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# Some New Upper Bounds for the Size of Partial Ovoids in Slim Generalized Polygons and Generalized Hexagons of Order $(s, s^3)$

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**Abstract.** From an elementary observation, we derive some upper bounds for the number of mutually opposite points in the classical generalized polygons having 3 points on each line. In particular, it follows that the Ree-Tits generalized octagon $\mathsf{O}(2)$ of order $(2, 4)$ has no ovoids. Also, we deduce from another observation a similar upper bound in any generalized hexagon of order $(s, s^3)$.

**Keywords:** generalized polygon, generalized hexagon, ovoid, partial ovoid, projective embedding

## 1. Introduction and statement of the main result

A *generalized polygon* $\Gamma$ of order $(s, t)$ is a rank 2 point-line geometry whose incidence graph has diameter $n$ and girth $2n$, for some $n \in \mathbb{N}\backslash\{0, 1\}$ (in which case the generalized polygon is also called a *generalized n-gon*), each vertex corresponding to a point has valency $t+1$ and each vertex corresponding to a line has valency $s+1$. If $s, t > 1$, then this geometry is usually called *thick*. Each non-thick generalized polygon can be obtained from a thick one, and so one usually only considers thick generalized polygons. If $s = 2$ and $t > 1$, then we call the generalized polygon *slim*. Generalized polygons were introduced by Tits [8]. More information is gathered in the monograph [9], to which we refer for a general introduction and basic properties. Here, we recall some notation. For an element $x$ of $\Gamma$, and a natural number $i$, we denote by $\Gamma_i(x)$ the set of elements of $\Gamma$ at distance $i$ from $x$ in the incidence graph of $\Gamma$. The distance function in that incidence graph is denoted by $\delta$. If two elements $x$ and $y$ are not at distance $n$, then there exists a unique element $\mathrm{proj}_y x$ incident with $y$ and at distance $\delta(x, y) - 1$ from $x$. We call that element the *projection of x onto y*. Also recall that the *dual* of $\Gamma$ is obtained by interchanging the words "point" and "line". The dual of a generalized $n$-gon is obviously again a generalized $n$-gon. Two elements at distance $n$ are called *opposite*. Now we call a set of mutually opposite points a *partial ovoid*. An *ovoid* $\mathcal{O}$ in $\Gamma$ is a partial ovoid such that every element of $\Gamma$ is at distance at most $n/2$ from some

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element of $\mathcal{O}$. Clearly ovoids in this sense only exist in generalized $n$-gons with $n$ even. In the finite case, this implies that $n = 4, 6, 8$ [4].

Now let $\Gamma$ be a finite slim generalized $n$-gon, $n \in \{4, 6, 8\}$. For $n = 4$, this means that $\Gamma$ has order $(2, 2)$ or $(2, 4)$ and there exist unique examples in each case (see [6](6.1)), denoted by $\mathsf{W}(2)$ (a *symplectic quadrangle*) and $\mathsf{Q}(5, 2)$ (an *orthogonal quadrangle*) respectively. For $n = 6$, $\Gamma$ has order $(2, 2)$ or $(2, 8)$ and there exist exactly two examples in case $(2, 2)$ (each one dual to the other) and one example in the case $(2, 8)$ (see [3]), and these are denoted by $\mathsf{H}(2)$ (a *split Cayley hexagon*), $\mathsf{H}(2)^D$ (the dual of the previous one) and $\mathsf{T}(2, 8)$ (a *twisted triality hexagon*). For $n = 8$, the order is necessarily $(2, 4)$, and there is such an example (the Ree-Tits octagon $\mathsf{O}(2)$), but it is not yet known to be unique.

The first aim of the present paper is to derive some upper bounds for the number of points of a partial ovoid in $\Gamma$. A trivial upper bound is the number of points of an ovoid. Hence, if $\Gamma$ admits ovoids, then the trivial upper bound can be reached and nothing else can be said. This is the case for $\mathsf{W}(2)$ (ovoids have 5 points) and for $\mathsf{H}(2)^D$ (ovoids have 9 points) and this is well known (see Chapter 7 of [9] for more details).

Our first main result reads as follows.

**Theorem 1** *If $\Gamma$ contains a partial ovoid, with $\Gamma \in \{\mathsf{Q}(5, 2), \mathsf{H}(2), \mathsf{T}(2, 8), \mathsf{O}(2)\}$, then the number $k$ of points of that partial ovoid satisfies*
(i) $k \le 7$ *if* $\Gamma = \mathsf{Q}(5, 2)$;
(ii) $k \le 7$ *if* $\Gamma = \mathsf{H}(2)$;
(iii) $k \le 27$ *if* $\Gamma = \mathsf{T}(2, 8)$ (*but see Theorem* 2);
(iv) $k \le 27$ *if* $\Gamma = \mathsf{O}(2)$.

Note that the upper bound in (i) is worse than the upper bound $k \le 6$ following from 2.7.1 of [6]. Regarding cases (ii), (iii) and (iv), the previously known upper bounds were respectively $k \le 8$, $k \le 43$ and $k \le 65$. The first one follows from the fact that there are no ovoids in $\mathsf{H}(2)$ (see [7]); the second one follows from an elementary counting argument (see 7.2.3 of [9] or below); the third one follows from the fact that an ovoid in $\mathsf{O}(2)$ must contain 65 points (see also 7.2.3 of [9]). Hence cases (iii) and (iv) of our Main Result are drastic improvements of the earlier bounds (and there is a further improvement of (iii) below). Also, the nonexistence of ovoids in $\mathsf{O}(2)$ is the first result on existence of ovoids in general in finite generalized octagons. It suggests the conjecture that no finite Ree-Tits octagon has an ovoid.

Now let $\Gamma$ be a generalized hexagon of order $(s, s^3)$. Examples are the dual twisted triality hexagons $\mathsf{T}(s, s^3)$; see [8] (we use the notation of [9]; in the literature, this hexagon is sometimes called the ${}^3D_4(s)$-hexagon). It is well known that $\Gamma$ cannot have an ovoid (see 7.2.4 of [9]), and that an upper bound for the maximal number of mutually opposite points in $\Gamma$ is given by the largest integer smaller than

$$\frac{(s+1)(s^8+s^4+1)}{s^4+s+1} = s^5 + s^4 - s^2 - s + \frac{s^3+2s^2+2s+1}{s^4+s+1},$$

which is $s^5 + s^4 - s^2 - s$ if $s > 2$. For $s = 2$, this is 43. Our second main results reads as follows.

**Theorem 2** *Let* $\Gamma$ *be a generalized hexagon of order* $(s, s^3)$, *for some integer* $s > 1$. *Then a partial ovoid has at most* $s^5 - s^3 + s - 1$ *points. In particular, putting* $s = 2$, *the hexagon* $\mathsf{T}(2, 8)$ *has no partial ovoids of size* $k \geq 26$, *thus improving the bound of Theorem* 1(iii) *by* 2.

## 2. Proof of Theorem 1

The crucial observation is contained in the next lemma. We first need a definition. A generalized polygon $\Gamma$ is *fully embedded* in a projective space $\mathbf{PG}(d, \mathbb{K})$ if the point set of $\Gamma$ is a subset of the point set of $\mathbf{PG}(d, \mathbb{K})$, and if for each line $L$ of $\Gamma$, the set of points of $\Gamma$ incident with $L$ forms a (complete) line of $\mathbf{PG}(d, \mathbb{K})$.

**Lemma 1** *Let* $\Gamma$ *be a slim polygon fully embedded in the finite projective space* $\mathbf{PG}(d, 2)$ *over the Galois field* $\mathbf{GF}(2)$. *Suppose that there is a symmetric bilinear form* $B$ *on the point set of* $\mathbf{PG}(d, 2)$ (*with values in* $\mathbf{GF}(2)$) *with the property that*, *for all pairs* $(x, y)$ *of points of* $\Gamma$, $B(x, y) \neq 0$ *whenever* $x$ *and* $y$ *are opposite. Then for every partial ovoid* $\mathcal{C}$ *of* $\Gamma$ *we have* $|\mathcal{C}| \leq d + 1$ (*if* $d$ *is even*), *or* $|\mathcal{C}| \leq d + 2$ (*if* $d$ *is odd*).

**Proof:** Put $k = |\mathcal{C}|$. Two distinct points $x$ and $y$ of $\mathcal{C}$ satisfy $B(x, y) = 1$. Hence the matrix $M$ with lines and columns indexed by the points of $\mathcal{C}$ and with $(x, y)$-entry equal to $B(x, y)$ is equal to $J - I$, where $J$ is the all-one matrix, and $I$ is the identity matrix of the appropriate size (namely, $k \times k$). Since $M$ can be written as $XAX^{\mathrm{t}}$, with $X$ the matrix whose lines are indexed by the points of $\mathcal{C}$ and line $x$ is just the $(d + 1)$-coordinate-tuple of $x$, with $A$ the matrix of the bilinear form $B$, and with $X^{\mathrm{t}}$ the transposed of $X$, we see that the rank of $M$ is at most $d + 1$. In particular, if $k > d + 1$, then $\det M = 0$. But it is readily checked that $\det M = \det(J - I) = 1$, whenever $k$ is even. Hence, if $k > d + 1$, then $k$ must be odd. Since every subset of $\mathcal{C}$ is again a partial ovoid, this implies that, if $k > d + 1$, only $d + 2$ can be odd and in that case $k = d + 2$. The lemma is proved. □

Now suppose $\Gamma = \mathsf{Q}(5, 2)$. Then, as an elliptic quadric in $\mathbf{PG}(5, 2)$, $\Gamma$ is fully embedded in $\mathbf{PG}(5, 2)$ and there is a natural bilinear form (namely, the one defining the quadric) $B$ with $B(x, y) = 0$ if and only if $x$ and $y$ are collinear in $\Gamma$. This proves (i) of the Main Result.

Next, suppose that $\Gamma = \mathsf{H}(2)$. Then $\Gamma$ is fully embedded in $\mathbf{PG}(5, 2)$. In fact, all points of $\mathbf{PG}(5, 2)$ are points of $\Gamma$, and the lines of $\Gamma$ are certain lines of a symplectic polarity in $\mathbf{PG}(5, 2)$ (see for instance 2.4.14 of [9]). Moreover, the bilinear form associated with that symplectic polarity has the required property to apply Lemma 1 (see the same reference). This shows (ii) of the Main Result.

Now let $\Gamma$ be equal to $\mathsf{O}(2)$. Then $\Gamma$ can be viewed as a sub building of a building $\Delta$ of type $F_4$, having itself 3 points per line. In fact, it is well known that the point set of $\Gamma$ is the set of absolute points of any polarity in $\Delta$ (a polarity is here a type reversing automorphism of order 2), and it follows easily from e.g. 2.5.2 of [9] that two points of $\Gamma$ are opposite in $\Gamma$ if and only if they are opposite in $\Delta$ (with the usual notion of opposition in buildings). Now, the point-line space of type $F_{4,1}$ related to $\Delta$ admits an embedding in $\mathbf{PG}(25, 2)$, and there is a symmetric bilinear form $B$ with values in $\mathbf{GF}(2)$ and such that $B(x, y) = 1$ (for points $x, y$ of $\Delta$) if and only if $x$ and $y$ are opposite in $\Delta$ (see 5.3 of [2]; the bilinear form is denoted there by $(\cdot, \cdot)$). Hence, it follows that $\Gamma$ is embedded in $\mathbf{PG}(25, 2)$ with appropriate bilinear form $B$. This shows (iv) of the Main Result.

Finally, we show (iii) of the Main Result. The universal embedding dimension of $\mathsf{T}(2, 8)$ is equal to 28, i.e., $\mathsf{T}(2, 8)$ can be fully embedded in $\mathbf{PG}(27, 2)$ and every other embedding arises from that one by projecting down (see [5]). But we are looking for an embedding in $\mathbf{PG}(25, 2)$ and moreover, we want a suitable bilinear form. We will establish this in full generality, that is, we will describe a full embedding of $\mathsf{T}(q, q^3)$. Since this can be of some interest on its own, we will do this in a separate section. This is also our motivation for proving (iii), although Theorem 2 for $s = 2$ gives a better result.

## 3. A full embedding of $\mathsf{T}(q, q^3)$

First, we need a description of $\mathsf{T}(q, q^3)$. We use the original description of Tits [8]. Explicit coordinates in $\mathbf{PG}(7, q^3)$ of the points and lines of the dual $\mathsf{T}(q^3, q)$ are calculated in 3.5.8 of [9]. We are especially interested in the lines of $\mathsf{T}(q^3, q)$, since these are the points of $\mathsf{T}(q, q^3)$. We list the lines and label them as in 3.5.8 of [9] (see Table 1, where $k, k', k'', l, l' \in \mathbf{GF}(q)$ and $a, a', b, b' \in \mathbf{GF}(q^3)$, and where $\sigma : \mathbf{GF}(q^3) \to \mathbf{GF}(q^3) : x \mapsto x^q$); the points are obtained by taking all the points of $\mathbf{PG}(7, q^3)$ on these lines. Note that the points of $\mathsf{T}(q^3, q)$ are contained in the quadric $Q^+(7, q^3)$ with equation $X_0X_4 + X_1X_5 + X_2X_6 + X_3X_7 = 0$.

It is easy to see that a line is opposite $[\infty]$ if and only if it is labeled $[k, b, k', b', k'']$, for some $k, k', k'' \in \mathbf{GF}(q)$ and $b, b' \in \mathbf{GF}(q^3)$. It is now an elementary exercise to calculate the Grassmannian coordinates of the lines of $\mathsf{T}(q^3, q)$. Without explicitly writing down the result of these calculations, we notice that the Grassmannian coordinates $(x_{0,0}, x_{0,1}, x_{0,2}, \ldots, x_{5,7}, x_{6,7})$ of an arbitrary line of $\mathsf{T}(q^3, q)$ satisfy, up to a scalar multiple and up to changing the sign of some coordinates, the following conditions:

(a) $x_{0,5}, x_{0,6}, x_{1,4}, x_{1,6}, x_{2,4}, x_{2,5} \in \mathbf{GF}(q)$,
(b) $x_{i,3} = x_{i,7}^{\sigma} = x_{j,k}^{\sigma^2}$, $i = 0, 1, 2$, $\{i + 4, j, k\} = \{4, 5, 6\}$, $j < k$,

*Table 1.* Coordinatization of $\mathsf{T}(q^3, q)$.

| Labels in $\mathsf{T}(q, q^3)$ | Coordinates in $\mathbf{PG}(7, q^3)$ |
|---|---|
| $[\infty]$ | $\langle(1, 0, 0, 0; 0, 0, 0, 0), (0, 0, 0, 0; 0, 0, 1, 0)\rangle$ |
| $[k]$ | $\langle(1, 0, 0, 0; 0, 0, 0, 0), (0, 0, 0, 0; 0, 1, -k, 0)\rangle$ |
| $[a, l]$ | $\langle(a, 0, 0, 0; 0, 0, 1, 0), (-l, 1, 0, a^{\sigma}; 0, a^{\sigma+\sigma^2}, 0, -a^{\sigma^2})\rangle$ |
| $[k, b, k']$ | $\langle(b, 0, 0, 0; 0, 1, -k, 0), (k', k, 1, -b^{\sigma}; 0, 0, b^{\sigma+\sigma^2}, b^{\sigma^2})\rangle$ |
| $[a, l, a', l']$ | $\langle(-l - aa', 1, 0, a^{\sigma}; 0, a^{\sigma+\sigma^2}, -a', -a^{\sigma^2}),$ $(a'^{\sigma+\sigma^2} - al', 0, -a, a'^{\sigma^2}; 1, l + (aa')^{\sigma} + (aa')^{\sigma^2}, -l', -a'^{\sigma})\rangle$ |
| $[k, b, k', b', k'']$ | $\langle(k' + bb', k, 1, -b^{\sigma}; 0, b', b^{\sigma+\sigma^2} - b'k, b^{\sigma^2}),$ $(b'^{\sigma+\sigma^2} + k''b, -b, 0, b'^{\sigma^2}; 1, k'', -kk'' - k' - (bb')^{\sigma} - (bb')^{\sigma^2}, -b'^{\sigma})\rangle$ |

(c) $x_{i,7} = x_{3,i}^{\sigma} = x_{j,k}^{\sigma^2}$, $i = 4, 5, 6$, $\{i-4, j, k\} = \{0, 1, 2\}$, $j < k$,
(d) $x_{0,4} - x_{1,5} \in \mathbf{GF}(q)$, $x_{2,6} + x_{3,7} = (x_{2,6} - x_{3,7})^{\sigma} = (x_{0,4} + x_{1,5})^{\sigma^2}$,
(e) (if $q$ is even) $x_{0,4} + x_{i,i+4} \in \mathbf{GF}(q)$, $i \in \{1, 2, 3\}$, and $x_{0,4} + x_{1,5} + x_{2,6} + x_{3,7} = 0$.

Moreover, it is easy to check that two lines of $Q^+(7, q^3)$ with Grassmannian coordinates $(x_{0,1}, x_{0,2}, \ldots, x_{6,7})$ and $(y_{0,1}, y_{0,2}, \ldots, y_{6,7})$, respectively, are opposite if and only if

$$\sum_{i<j\le 3} x_{i,j} y_{i+4,j+4} - \sum_{i\le 3<j} x_{i,j} y_{j-4,i+4} + \sum_{4\le i<j} x_{i,j} y_{i-4,j-4} \neq 0 \qquad (1)$$

Since two lines of $\mathsf{T}(q^3, q)$ are opposite in $\mathsf{T}(q^3, q)$ if and only if they are opposite on $Q^+(7, q^3)$ (as a building; or just think about opposition as being at maximal distance), the left hand side of Eq. (1) defines a bilinear form $B$ on the point set of $\mathsf{T}(q, q^3)$ vanishing on pairs of non-opposite points. Moreover, it is readily checked that coordinates can be chosen such that $B(x, y) \in \mathbf{GF}(q)$ for all pairs $x, y$ of points of $\mathsf{T}(q, q^3)$. Now let $q$ be odd. We choose two fixed elements $u, v \in \mathbf{GF}(q^3)$ such that the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ u & u^{\sigma} & u^{\sigma^2} \\ v & v^{\sigma} & v^{\sigma^2} \end{pmatrix}$$

is non-singular (this is always possible; it suffices to choose $\frac{u^{\sigma}-u}{v^{\sigma}-v}$ outside $\mathbf{GF}(q)$, which can be done because the $\mathbf{GF}(q)$-linear map $u \mapsto \frac{u^{\sigma}-u}{v^{\sigma}-v}$, for fixed $v$, has a 1-dimensional kernel, and hence a 2-dimensional image). The coordinate changes

$$\begin{cases} x'_{j,k} = x_{j,k} + x_{i,7} + x_{i,3}, \\ x'_{i,7} = u x_{j,k} + u^{\sigma} x_{i,7} + u^{\sigma^2} x_{i,3}, \\ x'_{i,3} = v x_{j,k} + v^{\sigma} x_{i,7} + v^{\sigma^2} x_{i,3}, \end{cases}$$

with $i, j, k$ as in (b) above, together with the analogous coordinate changes for the situations in (c) and (d) above, and also with $x_{0,4}$ substituted by $x_{0,4} - x_{1,5}$, embeds $\mathsf{T}(q, q^3)$ into $\mathbf{PG}(27, q)$, and moreover, the bilinear form $B$ has all its coefficients in $\mathbf{GF}(q)$ in the new coordinates. For a given point $x$ of $\mathsf{T}(q, q^3)$, the set of points $y$ of $\mathsf{T}(q, q^3)$ such that $B(x, y) = 0$ is exactly the set of points of $\mathsf{T}(q, q^3)$ not opposite $x$. One can check that this set always generates a hyperplane in $\mathbf{PG}(27, q)$ (for instance, if $x$ corresponds to the line $[\infty]$ of $\mathsf{T}(q^3, q)$ above, then this hyperplane has equation $X_{2,4} = 0$), which we call the *tangent hyperplane* at $x$. It can be checked that the set of points of $\mathsf{T}(q, q^3)$ actually generates $\mathbf{PG}(27, q)$ and that no point of $\mathbf{PG}(27, q)$ is contained in all tangent hyperplanes.

Now suppose that $q$ is even. We can still perform the coordinate changes related to (b) and (c) as above. Moreover, we can put $x'_{i,i+4} = x_{i,i+4} + x_{0,4}$, $i \in \{1, 2, 3\}$. Now, it is clear that the points of $\mathsf{T}(q, q^3)$ are contained in the hyperplane $H$ with old equation $X_{0,4} + X_{1,5} + X_{2,6} + X_{3,7} = 0$ (in fact, all points corresponding to the lines of the quadric $Q^+(7, q^3)$ are contained in that hyperplane as can be seen immediately from the bilinear form corresponding to

$Q^+(7, q^3)$). Moreover, the point $w$ with old coordinates $x_{0,4} = x_{1,5} = x_{2,6} = x_{3,7} = 1$, and all other coordinates equal to 0, lies in $H$ and in every tangent hyperplane. Hence we can project from $w$ onto the $\mathbf{PG}(25, q^3) \subseteq H$ with (old) equations $X_{0,4} = X_{1,5} + X_{2,6} + X_{3,7} = 0$, and we obtain a full embedding of $\mathsf{T}(q, q^3)$ into $\mathbf{PG}(25, q)$, obtained from $\mathbf{PG}(25, q^3)$ by restricting coordinates to $\mathbf{GF}(q)$. The bilinear form $B'$, obtained from $B$ by the same coordinate changes and projection as above, has its values in $\mathbf{GF}(q)$ when restricted to $\mathbf{PG}(25, q)$ (indeed, the effect of the projection is just the deletion of the terms with $X_{0,4}$ and $Y_{0,4}$; but after the coordinate changes and the restriction to $H$, there are none). Putting $q = 2$, (iii) of the Main Result follows.

**Remark** The previous construction of the full embedding of $\mathsf{T}(q, q^3)$ in $\mathbf{PG}(27, q)$ (for $q$ odd) or $\mathbf{PG}(25, q)$ (for $q$ even) provides an elementary way of seeing the group ${}^3D_4(q)$ included in an orthogonal group defined over $\mathbf{GF}(q)$. Also, the finiteness assumption is not essential, and everything works in the infinite case as well (treating characteristic 0 in the same way as odd characteristic).

## 4. Proof of Theorem 2

The crucial observation here is contained in Lemma 2.

**Lemma 2** *Let $\Gamma$ be a finite generalized hexagon of order $(s, s^3)$ and define the matrix $E$ whose rows and columns are indexed by the points of $\Gamma$ as follows. The $(x, y)$-entry of $E$ is equal to $(-s)^{3-d}$, where $d$ is the distance between the points $x$ and $y$ in the collinearity graph of $\Gamma$. Then the rank of $E$ is equal to $s^5 - s^3 + s$.*

**Proof:** The matrix $E$ is nothing other than a scalar multiple of the minimal idempotent of the Bose-Mesner algebra of the collinearity graph (viewed as an association scheme) corresponding to the eigenvalue $-s^3 - 1$, and the lemma follows from 2.2 of [1]. □

Lemma 2 can be stated in general for any finite generalized polygon. The rank of $E$ is then the multiplicity of the eigenvalue $-t - 1$ of the adjacency matrix of the collinearity graph of the generalized polygon in question. But only in the case of generalized hexagons of order $(s, s^3)$ will this observation give new bounds.

Now let $\mathcal{C}$ be a partial ovoid in the generalized hexagon $\Gamma$ of order $(s, s^3)$ and put $|\mathcal{C}| = k$. Suppose that $k > s^5 - s^3 + s - 1$. The sub matrix $D$ of $E$ indexed by the elements of $\mathcal{C}$ has $-s^3$ on the diagonal and everywhere else 1. Hence it is singular if and only if $s = -1$ or $s^3 = k - 1$, clearly both contradictions. Hence $D$ is nonsingular and hence its size cannot exceed the rank of $E$. This implies by Lemma 2 that $k = s^5 - s^3 + s = \mathrm{rk}E$. Since $\mathcal{C}$ is not an ovoid, there exists a point $p$ of $\Gamma$ not collinear with any point of $\mathcal{C}$. We consider the (symmetric) sub matrix $D'$ of $E$ indexed by $\mathcal{C} \cup \{p\}$. Define the natural numbers $\ell_2$ and $\ell_3$ as the number of points of $\mathcal{C}$ at distance 2 and 3, respectively, of $p$ in the collinearity graph of $\Gamma$. If we order the rows and columns of $D'$ such that the points of $\mathcal{C}$ not opposite $p$ correspond to the first $\ell_2$ rows and columns, the points of $\mathcal{C}$ opposite $p$ correspond to the next $\ell_3$ rows and columns, and the last row and last column correspond to $p$, then we perform the following operation on $D'$. Put

$$\begin{cases} k_1 = -s\ell_2 + \ell_3 + s(k - s^3 - 1), \\ k_2 = -s\ell_2 + \ell_3 - (k - s^3 - 1), \\ k_3 = -(s^3 + 1)(k - s^3 - 1). \end{cases}$$

Now we multiply the first $\ell_2$ rows of $D'$ by $k_1$, the next $\ell_3$ rows by $k_2$, and the last row by $k_3$, add all rows thus obtained together to get the row $r$ and replace the last row of $D'$ by this one. One can compute that $r$ has 0 in all positions, except possibly the last one, and this last entry is equal to (after some calculations)

$$r_p = (s+1)^2\big(\ell_2^2 - (s^5 - 2s^3 + 2s^2 - s + 1)\ell_2 + s(s^2 - s + 1)^2(s^2 - 1)^2\big).$$

Since the rank of $E$ is $k$, the determinant of $D'$ must be zero, and since the determinant of $D$ is not zero, it follows that $r_p = 0$. This determines a quadratic equation in $\ell_2$ with discriminant

$$(s^5 - 2s^4 - 3s - 1)^2 - (4s^4 + 12s^3 - 4s^2 + 12) := A(s)^2 - B(s).$$

Clearly for $s = 2, 3, 4, 5$, this is not a square. For $s > 5$, we have $B(s) < (2A(s) - 1)$, and hence $(A(s)-1)^2 < A(s)^2 - B(s) < A(s)^2$. So $\ell_2$ can never be an integer, consequently Theorem 2 is proved.

## References

1. A.E. Brouwer, A.M. Cohen, and A. Neumaier, “Distance-regular graphs,” *Ergeb. Math. Grenzgeb*. **18**(3), (1989). Springer-Verlag, Berlin.
2. A.M. Cohen, “Point-line geometries related to buildings,” in *Handbook of Incidence Geometry, Buildings and Foundations*, F. Buekenhout (Ed.), North-Holland, Amsterdam, Chapter 9, 1995, pp. 647–737.
3. A.M. Cohen and J. Tits, “On generalized hexagons and a near octagon whose lines have three points,” *European J. Combin*. **6** (1985), 13–27.
4. W. Feit and G. Higman, “The nonexistence of certain generalized polygons,” *J. Algebra* **1** (1964), 114–131.
5. D.E. Frohardt and S.D. Smith, “Universal embeddings for the ${}^3D_4(2)$ hexagon and the $J_2$ near-octagon,” *European J. Combin*. **13** (1992), 455–472.
6. S.E. Payne and J.A. Thas, *Finite Generalized Quadrangles*, Pitman, Boston, 1984.
7. J.A. Thas, “Ovoids and spreads of finite classical polar spaces,” *Geom. Dedicata* **10** (1981), 135–144.
8. J. Tits, “Sur la trialité et certains groupes qui s’en déduisent,” *Inst. Hautes Études Sci. Publ. Math.* **2** (1959), 13–60.
9. H. Van Maldeghem, *Generalized Polygons*, Birkhäuser Verlag, Basel, 1998. Monographs in Mathematics, Vol. 93.