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A NOTE ON NEAR-BARBILIAN PLANES

Near-Barbilian Planes (NBPs) and Strong Near-Barbilian Planes (SNBPs) were introduced in [2] as a variation of Barbilian Planes (BPs). In this short note we show that any NBP is an SNBP; we classify all NBPs up to the classification of linear spaces (many examples follow as a result of a universal construction) and we show that the only NBPs that are also BPs are those mentioned in [2], namely the projective planes.

DEFINITIONS. A *neighbour-incidence structure* is a quadruple $(\mathscr{P}, \mathscr{L}, |, \cong)$, where $\mathscr{P}$ is a set of points, $\mathscr{L}$ is a set of lines, $|$ is a symmetric binary relation between $\mathscr{P}$ and $\mathscr{L}$ called *incidence* and $\cong$ is a symmetric binary relation between $\mathscr{P}$ and $\mathscr{L}$ called *neighbour*. We shall denote arbitrary points by x, y, z and arbitrary lines by l, m, n, k.

Given a neighbour-incidence structure, we define a binary relation in $\mathscr{P}$ and $\mathscr{L}$ (also called *neighbour*) as follows.

(B.1) $x \cong y \Leftrightarrow \forall l | y \colon x \cong l.$
(B.2) $l \cong m \Leftrightarrow \forall x | m \colon l \cong x.$

Consider the following conditions.

(1) If $x | l$, then $x \cong l$.
(2) For all x and y such that $x \not\cong y$, there exists a unique l with $l | x$ and $l | y$. Notation: $l = x \vee y$.
(2′) For all l and m such that $l \not\cong m$, there exists a unique x with $x | l$ and $x | m$. Notation: $x = l \wedge m$.
(2*) If $l \not\cong m$, then there exists a unique x such that $x \cong l$ and $x \cong m$. Notation: $x = N(l, m)$.
(3) If $l \not\cong$ m, $x | l$ and $l \wedge m \not\cong x$, then $x \not\cong m$.
(4) There exists at least one line; for each l, there exists an $x | l$; for any x and y, there exists an l with $l \not\cong x$ and $l \not\cong y$.
(5) If $l \not\cong m$, then there exists a $z | l$ such that $N(l, m) \not\cong z$.
(6) If $x \cong l$ and $l \cong m$, then $x \cong m$.

A neighbour-incidence structure is called a (*projective*) *Barbilian plane* (BP) if it satisfies (1), (2), (2′), (3) and (4). It is called a *near-Barbilian plane* (NBP) if it satisfies (1), (2), (2*), (4) and (5). An NBP is called a *strong* NBP if it also

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satisfies (6) (for all this, see [2]). Now R. Spanicciati [2] proves the following properties of NBPs.

PROPERTIES

(P.1) *Any NBP gives rise to a linear space* ([1]) $(\mathscr{P}, \mathscr{L}, |)$.

(P.2) $x \cong y \Leftrightarrow x = y$.

(P.3) *The neighbour relation is an equivalence relation when restricted to the set of lines.*

(P.4) *If* $x \not\cong y$, $x \cong l$ *and* $y \cong l$, *then* $x \vee y \cong l$.

We now show that any NBP is also an SNBP.

THEOREM. *Every near-Barbilian plane is a strong near-Barbilian plane.*

Proof. We prove condition (6). Let $x \cong l$ and $l \cong m$. If $x|m$, then by (1), $x \cong m$. So suppose $x \not| m$. Choose $y|m$ (cf. (4)). Since $x \neq y$, by (2) and (P.2), $n = x \vee y$ is uniquely defined. Since $l \cong m$, we have by (B.2) $y \cong l$. Hence by (P.4), $n \cong l$. By (P.3), $n \cong m$. Since $x|n$, we have again by (B.2) $x \cong m$Q.E.D.

CONSTRUCTION. Choose any projective plane $\Pi = (\mathscr{P}, \mathscr{L}, \in)$ (viewed as a shadow space, i.e. every line is a subset of the set $\mathscr{P}$ of points (see [1]). Choose for any $l \in \mathscr{L}$ an arbitrary linear space $\Pi_l = (l, \mathscr{L}_l, |)$ (always existing since we can take for $\mathscr{L}_l$ the set of 2-subsets of l and $| = \in \cup \ni$). This linear space may be *trivial* (i.e. $|\mathscr{L}_l| = 1$). Now define $\Pi^* = (\mathscr{P}, \cup \{\mathscr{L}_l : l \in \mathscr{L}\}, |, \cong)$, where $x \cong m$ $(m \in \mathscr{L}_l) \Leftrightarrow x \in l$. Then it is easy to check that Π^* is an NBP. We verify, for example, the main axioms (2) and (2*). The other axioms are non-degeneracy axioms and follow directly from the construction, the non-degeneracy conditions of a projective plane and the properties of a linear space. Note also that $m \cong n \Leftrightarrow m, n \in \mathscr{L}_l$ for some $l \in \mathscr{L}$, and $x \cong y \Leftrightarrow x = y$.

(2) If $x, y \in \mathscr{P}$, then there is a unique l in $\mathscr{L}$ containing both x and y. Moreover, there exists a unique m in $\mathscr{L}_l$ incident with both x and y.

(2*) Let $m \not\cong n$ and let $m \in \mathscr{L}_l$ and $n \in \mathscr{L}_k$. Then $l \neq k$ and so, there exists unique x in $l \cap k$. By definition, x is the unique point neighbour to both m and n.

Conversely, it follows from [2, Prop. (3.9) and Th. (4.5)] that every SNBP, and hence every NBP, must be constructed in this manner. So the above construction is universal. Note that, with the above notation, Π_l is a maximal linear subspace of Π^* for every $l \in \mathscr{L}$.

Now, if (2′) holds in Π^*, then clearly every line of Π_l must be incident with all points of l; hence Π_l is trivial and Π^* is a projective plane. So this answers two open questions in [2] and shows that the class of NBPs is a well-defined subclass of the class of linear spaces.

REFERENCES

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