An elementary construction of the split Cayley Hexagon $\mathsf{H}(2)$

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Abstract

I present an elementary description of the smallest generalized hexagon using only the elements of a Fano projective plane. Representations of the symplectic space $\mathsf{W}_5(2)$ and of the projective space $\mathbf{PG}(5,2)$ using only elements of $\mathbf{PG}(2,2)$ emerge. As an application, I produce an embedding of the dual $\mathsf{H}(2)^D$ of the split Cayley hexagon $\mathsf{H}(2)$ in $\mathbf{PG}(7,2)$.

1 Introduction

A *generalized hexagon* Γ of order (s,t) is a rank 2 point-line geometry whose incidence graph has diameter 6 and girth 12, each vertex corresponding to a point has valency $t+1$ and each vertex corresponding to a line has valency $s+1$. These objects arise in the context of triality and were discovered by Tits [1959], who also constructed the main examples, and in fact, all known finite examples. For a general introduction, see the monograph *Generalized Polygons* by Van Maldeghem [1998] (which we abbreviate by [GP] from now on), or, emphasizing the finite case, the chapter by Thas [1995] in the Handbook of Incidence Geometry. In this note, we will use the notation and terminology of [GP] (in particular, distance between elements in Γ is the distance in the incidence graph, opposite elements are elements at maximal distance 6, collinear elements — denoted by "$\perp$" — are elements at distance 2, etc).

The finite examples known at present all have a large automorphism group, more exactly, the groups $G_2(q)$ and ${}^3D_4(q)$ arise here. The corresponding generalized hexagons are, up to duality, the geometries naturally associated with these groups and their parabolic subgroups (as groups with a BN-pair of type G_2). The hexagons

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related to $G_2(q)$ are called the *split Cayley hexagons* — denoted by $\mathsf{H}(q)$ — and we will focus on them for the moment.

Since $G_2(q)$ is a subgroup of $O_7(q)$, there is a representation of $\mathsf{H}(q)$ on the parabolic quadric $\mathsf{Q}(6,q)$ in the projective space $\mathbf{PG}(6,q)$. The points of $\mathsf{H}(q)$ are all points of $\mathsf{Q}(6,q)$ and the lines of $\mathsf{H}(q)$ are some lines of $\mathsf{Q}(6,q)$, namely those whose Grassmannian coordinates satisfy certain linear equations, see TITS [1959] for more details (or see [GP],2.4.13). If q is even, then $Q(6,q)$ has a kernel k (a point contained in every tangent hyperplane) and projecting $\mathsf{Q}(6,q)$ from k onto a hyperplane not incident with k, we obtain a representation of $\mathsf{H}(q)$ in $\mathbf{PG}(5,q)$, where the points of $\mathsf{H}(q)$ are all points of $\mathbf{PG}(5,q)$, and the lines of $\mathsf{H}(q)$ are some lines of a symplectic space $\mathsf{W}_5(q)$. This is well-known, see [GP],2.4.14.

Moreover, the group $G_2(2)$ is isomorphic to $\mathbf{PGU}_3(3)$, and this isomorphism translates into an elementary description of $\mathsf{H}(2)$ in the projective plane $\mathbf{PG}(2,9)$ (also due to TITS [1959]) as follows: the points of $\mathsf{H}(2)$ are the non-self-conjugate points of $\mathbf{PG}(2,9)$ with respect to a fixed unitary polarity, and three such points form a line if they constitute a self-conjugate triangle.

Also, PAYNE [1971] has represented $\mathsf{H}(q)$ using only elements of $\mathbf{PG}(3,q)$. In the special case of $s=2$, we will show that the Fano plane $\mathbf{PG}(2,2)$ already "contains" in an elementary way the hexagon $\mathsf{H}(2)$. As a consequence, we can also represent $\mathsf{W}_5(2)$ and $\mathbf{PG}(5,2)$ only using elements of $\mathbf{PG}(2,2)$. This will be done in the next sections. At the end of the paper, we give some motivation and background, as well as an application.

Note that the hexagon $\mathsf{H}(2)$ is, up to duality, the unique hexagon of order $(2,2)$ (see TITS [1959] or COHEN & TITS [1985]). It is not isomorphic to its dual (the dual is obtained by interchanging the roles of points and lines).

2 The construction

Let there be given the Fano plane $\mathbf{PG}(2,2)$. We define the following geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$. The set of points $\mathcal{P}$ is the set of points, lines, flags and anti flags of $\mathbf{PG}(2,2)$. Recall that a flag is an incident point-line pair, and an anti flag is a non-incident point-line pair. The elements of $\mathcal{L}$, the lines of Γ, are of two types. We define them by giving the 3 points incident with each line. For any flag $\{p,L\}$, with p a point of $\mathbf{PG}(2,2)$ and L a line of $\mathbf{PG}(2,2)$, the set $\{p, L, \{p,L\}\}$ is a line of Γ, and for every flag $\{p,L\}$ of $\mathbf{PG}(2,2)$, the set $\{\{p,L\},\{x_1,M_1\},\{x_2,M_2\}\}$ (where $\{p,x_1,x_2\}$ is the set of points of $\mathbf{PG}(2,2)$ incident with L, and dually, $\{L,M_1,M_2\}$ is the set of lines of $\mathbf{PG}(2,2)$ incident with p) is a line of Γ.

Proposition 1. *The geometry Γ is a generalized hexagon isomorphic to $\mathsf{H}(2)$.*

Proof. We define a bijection Θ between the points of $\mathsf{H}(2)$ and those of Γ and show that collinearity is preserved. Since the number of lines of $\mathsf{H}(2)$ equals the number of lines of Γ (namely, 63), it follows that also θ^{-1} preserves collinearity and hence Θ is an isomorphism.

For clarity's sake, we will denote elements of $\mathsf{H}(2)$ with lower case Greek letters, and elements of $\mathbf{PG}(2,2)$ with Latin letters.

We know that $\mathsf{H}(2)$ contains a sub geometry Δ isomorphic to a generalized hexagon of order $(1,2)$, see 1.9.10 and 2.4.15 of [GP]. Δ can be described as follows. Its points are the points and lines of $\mathbf{PG}(2,2)$, and its lines are the incident point-line pairs (hence the flags) of $\mathbf{PG}(2,2)$ with obvious incidence relation. We define Θ on the points of Δ in the natural way: a point of Δ is mapped onto to corresponding point or line of $\mathbf{PG}(2,2)$.

On each line λ of Δ, there is a unique point of $\mathsf{H}(2)$ not belonging to Δ. We map this point under θ to the flag of $\mathbf{PG}(2,2)$ corresponding with λ. Now consider an anti flag $\{p,L\}$ in $\mathbf{PG}(2,2)$. Then $\pi := \Theta^{-1}(p)$ and $\rho := \Theta^{-1}(L)$ are two opposite points of $\mathsf{H}(2)$. There are exactly three lines in $\mathsf{H}(2)$ at distance 3 from both these points, and there are exactly three points π,ρ,σ of $\mathsf{H}(2)$ at distance 3 from these three lines (by the *regulus condition*, see Ronan [1980], or in the terminology of [GP], by the *distance-3-regularity*). By definition, we put $\Theta(\sigma) := \{p,L\}$. This is well defined since it is readily seen that σ cannot be at distance 3 from at least 4 lines of Δ (see also Thas [1976] and 1.8.11 of [GP]). Since Θ is now a surjective map from the set of 63 points of $\mathsf{H}(2)$ onto a set of 63 elements (7 points, 7 lines, 21 flags and 28 anti flags of $\mathbf{PG}(2,2)$), we see that Θ is bijective.

It is clear that, if $\{p,L\}$ is a flag of $\mathbf{PG}(2,2)$, the points $\Theta^{-1}(p)$, $\Theta^{-1}(L)$ and $\Theta^{-1}(\{p,L\})$ form a line in $\mathsf{H}(2)$. It is also clear from the construction that, for any anti flag $\{p,L\}$ of $\mathbf{PG}(2,2)$, the point $\Theta^{-1}(\{p,L\})$ is collinear with the point $\Theta^{-1}(\{x,M\})$, where p is incident with M, M with x and x with L in $\mathbf{PG}(2,2)$. Hence it remains to show that, if $\{p,L\}$ is a flag of $\mathbf{PG}(2,2)$, if the set of points incident with L is p,x_1,x_2, and if the set of lines incident with p is L,M_1,M_2 (in $\mathbf{PG}(2,2)$), then the third point of $\mathsf{H}(2)$ on the line through $\Theta^{-1}(\{p,L\})$ and $\Theta^{-1}(\{x_1,M_1\})$ is $\Theta^{-1}(x_2,M_2\})$.

By definition of Θ, the only points of $\mathsf{H}(2)$ collinear with $\Theta^{-1}(\{p,L\})$ are the four points $\Theta^{-1}(\{x_i,M_j\})$, $i,j \in \{1,2\}$. Suppose by way of contradiction that $\Theta^{-1}(\{x_1,M_2\})$ is collinear with $\Theta^{-1}(\{x_1,M_1\})$. Let M be any line of $\mathbf{PG}(2,2)$ through x_1, $M \neq L$, and let y_i be the intersection of M with M_i, $i=1,2$. Then in $\mathsf{H}(2)$, we have

$$\begin{aligned}&\Theta^{-1}(M) \perp \Theta^{-1}(\{y_1,M\}) \perp \Theta^{-1}(\{x_1,M_1\}) \perp \\ &\perp \Theta^{-1}(\{x_1,M_2\}) \perp \Theta^{-1}(\{y_2,M\}) \perp \Theta^{-1}(M),\end{aligned}$$

which defines an ordinary 5-gon, contradicting the fact that the incidence graph of $\mathsf{H}(2)$ has girth 12.

The proposition is proved. $\square$

Remark. It is also easy to show directly that the geometry Γ is a generalized hexagon. Indeed, by counting the elements at distance ≤ 6 from a given element, one obtains that the diameter of the incidence graph of Γ is equal to 6. But now the girth must be 12, otherwise we cannot have 126 vertices at distance ≤ 6 from a given vertex in the incidence graph, which has valency 3.

If we define the *type* of a 3-set of $\mathcal{P}$ as the set of types of its elements, where P stands for point, L for line, F for flag, and A for anti flag, then the types of the lines of Γ are PLF and FAA.

3 The symplectic space $\mathsf{W}_5(2)$

By adding the image under Θ of the *ideal* lines (or the *hyperbolic lines*) of $\mathsf{H}(q)$ to $\mathcal{L}$, we obtain a presentation of $\mathsf{Q}(6,2)$ (see 2.4.16 of [GP]), and hence of $W_5(2)$. A hyperbolic line in $\mathsf{H}(2)$ is just a set of 3 points collinear with a fixed point π, and at distance 4 from another fixed point π' opposite π, see 6.5.1 of [GP].

It is now an elementary exercise to determine the hyperbolic lines in terms of the elements of $\mathbf{PG}(2,2)$, but it is rather tedious to write all arguments down. So we will just give the result.

Proposition 2. *Let the point-line geometry $\Gamma' = (\mathcal{P}, \mathcal{L}', \mathbf{I}\,)$ be defined as follows. The point set $\mathcal{P}$ is, as above, the set of points, lines, flags and anti flags of $\mathbf{PG}(2,2)$. The elements of $\mathcal{L}'$ are the elements of $\mathcal{L}$ completed with the following 3-sets (and for the convenience of the reader we give the type of each 3-set):*

(i) $\{x_1, x_2, x_3\}$, *with x_1, x_2, x_3 three elements incident with the same element of $\mathbf{PG}(2,2)$. There are 14 such sets, and they are precisely the point rows and line pencils of $\mathbf{PG}(2,2)$. They have types* PPP *and* LLL.

(ii) $\{x_1, \{x_2, y\}, \{x_3, y\}\}$, *with x_1, x_2, x_3 the three elements incident with the element y. There are 42 such sets. They have types* PFF *and* LFF.

(iii) $\{\{p_1, L_1\}, \{p_2, L_2\}, \{p_3, L_3\}\}$, *with p_1, p_2, p_3 three collinear points of $\mathbf{PG}(2,2)$, with L_1, L_2, L_3 three concurrent lines of $\mathbf{PG}(2,2)$, and with p_i incident with L_i, $i = 1, 2, 3$. There are 28 such sets. They have type* FFF.

(*iv*) $\{x_1, \{x_2, y\}, \{x_3, y\}\}$, *with* x_1, x_2, x_3 *three elements incident with a common element* z *of* $\mathbf{PG}(2,2)$, *and with* $y \neq z$ *an element incident with* x_1. *There are* 84 *such sets. They have types* PAA *and* LAA.

(*v*) $\{\{p_1, L_1\}, \{p_2, L_3\}, \{p_3, L_2\}$, *with* $p_1, p_2, p_3, L_1, L_2, L_3$ *as in* (*iii*) *above. There are* 84 *such sets. They have type* FAA.

Then Γ' *is isomorphic to the point-line geometry of* $\mathsf{W}_5(2)$.

Note that, indeed, $\mathsf{W}_5(2)$ has $315 = 63 + (14 + 42 + 28 + 84 + 84)$ lines.

4 The projective space $\mathbf{PG}(5,2)$

By adjoining the images under Θ of the *imaginary lines* of $\mathsf{H}(2)$, see 6.5.5 of [GP], to Γ', we obtain a geometry Γ'' which is isomorphic to the point-line geometry of $\mathbf{PG}(5,2)$. An imaginary line is the set of points at distance 3 from two opposite lines of $\mathsf{H}(2)$. The proof of the next proposition is again elementary but tedious to write down. We omit it, but the reader can easily do it for himself.

Proposition 3. *Let the point-line geometry* $\Gamma'' = (\mathcal{P}, \mathcal{L}'', \mathbf{I}\,)$ *be defined as follows. The point set* $\mathcal{P}$ *is, as above, the set of points, lines, flags and anti flags of* $\mathbf{PG}(2,2)$. *The elements of* $\mathcal{L}''$ *are the elements of* $\mathcal{L}'$ *completed with the following* 3*-sets:*

(*a*) $\{p, L, \{p, L\}\}$, *with* $\{p, L\}$ *an anti flag of* $\mathbf{PG}(2,2)$. *There are* 28 *such sets. They have type* PLA.

(*b*) $\{\{p_1, L_2\}, \{p_2, L_3\}, \{p_3, L_1\}$, *with* $p_1, p_2, p_3, L_1, L_2, L_3$ *as in* (*iii*) *above (see Proposition* 2*). There are* 56 *such sets. They have type* AAA.

(*c*) $\{\{p, L\}, \{q_1, M_1\}, \{q_2, M_2\}\}$, *with* M_1 *a line of* $\mathbf{PG}(2,2)$ *containing the three (distinct) points* p, q_1, q_2, *and* q_2 *a point incident with the three (distinct) lines* L, M_1, M_2. *There are* 84 *such sets. They have type* FFA

(*d*) $\{x_1, \{x_2, y\}, \{x_3, y\}\}$, *with* x_1, x_2, x_3 *three elements incident with a common element* z, *and* $y \neq z$ *is incident with* x_3. *There are* 168 *such sets. They have types* PFA *and* LFA.

5 Motivation and Remarks

The main motivation for the construction of $\mathsf{H}(2)$ in this paper is the simplicity of that construction. It can be given in any introductory lecture about generalized polygons, since the Fano projective plane is very easily introduced by a picture. There has lately been some interest in pictures of the hexagon $\mathsf{H}(2)$ and its dual. The pictures of SCHROTH [19**] are very nice, but slightly too complicated to serve as a definition of $\mathsf{H}(2)$. POLSTER [1998], on the other hand, makes pictures that can really be used as a definition of certain geometries. His idea is to let one group act on different configurations (defining the various elements and incidence of the geometry) given by pictures, and this way generate the full geometry. In fact, what he does is encode the full geometry in a small number of pictures. This is also the exercise he carries out in POLSTER [19**] to construct $\mathsf{H}(2)$. Unfortunately, he does not arrive at the construction of the present paper, but at a rather more complicated one (there are also four types of points, vaguely related to the Fano plane, or, more precisely, to a pair of Fano planes with common point set, so that only a dihedral group of order 14 acts on his set of points instead of the group Aut $\mathbf{PGL}_3(2) \cong \mathbf{PGL}_2(7)$ as in our case, and there are six types of lines; moreover, to obtain $\mathsf{W}_5(2)$, he has to define an additional 36 types of lines; not less than a total of 93 different types of lines are needed to define $\mathbf{PG}(5,2)$!).

Some special substructures of $\mathsf{H}(2)$, of $\mathsf{W}_5(2)$ or $\mathbf{PG}(5,2)$ can be seen in the representation we gave. For instance, there is a unique line of Γ through every point of type F. Hence the points of type F define a partition of $\mathcal{L}$. Deleting the points of type F, we see that there remain the points of type P and L, with collinearity defining the incidence graph of $\mathbf{PG}(2,2)$, and the points of type A, where collinearity precisely defines the Coxeter graph, as can be easily seen (see e.g. 12.3 of BROUWER, COHEN & NEUMAIER [1989] for this definition of Coxeter graph). This is also remarked by POLSTER [19**], although in his representation, the Coxeter graph is slightly harder to identify. Also, the thirty five points of type P, L and F form the point set of a Klein quadric in $\mathbf{PG}(5,2)$ (or rather in Γ''). Every passant (non-intersecting line) is of type AAA, and it is easily seen that they are all equivalent under the group $\mathbf{PGL}_3(2)$ of $\mathbf{PG}(2,2)$ (which of course acts on Γ, Γ' and Γ'').

We leave it to the reader to discover other side-accidents of this construction.

We content ourselves by mentioning one further application in the next section.

6 Application: an embedding of $\mathsf{H}(2)^D$ in $\mathbf{PG}(7,2)$

We now define a mapping Ψ from the set of lines of $\mathsf{H}(2)$ to the set of points of $\mathbf{PG}(7,2)$. Therefore, we view $\mathbf{PG}(7,2)$ as the hyperplane in $\mathbf{PG}(8,2)$ (where we denote a general point by the 9-tuple $(x_{00}, x_{01}, x_{02}, x_{10}, x_{11}, x_{12}, x_{20}, x_{21}, x_{22})$) with equation $X_{00} + X_{11} + X_{22} = 0$. We identify the lines of $\mathsf{H}(2)$ with the sets of flags, points, lines and anti-flags of $\mathbf{PG}(2,2)$ given in Section 2 above. We consider any coordinatization of $\mathbf{PG}(2,2)$ and denote the points with parentheses and the lines with square brackets. For any point-line pair $\{p,L\}$ of $\mathbf{PG}(2,2)$, with $p=(x_0,x_1,x_2)$ and $L=[a_0,a_1,a_2]$, $a_i,x_i\in\mathbf{GF}(2)$, $i=1,2,3$, we define the point $\{p,L\}^\theta$ of $\mathbf{PG}(8,2)$ by

$$(a_0x_0,a_1x_0,a_2x_0,a_0x_1,a_1x_1,a_2x_1,a_0x_2,a_1x_2,a_2x_2).$$

Now for any flag $\{p,L\}$ of $\mathbf{PG}(2,2)$, we define

$$\{p,L,\{p,L\}\}^\Psi=\{p,L\}^\theta.$$

Also, for the line $\{\{p,L\},\{x_1,M_1\},\{x_2,M_2\}\}$ (where $\{p,x_1,x_2\}$ is the set of points of $\mathbf{PG}(2,2)$ incident with L, and dually, $\{L,M_1,M_2\}$ is the set of lines of $\mathbf{PG}(2,2)$ incident with p) of $\mathsf{H}(2)$, we define $\{\{p,L\},\{x_1,M_1\},\{x_2,M_2\}\}^\Psi$ as the nucleus of the conic with point set

$$\{\{p,L\}^\theta,\{x_1,M_1\}^\theta,\{x_2,M_2\}^\theta\}.$$

Equivalently, this point is equal to the coordinate wise addition

$$\{p,L\}^\theta+\{x_1,M_1\}^\theta+\{x_2,M_2\}^\theta.$$

It is now an elementary exercise to see that $\mathcal{L}^\Psi$ is contained in $\mathbf{PG}(7,2)$. Moreover, one can check that concurrent lines in $\mathsf{H}(2)$ are mapped under Ψ to collinear points (the calculations boil down to checking this only once for each of the four types of sets of concurrent lines — corresponding with the four types of points of $\mathsf{H}(2)$ — using the action of the group $\mathbf{PGL}_3(2)$ acting with four orbits on the embedding). Hence we obtain a full embedding Γ of the dual of $\mathsf{H}(2)$ in $\mathbf{PG}(7,2)$.

We claim that the full automorphism group of the embedding is the full automorphism group of $\mathbf{PGL}_3(2)$. Indeed, clearly, this group acts on the embedding. But the images of the two types of lines of $\mathsf{H}(2)$ form two orbits under the automorphism group of the embedding because the lines of $\mathsf{H}(2)$ concurrent with a line of the form $\{p,L,\{p,L\}$, with $\{p,L\}$ a flag of $\mathbf{PG}(2,2)$, are mapped under Ψ to a set of 6 coplanar points, while this is not true for the other lines of $\mathsf{H}(2)$. The claim now follows.

Further, it is easy to check that the points x corresponding with lines of $\mathsf{H}(2)$ of type $\{p,L,\{p,L\}\}$, with $\{p,L\}$ a flag of $\mathbf{PG}(2,2)$, have the following property: the set of points of Γ collinear with x is contained in a plane of $\mathbf{PG}(7,2)$ and the set of points of Γ not opposite x is contained in a hyperplane of $\mathbf{PG}(7,2)$. None of these is true for the other points of Γ.

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