x'' \rangle$, $\gamma_{x'} \cap \langle x, x', x'' \rangle$ and $\gamma_{x''} \cap \langle x, x', x'' \rangle$ contain the tangent lines of $\mathscr{C}$ at the respective points x, x', x''. Hence it is now easily seen that $\mathscr{C}$ is completely determined by $\mathscr{Q}$ and x, x', x''. It follows that, if we denote by π the plane over $\mathbf{GF}(q)$ containing $\mathscr{C}$, then $\mathscr{C}$ is precisely the set of points x^* of $\pi \cap \mathscr{Q}$ for which the cross-ratio $(x, x'; x'', x^*)$ on the conic $\pi \cap \mathscr{Q}$ belongs to $\mathbf{GF}(q')$.

We now coordinatize the situation (in the 3-dimensional space $\eta_{L,M}$). Let y have coordinates $(x_0, x_1, x_2, x_3) = (0, 0, 0, 1)$ and let $\mathscr{Q}$ have equation $X_1^2 = X_0 X_2$. Without loss of generality, we may suppose that the unique conic over $\mathbf{GF}(q')$ on $\mathscr{A}_{L,M}$ containing $x_0, x_1, ..., x_{q'}$ lies in the plane with equation $X_3 = 0$ and is the set of points $\{(1, a, a^2, 0) \mid a \in \mathbf{GF}(q')\} \cup \{(0, 0, 1, 0)\}$. We may also assume that x_0 has coordinates $(1, 0, 0, 0)$, x_1 has coordinates $(0, 0, 1, 0)$ and x_2 has coordinates $(1, 1, 1, 0)$. Now let r be a generator of the multiplicative group of $\mathbf{GF}(q)$ and let x be the unique point of $\mathscr{A}_{L,M}$ lying on the lines through y and the point $(1, r, r^2, 0)$. We may choose coordinates such that x is represented by $(1, r, r^2, 1)$.

For every element t of $\mathbf{GF}(q)$, written as $t = a_0 + a_1 r + a_2 r^2 + \cdots + a_\ell r^\ell$, with $a_i \in \mathbf{GF}(q')$, $i = 0, 1, ..., \ell$ and $\ell \in \mathbb{N}$, we denote by ∂t the element $a_1 + 2a_2 r + \cdots + \ell a_\ell r^{\ell-1}$. Note that ∂t strongly depends on the way one writes t as a polynomial in r over $\mathbf{GF}(q')$.

Now we will use the following identity (which also holds in general for derivable functions of one variable over $\mathbb{R}$ or $\mathbb{C}$, with ∂f the derivative of the function f). Let t_1, t_2, t_3, t_4 be four distinct elements of $\mathbf{GF}(q)$, written as a polynomial in r over $\mathbf{GF}(q')$. Then we have

$$\begin{vmatrix} 1 & t_1 & t_1^2 & \partial t_1 \\ 1 & t_2 & t_2^2 & \partial t_2 \\ 1 & t_3 & t_3^2 & \partial t_3 \\ 1 & t_4 & t_4^2 & \partial t_4 \end{vmatrix} = 0 \Leftrightarrow \partial(t_1, t_2; t_3, t_4) = 0, \tag{1}$$

where the cross-ratio $(t_1, t_2; t_3, t_4)$ is viewed as a rational function in r with coefficients in $\mathbf{GF}(q')$ in the obvious way. Moreover, if $\partial(t_1, t_2; t_3, t_4) = 0$, then the rank of the matrix corresponding with the determinant above is 3. The proof of the equivalence (1) is an easy and elementary, but tedious calculation. This equivalence can also be derived from the theory of the Riccati differential equations. Note that for $t_1 = \infty$ the equivalence (1) remains true if we rewrite the first line of the determinant in (1) as 0 0 1 0.

From the equivalence (1), it now follows that, if $(1, t_i, t_i^2, \partial t_i)$, $i = 1, 2, 3$, are three distinct points of $\mathscr{A}_{L,M}$, and if $(t_1, t_2; t_3, t_4) \in \mathbf{GF}(q')$ (where the cross-ratio is calculated as a function of r depending on the polynomials t_1, t_2, t_3, t_4 in r over $\mathbf{GF}(q')$), for some $t_4 \in \mathbf{GF}(q)$, then the point $(1, t_4, t_4^2, \partial t_4)$ belongs to $\mathscr{A}_{L,M}$.

Let q be odd. We claim that, for any positive integer ℓ, the point $(1, r^{\ell}, r^{2\ell}, \ell r^{\ell-1})$ belongs to $\mathscr{A}_{L,M}$. We show our claim by induction. For $\ell = 1$, this is a consequence of our choice of coordinates above. So let $\ell > 1$. From the equality $(\infty, 0; r^{\ell-2}, -3r^{\ell-2}) = -3 \in \mathbf{GF}(q')$ it follows that $(1, -3r^{\ell-2}, 9r^{2\ell-4}, -3\partial r^{\ell-2}) \in \mathscr{A}_{L,M}$. Also, from $(\infty, -3r^{\ell-2}; r^{\ell-1}, 2r^{\ell-1}+3r^{\ell-2}) = 2$, it follows that $(1, 2r^{\ell-1}+3r^{\ell-2}, (2r^{\ell-1}+3r^{\ell-2})^2, \partial(2r^{\ell-1}+3r^{\ell-2}))$ belongs to $\mathscr{A}_{L,M}$. Similarly, from $(\infty, r^{\ell-2}; r^{\ell-1}, 4r^{\ell-1}-3r^{\ell-2}) = 4$, we derive that $(1, 4r^{\ell-1}-3r^{\ell-2}, (4r^{\ell-1}-3r^{\ell-2})^2, \partial(4r^{\ell-1}-3r^{\ell-2}))$ belongs to $\mathscr{A}_{L,M}$. But now we have the equality

$$(r^{\ell-2}, 2r^{\ell-1}+3r^{\ell-2}; 4r^{\ell-1}-3r^{\ell-2}, r^{\ell}) = 2,$$

and our claim follows. Now we put $\ell = q$ and, noting that $r^q = r$, we see that the point $(1, r, r^2, 0)$ must belong to $\mathscr{A}_{L,M}$, a contradiction. Note that, despite the constants 3 and -3 appearing in this argument, the proof is also valid in characteristic 3.

Let q be even. Since $q' > 2$, there is some $\omega \in \mathbf{GF}(q')$ with $0 \neq \omega \neq 1$. From the identities

$$(\infty, 0; r, (\omega+1)r) = \omega + 1 \in \mathbf{GF}(q'),$$

$$(1, r; \infty, \omega r + (\omega+1)) = \frac{\omega+1}{\omega} \in \mathbf{GF}(q'),$$

$$(r, (\omega+1)r; \omega r + (\omega+1), r^2) = \omega + 1 \in \mathbf{GF}(q'),$$

we derive that $(1, r^2, r^4, 2r) = (1, r^2, r^4, 0)$ belongs to $\mathscr{A}_{L,M}$, hence $r \in \mathbf{GF}(q')$, a contradiction.

The lemma is proved. ∎

So we may assume that the lines L_N and $L_{N'}$ (see above) of $\eta_{L,M}$ do not meet. Our next aim is to derive an explicit equation for the set $\mathscr{A}_{L,M}$ in that case. Therefore, we need a lemma.

Lemma 15. *Let σ be a permutation of $\mathbf{PG}(1, q) = \mathbf{GF}(q) \cup \{\infty\}$ with the property that $(a, b; c, d) = (a^\sigma, b^\sigma; c^\sigma, d^\sigma)$ if and only if $a, b, c, d \in \mathbf{GF}(q') \cup \{\infty\}$. Then σ is a semi-linear map of $\mathbf{PG}(1, q)$. If σ fixes 0, 1 and ∞, then its restriction to $\mathbf{GF}(q)$ is a field automorphism with as fixed point set $\mathbf{GF}(q')$.*

Proof. By the 3-transitivity of $\mathbf{PGL}_2(q)$ on $\mathbf{PG}(1, q)$, it suffices to prove the second part of the lemma. So we may assume that σ fixes 0, 1 and ∞, and preserves cross-ratios belonging to $\mathbf{GF}(q')$. As an immediate consequence, σ fixes all points of $\mathbf{PG}(1, q') := \mathbf{GF}(q') \cup \{\infty\}$.

In the rest of the proof, we forget our general notation. Let $h \in \mathbb{N}$ be defined as $q'^h = q$.

Consider $\mathbf{PG}(2h-1, q')$, its extension $\mathbf{PG}(2h-1, q)$, and h lines L_1, $L_2, ..., L_h$ of $\mathbf{PG}(2h-1, q)$ which are conjugate with respect to the extension $\mathbf{GF}(q)$ of $\mathbf{GF}(q')$. We identify L_1 with $\mathbf{PG}(1, q)$. Let $y_1 \in L_1$ and let $\{y_1, y_2, ..., y_h\}$ be the corresponding set of conjugate points. Further, consider $\langle y_1, y_2, ..., y_h \rangle = \mathbf{PG}(h-1, q)$. Then $\mathbf{PG}(h-1, q)$ extends some subspace $\mathbf{PG}(h-1, q')$ of $\mathbf{PG}(2h-1, q')$. There arise $q'^h+1=q+1$ of these $\mathbf{PG}(h-1, q')$'s; they form a regular $(h-1)$-spread $\mathscr{S}$ of $\mathbf{PG}(2h-1, q')$ (see Hirschfeld and Thas [1]). Let D be a subline over $\mathbf{GF}(q')$ of $\mathbf{PG}(1, q)$. With D there correspond $q'+1$ elements of $\mathscr{S}$. These elements are generators of some Segre variety $\mathscr{S}_{h-1;1}$ (see Hirschfeld and Thas [1]). The Segre variety $\mathscr{S}_{h-1;1}$ has degree h.

On $\mathbf{PG}(1, q)$ we introduce affine coordinates. The point with affine coordinate $x \in \mathbf{GF}(q) \cup \{\infty\}$ will be denoted by p_x. Assume that $p_\infty \in \mathbf{PG}(1, q')$ and that $\mathbf{PG}(1, q') = \{p_x \mid x \in \mathbf{GF}(q') \cup \{\infty\}\}$. The element of $\mathscr{S}$ corresponding to p_x, $x \in \mathbf{GF}(q) \cup \{\infty\}$, will be denoted by π_x. Let Σ be an h-dimensional space in $\mathbf{PG}(2h-1, q')$ containing π_∞. Further, let $\Sigma \cap \pi_x = r_x$, $x \in \mathbf{GF}(q)$. For a subline over $\mathbf{GF}(q')$ containing p_∞ of $\mathbf{PG}(1, q)$, the corresponding points r_x, $x \neq \infty$, form an affine line of $\tilde{\Sigma} := \Sigma \backslash \pi_\infty$. In this way we find all affine lines of the space $\tilde{\Sigma}$. Let $i_1, i_2, ..., i_h$ be a basis of $\mathbf{GF}(q)$, viewed as a vector space over $\mathbf{GF}(q')$. Let $x = x_1 i_1 + x_2 i_2 + \cdots + x_h i_h$, $x \in \mathbf{GF}(q)$, $x_i \in \mathbf{GF}(q')$, $i = 1, 2, ..., h$. If $\mathbf{AG}(h, q')$ is any coordinatized h-dimensional affine space over $\mathbf{GF}(q')$, then $r_x \mapsto (x_1, x_2, ..., x_h)$ defines a linear projectivity from $\tilde{\Sigma}$ onto $\mathbf{AG}(h, q')$ (lines are mapped onto lines and cross-ratios are preserved).

Let D be a subline of $\mathbf{PG}(1, q)$ over $\mathbf{GF}(q')$ which does not contain p_∞. With D corresponds a Segre variety $\mathscr{S}_{h-1;1}$. Extended to $\mathbf{GF}(q)$, this variety contains the h points $\pi_\infty \cap L_1 = y_1, ..., \pi_\infty \cap L_h = y_h$. Then $\mathscr{S}_{h-1;1} \cap \Sigma$ is an algebraic curve of degree h containing over $\mathbf{GF}(q)$ the points y_1, $y_2, ..., y_h$. In fact, in this way, all normal rational curves of degree h of Σ containing over $\mathbf{GF}(q)$ the points $y_1, y_2, ..., y_h$ arise.

With σ there corresponds a permutation ξ of $\tilde{\Sigma}$ mapping lines onto lines and fixing each point of the line $P := \{r_x \mid x \in \mathbf{GF}(q')\}$. Hence we have a linear projectivity ξ of $\tilde{\Sigma}$. Clearly ξ maps a normal rational curve of Σ containing over $\mathbf{GF}(q)$ the points $y_1, y_2, ..., y_h$, onto a normal rational curve with the same property. Hence the extension $\tilde{\xi}$ of ξ to the extension of Σ over $\mathbf{GF}(q)$, stabilizes the set $\{y_1, y_2, ..., y_h\}$.

Consider the element π_0 and also the points $\pi_0 \cap L_i = z_i$, $i = 1, 2, ..., h$. Let $y_i^{\tilde{\xi}} = y_{i^\eta}$. Then $\mathbf{PG}(2h-1, q)$ admits just one linear projectivity $\bar{\xi}$ inducing $\tilde{\xi}$ on the extension of Σ and mapping z_i onto z_{i^η}. Clearly $\bar{\xi}$ induces a linear projectivity ξ^* of $\mathbf{PG}(2h-1, q')$. We have $L_i^{\bar{\xi}} = L_{i^\eta}$, $i = 1, 2, ..., h$. Clearly $\mathscr{S}^{\xi^*} = \mathscr{S}$ and ξ^* is the permutation of $\mathscr{S}$ induced by σ.

Now we embed $\mathbf{PG}(2h-1, q')$ into a $\mathbf{PG}(2h, q')$ and consider the Desarguesian plane Π over $\mathbf{GF}(q)$ defined by the regular spread $\mathscr{S}$.

Further, we embed the given line $\mathbf{PG}(1, q)$ into a plane $\mathbf{PG}(2, q)$. We extend ξ^* to a linear projectivity of $\mathbf{PG}(2h, q')$. With ξ^* corresponds a collineation ζ of Π, hence a collineation θ of $\mathbf{PG}(2, q)$. It is clear that θ induces σ on $\mathbf{PG}(1, q)$, hence σ is an automorphism of $\mathbf{PG}(1, q)$. As σ fixes 0, 1 and ∞, its restriction to $\mathbf{GF}(q)$ is a field automorphism.

The lemma is proved. ∎

We can now show the following result (and we again use the notation introduced before the previous lemma).

LEMMA 16. *In $\eta_{L,M}$, coordinates can be chosen such that $\mathscr{A}_{L,M}$ is the set of points $\{(1, \lambda, \lambda^\sigma, \lambda^{\sigma+1}) \mid \lambda \in \mathbf{GF}(q)\} \cup \{(0, 0, 0, 1)\}$, for some automorphism σ of $\mathbf{GF}(q)$ fixing the subfield $\mathbf{GF}(q')$ elementwise and fixing no element of $\mathbf{GF}(q) \backslash \mathbf{GF}(q')$.*

Proof. We use the previously introduced notation. By Lemma 14, we know that the lines L_N and $L_{N'}$ do not meet. Hence the $q-1$ lines $\langle L_N, x_i\rangle \cap \langle L_{N'}, x_i\rangle$ belong to a regulus $\mathscr{R}$, together with the lines $\gamma_{x_0} \cap \langle L_{N'}, x_0\rangle$ and $\gamma_{x_1} \cap \langle L_N, x_1\rangle$. Every line of the regulus $\mathscr{R}$ contains exactly one point of $\mathscr{A}_{L,M}$ and every point of $\mathscr{A}_{L,M}$ is contained in exactly one element of $\mathscr{R}$.

Suppose now that two points of $\mathscr{A}_{L,M}$ are contained in the same line P of the complementary regulus $\mathscr{R}'$. Choosing an arbitrary third point z of $\mathscr{A}_{L,M}$, we obtain three points of $\mathscr{A}_{L,M}$ contained in a unique plane π_0 of $\eta_{L,M}$. If R is the unique element of $\mathscr{R}$ incident with z, then $\pi = \langle R, P\rangle$ and hence $|\pi \cap \mathscr{A}_{L,M}| = 3$, contradicting our assumption $q' > 2$. Hence every line of the regulus $\mathscr{R}'$ contains exactly one point of $\mathscr{A}_{L,M}$ and every point of $\mathscr{A}_{L,M}$ is contained in exactly one element of $\mathscr{R}'$.

We can choose coordinates such that $\mathscr{R}$ contains the lines

$$A_\lambda \begin{cases} X_3 = \lambda X_2, \\ X_0 = \dfrac{1}{\lambda} X_1, \end{cases}$$

for $\lambda \in \mathbf{GF}(q) \cup \{\infty\}$, and such that $\mathscr{R}'$ contains the lines

$$B_\mu \begin{cases} X_3 = \mu X_1, \\ X_0 = \dfrac{1}{\mu} X_2, \end{cases}$$

with $\mu \in \mathbf{GF}(q) \cup \{\infty\}$. By the previous paragraph, there is a permutation σ of $\mathbf{GF}(q) \cup \{\infty\}$, such that $\lambda^\sigma = \mu$ if and only if $A_\lambda \cap B_\mu \in \mathscr{A}_{L,M}$. Coordinates may be chosen such that σ fixes 0, 1 and ∞. Moreover, the four points of $\mathscr{A}_{L,M}$ corresponding with $\lambda = \lambda_1, \lambda_2, \lambda_3, \lambda_4$ are contained in a plane if and only if $(\lambda_1, \lambda_2; \lambda_3, \lambda_4) = (\lambda_1^\sigma, \lambda_2^\sigma; \lambda_3^\sigma, \lambda_4^\sigma)$, in which case this common cross-ratio is equal to the cross-ratio of the four corresponding points on the corresponding conic over $\mathbf{GF}(q')$, and hence belongs to $\mathbf{GF}(q')$. Conversely, if $(\lambda_1, \lambda_2; \lambda_3, \lambda_4) \in \mathbf{GF}(q')$, then the previous reasoning implies that the unique point x^* of $\mathscr{A}_{L,M}$ coplanar with the points x, x', x'' corresponding with λ_1, λ_2, λ_3, respectively, and such that the cross-ratio $(x, x'; x'', x^*)$ (on the unique conic over $\mathbf{GF}(q')$ contained in $\mathscr{A}_{L,M}$ and containing x, x', x'') is equal to $(\lambda_1, \lambda_2; \lambda_3, \lambda_4)$, corresponds with λ_4. Hence $(\lambda_1, \lambda_2; \lambda_3, \lambda_4) = (\lambda_1^\sigma, \lambda_2^\sigma; \lambda_3^\sigma, \lambda_4^\sigma)$.

Consequently, the assumptions of Lemma 15 are satisfied, and we conclude that σ is an automorphism of $\mathbf{GF}(q)$, additionally mapping ∞ to ∞. It is now easy to deduce from the explicit form of the elements of $\mathscr{R}$ and $\mathscr{R}'$, and the definition of σ that $\mathscr{A}_{L,M}$ consists of the points with coordinates $(1, \lambda, \lambda^\sigma, \lambda^{\sigma+1})$, and $(0, 0, 0, 1)$.

The lemma is proved. ∎

Remark 17. Remark that, if x_i, x_j, x_k, x_ℓ are four distinct points of $\mathscr{A}_{L,M}$, $i, j, k, \ell \in \{0, 1, ..., q\}$, corresponding with the values λ_i, λ_j, λ_k, λ_ℓ as at the end of the previous proof, and if v_i, v_j, v_k, v_ℓ are the unique points of N at distance 4 from x_i, x_j, x_k, x_ℓ, respectively (for $i, j, k, \ell \neq 0$; if one of them is 0, then the corresponding point on N is s), then $(\lambda_i, \lambda_j; \lambda_k, \lambda_\ell) = (\zeta_{v_i}, \zeta_{v_j}; \zeta_{v_k}, \zeta_{v_\ell})$. It follows that, if z_i, z_j, z_k, z_ℓ are the unique points on M collinear with x_i, x_j, x_k, x_ℓ, respectively, then $(\lambda_i, \lambda_j; \lambda_k, \lambda_\ell) = (z_i, z_j; z_k, z_\ell)$.

4. THE CASE $d = 7$

Now we put $d = 7$ and prove our Main Result. We may choose coordinates in $\mathbf{PG}(7, q)$ as

$$(0, 1, 0, 0; 0, 0, 0, 0) := M \cap M_0,$$

$$(0, 0, 1, 0; 0, 0, 0, 0) := x_0,$$

$$(0, 0, 0, 1; 0, 0, 0, 0) := y_0,$$

$$(0, 0, 0, 0; 1, 0, 0, 0) := y_1,$$

$$(0, 0, 0, 0; 0, 1, 0, 0) := x_1,$$

$$(0, 0, 0, 0; 0, 0, 1, 0) := M \cap M_1.$$

Furthermore, we choose $(1, 0, 0, 0; 0, 0, 0, 0)$ and $(0, 0, 0, 0; 0, 0, 0, 1)$ inside $\eta_{L,M}$, and we can arrange it so that the points of $\mathscr{A}_{L,M}$ have coordinates $(x, 0, 1, 0; 0, x^{\sigma+1}, 0, x^\sigma)$, with σ as above and $x \in \mathbf{GF}(q)$, and $(0, 0, 0, 0; 0, 1, 0, 0)$. Moreover, we may assume that the point $(0, 1, 0, 0; 0, 0, 1, 0)$

of M is collinear in Γ with the point $(1, 0, 1, 0; 0, 1, 0, 1)$. By Remark 17, the point $(0, 1, 0, 0; 0, 0, x, 0)$ is collinear in Γ with $(x, 0, 1, 0; 0, x^{\sigma+1}, 0, x^{\sigma})$. Finally, we may assume that the point $s =: (0, 0, 1, 1; 0, 0, 0, 0)$ is at distance 4 in Γ of the point $s' := (0, 0, 0, 0; 0, 1, 1, 0)$ on M_1.

Now we look for coordinates for N. It is easily calculated that the tangent planes of $\mathscr{A}_{L, M}$ at x_0 and x_1 meet in the line containing the points $u := (1, 0, 0, 0; 0, 0, 0, 0)$ and $u' := (0, 0, 0, 0; 0, 0, 0, 1)$. Since this line is contained in ζ_{x_0}, ζ_{x_1}, ζ_{y_0} and ζ_{z_1}, with $\{z_1\} = \Gamma_1(M) \cap \Gamma_2(x_1)$, we see that u and u' are contained in $\eta_{L_0, M_1} := \xi_{L_0} \cap \xi_{M_1}$. We therefore can choose coordinates such that N contains the point $(a, 1, 0, 0; 1, 0, 0, b)$, for some $a, b \in \mathbf{GF}(q)$ (this point is exactly $\Gamma_2(s) \cap \Gamma_2(s')$).

Let z_i, $0 \leqslant i \leqslant q$, be the unique point on M collinear in Γ with x_i. The set of points $\mathscr{A}_{M, N}$ is contained in the 3-dimensional space $\eta_{M, N}$ and contains exactly one point of every line $x_i z_i$, $i = 0, 1, ..., q$. Moreover, we assume that the set of points of $\mathscr{A}_{M, N}$ on the lines $x_i z_i$, $i = 0, 1, ..., q'$, is contained in a plane π''. We may assume that z_2 has coordinates $(0, 1, 0, 0; 0, 0, 1, 0)$. Then the points z_i, $i \in \{2, 3, ..., q'\}$, have coordinates $(0, 1, 0, 0; 0, 0, x, 0)$, with $x \in \mathbf{GF}(q')^{\times}$. Let u_i, $0 \leqslant i \leqslant q$, be the unique point of $\mathscr{A}_{M, N}$ on the line $x_i z_i$ (note that $s' = u_1$ and $x_0 = u_0$). Then there is a $c \in \mathbf{GF}(q)$ such that u_2 has coordinates $(1, c, 1, 0; 0, 1, c, 1)$. Furthermore, the plane $\langle u_0, u_1, u_2 \rangle$ must meet every line $x_i z_i$, $0 \leqslant i \leqslant q'$, in a point. Hence, for every $f \in \mathbf{GF}(q')$, the line

$$\langle (0, 1, 0, 0; 0, 0, f, 0), (f, 0, 1, 0; 0, f^2, 0, f) \rangle$$

must meet the plane

$$\langle (0, 0, 0, 0; 0, 1, 1, 0), (0, 0, 1, 0; 0, 0, 0, 0), (1, c, 1, 0; 0, 1, c, 1) \rangle.$$

One easily calculates that this implies $c = 1$ and the intersection point u_i (for $i \in \{2, 3, ..., q'\}$) has coordinates $(f, f, 1, 0; 0, f^2, f^2, f)$ (and f runs through $\mathbf{GF}(q')^{\times}$ as i runs through $\{2, 3, ..., q'\}$). Now let $i \in \{q'+1, q'+2, ..., q\}$. The point u_i has coordinates $(x, c, 1, 0; 0, x^{\sigma+1}, cx, x^{\sigma})$, for some $x, c \in \mathbf{GF}(q)$. For every $y \in \mathbf{GF}(q) \backslash \mathbf{GF}(q')$, the space $\langle u_0, u_1, u_2, u_i \rangle$ must intersect the line $\langle (0, 1, 0, 0; 0, 0, y, 0), (y, 0, 1, 0; 0, y^{\sigma+1}, 0, y^{\sigma}) \rangle$. This implies that

$$\begin{vmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ x & c & 1 & x^{\sigma+1} & cx & x^{\sigma} \\ 0 & 1 & 0 & 0 & y & 0 \\ y & 0 & 1 & y^{\sigma+1} & 0 & y^{\sigma} \end{vmatrix} = 0,$$

and one calculates that this means $c = x^\sigma$. So, in conclusion, the set $\mathscr{A}_{M,N}$ is the set $\{p_x := (x, x^\sigma, 1, 0; 0, x^{\sigma+1}, x^{\sigma+1}, x^\sigma) \mid x \in \mathbf{GF}(q)\} \cup \{(0, 0, 0, 0; 0, 1, 1, 0)\}$. Note that the latter can be viewed as belonging to the former set with $x = \infty$.

The line N_x of Γ incident with p_x and meeting N has a unique point q_x in $\Gamma_3(L_1)$. Hence the point q_x is contained in the intersection of $\langle p_x, N\rangle$ with ξ_{L_1}. We previously mentioned that uu' is contained in $\xi_{L_0} \cap \xi_{M_1}$; analogously uu' is contained in $\xi_{L_1} \cap \xi_{M_0}$. So $\xi_{L_1} = \langle u, u', x_1, z_1, y_0, y_1\rangle$. It is easily calculated that q_x has coordinates $(ax^\sigma - x, 0, 0, 1; x^\sigma, -x^{\sigma+1}, -x^{\sigma+1}, bx^\sigma - x^\sigma)$. Furthermore, the unique line of Γ through q_x meeting L_1 has a point q'_x in common with $\mathscr{A}_{L_1, M_0}$ (which lies in the space with equations $X_1 = X_2 = X_4 = X_5 = 0$); q'_x is contained in $\langle q_x, L_1\rangle$, and this implies that q'_x has coordinates $(ax^\sigma - x, 0, 0, 1; 0, 0, -x^{\sigma+1}, bx^\sigma - x^\sigma)$. One now easily deduces that the unique point q''_x on L_1 collinear in Γ with q_x has coordinates $(0, 0, 0, 0; 1, -x, 0, 0)$. Note also that the unique point p''_x on M collinear in Γ with p_x has coordinates $(0, 1, 0, 0; 0, 0, x, 0)$.

We now calculate the coordinates of the points of the set $\mathscr{A}_{M, q'_x q''_x}$, for $x \in \mathbf{GF}(q)^\times$. As before, we have to find a suitable 3-space in $\langle \xi_M\rangle$ containing the points x_1 and p_x. Similarly as above, but here considering the points x_1, p_x and $\Gamma_2(z_0) \cap \Gamma_2(q'_x)$ of $\mathscr{A}_{M, q'_x q''_x}$, one first computes that the points $p_{x, rx}$ with coordinates $(rx, x^\sigma, 1, 0; 0, r^2 x^{\sigma+1}, rx^{\sigma+1}, rx^\sigma)$, $r \in \mathbf{GF}(q')$, together with the point $p_{x,\infty} := (0, 0, 0, 0; 0, 1, 0, 0)$ belong to $\mathscr{A}_{M, q'_x q''_x}$ (and all lie in a plane). In order to compute the coordinates $(y, c, 1, 0; 0, y^{\sigma+1}, cy, y^\sigma)$ of the unique point $p_{x,y}$ of $\mathscr{A}_{M, q'_x q''_x}$ collinear in Γ with $(0, 1, 0, 0; 0, 0, y, 0)$, $y \in \mathbf{GF}(q) \backslash \{rx \mid r \in \mathbf{GF}(q')\}$, we express, as before, that the space $\langle x_1, p_{x,0}, p_x, p_{x,y}, (0, 1, 0, 0; 0, 0, z, 0), (z, 0, 1, 0; 0, z^{\sigma+1}, 0, z^\sigma)\rangle$ is 4-dimensional, for all $z \in \mathbf{GF}(q)$. This is equivalent with requiring that (deleting the fourth and fifth coordinates)

$$\begin{vmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & x^\sigma & 1 & 0 & 0 & 0 \\ x & x^\sigma & 1 & x^{\sigma+1} & x^{\sigma+1} & x^\sigma \\ y & c & 1 & y^{\sigma+1} & cy & y^\sigma \\ 0 & 1 & 0 & 0 & z & 0 \\ z & 0 & 1 & z^{\sigma+1} & 0 & z^\sigma \end{vmatrix} = 0,$$

which implies after a tedious calculation that $c = x^\sigma$ (and this apparently is also valid for $y = rx$, with $r \in \mathbf{GF}(q')$; so from now on $y \in \mathbf{GF}(q)$). Now we can calculate the coordinates of the point $r_{x,y} \in \Gamma_3(L_0) \cap \Gamma_2(p_{x,y})$. It is precisely the intersection of the plane $\langle p_{x,y}, q'_x q''_x\rangle$ with the space $\langle \xi_{L_0}\rangle$. We obtain

$$r_{x,y} = (ax^\sigma y, x^{\sigma+1}, x, y; y^{\sigma+1}, 0, 0, xy^\sigma - x^\sigma y + bx^\sigma y).$$

Note that $r_{x,y}$ is collinear in Γ with $r'_{x,y}=(0,0,x,y;0,0,0,0)$ (this is obtained by expressing that $\langle r_{x,y}, L_0\rangle$ meets η_{L_0,M_1}).

We will now derive a contradiction. Therefore, we consider the point $r_{x,y}$ (for arbitrary $x, y\in \mathbf{GF}(q)^\times$). We claim that $U:=\langle \Gamma_4(r_{x,y})\rangle=\mathbf{PG}(7,q)$. First we note that U contains the 5-dimensional space $W:=\xi_{r_{x,y}r'_{x,y}}$. Put $c=y/x$. Choose $r\in\mathbf{GF}(q')\backslash\{0,1\}$ and $z\in\mathbf{GF}(q)\backslash\mathbf{GF}(q')$. As the line $r_{x,y}r'_{x,y}$ is only dependent of y/x, the space W contains x_0, y_0, $r_{x,y}$, $p_{1,c}$, $p_{r,rc}$ and $p_{z,zc}$. Moreover, U contains the points q''_x and $p''_y=p''_{cx}$ (because these points belong to $\Gamma_4(r_{x,y})$). This implies that

$$\begin{vmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
acx^{\sigma+1} & x^{\sigma+1} & x & cx & (cx)^{\sigma+1} & 0 & 0 & (c^\sigma-c+bc)\,x^{\sigma+1}\\
c & 1 & 1 & 0 & 0 & c^{\sigma+1} & c & c^\sigma\\
cr & r & 1 & 0 & 0 & c^{\sigma+1}r^2 & cr^2 & c^\sigma r\\
cz & z^\sigma & 1 & 0 & 0 & (cz)^{\sigma+1} & cz^{\sigma+1} & (cz)^\sigma\\
0 & 0 & 0 & 0 & 1 & -x & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & cx & 0
\end{vmatrix}=0.$$

After a tedious calculation, one finds $b=(c+c^\sigma)/c$. But c was arbitrary, and $c^{\sigma-1}+1$ is fixed for all $c\in\mathbf{GF}(q)^\times$, hence $\sigma=1$, a contradiction.

Our Main Result is proved.

REFERENCES

1. J. W. P. Hirschfeld and J. A. Thas, "General Galois Geometries," Oxford Univ. Press, Oxford, 1991.
2. J. A. Thas and H. Van Maldeghem, Embedded thick finite generalized hexagons in projective space, *Proc. London Math. Soc. (2)* **54** (1996), 566–580.
3. J. A. Thas and H. Van Maldeghem, Flat lax and weak lax embeddings of finite generalized hexagons, *European J. Combin.* **19** (1998), 733–751.
4. J. A. Thas and H. Van Maldeghem, On embeddings of the flag geometries of projective planes in finite projective spaces, *Des. Codes Cryptogr.* **17** (1999), 97–104.
5. J. A. Thas and H. Van Maldeghem, Classification of embeddings of the flag geometries of projective planes in finite projective spaces, part 1, *J. Combin. Theory Ser. A* **89** (2000).
6. J. A. Thas and H. Van Maldeghem, Classification of embeddings of the flag geometries of projective planes in finite projective spaces, part 3, *J. Combin. Theory Ser. A* **89** (2000).
7. J. Tits, Sur la trialité et certains groupes qui s'en déduisent, *Inst. Hautes Études Sci. Publ. Math.* **2** (1959), 13–60.
8. H. Van Maldeghem, "Generalized Polygons," Monographs in Mathematics, Vol. 93, Birkhäuser Verlag, Basel, 1998.