On Opposition in Spherical Buildings and Twin Buildings

Peter Abramenko[1*] Hendrik Van Maldeghem[2†‡]

[1] *Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D – 33501 Bielefeld,* `abramenk@@mathematik.uni-bielefeld.de`

[2] *Department of Pure Mathematics and Computer Algebra, University of Ghent, Galglaan 2, B – 9000 Ghent,* `hvm@@cage.rug.ac.be`

Abstract

In this paper, we prove a combinatorial property of twin apartments and opposition of chambers in twin buildings. We then characterize adjacency of chambers in twin buildings by means of opposition of chambers. As an application, we study maps which satisfy certain conditions related to opposition of chambers, e.g. maps that preserve opposition. Applied to the special case of spherical buildings, all our main results as well as their corollaries are new.

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1 Introduction and Statement of the Main Results

Twin buildings are natural generalizations of spherical buildings, which play a central role in the modern theories of incidence geometry, group theory, finite geometry, etc. Where

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spherical buildings are associated to groups of Lie type, twin buildings arise from Kac-Moody groups. The crucial notion in a twin building is that of *opposition* (of chambers). Roughly speaking, and denoting the set of chambers of a building Ω by $\mathrm{Ch}(\Omega)$, a twin building $\Delta = (\Delta_+, \Delta_-, \delta^*)$ consists of a pair of buildings (Δ_+, Δ_-) of the same type, say determined by the Coxeter System (W, S) (standard notation), and a "codistance" $\delta^* : (\mathrm{Ch}(\Delta_+) \times \mathrm{Ch}(\Delta_-)) \cup (\mathrm{Ch}(\Delta_-) \times \mathrm{Ch}(\Delta_+)) \to W$. The axioms are modelled on the situation in spherical buildings. There, the codistance is in a sense "complementary" to the Weyl distance, in particular the codistance between two chambers is equal to 1 (the identity in W) if and only if the Weyl distance between these two chambers is equal to the longest word. Correspondingly, if Δ is a twin building as before, then two chambers $C, D \in \mathrm{Ch}(\Delta) := \mathrm{Ch}(\Delta_+) \cup \mathrm{Ch}(\Delta_-)$ are called *opposite* precisely when they are in different components and $\delta^*(C, D) = 1$ (there is also a numerical codistance, which is obtained from δ^* by taking the length of the words in (W, S)). Hence spherical buildings can be twinned to themselves and constitute a subclass of the class of twin buildings (cf. Tits [1990], Proposition 1).

Twin buildings were originally invented by Mark Ronan and Jacques Tits. They were introduced in the literature by Tits [1990] (see also Tits [1989]), where many elementary properties are given, but proofs are only sketched or omitted. An introduction and an exposition of many elementary properties with proofs is provided by Abramenko [1996], to which we refer for definitions and notation.

The aim of the present paper is to use the opposition of chambers to characterize two fundamental notions in the theory of (twin) buildings: that of "twin apartments" and that of "adjacency of chambers". A twin apartment in a twin building Δ as above is a pair of apartments $\Sigma = (\Sigma_+, \Sigma_-)$, $\Sigma_\epsilon \subseteq \Delta_\epsilon$, $\epsilon \in \{+, -\}$, such that the restriction of the codistance to that pair induces the structure of a (thin) twin building. Applied to the spherical case, every usual apartment can be seen as a twin apartment, unlike the general situation (there are many apartments in Δ_+ which do not have a twin in Δ_- and vice versa).

The characterization of twin apartments presented below is the consequence of a combinatorial parity pattern which we discovered first in spherical and then also in twin buildings. The surprising general fact is that the number of chambers in a given twin apartment which are opposite a fixed chamber outside this twin apartment is always even (see Theorem 1.1 and Corollary 1.3). As might be expected, such a general statement has interesting applications in building theory. As one important example, we shall deduce below a characterization of the adjacency of chambers in thick twin buildings by means of the opposition relation (see Theorem 1.2 and Corollary 1.4). It is perhaps not surprising that such a characterization can be given. However, only by applying our first theorem we were able to give a short and elegant proof for it, which will be presented in Section 4; our first approach was much more laborious and needed a separate investigation of generalized polygons and twin trees before spherical and twin buildings could be treated.

We would like to emphasize the fact that our main results are also new for the case of spherical buildings. Hence we do not "just generalize" known facts from spherical building theory to twin building theory. For people only interested in the results restricted to the spherical case, we will phrase our main results in the classical language of spherical buildings as two corollaries. We have also tried to write down our proofs avoiding as much as possible the specific notation of twin buidings. Therefore, it should be possible for someone interested in spherical buildings to read the proofs without much knowledge of twin buildings.

Some further notation that we will use in the sequel: for $\epsilon \in \{+,-\}$, we define $-\epsilon$ as $\{\epsilon, -\epsilon\} = \{+,-\}$. Also, $C \,\mathrm{op}\, D$ means that C is opposite D; $C \,\overline{\mathrm{op}}\, D$ means that C is not opposite D. A twin building $\Delta = (\Delta_+, \Delta_-, \delta^*)$ is called *thick* if the buildings Δ_+ and Δ_- are both thick.

Theorem 1.1 *Let $\Delta = (\Delta_+, \Delta_-, \delta^*)$ be a thick twin building. Let $\mathcal{M}$ be a non-empty set of chambers of Δ. Then $\mathcal{M}$ is the chamber set of a twin apartment Σ if and only if for every chamber C of $\mathcal{M}$, there is a unique chamber $C' \in \mathcal{M}$ opposite C, and for every chamber C in $\mathrm{Ch}(\Delta) \setminus \mathcal{M}$, the number of chambers of $\mathcal{M}$ which are opposite C is even.*

Theorem 1.2 *Let $(\Delta_+, \Delta_-, \delta^*)$ be a thick twin building. Let C and D be arbitrary distinct chambers of Δ. Then C and D are adjacent if and only if there exists a chamber E in Δ such that no chamber of Δ is opposite exactly one of $\{C, D, E\}$.*

For the convenience of the reader only interested in spherical buildings, we restate these results for spherical buildings.

Corollary 1.3 *Let Δ be a thick spherical building. Let $\mathcal{M}$ be a non-empty set of chambers. Then $\mathcal{M}$ is the chamber set of an apartment of Δ if and only if every chamber of $\mathcal{M}$ has a unique opposite in $\mathcal{M}$ and every chamber outside $\mathcal{M}$ has an even number of opposites in $\mathcal{M}$.*

Corollary 1.4 *Let Δ be a thick spherical building. Let C, D be arbitrary distinct chambers of Δ. Then C and D are adjacent if and only if there is a chamber E such that no chamber of Δ is opposite a unique member of $\{C, D, E\}$.*

There are some corollaries to these results, and they are gathered in Section 5. Here, we just mention one of them.

Corollary 1.5 *Let $\Delta = (\Delta_+, \Delta_-, \delta^*)$ be a thick twin building without any non-spherical rank 2 residues. Then the distance and codistance functions are completely determined by the opposition relation. Hence a thick 2-spherical twin building (in particular any spherical building) is completely determined by its set of chambers and the opposition relation.*

We emphasize the fact that in the previous corollary, the twin building is determined only by the *set* of chambers and the opposition relation, unlike the classical well known result, where in addition the whole structure of the two separate buildings must be given.

Finally we remark that the results presented in this paper are also motivated by Bernhard Mühlherr's question, which "1-twinnings" (for a definition, see Mühlherr [1998]) could be extended to twinnings. In a forthcoming paper we shall, applying Theorem 1.2, answer this question, thereby obtaining new characterizations of twin buildings.

2 Twin Apartments and Twin Roots

We recall some notation and basic facts concerning twin buildings. The results summarized below are due to Tits [1989], [1990]; proofs can be found in Abramenko [1996], Ch.I, §2.

As in the introduction, we denote by $\Delta = (\Delta_+, \Delta_-, \delta^*)$ a (not necessarily thick) *twin building* with W-codistance δ^* as introduced in Tits [1990] and Abramenko [1996], Definition 3, and by "op" the corresponding opposition relation. By a *twin apartment* Σ of Δ we understand a pair (Σ_+, Σ_-) of apartments in Δ_+, Δ_-, respectively, such that $(\Sigma_+, \Sigma_-, {\delta^*}_{|\Sigma})$ is itself a (thin) twin building (cp. Definition 4 and Lemma 2(*iii*) in Abramenko [1996]). From the very definition of a twin building we immediately deduce the following

Fact 2.1 *If $\Sigma = (\Sigma_+, \Sigma_-)$ is a twin apartment, then for any chamber C in $\Sigma_+ \cup \Sigma_-$, there exists exactly one chamber C' in $\Sigma_+ \cup \Sigma_-$ opposite C, i.e., such that $C \operatorname{op} C'$.*

Remark 2.2 One can easily show the converse, namely, that any pair $\Sigma = (\Sigma_+, \Sigma_-)$ of apartments in Δ_+, Δ_- satisfying the statement of Fact 2.1 is a twin apartment of Δ.

If Σ, C and C' are as in Fact 2.1, we write $C' =: \operatorname{op}_\Sigma(C)$ (and hence also $C = \operatorname{op}_\Sigma(C')$).

Another immediate consequence of the definition of twin apartments is the following

Fact 2.3 *If $\Sigma = (\Sigma_+, \Sigma_-)$ is a twin apartment, then the opposition involution op_Σ induces an isomorphism of Coxeter complexes between Σ_+ and Σ_-.*

For the next basic result, we again refer to Abramenko [1996], Lemma 2.

Fact 2.4 *Given chambers C, D in $\Delta_+ \cup \Delta_-$, there exists a twin apartment $\Sigma = (\Sigma_+, \Sigma_-)$ such that C, D belong to $\Sigma_+ \cup \Sigma_-$. If additionally $C \operatorname{op} D$, then this twin apartment Σ is unique.*

If $C \operatorname{op} D$, then the unique twin apartment Σ of Fact 2.4 will be denoted by $\mathbf{\Sigma}\{C, D\}$.

We shall also need generalizations of the classical notion in spherical buildings of a "root" (= a half apartment). If β is a root in a Coxeter complex Θ, we shall denote by $-\beta$ the unique root in Θ satisfying $\Theta = \beta \cup (-\beta)$ such that $\beta \cap (-\beta)$ does not contain any chambers.

Now let $\Delta = (\Delta_+, \Delta_-, \delta^*)$ be a twin building as above. A pair $\alpha = (\alpha_+, \alpha_-)$ is called a *twin root* of Δ if there exists a twin apartment $\Sigma = (\Sigma_+, \Sigma_-)$ of Δ such that α_ϵ is a root in Σ_ϵ for $\epsilon \in \{+, -\}$, and $\alpha_- = -\operatorname{op}_\Sigma(\alpha_+)$. Note that $\operatorname{op}_\Sigma(\alpha_+)$ is a root in Σ_- by Fact 2.3.

Any twin root is the "coconvex hull" (in the sense of Abramenko [1996], Chapter I, §4, Appendix) of two properly selected chambers C, D of Δ which are at numerical codistance 1, i.e., $\delta^*(C, D) \in S$, where the Coxeter system (W, S) defines the type of Δ. This alternative characterization of twin roots follows immediately from Proposition 5 of Abramenko [1996]. However, the proof of that proposition is not yet published, and therefore we shall now derive this property of twin roots directly in the form it is needed later on.

Lemma 2.5 *Let C, D be non-opposite chambers of Δ such that D is adjacent to a chamber $C' \operatorname{op} C$. Then there exists a unique twin root $\alpha = (\alpha_+, \alpha_-)$ of Δ such that C, D belong to $\alpha_+ \cup \alpha_-$. This twin root automatically satisfies $\alpha \subseteq \Sigma\{C, C'\}$.*

Proof. We may assume $C \in \Delta_+$ and $C' \in \Delta_-$. We also assume that Δ is of type (W, S) and $S = \{s_i \mid i \in I\}$ for some (finite) index set I. Then C' and D are i-adjacent for some $i \in I$, and, since $C \,\overline{\operatorname{op}}\, D$, $\delta^*(C, D) = s_i$. Setting $\Sigma := \Sigma\{C, C'\} = (\Sigma_+, \Sigma_-)$, we observe $D \in \Sigma_-$ since Σ_- must contain a chamber i-adjacent to C' and at codistance s_i from C (and there is only one such chamber in Δ_-; with the notation of Definition 6 of Abramenko [1996], we have $D = \operatorname{proj}^*_{C' \cap D} C$, the "coprojection" of C onto the panel $C' \cap D$). Denote by α_- the root in Σ_- containing D but not C', and set $\alpha_+ := -\operatorname{op}_\Sigma(\alpha_-) = \operatorname{op}_\Sigma(-\alpha_-)$. Then $C = \operatorname{op}_\Sigma(C') \in \Sigma_+$, and $\alpha := (\alpha_+, \alpha_-) \subseteq \Sigma$ is a twin root containing both C and D.

Assume now that there is a second twin root $\widetilde{\alpha} = (\widetilde{\alpha}_+, \widetilde{\alpha}_-)$ containing C and D. Let $\widetilde{\Sigma} = (\widetilde{\Sigma}_+, \widetilde{\Sigma}_-)$ be a twin apartment containing $\widetilde{\alpha}$, and let $\widetilde{C}$ be the chamber i-adjacent to D in $\widetilde{\Sigma}_-$. Then $\delta^*(C, D) = s_i$ implies $C \operatorname{op} \widetilde{C}$, hence $\widetilde{\Sigma} = \Sigma\{C, \widetilde{C}\}$ and $\widetilde{C} \notin \widetilde{\alpha}_-$ (because $C \in \widetilde{\alpha}_+$). We shall show now by induction on the gallery distance between D and X that any chamber X of α_- is also contained in $\widetilde{\alpha}_-$. So let $(D = D_0, D_1, \ldots, D_m = X)$ be a minimal gallery connecting D and X, hence automatically contained in α_-. Let $D_{\ell-1}$ be i_ℓ-adjacent to D_ℓ ($i_\ell \in I$) for all positive $\ell \le m$. Observe that $\gamma := (C', D, D_1, \ldots, D_m)$ is also a minimal gallery since $C' \notin \alpha_-$ and $D \in \alpha_-$. Hence $s_i s_{i_1} \cdots s_{i_\ell}$ is reduced with respect to the Coxeter system (W, S). Because $\Sigma = \Sigma\{C, C'\}$ and $\gamma \subseteq \Sigma_-$, we obtain $\delta^*(C, D_\ell) = s_i s_{i_1} \cdots s_{i_\ell}$, for all positive $\ell \le m$ (and all these expressions are reduced). In particular, D_m is the coprojection of C onto the panel $p_- := D_{m-1} \cap D_m$. Now, by induction we have $D_{m-1} \in \widetilde{\alpha}_- \subseteq \widetilde{\Sigma}_-$. However, $C \in \widetilde{\Sigma}_+$ and $p_- \in \widetilde{\Sigma}_-$ imply $D_m = \mathrm{proj}^*_{p_-} C \in \widetilde{\Sigma}_-$ (cp. ABRAMENKO [1996], Corollary 3).

Hence $(\widetilde{C}, D, D_1, \ldots, D_m)$, being of reduced type $(i, i_1, \ldots, i_m)$, is a minimal gallery in $\widetilde{\Sigma}_-$. Since $\widetilde{C} \notin \widetilde{\alpha}_-$ and $D \in \widetilde{\alpha}_-$, this implies $X = D_m \in \widetilde{\alpha}_-$. So we have shown $\alpha_- \subseteq \widetilde{\alpha}_-$. By symmetry, we also get $\widetilde{\alpha}_- \subseteq \alpha_-$. Reversing the roles of $+$ and $-$ (and using the chambers $D' \in \Sigma_+$ and $\widetilde{D} \in \widetilde{\Sigma}_+$ i-adjacent to C instead of C' and $\widetilde{C}$, respectively), one finally obtaines $\alpha_+ = \widetilde{\alpha}_+$, hence $\alpha = \widetilde{\alpha}$. $\square$

Lemma 2.6 *Let C_1, C_2, C_3 be three different pairwise adjacent chambers of Δ. Let Σ be a twin apartment of Δ which contains C_1 and C_2. Then there exist three twin roots $\alpha_1, \alpha_2, \alpha_3$ satisfying the following properties.*

(1) $\mathrm{Ch}(\alpha_i) \cap \{C_1, C_2, C_3\} = \{C_i\}$, *for* $i = 1, 2, 3$.

(2) *The sets* $\mathrm{Ch}(\alpha_1)$, $\mathrm{Ch}(\alpha_2)$, $\mathrm{Ch}(\alpha_3)$ *are pairwise disjoint.*

(3) *For any $i, j \in \{1, 2, 3\}$ with $i < j$, the set* $\mathrm{Ch}(\alpha_i) \cup \mathrm{Ch}(\alpha_j)$ *is the set of chambers of a twin apartment Σ_{ij}, with $\Sigma_{12} = \Sigma$.*

Proof. Denote by p the panel $C_1 \cap C_2 = C_2 \cap C_3 = C_3 \cap C_1$, and set $C_i' := \mathrm{op}_\Sigma(C_i)$, $i = 1, 2$, and $p' := C_1' \cap C_2'$. Let D_3 be the unique chamber of Δ containing p' and not opposite C_3. Note that by Fact 2.1 C_1' is not opposite C_2 and C_2' is not opposite C_1. Hence C_1' and C_2' are both opposite C_3, and $D_3 \neq C_1', C_2'$. This implies $D_3 \,\mathrm{op}\, C_1, C_2$. According to Fact 2.4, we have uniquely defined twin apartments $\Sigma_{12} := \Sigma\{C_1, C_1'\} = \Sigma\{C_2, C_2'\} = \Sigma$, $\Sigma_{13} := \Sigma\{C_1, D_3\}$ and $\Sigma_{23} := \Sigma\{C_2, D_3\}$. Applying Lemma 2.5 repeatedly, we obtain (uniquely determined) twin roots α_1, α_2, α_3 such that $\{C_1, C_2'\} \subseteq \mathrm{Ch}(\alpha_1)$, $\{C_2, C_1'\} \subseteq \mathrm{Ch}(\alpha_2)$, $\{C_3, D_3\} \subseteq \mathrm{Ch}(\alpha_3)$ satisfying additionally $\alpha_1 \subseteq \Sigma \cap \Sigma_{13}$, $\alpha_2 \subseteq \Sigma \cap \Sigma_{23}$ and $\alpha_3 \subseteq \Sigma_{13} \cap \Sigma_{23}$.

Without loss of generality, we may assume $C_1, C_2, C_3 \in \Delta_+$ and $C_1', C_2', D_3 \in \Delta_-$. By definition, a twin root never contains opposite chambers. Hence the above construction implies $(\alpha_i)_+ \cap \{C_1, C_2, C_3\} = \{C_i\}$, $i = 1, 2, 3$, and, with $\mathcal{M} := \{C_1', C_2', D_3\}$, $(\alpha_1)_- \cap \mathcal{M} = \{C_2'\}$, $(\alpha_2)_- \cap \mathcal{M} = \{C_1'\}$, $(\alpha_3)_- \cap \mathcal{M} = \{D_3\}$. Now it follows immediately from basic properties of roots in Coxeter complexes that $(\alpha_i)_\epsilon = -(\alpha_j)_\epsilon$ in $(\Sigma_{ij})_\epsilon$, for $i, j \in \{1, 2, 3\}$, $i < j$, and $\epsilon \in \{+, -\}$. This implies (2) and (3), and we already deduced (1) above. $\square$

Remark 2.7 It is easy to see that Σ, C_1, C_2, C_3 and properties (1), (2) and (3) determine $\alpha_1, \alpha_2, \alpha_3$ uniquely, but we will not need this fact in the sequel.

Remark 2.8 If the reader is only interested in spherical buildings, he may restrict to the special case where $\Delta = \Delta_+ = \Delta_-$ is a spherical building in which ordinary apartments $\Sigma = \Sigma_+ = \Sigma_-$, respectively ordinary roots $\alpha = \alpha_+ = \alpha_-$ are considered. In this case, all the facts stated above (including the lemmas) are well known.

3 A Characterization of Twin Apartments

In this section, we prove Theorem 1.1. First we introduce some notation.

Let us briefly denote the set $\mathrm{Ch}(\Delta_\epsilon)$ of chambers of Δ_ϵ by $\mathcal{C}_\epsilon$, $\epsilon \in \{+,-\}$, and put $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_- = \mathrm{Ch}(\Delta)$. For a pair Ω_+, Ω_- of subcomplexes of Δ_+, Δ_-, respectively, we write $\mathrm{Ch}(\Omega) = \mathrm{Ch}(\Omega_+) \cup \mathrm{Ch}(\Omega_-)$, where $\Omega = (\Omega_+, \Omega_-)$. For any $C \in \mathcal{C}$ and any subset $\mathcal{M} \subseteq \mathcal{C}$, we define the number $\mathbf{n}(C, \mathcal{M})$ as the number of elements of $\mathcal{M}$ opposite C. If Σ is a twin apartment, we also briefly denote $n(C,\Sigma) := n(C, \mathrm{Ch}(\Sigma))$. Similarly, we write $n(C,\alpha) := n(C, \mathrm{Ch}(\alpha))$ for a twin root α.

Remark 3.1 Let $\Sigma = (\Sigma_+, \Sigma_-)$ be a twin apartment and let $\alpha = (\alpha_+, \alpha_-)$ be a twin root of Δ.

(*i*) According to Fact 2.1, $n(C,\Sigma) = 1$ if $C \in \Sigma_+ \cup \Sigma_-$.

(*ii*) It is an easy exercise to show $n(C,\Sigma) < \infty$ for all $C \in \mathcal{C}$. However, this also follows from Proposition 3.2 below.

(*iii*) As an immediate consequence of the axiom (Tw2) in Tits [1990], Section 2.2, one obtains $n(C,\Sigma) \geq 1$ for all $C \in \mathcal{C}$. It is also known that $n(C,\Sigma) > 1$ if $C \notin \Sigma_+ \cup \Sigma_-$ (cp. Tits [1990], Proposition 3). Again, this last statement independently follows from the proof of Proposition 3.2 below.

(*iv*) By the definition of twin roots, we have $n(C,\alpha) = 0$, for all chambers $C \in \alpha_+ \cup \alpha_-$.

We will now prove the following slightly more detailed version of Theorem 1.1.

Proposition 3.2 *For a twin building Δ and a non-empty set of chambers $\mathcal{M} \subseteq \mathcal{C}$, we consider the following two statements:*

(a) *There exists a twin apartment Σ of Δ such that $\mathcal{M} = \mathrm{Ch}(\Sigma)$.*

(b) $n(C,\mathcal{M}) = 1$ *for all* $C \in \mathcal{M}$ *and* $n(C,\mathcal{M}) \equiv 0 \bmod 2$, *for all* $C \in \mathcal{C} \setminus \mathcal{M}$.

Then (a) *always implies* (b), *and* (b) *implies* (a) *whenever Δ is thick.*

Proof. We first show that (a) always implies (b). Given a twin apartment $\Sigma = (\Sigma_+, \Sigma_-)$, we already know from Fact 2.1 that $n(C, \Sigma) = 1$ for $C \in \mathrm{Ch}(\Sigma)$. So we have to show $n(C, \Sigma) \equiv 0 \bmod 2$ for $C \in \mathcal{C} \setminus \mathrm{Ch}(\Sigma)$.

Choose a minimal gallery γ between C and Σ, i.e., a gallery $\gamma = (C = X_0, X_1, \ldots, X_m)$ with X_m a chamber in Σ and m minimal. Eventually, the claim will be proved by induction on $m \geq 1$. We now apply Lemma 2.6 with $C_1 := X_m$, $C_3 := X_{m-1}$ and with C_2 being the unique neighbour of C_1 in Σ containing $C_1 \cap C_3$. So let the twin apartments $\Sigma_{12} = \Sigma$, Σ_{13}, Σ_{23} as well as the twin roots α_1, α_2 and α_3 satisfy the properties (1)–(3) of Lemma 2.6. Then we distinguish two cases.

(1) $C = X_0 \in \mathrm{Ch}(\alpha_3)$.
Then also $C \in \mathrm{Ch}(\Sigma_{i3})$, for $i = 1, 2$, and $\mathrm{Ch}(\Sigma_{i3}) = \mathrm{Ch}(\alpha_i) \dot{\cup} \mathrm{Ch}(\alpha_3)$ implies $1 = n(C, \Sigma_{i3}) = n(C, \alpha_i) + n(C, \alpha_3) = n(C, \alpha_i)$, $i = 1, 2$, hence $n(C, \Sigma) = n(C, \alpha_1) + n(C, \alpha_2) = 1 + 1 = 2$.

Note that this case contains the case $m = 1$, where $C = X_0 = C_3$; hence we can start the induction alluded to above.

(2) $C \notin \mathrm{Ch}(\alpha_3)$.
Then, because $C \notin \mathrm{Ch}(\Sigma) = \mathrm{Ch}(\alpha_1) \cup \mathrm{Ch}(\alpha_2)$, also $C \notin \mathrm{Ch}(\Sigma_{i3})$ for $i = 1, 2$. However, since $X_{m-1} = C_3 \in \mathrm{Ch}(\Sigma_{13}) \cap \mathrm{Ch}(\Sigma_{23})$, the chamber C is now nearer (in the sense of galleries) to Σ_{13} and Σ_{23} than to Σ. Therefore, the induction hypothesis yields $n(C, \alpha_i) + n(C, \alpha_3) = n(C, \Sigma_{i3}) \equiv 0 \bmod 2$, for $i = 1, 2$. This implies $n(C, \Sigma) = n(C, \alpha_1) + n(C, \alpha_2) = n(C, \Sigma_{13}) + n(C, \Sigma_{23}) - 2n(C, \alpha_3) \equiv 0 \bmod 2$.

Now we show that (b) implies (a) if Δ is thick. Choose $C \in \mathcal{M}$, let C' be the unique element of $\mathcal{M}$ opposite C, and set $\Sigma := \Sigma\{C, C'\}$. We will show $\mathcal{M} = \mathrm{Ch}(\Sigma)$.

Let D be a neighbour of C in Σ and let E be an arbitrary chamber distinct from both C and D containing the panel $C \cap D$ (E exists by the thickness of Δ). Now D is not opposite C'. Hence $E \operatorname{op} C'$, and therefore $E \notin \mathcal{M}$ because $n(C', \mathcal{M}) = 1$.

Now consider an arbitrary chamber $X \in \mathcal{M} \setminus \{C'\}$. Since $n(C, \mathcal{M}) = 1$, the chamber X is not opposite C. Because the three chambers C, D, E are pairwise adjacent, this implies that $X \operatorname{op} D$ if and only if $X \operatorname{op} E$. Therefore $n(E, \mathcal{M}) = n(D, \mathcal{M}) + 1$. Since $E \notin \mathcal{M}$, our assumption (b) implies that $n(E, \mathcal{M})$ is even. Hence $n(D, \mathcal{M})$ is odd which forces, again by assumption (b), $D \in \mathcal{M}$. So any neighbour of C in Σ is a member of Σ. Analogously, any neighbour of C' in Σ is an element of $\mathcal{M}$. Hence a straightforward induction using gallery distances shows that $\mathrm{Ch}(\Sigma) \subseteq \mathcal{M}$. In view of the implication "(a)$\Longrightarrow$(b)" and the first statement of Remark 3.1(*iii*), this yields $n(Z, \mathcal{M}) \geq 2$ for all $Z \in \mathcal{C} \setminus \mathrm{Ch}(\Sigma)$. Therefore the first part of assumption (b) implies now $\mathrm{Ch}(\Sigma) = \mathcal{M}$. The proposition is proved. □

Remark 3.3 The last statement in Remark 3.1(*iii*) might give hope to characterize twin apartments more generally as corresponding to sets $\mathcal{M}$ of chambers such that

(*) $n(X,\mathcal{M}) = 1$ for $X \in \mathcal{M}$, and $n(X,\mathcal{M}) > 1$ for $X \in \mathcal{C} \setminus \mathcal{M}$.

This is not true. Indeed, consider a thick building Γ of type A_2, i.e., the flag complex of a projective plane. Let p and l be two opposite panels and let $\{C_i \,|\, i \in I\}$ (respectively $\{D_i \,|\, i \in I\}$) be the set of chambers containing p (respectively l), where I is some appropriate index set, which may be assumed to contain the elements 0 and 1. We may also assume that $C_i \,\overline{\mathrm{op}}\, D_i$, for all $i \in I$. Let E_i, $i \in I$, be the unique chamber adjacent to both C_i and D_i. Take any $i, j \in I$, $i \neq j$. Then the set $\{C_0, C_1, D_0, D_1, E_i, E_j\}$ satisfies condition (*) above. Note that only for $\{i, j\} = \{0, 1\}$, this set is the set of chambers of an apartment.

In fact, one can show that every set of chambers of a projective plane satisfying (*) is constructed in the above way. An interesting conjecture is that every set of chambers of some thick spherical building Δ satisfying (*) has the same number of chambers as any apartment in Δ.

The above counterexample may be generalized to an arbitrary twin building as follows. Consider the set of chambers $\mathrm{Ch}(\Sigma)$ of some twin apartment Σ of Δ. Let p_+, p_- be two opposite panels in Σ, and let C_+, C_-, D_+, D_- be the four chambers of Σ containing these panels, with $C_+ \,\mathrm{op}\, C_-$ and $D_+ \,\mathrm{op}\, D_-$. Consider two arbitrary chambers X_+ and Y_+ on p_+ and let Y_- respectively X_- be the unique chamber on p_- not opposite X_+ respectively Y_+. Put $\mathcal{M} = (\mathrm{Ch}(\Sigma) \cup \{X_+, X_-, Y_+, Y_-\}) \setminus \{C_+, C_-, D_+, D_-\}$. Clearly $n(X,\mathcal{M}) = 1$ for $X \in \mathcal{M}$, and it follows easily from basic facts in the theory of twin buildings that $n(X,\mathcal{M}) > 1$ for $X \in \mathcal{C} \setminus \mathcal{M}$. So $\mathcal{M}$ satisfies (*).

Clearly, one can repeat this "replacement" procedure for other pairs of opposite panels to obtain very different sets $\mathcal{M}$ satisfying (*).

Remark 3.4 The implication (b)$\Longrightarrow$(a) is not true for weak non-thick buildings. Even if we require that

(**) $n(C,\mathcal{M}) > 1$ for every chamber $C \in \mathcal{C} \setminus \mathcal{M}$.

Indeed, let Δ be a finite non-thick building of type C_2. We may describe Δ as follows. Let $L_0, L_1 \ldots, L_m, M_0, M_1, \ldots, M_n$ be arbitrary distinct symbols, $m, n > 1$. These symbols contitute one type of panels of Δ. The other type is formed by the set $\{p_{i,j} \,|\, 0 \leq i \leq m, 0 \leq j \leq n\}$. The set of chambers is $\mathrm{Ch}(\Delta) = \{\{L_i, p_{i,j}\}, \{M_j, p_{i,j}\} \,|\, 0 \leq i \leq m, 0 \leq j \leq n\}$. Choose arbitrarily ℓ and k such that $0 \leq \ell < k \leq m$ and ℓ' and k' such that $0 \leq \ell' < k' \leq n$ and consider the set $\mathcal{M} = \{\{L_0, p_{0,0}\}, \{L_0, p_{0,1}\}, \{L_1, p_{1,0}\}, \{L_1, p_{1,1}\}, \{M_{\ell'}, p_{\ell,\ell'}\}, \{M_{\ell'}, p_{k,\ell'}\}, \{M_{k'}, p_{\ell,k'}\}, \{M_{k'}, p_{k,k'}\}\}$. Then $\mathcal{M}$ satisfies condition (b) and (**), and it is not the chamber set of an apartment whenever $\{k, \ell, k', \ell'\} \neq \{0, 1\}$. This can be checked very easily.

4 A Characterization of Adjacency by Opposition

We keep the notation of the previous section. In addition, we denote by $C^{\,\mathrm{op}}$ the set of chambers opposite the chamber $C \in \Delta$. It is our aim to show now the following proposition, which is obviously a reformulation of Theorem 1.2 stated in the Introduction.

Proposition 4.1 *In a thick twin building Δ, two different chambers C and D are adjacent if and only if there exists a third chamber E such that $n(X, \{C, D, E\}) \neq 1$ for all chambers $X \in \mathcal{C}$.*

Proof. If C and D are adjacent, then $C \cap D$ is a panel. In view of thickness, there exists a chamber E containing $C \cap D$ and different from C, D. By the basic properties of the opposition relation, any chamber $X \in \mathcal{C}$ is opposite 0, 2 or 3 members of the set $\{C, D, E\}$.

Let us now assume that the following is satisfied for some $E \in \mathcal{C}$:

(A) $n(X, \{C, D, E\}) \neq 1$ for all $X \in \mathcal{C}$.

Choose (using Fact 2.4) a twin apartment Σ containing C and D. Set $C' := \mathrm{op}_{\Sigma}(C)$ and $D' := \mathrm{op}_{\Sigma}(D)$. Since $C \neq D$, we also have (by Fact 2.1) $C' \neq D'$, $C' \,\overline{\mathrm{op}}\, D$ and $D' \,\overline{\mathrm{op}}\, C$. Hence (A) implies $E \,\mathrm{op}\, C'$ as well as $E \,\mathrm{op}\, D'$. Note that this is only possible if $E \neq D, C$ and if C', D' (respectively C, D, E) lie in the same "half" of Δ, say $C', D' \in \Delta_-$ (respectively $C, D, E \in \Delta_+$). In particular, there exists a minimal gallery $\gamma = (C = Z_1, Z_2, \ldots, Z_\ell = D)$ connecting C and D in Δ. Note that γ is contained in Σ.

Now assume that C and D are not adjacent, i.e., $\ell > 2$. We seek a contradiction. Set $C_i := Z_i$ for $i = 1, 2$. Since Δ is thick, there exists a chamber $C_3 \neq C_1, C_2$ which contains $C_1 \cap C_2$. Now let $\alpha_1, \alpha_2, \alpha_3$ and Σ_{13}, Σ_{23} be the twin roots and twin apartments, respectively, satisfying the properties (1)–(3) of Lemma 2.6 and guaranteed to exist by Lemma 2.6.

Since γ is minimal and $C = C_1 = Z_1 \notin \alpha_2$ but $C_2 = Z_2 \in \alpha_2$, we also have $D = Z_\ell \in \alpha_2$. In view of $\Sigma = \alpha_1 \cup \alpha_2$, we obtain $C' \in \alpha_2$ and $D' \in \alpha_1$. Furthermore, $C \in \alpha_1 \subseteq \Sigma_{13} = \alpha_1 \cup \alpha_3$ implies that $C'' := \mathrm{op}_{\Sigma_{13}}(C) \in \alpha_3$, and $D \in \alpha_2 \subseteq \Sigma_{23} = \alpha_2 \cup \alpha_3$ implies that $D'' := \mathrm{op}_{\Sigma_{23}}(D) \in \alpha_3$.

Note that C'', being opposite $C = C_1$ but not opposite C_3 (because C_3 and C'' lie in the same twin root α_3), is necessarily opposite C_2. Since both Z_2 and $Z_\ell = D$ belong to Σ_{23}, and $C_2 = Z_2 \neq D$ (because $\ell > 2$), we have $D'' = \mathrm{op}_{\Sigma_{23}}(D) \neq \mathrm{op}_{\Sigma_{23}}(C_2) = C''$. Hence C', C'', D', D'' are pairwise different, $C^{\,\mathrm{op}} \cap (\alpha_1 \cup \alpha_2 \cup \alpha_3) = \{C', C''\}$, and $D^{\,\mathrm{op}} \cap (\alpha_1 \cup \alpha_2 \cup \alpha_3) = \{D', D''\}$. Applying now our assumption (A) for all chambers $X \in \alpha_1 \cup \alpha_2 \cup \alpha_3$, we obtain $E^{\,\mathrm{op}} \cap (\alpha_1 \cup \alpha_2 \cup \alpha_3) = \{C', C'', D', D''\}$, implying $E^{\,\mathrm{op}} \cap (\alpha_1 \cup \alpha_3) = \{C'', D', D''\}$ and so $n(E, \Sigma_{13}) = 3$.

However, this contradicts Proposition 3.2 which says that $n(E, \Sigma_{13})$ is either 1 or even. Hence C and D must be adjacent. $\square$

As an immediate corollary to this proof, we have the following result:

Corollary 4.2 *Three distinct chambers C, D, E of a thick twin building Δ are pairwise adjacent if and only if $n(X, \{C, D, E\}) \neq 1$ for all chambers X of Δ.*

If $\mathcal{C}$ is the set of chambers of a thick twin building, and if $\mathcal{M} \subseteq \mathcal{C}$ is such that for any $X \in \mathcal{C}$, there exists a chamber $C \in \mathcal{M}$ satisfying $C \,\overline{\mathrm{op}}\, X$, then $|\mathcal{M}| > 2$ (cf. Remark 5.1 below). Hence Corollary 4.2 implies the following

Corollary 4.3 *A non-empty set $\mathcal{M}$ of chambers is the set of chambers through some panel of the thick twin building Δ if and only if for all chambers X of Δ, either all elements of $\mathcal{M}$ are not opposite X, or exactly one chamber in $\mathcal{M}$ is not opposite X.*

5 Maps defined on the Set of Chambers of a Twin Building

In the following, we will consider **thick** twin buildings $\Delta = (\Delta_+, \Delta_-, \delta^*)$, $\widetilde{\Delta} = (\widetilde{\Delta}_+, \widetilde{\Delta}_-, \widetilde{\delta}^*)$ and maps $\varphi : \mathcal{C} \to \widetilde{\mathcal{C}}$ (where $\widetilde{\mathcal{C}} := \mathrm{Ch}(\widetilde{\Delta}_+) \cup \mathrm{Ch}(\widetilde{\Delta}_-)$) which *preserve opposition and non-opposition*, i.e., $C \,\mathrm{op}\, D$ (in Δ) if and only if $\varphi(C) \,\mathrm{op}\, \varphi(D)$ (in $\widetilde{\Delta}$).

We will say that Δ is 2-*spherical* if every rank 2-residue in Δ is of spherical type.

Remark 5.1 Choose $C \in \mathcal{C}_+$ and assume $\varphi(C) \in \widetilde{\mathcal{C}}_+$ (otherwise we just change signs in $\widetilde{\Delta}$). Let $D \in \mathcal{C}_+$ be arbitrary. Then there exists a chamber $E \in \mathcal{C}_-$ which is opposite both C and D. Indeed, take any $E \in \mathcal{C}_-$ opposite C such that the codistance $\delta^*(E, D)$ is of minimal length. If $\delta^*(E, D)$ were not equal to 1, then the thickness of Δ and the twin building axioms would yield a neighbour F of E with $F \,\mathrm{op}\, C$ and the length of $\delta^*(F, D)$ strictly smaller than the length of $\delta^*(E, D)$.

Hence $\varphi(E) \,\mathrm{op}\, \varphi(C), \varphi(D)$, showing that $\varphi(E) \in \widetilde{\mathcal{C}}_-$ and $\varphi(D) \in \widetilde{\mathcal{C}}_+$. The same argument applied to E and $\mathcal{C}_-$ instead of C and $\mathcal{C}_+$ yields $\varphi(\mathcal{C}_-) \subseteq \widetilde{\mathcal{C}}_-$. Therefore we may and will assume $\varphi(\mathcal{C}_+) \subseteq \widetilde{\mathcal{C}}_+$ and $\varphi(\mathcal{C}_-) \subseteq \widetilde{\mathcal{C}}_-$. We shall denote the restriction of φ to $\mathcal{C}_+$ and $\mathcal{C}_-$ by φ_+ and φ_- respectively.

Our second main theorem has the following consequences:

Corollary 5.2 *Let Δ and $\widetilde{\Delta}$ be as above. Assume additionally that Δ is 2-spherical. Then any surjective map $\varphi : \mathcal{C} \to \widetilde{\mathcal{C}}$ preserving opposition and non-opposition extends (uniquely), after an appropriate adjustment of the numberings (i.e. the type functions) of $\widetilde{\Delta}_+$ and $\widetilde{\Delta}_-$, to an isomorphism of twin buildings between Δ and $\widetilde{\Delta}$.*

Proof. First we claim that φ is injective. Indeed, given two distinct chambers $C, D \in \mathcal{C}$, we can find (in any twin apartment containing both C and D) a chamber $E \in \mathcal{C}$ opposite C but not opposite D. Then $\varphi(E)$ is opposite $\varphi(C)$ but not opposite $\varphi(D)$, implying $\varphi(C) \neq \varphi(D)$. Hence the maps $\varphi_\epsilon : \mathcal{C}_\epsilon \to \widetilde{\mathcal{C}}_\epsilon$, $\epsilon \in \{+,-\}$, introduced in Remark 5.1, are bijections. Now Proposition 4.1 implies that φ_+ and φ_- as well as their inverses preserve adjacency. Therefore Theorem 3.21 in Tits [1974] yields that φ_+ and φ_- (uniquely) extend to simplicial isomorphisms $\phi_\epsilon : \Delta_\epsilon \to \widetilde{\Delta}_\epsilon$, for $\epsilon \in \{+,-\}$. In particular, all these buildings are of the same type, and we may assume that the type functions on $\widetilde{\Delta}_+, \widetilde{\Delta}_-$ are such that ϕ_+ and ϕ_- are type-preserving. This implies that the Weyl distances are also preserved by ϕ_+ and by ϕ_-. Since the opposition relation is of course still preserved by $\phi = (\phi_+, \phi_-)$, the characterization of the Weyl codistance given in Remark 3 of Abramenko [1996] finally shows that $\delta^*(C, D) = \widetilde{\delta}^*(\phi_+(C), \phi_-(D))$, for all $C \in \mathcal{C}_+$ and all $D \in \mathcal{C}_-$. Hence ϕ is indeed an isomorphism of twin buildings. $\square$

If Δ is not 2-spherical, then Theorem 3.21 in Tits [1974] cannot be applied any longer. Also, if φ is not surjective, our characterizations using the opposition relation do not help any longer, because our results involve *all* chambers of a twin building.

Finally, the thickness assumption in Corollary 5.2 is necessary as well. Indeed, consider two thin twin buildings $\Sigma = (\Sigma_+, \Sigma_-, \delta^*)$, $\widetilde{\Sigma} = (\widetilde{\Sigma}_+, \widetilde{\Sigma}_-, \widetilde{\delta}^*)$, not necessarily of the same type, such that the cardinalities of $\mathrm{Ch}(\Sigma_+)$ and $\mathrm{Ch}(\widetilde{\Sigma}_+)$ (and hence also of $\mathrm{Ch}(\Sigma_-)$ and $\mathrm{Ch}(\widetilde{\Sigma}_-)$) are equal. Then any bijection between $\mathrm{Ch}(\Sigma_+)$ and $\mathrm{Ch}(\widetilde{\Sigma}_+)$ can be extended to a bijection between $\mathrm{Ch}(\Sigma)$ and $\mathrm{Ch}(\widetilde{\Sigma})$ preserving opposition and non-opposition.

Corollary 5.3 *If Δ is a thick 2-spherical twin building, then the Weyl distances and the Weyl codistance are uniquely determined (up to the notation of types) by the opposition relation on the set of chambers.*

Proof. Let $((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta^*)$ and $((\mathcal{C}_+, \widetilde{\delta}_+), (\mathcal{C}_-, \widetilde{\delta}_-), \widetilde{\delta}^*)$ be two "realizations" of the same opposition relation "op" on Δ. Then Corollary 5.2 shows that the identity id: $\mathcal{C} \to \mathcal{C}$ extends to a twin building isomorphism. This implies $\delta_+ \equiv \widetilde{\delta}_+$, $\delta_- \equiv \widetilde{\delta}_-$ and $\delta^* \equiv \widetilde{\delta}^*$ (if the types in $\widetilde{\Delta}$ are denoted appropriately). $\square$

In case of twin buildings of the same type, we do not know in general whether the surjectivity assumption in Corollary 5.2 can be deleted in order to conclude that φ extends to an isomorphism of twin buildings between Δ and a uniquely defined subbuilding of $\widetilde{\Delta}$.

That appears to be the case for buildings of type A_2 (projective planes), but if the twin buildings do not have the same type, then there are counterexamples. Indeed, consider the building associated with a symplectic quadrangle $\mathsf{W}(\mathbb{K})$ (i.e., the natural geometry of the symplectic group $\mathrm{Sp}_4(\mathbb{K})$) over a perfect field $\mathbb{K}$ of characteristic 2 admitting an automorphism σ whose square is the Frobenius automorphism. Then the building associated with the double $2\mathsf{W}(\mathbb{K})$ of it (in the sense of Van Maldeghem [1998]; this just means the non-thick generalized octagon with point set the panels of $\mathsf{W}(\mathbb{K})$ (as a building), with line set the set of chambers of $\mathsf{W}(\mathbb{K})$ and with inclusion as incidence relation) can be viewed as a subbuilding of the thick generalized octagon $\mathsf{O}(\mathbb{K},\sigma)$ associated with the group ${}^2F_4(\mathbb{K},\sigma)$. The map sending the chamber $\{p, L\}$ (with p a point of $\mathsf{W}(\mathbb{K})$ and L a line) of $\mathsf{W}(\mathbb{K})$ to the chamber $\{p, \{p, L\}\}$ of $\mathsf{O}(\mathbb{K},\sigma)$ preserves opposition and non-opposition of chambers, but it can of course never preserve adjacency.

For finite buildings, however, we have the following interesting result.

Corollary 5.4 *Let Δ be a thick, finite (hence spherical) building, and let $\varphi : \mathrm{Ch}(\Delta) \to \mathrm{Ch}(\Delta)$ be a map preserving opposition and non-opposition. Then φ extends uniquely to an automorphism of Δ.*

Proof. As remarked in the beginning of the proof of Corollary 5.2, φ is injective and hence automatically surjective in the present situation. Now we apply, as in the proof of Corollary 5.2, our Proposition 4.1 together with Theorem 3.21 in Tits [1974] in order to see that φ extends (uniquely) to a simplicial automorphism of Δ. Note that automorphisms are just simplicial automorphisms in the category of spherical buildings so that we need not consider types here. $\square$

Our results also imply a very elegant (and unexpected) characterization of involutions in spherical buildings.

Corollary 5.5 *Let Δ be a thick spherical building. A map $\theta : \mathrm{Ch}(\Delta) \to \mathrm{Ch}(\Delta)$ satisfies*

(OPP) $C \,\mathrm{op}\, \theta(D) \Rightarrow \theta(C) \,\mathrm{op}\, D$, *for all chambers $C, D \in \mathrm{Ch}(\Delta)$,*

if and only if θ extends uniquely to an involutive automorphism of Δ.

Proof. If θ extends to an involutive automorphism of Δ, then it preserves opposition and hence we have $C \,\mathrm{op}\, \theta(D) \Rightarrow \theta(C) \,\mathrm{op}\, \theta^2(D) = D$.

We now assume that θ is a transformation of $\mathrm{Ch}(\Delta)$ satisfying (OPP).

First we claim that θ maps distinct adjacent chambers to distinct adjacent chambers. To prove this, let C and D be adjacent chambers in Δ, with $C \neq D$. Suppose first that $\theta(C) = \theta(D)$. In view of the thickness assumption, there is some chamber E adjacent to both C and D, and distinct form both C and D. Assume by way of contradiction that $\theta(E) \neq \theta(C)$. Then there exists a chamber F opposite $\theta(E)$, but not opposite $\theta(C) = \theta(D)$. Applying (OPP), we deduce that $\theta(F)$ is opposite E but not opposite both C and D, a contradiction. Hence $\theta(C) = \theta(E)$, for every chamber containing the panel $C \cap D$. Now consider any chamber F' opposite $\theta(C)$. Then (OPP) implies that $\theta(F')$ is opposite every chamber containing the panel $C \cap D$, a contradiction. We conclude $\theta(C) \neq \theta(D)$.

Let E be as above. Suppose some chamber F'' is opposite exactly one member of $\{\theta(C), \theta(D), \theta(E)\}$. Then, by (OPP), $\theta(F'')$ is opposite exactly one member of $\{C, D, E\}$, a contradiction. Now it follows from Proposition 4.1 that $\theta(C)$ and $\theta(D)$ are adjacent.

Let C and D again be adjacent chambers of Δ. We claim that θ induces a bijection from the set of chambers containing $C \cap D$ to the set of chambers containing $\theta(C) \cap \theta(D)$. Indeed, by the previous paragraph, θ already induces an injection between these two sets. Now suppose that E' is some chamber containing $\theta(C) \cap \theta(D)$ and which is not the image of some chamber containing $C \cap D$. We select a chamber F of Δ opposite $\theta(C)$, but not opposite E'. Then F is opposite $\theta(E)$, for all $E \supseteq (C \cap D)$. Hence, by (OPP), $\theta(F)$ is opposite all chambers containing the panel $C \cap D$, a contradiction. Our claim follows.

Now suppose C is some fixed chamber of Δ, and suppose by way of contradiction that there exists a panel $p \subseteq \theta(C)$ such that no chamber adjacent to C is mapped onto a chamber containing p (except for C). Let C' be any chamber opposite $\theta(C)$. Then the apartment $\Sigma\{\theta(C), C'\}$ contains a unique chamber F' opposite the unique chamber F of $\Sigma\{\theta(C), C'\}$, with F containing p and $F \neq \theta(C)$. Hence F' is opposite no chamber $\theta(X)$, with X adjacent to $\theta(C)$. By (OPP), $\theta(F')$ is not opposite any chamber adjacent to C. However, we already showed that $\theta(F')$ is adjacent to $\theta(C')$, which is opposite C. But this implies that $\theta(F')$ must be opposite at least one neighbour of C, a contradiction. Hence we have shown that, if a chamber D belongs to the image of θ, then every chamber adjacent to D does. An obvious induction argument on the gallery distance of an arbitrary chamber to D now implies that θ is surjective.

Now we claim that θ is also injective. Suppose that C and D are two distinct chambers of Δ. Then there exists a chamber opposite C, but not opposite D, and by the surjectivity of θ, we may write this chamber as $\theta(F)$. Now (OPP) yields that F is opposite $\theta(C)$, but not opposite $\theta(D)$, implying $\theta(C) \neq \theta(D)$.

So θ is a bijection, mapping adjacent chambers onto adjacent chambers. Since θ^{-1} also satisfies (OPP), it preserves adjacency as well. Applying again (as in the proof of Corollary 5.2) Theorem 3.21 in Tits [1974], we conclude that θ extends uniquely to an automorphism of Δ. Now fix a chamber C, then we have $X \operatorname{op} \theta(C) \Leftrightarrow \theta(X) \operatorname{op} \theta^2(C)$, for all $X \in \mathrm{Ch}(\Delta)$ (because θ is an automorphism), but (OPP) implies that $X \operatorname{op} \theta(C) \Leftrightarrow \theta(X) \operatorname{op} C$. Hence $C^{\mathrm{op}} = (\theta^2(C))^{\mathrm{op}}$ by the surjectivity of θ. We infer $C = \theta^2(C)$, so θ is involutive.

The corollary is completely proved. $\square$

Remark 5.6 One might try to characterize involutions in 2-spherical twin buildings Δ similarly as in the previous corollary. However, especially in the case where the map θ interchanges the two components of Δ, condition (OPP) does not seem to be the right requirement. If θ preserves the components of $\Delta = (\Delta_+, \Delta_-, \delta^*)$, then we have the following result, which has, up to the first and the last paragraph, a proof that is identical to the proof of Corollary 5.5 (and we leave it to the reader as an easy exercise to adjust these paragraphs).

Let $\Delta = (\Delta_+, \Delta_-, \delta^*)$ be a thick twin building. A map $\theta = (\theta_+, \theta_-)$ with $\theta_\epsilon : \mathrm{Ch}(\Delta_\epsilon) \to \mathrm{Ch}(\Delta_\epsilon)$, $\epsilon \in \{+, -\}$, satisfies

(OPP) $C \operatorname{op} \theta(D) \Rightarrow \theta(C) \operatorname{op} D$, *for all chambers $C, D \in \mathrm{Ch}(\Delta)$,*

if and only if θ_+ and θ_- are adjacency-preserving bijections and $(\theta_+, \theta_-^{-1})$ preserves opposition and non-opposition.

Remark 5.7 There is an alternative and shorter argument to deduce surjectivity in the proof of Corollary 5.5, if one uses the finite rank of the building Δ. However, we have chosen to write down the longer argument because our proof is now also valid for appropriately defined (generalizations of) spherical buildings of infinite rank (infinite dimensional projective spaces and polar spaces).

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